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Inequalities for J-Hermitian matrices

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Abstract

Indefinite versions of classical results of Schur, Ky Fan and Rayleigh-Ritz on Hermitian matrices are stated to J-Hermitian matrices, $J = I_r \oplus -I_{n-r},\ 0 < r < n$. Spectral inequalities for the trace of the product of J-Hermitian matrices are presented. The inequalities are obtained in the context of the theory of numerical ranges of linear operators on indefinite inner product spaces.

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1. Introduction

The comparison of two vectors often leads to interesting inequalities that can be concisely expressed as *majorization* relations. Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ have the entries arranged in non-increasing order $x_1 \geqslant \cdots \geqslant x_n$, $y_1 \geqslant \cdots \geqslant y_n$. The vector x is said to be *majorized* by y, in symbols $x \prec y$, if

$$\sum_{j=1}^{k} x_j \leqslant \sum_{j=1}^{k} y_j,$$

for all $k = 1, \dots, n$ with equality for $k = n$. The first example of majorization in the history of matrix analysis is the famous *Theorem of Schur* (1923) [5, p. 193], which asserts that the vector of diagonal entries of a Hermitian matrix A is majorized by the vector of the eigenvalues of A.

Let M_n be the algebra of $n \times n$ complex matrices, and let H_n be the real space of $n \times n$ Hermitian matrices. It is well-known that the solutions of several optimization and variational problems are given in terms of the eigenvalues of Hermitian matrices. For $A \in H_n$, the physicists Rayleigh and Ritz [5, p. 176] proved that

$$\alpha_1 = \max_{x^*x=1} (x^* A x), \quad \alpha_n = \min_{x^*x=1} (x^* A x), \qquad x \in \mathbb{C}^n,$$

where α_1 and α_n are the largest and the smallest eigenvalue of A, respectively. Another famous variational result for Hermitian matrices is *Ky Fan's Maximum Principle* (1950) [6, p. 511], which establishes that

$$\max_{\Lambda} \sum_{j=1}^{k} x_j^* A x_j = \sum_{j=1}^{k} \alpha_j, \quad k = 1, \dots, n,$$

where $\alpha_1 \geqslant \cdots \geqslant \alpha_n$ are the eigenvalues of A and Λ is the set of the first k columns $x_1, \dots, x_k$ of an $n \times n$ unitary matrix. Ky Fan's Maximum Principle is a source of inspiration, often used as a fundamental tool for obtaining several results. For instance, Schur's Theorem can be easily derived from it.

Given a Hermitian involutive matrix J, that is, $J^* = J$, $J^2 = I_n$, let us consider $\mathbb{C}^n$ endowed with the indefinite inner product induced by J:

$$[x, y] = y^* J x, \quad x, y \in \mathbb{C}^n.$$

For a matrix $A \in M_n$, its J-adjoint $A^\#$ is defined by

$$[Ax, y] = [x, A^\# y], \quad x, y \in \mathbb{C}^n,$$

or equivalently, $A^\# = J A^* J$. A matrix $A \in M_n$ is said to be *J-Hermitian* if $A = A^\#$. A matrix $U \in M_n$ is said to be *J-unitary* if $UU^\# = U^\# U = I_n$. For a Hermitian involutive matrix J with *signature* $(r, n-r)$, $0 < r < n$ (that is, with r positive and $n - r$ negative eigenvalues), the J-unitary matrices form a non-compact group denoted by $\mathcal{U}_{r,n-r}$ and called the *J-unitary group*.

Our aim is the investigation of spectral inequalities for J-Hermitian matrices. We recall that the spectrum of a J-Hermitian matrix $A \in M_n$ is symmetric relatively to the real axis. In this vein, Ando [1] recently obtained a Löwner inequality of indefinite type. In this paper, indefinite type versions of Ky Fan's Maximum Principle, Rayleigh-Ritz Theorem, and Schur's Theorem are presented in Theorem 3.1, Corollary 3.2 and Theorem 3.3, respectively. These results will be derived from Theorem 1.1, whose Corollary 1.2 may be thought as an indefinite version of the following spectral tracial inequalities obtained by Richter [9]. For Hermitian matrices A and C with prescribed spectra $\alpha_1 \geqslant \cdots \geqslant \alpha_n$ and $c_1 \geqslant \cdots \geqslant c_n$, respectively, Richter proved that (cfr. the alternative proofs of Mirsky [7] and Theobald [10]):

$$\sum_{i=1}^{n} c_i \alpha_{n-i+1} \leqslant \operatorname{Tr}(CA) \leqslant \sum_{i=1}^{n} c_i \alpha_i. \tag{1}$$

Given an $n \times n$ Hermitian involutive matrix J and $A, C \in M_n$ consider the set of complex numbers denoted and defined by

$$W_C^J(A) = \{\operatorname{Tr}(CU^{-1}AU) : U \in M_n, U^*JU = J\} \tag{2}$$

called the J, C*-numerical range* of A.

From (2), it follows that $W_C^J(A) = W_A^J(C)$, that is, the roles of A and C are symmetric. Without loss of generality, in (2) we may consider $J = I_r \oplus -I_{n-r}$, r being the number of positive eigenvalues of J. Since $\mathscr{U}_{r,n-r}$ is connected and $W_C^J(A)$ is the range of the continuous map from $\mathscr{U}_{r,n-r}$ to $\mathbb{C}$ defined by $U \mapsto \operatorname{Tr}(CU^{-1}AU)$, $W_C^J(A)$ is a connected set in the complex plane, for all $A, C \in M_n$. For any $U \in \mathscr{U}_{r,n-r}$, $W_C^J(A) = W_C^J(U^{-1}AU)$.

Let $A \in M_n$ and let C be a J-Hermitian and J-unitarily diagonalizable matrix with eigenvalues $c_1, \ldots, c_n$. For $J = I_r \oplus -I_{n-r} = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, it can be seen that (2) may be written as

$$W_C^J(A) = \left\{ \sum_{i=1}^{r} c_i[Ax_i, x_i] - \sum_{i=r+1}^{n} c_i[Ax_i, x_i],\ x_i \in \mathbb{C}^n, \right. \\ \left. [x_i, x_l] = \delta_{il}\varepsilon_i,\ i, l = 1, \ldots, n \right\}. \tag{3}$$

If A is a J-Hermitian matrix, then $W_C^J(A)$ is a connected subset of the real line (cf. [2]).

We denote by $\sigma_J^{\pm}(A)$ the sets of the eigenvalues of A with eigenvectors x such that $x^*Jx = \pm 1$. We recall that a J-Hermitian matrix A is J-unitarily diagonalizable if and only if every eigenvalue of A belongs either to $\sigma_J^+(A)$ or to $\sigma_J^-(A)$. In this case, $\sigma_J^+(A)$ (respectively, $\sigma_J^-(A)$) consists of r (respectively, $n - r$) eigenvalues.

Let A be a J-Hermitian matrix whose eigenvalues $\alpha_1 \geqslant \cdots \geqslant \alpha_r$ belong to $\sigma_J^+(A)$ and $\alpha_{r+1} \geqslant \cdots \geqslant \alpha_n$ belong to $\sigma_J^-(A)$. The eigenvalues of A are said *to not interlace* if either $\alpha_r > \alpha_{r+1}$ or $\alpha_n > \alpha_1$. Otherwise, they are said *to interlace*.

Before the statement of Theorem 1.1, some observations are in order. If the eigenvalues of A and the eigenvalues of C do not interlace, then the following four possibilities may occur: (i) $\alpha_r > \alpha_{r+1}$ and $c_r > c_{r+1}$, (ii) $\alpha_r > \alpha_{r+1}$ and $c_n > c_1$, (iii) $\alpha_n > \alpha_1$ and $c_r > c_{r+1}$, (iv) $\alpha_n > \alpha_1$ and $c_n > c_1$. Then $(\alpha_k - \alpha_l)(c_{k'} - c_{l'}) < 0$ for all $1 \leqslant k, k' \leqslant r, r+1 \leqslant l, l' \leqslant n$ if and only if (ii) or (iii) occurs. In the same way, $(\alpha_k - \alpha_l)(c_{k'} - c_{l'}) > 0$ for all $1 \leqslant k, k' \leqslant r$, $r+1 \leqslant l, l' \leqslant n$ if and only if (i) or (iv) occurs.

Our main result is the following theorem.

Theorem 1.1. *Let $J = I_r \oplus -I_{n-r}$, $0 < r < n$, and let A, C be non-scalar J-Hermitian and J-unitarily diagonalizable matrices with eigenvalues α_i, c_i, $i = 1, \dots, n$, respectively. Let $\alpha_1 \geqslant \cdots \geqslant \alpha_r$ $(c_1 \geqslant \cdots \geqslant c_r)$ belong to $\sigma_J^+(A)$ $(\sigma_J^+(C))$ and let $\alpha_{r+1} \geqslant \cdots \geqslant \alpha_n$ $(c_{r+1} \geqslant \cdots \geqslant c_n)$ belong to $\sigma_J^-(A)$ $(\sigma_J^-(C))$. If the eigenvalues of A and the eigenvalues of C do not interlace, the statements* (i) *and* (ii) *hold*:

(i) $W_C^J(A) = \left(-\infty, \sum_{i=1}^n c_i\alpha_i\right]$ *if and only if* $(\alpha_k - \alpha_l)(c_{k'} - c_{l'}) < 0$, *for all* $1 \leqslant k, k' \leqslant r, r+1 \leqslant l, l' \leqslant n$.

(ii) $W_C^J(A) = \left[\sum_{i=1}^r c_i\alpha_{r-i+1} + \sum_{i=r+1}^n c_i\alpha_{n+r-i+1}, +\infty\right)$ *if and only if* $(\alpha_k - \alpha_l)(c_{k'} - c_{l'}) > 0$, *for all* $1 \leqslant k, k' \leqslant r, r+1 \leqslant l, l' \leqslant n$.

(iii) *If either the eigenvalues of A interlace and $\alpha_r \neq \alpha_{r+1}$, $\alpha_1 \neq \alpha_n$ or the eigenvalues of C interlace and $c_r \neq c_{r+1}$, $c_1 \neq c_n$, then $W_C^J(A)$ is the whole real line*.

As will be shown in Theorem 2.1, the converse of (iii) in Theorem 1.1 does not hold.

Corollary 1.2. *Under the same assumptions of Theorem 1.1 on J, A, C and assuming that the eigenvalues of A and C do not interlace, the statements* (i) *and* (ii) *hold*:

(i) *If* $(\alpha_k - \alpha_l)(c_{k'} - c_{l'}) < 0$ *for all* $1 \leqslant k, k' \leqslant r, r+1 \leqslant l, l' \leqslant n$, *then*

$$\operatorname{Tr}(CA) \leqslant \sum_{i=1}^n c_i\alpha_i.$$

(ii) *If* $(\alpha_k - \alpha_l)(c_{k'} - c_{l'}) > 0$ *for all* $1 \leqslant k, k' \leqslant r, r+1 \leqslant l, l' \leqslant n$, *then*

$$\sum_{i=1}^r c_i\alpha_{r-i+1} + \sum_{i=r+1}^n c_i\alpha_{n+r-i+1} \leqslant \operatorname{Tr}(CA).$$

2. Proof of Theorem 1.1

We present some lemmas needed for the proof of Theorem 1.1.

Lemma 2.1 [2]. *Let $A \in M_n$ and $J = I_r \oplus -I_{n-r}$, $0 < r < n$. Suppose that $C = c_1 I_{n_1} \oplus \cdots \oplus c_p I_{n_p} \in M_n$, $n_1 + \cdots + n_p = n$, and that $c_1, \ldots, c_p$ are distinct. If $z = \operatorname{Tr}(CU^{-1}AU)$, $U \in \mathscr{U}_{r,n-r}$, is a corner of $W_C^J(A)$, that is, if z is a boundary point of $W_C^J(A)$ and there exists a sufficiently small $\epsilon > 0$ such that the intersection of $W_C^J(A)$ and the circular disc $\mathscr{D} = \{\upsilon \in \mathbb{C} : |\upsilon - z| < \epsilon\}$ is contained in a sector of $\mathscr{D}$ of degree less than π, then $U^{-1}AU = A_1 \oplus \cdots \oplus A_p$, where $A_i \in M_{n_i}$, $i = 1, \ldots, p$, and $z = \sum_{i=1}^{p} c_i \operatorname{Tr}(A_i)$.*

Let S_n be the symmetric group of degree n, and let

$$S_n^r = \{\sigma \in S_n : \sigma(j) = j,\ j = r+1, \ldots, n\},$$
$$\widehat{S}_n^{n-r} = \{\sigma \in S_n : \sigma(j) = j,\ j = 1, \ldots, r\}.$$

Lemma 2.2. *Let $J = I_r \oplus -I_{n-r}$, $0 < r < n$. Let C be a diagonal matrix with principal entries $c_1 \geqslant \cdots \geqslant c_r > c_{r+1} \geqslant \cdots \geqslant c_n$ and let A be a J-Hermitian matrix. If $z \in W_C^J(A)$ is a corner of $W_C^J(A)$, then all eigenvalues $\alpha_1, \ldots, \alpha_n$ of A are real and there is a permutation $\sigma_1 \in S_n^r$ and $\sigma_2 \in \widehat{S}_n^{n-r}$ such that*

$$z = z_{\sigma_1\sigma_2} = \sum_{i=1}^{r} c_i \alpha_{\sigma_1(i)} + \sum_{i=r+1}^{n} c_i \alpha_{\sigma_2(i)}. \tag{4}$$

Proof. Write

$$c_1 = \cdots = c_{k_1} > c_{k_1+1} = \cdots = c_{k_2} > \cdots > c_{k_{p-1}+1} = \cdots = c_n.$$

Then

$$C = \bigoplus_{j=1}^{p} c_{k_j} I_{n_j}, \quad n_j = k_j - k_{j-1} \qquad (k_0 = 0, k_p = n).$$

By Lemma 2.1, $U^{-1}AU = A_1 \oplus \cdots \oplus A_p$ with $A_i \in M_{n_i}$, $i = 1 \ldots, p$, and $n_1 + \cdots + n_p = n$. Since $n_1 + \cdots + n_k = r$, for some k, each A_i is Hermitian. There are unitary matrices S_i such that $S_i^{-1} A_i S_i$ is a diagonal Hermitian matrix. Therefore, all the eigenvalues $\alpha_1, \ldots, \alpha_n$ of A are real. Let $V = S_1 \oplus \cdots \oplus S_p$. Then V is unitary as well as J-unitary and

$$V^{-1}U^{-1}AUV = P_\sigma^{\mathrm{T}} \operatorname{diag}(\alpha_1, \ldots, \alpha_n) P_\sigma,$$

for P_σ the permutation matrix associated with $\sigma = \sigma_1\sigma_2$, $\sigma_1 \in S_n^r$ and $\sigma_2 \in \widehat{S}_n^{n-r}$. By Lemma 2.1, we obtain (4). □

Lemma 2.3. *Let $J = I_1 \oplus -I_1$, and let $C, A \in M_2$ be non-scalar and J-Hermitian with eigenvalues c_1, c_2 and α_1, α_2, respectively. Suppose that C, A are J-unitarily diagonalizable. Then*:

(i) $W_C^J(A) = (-\infty, \alpha_1 c_1 + \alpha_2 c_2]$ *if and only if* $(\alpha_1 - \alpha_2)(c_1 - c_2) < 0$;
(ii) $W_C^J(A) = [\alpha_1 c_1 + \alpha_2 c_2, +\infty)$ *if and only if* $(\alpha_1 - \alpha_2)(c_1 - c_2) > 0$.

Proof. The matrix C is J-unitarily diagonalizable, therefore we may assume, without loss of generality, $C = \mathrm{diag}(c_1, c_2)$. Since $C = (c_1 - c_2)E_{11} + c_2 I_2$, we clearly have

$$W_C^J(A) = (c_1 - c_2)W_{E_{11}}^J(A) + c_2\mathrm{Tr}(A),$$

where $E_{11} = \mathrm{diag}(1, 0)$. The result follows from the Hyperbolical Range Theorem [2]. □

In the sequel, $A[kl]$ denotes the submatrix of A lying in rows and columns k, l.

Lemma 2.4. *Let* $J = I_r \oplus -I_{n-r}$, $0 < r < n$, *and let* $C \in M_n$ *be a non-scalar diagonal matrix. Given a J-unitarily diagonalizable matrix* $A \in M_n$, $W_C^J(A)$ *is a singleton if and only if A is a scalar matrix.*

Proof. The implication ($\Leftarrow$) is obvious.

($\Rightarrow$) (*By contradiction.*) Since $A \in M_n$ is J-unitarily diagonalizable, we may consider $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$. If A, C are non-scalar matrices, it is possible to find integers k and l, $1 \leqslant k < l \leqslant n$, such that $A' = A[kl]$ and $C' = C[kl]$ are non-scalar matrices. If $J' = J[kl] = I_1 \oplus -I_1$, then $k \leqslant r < r + 1 \leqslant l$. Obviously, $W_C^J(A)$ contains the subset $\Gamma = W_{C'}^{J'}(A') + \sum_{i \neq k,l} c_i\alpha_i$. By Lemma 2.3, Γ does not reduce to a point, contradicting the hypothesis. If $J' = I_2$, then $k < l \leqslant r$. By the Elliptical Range Theorem [4], the subset Γ of $W_C^J(A)$ is an elliptical disc possibly degenerate but, under our assumptions, never a point, a contradiction. If $J' = -I_2$, then $r + 1 \leqslant k < l$, and this case can be analogously treated. □

The proof of Lemma 2.5 is an adaptation of the proof of Proposition 3.1 in [8]. We start by fixing some notation.

Consider the affine space

$$A_{(n-1)^2} = \left\{ A = (a_{ij}) \in M_n(\mathbb{R}) : \sum_{i=1}^{n} a_{iq} = \sum_{j=1}^{n} a_{pj} = 1, 1 \leqslant p, q \leqslant n \right\}.$$

For $J = I_r \oplus -I_{n-r} = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, define the set of all J-doubly stochastic matrices

$$D_J(r, n-r) = \left\{ A = (a_{i\ell}) \in A_{(n-1)^2} : \ a_{i\ell}\varepsilon_i\varepsilon_\ell \geqslant 0, \ 1 \leqslant i, \ell \leqslant n \right\}.$$

This convex set is a subset of the closed convex cone

$$\widetilde{D}_J(r, n-r) = \{A = (a_{i\ell}) \in M_n(\mathbb{R}) : \ a_{i\ell}\varepsilon_i\varepsilon_\ell \geqslant 0, \ 1 \leqslant i, \ell \leqslant n\}.$$

Denote by $\Omega_J(r, n-r)$ the set of all J-*orthostochastic* matrices of size $n \times n$, that is, the set of matrices $T = (t_{ik}) \in D_J(r, n-r)$ defined by

$$t_{ik} = \varepsilon_i \varepsilon_k |u_{ik}|^2 = \varepsilon_i \varepsilon_k u_{ik} \overline{u_{ik}}, \quad 1 \leqslant i, k \leqslant n,$$

for $U = (u_{ik}) \in \mathscr{U}_{r,n-r}$.

Lemma 2.5. *Let* $J = I_r \oplus -I_{n-r} = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, $n \geqslant 2, 0 < r < n$, *and consider* $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$, $A = \mathrm{diag}(a_1, a_2, \ldots, a_n) \in M_n(\mathbb{R})$. *If there exists* $\beta > 0$ *for which*

$$c_k a_\ell \varepsilon_k \varepsilon_\ell \geqslant \beta, \quad 1 \leqslant k, \ell \leqslant n, \tag{5}$$

that is, if the convex hull of the n^2 *points* $c_k a_\ell \varepsilon_k \varepsilon_\ell$ $(k, \ell = 1, 2, \ldots, n)$ *is contained in the open positive half-axis, then* $W_C^J(A)$ *is a closed half-line in* $\mathbb{R}$.

Proof. Let $n \geqslant 2$ and $0 < r < n$. Consider the affine functional $\Phi : A_{(n-1)^2} \to \mathbb{C}$ defined by

$$\Phi(b_{ij}) = \sum_{i,j=1}^{n} c_i a_j b_{ij}.$$

If $B = (b_{kl}) \in D_J(r, n-r)$ and (5) is satisfied, then

$$\beta \sum_{k,\ell=1}^{n} b_{k\ell} \varepsilon_k \varepsilon_\ell \leqslant \Phi(B) \leqslant \max_{1 \leqslant p,q \leqslant n} c_q a_p \varepsilon_p \varepsilon_q \sum_{k,\ell=1}^{n} b_{k\ell} \varepsilon_k \varepsilon_\ell.$$

For every constant $M > 0$, the set

$$\left\{ B = (b_{k\ell}) \in D_J(r, n-r) : \sum_{k,\ell=1}^{n} b_{k\ell} \varepsilon_k \varepsilon_\ell \leqslant M \right\},$$

as well as its subset of J-orthostochastic matrices

$$\left\{ B = (b_{k\ell}) \in \Omega_J(r, n-r) : \sum_{k,\ell=1}^{n} b_{k\ell} \varepsilon_k \varepsilon_\ell \leqslant M \right\}, \tag{6}$$

are compact. We have

$$W_C^J(A) \subset \left\{ \Phi(B) : B \in \widetilde{D}_J(n-r, r) \right\} \subset [0, +\infty),$$

because if $U = (u_{ij})$ is J-unitary, then $U^{-1} = JU^*J = (\varepsilon_k \varepsilon_l \bar{u}_{lk})_{k,l}$ so that

$$\mathrm{Tr}(CU^{-1}AU) = \sum_{k,l=1}^{n} c_k a_l \varepsilon_k \varepsilon_l |u_{kl}|^2.$$

Let $(z_n)_{n=1}^{\infty}$ be an arbitrary sequence of points of $W_C^J(A)$ satisfying $z_n \to z_\infty \in \mathbb{R}$ as $n \to \infty$. Then, there exists $M_0 > 0$ such that

$$\{z_n : n = 1, 2, 3, \ldots\} \subset [0, M_0].$$

We set $M = M_0/\beta$. Hence, there exist J-orthostochastic matrices $B^{(n)} = (b_{k\ell}^{(n)})$ for which

$$z_n = \Phi(B^{(n)}), \qquad \sum_{k,\ell=1}^{n-1} b_{k\ell}^{(n)} \varepsilon_k \varepsilon_\ell \leqslant M.$$

By the compactness of (6), we can choose a subsequence n_k ($k = 1, 2, 3, \ldots$) for which

$$B^{(n_k)} \to B^{(\infty)}$$

as $k \to \infty$, for some J-orthostochastic matrix $B^{(\infty)}$, and so

$$z_\infty = \Phi(B^{(\infty)}).$$

Thus, $W_C^J(A)$ is a closed subset of $[0, +\infty)$. As a consequence of (5), A, C are non-scalar matrices and so, by Lemma 2.4, $W_C^J(A)$ does not reduce to a point. Since $W_C^J(A)$ is connected and unbounded, it must be a closed half-line and the proof is complete. □

Remark. If there exists $\beta < 0$ such that

$$c_k a_\ell \varepsilon_k \varepsilon_\ell \leqslant \beta, \quad 1 \leqslant k, \ell \leqslant n, \tag{7}$$

Lemma 2.5 is also valid.

Proof of Theorem 1.1. (i) ($\Leftarrow$) The J-Hermitian and J-unitarily diagonalizable matrices A, C may be assumed in diagonal form, say $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n)$ and $C = \mathrm{diag}(c_1, \ldots, c_r, c_{r+1}, \ldots, c_n)$. Since A and C are non-scalar matrices, Lemma 2.4 guarantees that $W_C^J(A)$, which is a connected subset of the real line, is not a singleton. Since the eigenvalues of A and the eigenvalues of C do not interlace, there exist $\alpha, c \in \mathbb{C}$ such that $\alpha'_p = \alpha_p + \alpha$, $c'_p = c_p + c$, $p = 1, \ldots, n$, satisfy (5) or (7). For instance, choose α such that $\alpha_r > -\alpha > \alpha_{r+1}$ if $\alpha_r > \alpha_{r+1}$ and $\alpha_n > -\alpha > \alpha_1$. Therefore, Lemma 2.5 ensures that $W_{C+cI}^J(A + \alpha I)$ is a closed half-line in $\mathbb{R}$. Having in mind that

$$W_{C+cI}^J(A + \alpha I) = W_C^J(A) + \alpha \mathrm{Tr}(C) + c\mathrm{Tr}(A) + n\alpha c,$$

$W_C^J(A)$ is also a closed half-line contained in $\mathbb{R}$.

Let $\widetilde{C} = C[1n]$, $\widetilde{A} = A[1n]$ and $\widetilde{J} = J[1n]$. Then

$$W_{\widetilde{C}}^{\widetilde{J}}(\widetilde{A}) + \sum_{j=2}^{n-1} c_j \alpha_j \subset W_C^J(A).$$

Assume that $(\alpha_k - \alpha_l)(c_{k'} - c_{l'}) < 0$, $1 \leqslant k, k' \leqslant r$, $r + 1 \leqslant l, l' \leqslant n$. This implies that $(\alpha_1 - \alpha_n)(c_1 - c_n) < 0$ and, by Lemma 2.3, $W_{\widetilde{C}}^{\widetilde{J}}(\widetilde{A})$ contains the half-line unbounded below generated by the $\widetilde{J}$-unitary subgroup

$$\left\{ \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}.$$

Thus, $W_C^J(A) = (-\infty, z]$ for a certain real number z. Obviously, the extremum z of the half-line is a corner of $W_C^J(A)$. By the hypothesis, $c_r \neq c_{r+1}$. If $c_r < c_{r+1}$, the condition $(\alpha_r - \alpha_{r+1})(c_r - c_{r+1}) < 0$ implies $\alpha_r > \alpha_{r+1}$. Since $W_C^J(A) = W_A^J(C)$, we can exchange the roles of C and A, and so we can assume $c_1 \geqslant \cdots \geqslant c_r > c_{r+1} \geqslant \cdots \geqslant c_n$. Now, from Lemma 2.2, it follows that z is a σ-point, z_σ, of type (4). The case $c_r > c_{r+1}$ may be treated similarly.

Suppose that $\sigma_1(i) = i$, $i = 1, \dots, l-1$, $\sigma_1(l) \neq l$ and consider $l < k \leqslant r$ such that $\sigma_1(k) = l$. We have

$$\sum_{\substack{i=1\\ i\neq k,l}}^{r} c_i\alpha_{\sigma_1(i)} + c_l\alpha_l + c_k\alpha_{\sigma_1(l)} + \sum_{i=r+1}^{n} c_i\alpha_{\sigma_2(i)} - z_\sigma$$
$$= (c_l - c_k)(\alpha_l - \alpha_{\sigma_1(l)}) \geqslant 0, \tag{8}$$

because $k > l$ and $\sigma_1(l) > l$. Let $\xi \in S_n$ be such that $\xi(l) = l$, $\xi(k) = \sigma_1(l)$, $\xi(j) = \sigma_1(j)$, for $1 \leqslant j \leqslant r$, $j \neq k$, $j \neq l$, and $\xi(j) = \sigma_2(j)$, $j = r+1, \dots, n$. If the equality does not hold in (8), then $z_\xi > z_\sigma$, a contradiction, since z_σ is the maximum of $W_C^J(A)$. Therefore, the equality in (8) holds and the point z_ξ is also the maximum. Hence, we can take ξ as new σ_1 in (8). Repeating this argument, we conclude that $\sigma_1(i) = i$, $i = 1, \dots, r-1$. Since $\sigma_1 \in S_n^r$, then $\sigma_1(r) = r$. Thus, σ_1 can be assumed the identity. Similarly, it can be shown that $\sigma_2 \in \widehat{S}_n^{n-r}$ is the identity, and so $z = \sum_{i=1}^n c_i\alpha_i$.

We prove (*by contradiction*) the direct implication in (i). Indeed, assume that there exist $1 \leqslant k, k' \leqslant r$, $r+1 \leqslant l, l' \leqslant n$, such that $(\alpha_k - \alpha_l)(c_{k'} - c_{l'}) > 0$ and

$$W_C^J(A) = \left(-\infty, \sum_{i=1}^{n} \alpha_i c_i\right].$$

Obviously, the points $z_\sigma = \sum_{i=1}^n c_i\alpha_{\sigma(i)}$, $\sigma = \sigma_1\sigma_2 \in S_n$, $\sigma_1 \in S_n^r$, $\sigma_2 \in \widehat{S}_n^{n-r}$, belong to $W_C^J(A)$. Consider any $\sigma_1 \in S_n^r$ and $\sigma_2 \in \widehat{S}_n^{n-r}$ such that $\sigma_1(k) = k'$ and $\sigma_2(l) = l'$. Consider the matrices $A'_{kl} = \operatorname{diag}(\alpha_{\sigma(k)}, \alpha_{\sigma(l)})$, $C_{kl} = \operatorname{diag}(c_k, c_l)$ and $J_{kl} = J[kl]$. We have

$$z_\sigma - \sum_{i\neq k,l} c_i\alpha_{\sigma(i)} = c_k\alpha_{\sigma(k)} + c_l\alpha_{\sigma(l)} = \operatorname{Tr}(C_{kl}A'_{kl}) \in W_{C_{kl}}^{J_{kl}}(A'_{kl}).$$

The set $W_{C_{kl}}^{J_{kl}}(A'_{kl}) + \sum_{i\neq k,l} c_i\alpha_{\sigma(i)}$ is contained in $W_C^J(A)$. Since $(\alpha_k - \alpha_l)(c_{k'} - c_{l'}) > 0$, by Lemma 2.3 (ii), we have

$$\left[\sum_{i=1}^{n} c_i\alpha_{\sigma(i)}, +\infty\right) = W_{C_{kl}}^{J_{kl}}(A'_{kl}) + \sum_{i\neq k,l} c_i\alpha_{\sigma(i)} \subset W_C^J(A),$$

a contradiction.

(ii) ($\Leftarrow$) Analogously to the proof of (i) ($\Leftarrow$), it can be proved that $W_C^J(A) = [w, +\infty)$, for a certain real number w. Thus, w is a corner of $W_C^J(A)$, and so

$w = z_\sigma$, for $\sigma = \sigma_1\sigma_2 \in S_n$, $\sigma_1 \in S_n^r$ and $\sigma_2 \in \widehat{S}_n^{n-r}$. Suppose that $\sigma_1(i) = r+1-i$, $i = 1, \ldots, l-1$, $\sigma_1(l) \neq r+1-l$, and consider $l < k \leqslant r$ such that $\sigma_1(k) = r+1-l$. Then

$$\begin{aligned}\sum_{\substack{i=1\\ i\neq k,l}}^{r} c_i\alpha_{\sigma_1(i)} + c_l\alpha_{r+1-l} + c_k\alpha_{\sigma_1(l)} + \sum_{i=r+1}^{n} c_i\alpha_{\sigma_2(i)} - z_\sigma \\ = (c_l - c_k)(\alpha_{r+1-l} - \alpha_{\sigma_1(l)}) \leqslant 0, \qquad (9)\end{aligned}$$

because $k > l$ and $\sigma_1(l) < r+1-l$. Let $\tau \in S_n$ be such that $\tau(i) = r+1-i$, $i = 1, \ldots, l$, $\tau(k) = \sigma_1(l)$ and $\tau(i) = \sigma_2(i)$, $i = r+1, \ldots, n$. Only the equality can occur in (9), otherwise we would have $z_\tau < z_\sigma$, a contradiction. That is, z_τ is also the minimum. Repeating this argument, we get $\sigma_1(i) = r+1-i$, $i = 1, \ldots, r-1$. Since $\sigma_1 \in S_n^r$, then $\sigma_1(r) = 1$. Analogously, we find that $\sigma_2(i) = n+r+1-i$, $i = r+1, \ldots, n$, and so $w = \sum_{i=1}^{r} c_i\alpha_{\sigma_1(i)} + \sum_{i=r+1}^{n} c_i\alpha_{\sigma_2(i)}$, $\sigma_1 \in S_n^r$, $\sigma_2 \in \widehat{S}_n^{n-r}$.

(ii) ($\Rightarrow$) The proof is analogous to the proof of (i) ($\Rightarrow$).

(iii) We take the matrices A, C in diagonal form. By the hypothesis, either the eigenvalues of A interlace and $\alpha_r \neq \alpha_{r+1}, \alpha_1 \neq \alpha_n$ or the eigenvalues of C interlace and $c_r \neq c_{r+1}, c_1 \neq c_n$. Suppose that the eigenvalues of A interlace and $\alpha_r \neq \alpha_{r+1}, \alpha_1 \neq \alpha_n$. Then $\alpha_r - \alpha_{r+1} < 0, \alpha_1 - \alpha_n > 0$.

Since C is non-scalar, $c_1 \neq c_n$ or $c_r \neq c_{r+1}$. We assume that $c_1 > c_n$. Hence we have

$$(\alpha_r - \alpha_{r+1})(c_1 - c_n) < 0, \quad (\alpha_1 - \alpha_n)(c_1 - c_n) > 0.$$

Consider the permutation matrix P' associated with the product of the transposition $(1r)$ and the transposition $(r+1n)$ and let $A' = P'AP'^{\mathrm{T}} = \mathrm{diag}(\alpha'_1, \ldots, \alpha'_n)$.

Let $A'_{1n} = \mathrm{diag}(\alpha'_1, \alpha'_n)$, $A_{1n} = \mathrm{diag}(\alpha_1, \alpha_n)$, $C_{1n} = \mathrm{diag}(c_1, c_n)$, and $J_{1n} = I_1 \oplus -I_1$. By Lemma 2.3 (i), the set

$$W_{C_{1n}}^{J_{1n}}(A'_{1n}) + \sum_{q\neq 1,n} c_q\alpha'_q$$

is a half-line $(-\infty, z'_1]$. This half-line is contained in $W_C^J(A)$. By Lemma 2.3 (ii), the set

$$W_{C_{1n}}^{J_{1n}}(A_{1n}) + \sum_{g\neq 1,n} c_g\alpha_g$$

is a half-line $[z'_2, \infty)$. This set is contained in $W_C^J(A)$. By the connectedness of $W_C^J(A)$, we conclude that $W_C^J(A) = (-\infty, +\infty)$. □

The study of $W_C^J(A)$, for J-Hermitian matrices A and C, such that A has a non-real spectrum and C is J-unitarily diagonalizable (and so C has a real spectrum), is treated in Theorem 2.1. This theorem uses the following lemma, an easy consequence of the Hyperbolical Range Theorem [2].

Lemma 2.6. *Let $J = I_1 \oplus -I_1$, and let $C \in M_2$ be a J-unitarily diagonalizable J-Hermitian matrix with distinct eigenvalues c_1, c_2. Suppose that $A \in M_2$ is a J-Hermitian matrix with eigenvalues $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\mathbb{R}$. Then $W_C^J(A) = \mathbb{R}$.*

Theorem 2.1. *Let $J = I_r \oplus -I_{n-r}$, $0 < r < n$, and let C be a non-scalar J-Hermitian and J-unitarily diagonalizable matrix. Let $A \in M_n$ be a J-Hermitian matrix with eigenvalues which are not all real. Then $W_C^J(A)$ is the whole real line.*

Proof. We use the fact that $W_C^J(A)$ may be defined by (3). Suppose that X is a non-degenerate linear subspace of $\mathbb{C}^n$ and Y is the orthogonal complement of X with respect to the inner product $[\cdot,\cdot]$. If X and Y are of the type (r_1, s_1) and (r_2, s_2), respectively, then we have $r = r_1 + r_2$ and $n - r = s_1 + s_2$. This is a consequence of Sylvester's Inertia Theorem and [3, Theorem 10.10, p. 23]. We consider the projection P defined by $P(x + y) = x$, for $x \in X$, $y \in Y$. Suppose that $\sigma_J^+(C) = \{c_1, \ldots, c_r\}$ and $\sigma_J^-(C) = \{c_{r+1}, \ldots, c_n\}$. Then we have the inclusion

$$\begin{aligned}&W^J_{(c_1,\ldots,c_{r_1},c_{r+1},\ldots,c_{r+s_1})}((JP^*JAP)|_X)\\ &\quad+W_{(c_{r_1+1},\ldots,c_r,c_{r+s_1+1},\ldots,c_n)}((J[I-P]^*JA[I-P])|_Y) \subset W_C^J(A).\end{aligned}$$

We show that there exists a 2-dimensional non-degenerate subspace X of $\mathbb{C}^n$ of type $(1, 1)$ for which $(JP^*JAP)|_X$ is J-Hermitian and its eigenvalues are imaginary. If we take such a space X, then we may suppose that $c_1 \in \sigma_J^+(C)$ and $c_{r+1} \in \sigma_J^-(C)$ satisfy the condition $c_1 \neq c_{r+1}$. Hence, the theorem follows from Lemma 2.6.

Suppose that $\alpha \in \mathbb{C}$ is an eigenvalue of A with $\Im(\alpha) > 0$. Let ξ be a non-zero eigenvector of A corresponding to the eigenvalue α. Then ξ satisfies $[\xi, \xi] = 0$. Set

$$X = \{x : x = \alpha\xi + \beta J\xi,\ \alpha, \beta \in \mathbb{C}\}.$$

Since $[J\xi, \xi] = (\xi, \xi) > 0$, the vectors ξ and $J\xi$ are linearly independent. Taking into account that

$$\begin{aligned}[a\xi + bJ\xi, c\xi + dJ\xi] &= a\bar{c}[\xi, \xi] + b\bar{d}[J\xi, J\xi] + b\bar{c}[J\xi, \xi] + a\bar{d}[\xi, J\xi]\\ &= (b\bar{c} + a\bar{d})(\xi, \xi),\end{aligned}$$

the space X is non-degenerate with respect to $[\cdot,\cdot]$. The operator $(JP^*JAP)|_X$ is J-Hermitian and has an eigenvalue α and, hence, $\overline{\alpha}$. Thus, the existence of the asserted linear subspace is proved. □

3. Consequences of Theorem 1.1

Theorem 3.1. *Let r be a given integer with $0 < r < n$ and $J = I_r \oplus -I_{n-r} =$ $\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$. Let $A \in M_n$ be J-Hermitian with non-interlacing real eigenvalues*

$\alpha_1 \geqslant \cdots \geqslant \alpha_r$ *in* $\sigma_J^+(A)$ *and* $\alpha_{r+1} \geqslant \cdots \geqslant \alpha_n$ *in* $\sigma_J^-(A)$. *Then statements* (i)–(iv) *hold*:

(i) *If* $\alpha_1 < \alpha_n$ *and* $1 \leqslant k \leqslant r$, *then*

$$\sum_{j=1}^{k} x_j^* J A x_j \leqslant \sum_{j=1}^{k} \alpha_j \tag{10}$$

for all $x_j \in \mathbb{C}^n$ *such that* $x_j^* J x_l = \delta_{jl}$, *and conversely.*

(ii) *If* $\alpha_1 < \alpha_n$ *and* $r + 1 \leqslant k \leqslant n$, *then*

$$\sum_{j=1}^{r} x_j^* J A x_j - \sum_{j=r+1}^{k} x_j^* J A x_j \leqslant \sum_{j=1}^{k} \alpha_j \tag{11}$$

for all $x_j \in \mathbb{C}^n$ *such that* $x_j^* J x_l = \delta_{jl}\varepsilon_l$, *and conversely.*

(iii) *If* $\alpha_r > \alpha_{r+1}$ *and* $1 \leqslant k \leqslant r$, *then*

$$\sum_{j=r-k+1}^{r} \alpha_j \leqslant \sum_{j=1}^{k} x_j^* J A x_j \tag{12}$$

for all $x_j \in \mathbb{C}^n$ *such that* $x_j^* J x_l = \delta_{jl}$, *and conversely.*

(iv) *If* $\alpha_r > \alpha_{r+1}$ *and* $r + 1 \leqslant k \leqslant n$, *then*

$$\sum_{j=1}^{r} \alpha_j + \sum_{j=n-k+r+1}^{n} \alpha_j \leqslant \sum_{j=1}^{r} x_j^* J A x_j - \sum_{j=r+1}^{k} x_j^* J A x_j \tag{13}$$

for all $x_j \in \mathbb{C}^n$ *such that* $x_j^* J x_l = \delta_{jl}\varepsilon_l$, *and conversely.*

(v) *If* $A \in M_n$ *is* J*-Hermitian with interlacing real eigenvalues,* $\alpha_r \neq \alpha_{r+1}, \alpha_n \neq \alpha_1$, *then*

$$\sum_{j=1}^{k} x_j^* J A x_j, \quad 1 \leqslant k \leqslant r$$

and

$$\sum_{j=1}^{r} x_j^* J A x_j - \sum_{j=r+1}^{k} x_j^* J A x_j, \quad r + 1 \leqslant k \leqslant n$$

with $x_j \in \mathbb{C}^n$ *such that* $x_j^* J x_l = \delta_{jl}\varepsilon_l$, *may assume any real value.*

Proof. Let $C = \operatorname{diag}(c_1, \ldots, c_n)$ and let $U = [x_1\ x_2 \cdots x_n]$ be a J-unitary matrix. Then

$$\mathrm{Tr}(CU^{-1}AU) = \sum_{j=1}^{n} e_j^* CU^{-1}AUe_j = \sum_{j=1}^{r} c_j[Ax_j, x_j] - \sum_{j=r+1}^{n} c_j[Ax_j, x_j],$$

since $U^{-1} = JU^*J$ and $Ce_j = \varepsilon_j c_j e_j$ $(j = 1, \dots, n)$, where e_j denote the vectors of the standard basis in $\mathbb{C}^n$. When $C = I_k \oplus 0_{n-k}$, this becomes

$$\mathrm{Tr}(CU^{-1}AU) = \begin{cases} \sum_{j=1}^{k}[Ax_j, x_j], & \text{if } 1 \leqslant k \leqslant r, \\ \sum_{j=1}^{r}[Ax_j, x_j] - \sum_{j=r+1}^{k}[Ax_j, x_j], & \text{if } r+1 \leqslant k \leqslant n. \end{cases}$$

Conversely, any sequence $x_1, \dots, x_k$ such that $[x_j, x_l] = \varepsilon_l \delta_{jl}$ $(j, l = 1, \dots, k)$ can be completed to a sequence $x_1, \dots, x_k, x_{k+1}, \dots, x_n$ such that $[x_j, x_l] = \varepsilon_l \delta_{jl}$ $(j, l = 1, \dots, n)$. To prove the direct implications in (i) and (iii), we consider $C_\epsilon = I_k \oplus \epsilon I_{r-k} \oplus 0_{n-r}$, $1 \leqslant k < r$, $0 < \epsilon < 1$, and take the limit as $\epsilon \to 0$. The result easily follows from Theorem 1.1.

To prove the converse implication in (i), we observe that by Theorem 1.1 (i) the inequality (9) implies $\alpha_k - \alpha_l < 0$ for all $1 \leqslant k \leqslant r$ and $r+1 \leqslant l \leqslant n$. Therefore, for $k = 1$ and $l = n$ we get $\alpha_1 < \alpha_n$.

To prove the converse implication in (iii), we proceed analogously.

To prove the direct implications in (ii) and (iv), we consider $C_\epsilon = I_r \oplus (1-\epsilon)I_{k-r} \oplus 0_{n-k}$, $r+1 \leqslant k \leqslant n$, $0 < \epsilon < 1$, and take the limit for $\epsilon \to 0$. The converse implications easily follow from Theorem 1.1. □

Remarks. The equality holds in the right hand side inequality in (10) if the x_j are chosen to be J-orthonormal eigenvectors corresponding to the k greatest eigenvalues of A. Similar choices yield equalities in the other inequalities.

The converse of Theorem 3.1 (v) does not hold, as a consequence of Theorem 2.1.

Corollary 3.2. *Let $J = I_r \oplus -I_{n-r}$, $0 < r < n$, and let $A \in M_n$ be J-Hermitian with non-interlacing eigenvalues $\alpha_1 \geqslant \cdots \geqslant \alpha_r$ in $\sigma_J^+(A)$ and $\alpha_{r+1} \geqslant \cdots \geqslant \alpha_n$ in $\sigma_J^-(A)$. The following holds*:

(i) *If $\alpha_1 < \alpha_n$, then*

$$\frac{x^*JAx}{x^*Jx} \leqslant \alpha_1, \quad \text{for all } x \in \mathbb{C}^n \text{ such that } x^*Jx > 0;$$

$$\alpha_n \leqslant \frac{x^*JAx}{x^*Jx}, \quad \text{for all } x \in \mathbb{C}^n \text{ such that } x^*Jx < 0;$$

$$\alpha_1 = \max_{x^*Jx=1}(x^*JAx); \quad \alpha_n = \min_{x^*Jx=-1}(-x^*JAx);$$

and conversely.

(ii) *If* $\alpha_r > \alpha_{r+1}$, *then*

$$\frac{x^*JAx}{x^*Jx} \leqslant \alpha_{r+1}, \quad \text{for all } x \in \mathbb{C}^n \text{ such that } x^*Jx < 0;$$
$$\alpha_r \leqslant \frac{x^*JAx}{x^*Jx}, \quad \text{for all } x \in \mathbb{C}^n \text{ such that } x^*Jx > 0;$$
$$\alpha_r = \min_{x^*Jx=1}(x^*JAx); \quad \alpha_{r+1} = \max_{x^*Jx=-1}(-x^*JAx);$$

and conversely.

Proof. (i) ($\Rightarrow$) The first inequality in (i) is a straightforward consequence of Theorem 3.1 (i) with $k = 1$. For the second inequality, we consider $-J$ instead of J. In this case, $\alpha_{r+1}, \dots, \alpha_n \in \sigma^+_{-J}(A)$ and $\alpha_1, \dots, \alpha_r \in \sigma^-_{-J}(A)$. Therefore, using Theorem 3.1 (iii) with $k = 1$, we have that

$$\alpha_n \leqslant \frac{x^*(-J)Ax}{x^*(-J)x} = \frac{x^*JAx}{x^*Jx}.$$

(i) ($\Leftarrow$) It is an obvious consequence of the converse implication in Theorem 3.1 (i).

The proof of (ii) follows analogously. □

Theorem 3.3. *Let* $J = I_r \oplus -I_{n-r}$, $0 < r < n$, *and let* $A = (a_{ij}) \in M_n$ *be a* J*-Hermitian matrix with non-interlacing eigenvalues* $\alpha_1 \geqslant \cdots \geqslant \alpha_r$ *and* $\alpha_{r+1} \geqslant \cdots \geqslant \alpha_n$ *in* $\sigma^+_J(A)$ *and* $\sigma^-_J(A)$, *respectively. Let* $a'_{11} \geqslant \cdots \geqslant a'_{rr}$ *and* $a'_{r+1,r+1} \geqslant \cdots \geqslant a'_{nn}$ *be a rearrangement of the diagonal entries* $a_{11}, \dots, a_{rr}$ *and* $a_{r+1,r+1}, \dots, a_{nn}$, *respectively. Then*:

(i) $\sum_{j=1}^k a'_{jj} \leqslant \sum_{j=1}^k \alpha_j$, *for all* $1 \leqslant k \leqslant n$, *with equality for* $k = n$, *if and only if* $\alpha_1 < \alpha_n$;

(ii) $\sum_{j=1}^k a'_{jj} \geqslant \sum_{j=r-k+1}^r \alpha_j$, *for all* $1 \leqslant k \leqslant r$, *or* $\sum_{j=1}^r a'_{jj} + \sum_{j=n-k+r+1}^n a'_{jj} \geqslant \sum_{j=1}^r \alpha_j + \sum_{j=n-k+r+1}^n \alpha_j$, *for all* $r \leqslant k \leqslant n$, *with equality for* $k = n$, *if and only if* $\alpha_r > \alpha_{r+1}$.

Proof. (i) ($\Leftarrow$) There exists a permutation matrix P_σ associated with $\sigma = \sigma_1\sigma_2 \in S_n$, $\sigma_1 \in S^r_n$ and $\sigma_2 \in \widehat{S}^{n-r}_n$, such that the diagonal entries of $A' = JP_\sigma JAP^{\mathrm{T}}_\sigma = (a'_{ii})$ are arranged in the following order: $a'_{11} \geqslant \cdots \geqslant a'_{rr}$ and $a'_{r+1,r+1} \geqslant \cdots \geqslant a'_{nn}$. Consider the k first vectors of the standard basis of $\mathbb{C}^n$, $x_j = e_j$, $j = 1, \dots, k$. Since $W^J_C(A) = W^J_C(A')$ and $\alpha_1 < \alpha_n$, by (10) and (11) we obtain

$$\sum_{j=1}^k \alpha_j \geqslant \sum_{i=1}^k e_i^* JA'e_i = \sum_{j=1}^k a'_{jj}, \quad k = 1, \dots, r$$

and

$$\sum_{j=1}^{k}\alpha_j \geqslant \sum_{j=1}^{r}e_j^* JA' e_j - \sum_{i=r+1}^{k} e_i^* JA' e_i = \sum_{j=1}^{k} a'_{jj}, \quad k = r+1, \ldots, n$$

and equality holds for $k = n$, because $a'_{11} + \cdots + a'_{nn} = \mathrm{Tr}(A)$.

(i) (⇒) It is an obvious consequence of the converse implications in Theorem 3.1 (i) and (ii).

(ii) The proof follows analogously to (i). □

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