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Action of the symmetric group on sets of skew-tableaux with prescribed matrix realization ☆

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Abstract

Let M be the set of all rearrangements of t fixed integers in $\{1,\ldots,n\}$. We consider those Young tableaux $\mathscr{T}$, of weight $(m_1,\ldots,m_t)$ in M, arising from a sequence of products of matrices over a local principal ideal domain, with maximal ideal (p),

$$\left(\Delta_a, \Delta_a U(pI_{m_1}\oplus I_{n-m_1}), \Delta_a U\prod_{k=1}^{2}(pI_{m_k}\oplus I_{n-m_k}),\right.$$
$$\left.\ldots, \Delta_a U\prod_{k=1}^{t}(pI_{m_k}\oplus I_{n-m_k})\right),$$

where Δ_a is an $n\times n$ nonsingular diagonal matrix, with invariant partition a, and U is an $n\times n$ unimodular matrix. Given a partition a and an $n\times n$ unimodular matrix U, we consider the set $T_{(a,M)}(U)$ of all sequences of matrices, as above, with $(m_1,\ldots,m_t)$ running over M. The symmetric group acts on $T_{(a,M)}(U)$ by place permutations of the tuples in M. When $t=2,3$, the action of the symmetric group on the set of Young tableaux, having the set $T_{(a,M)}(U)$ as matrix realization, is described by a decomposition of the indexing sets of

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the Littlewood–Richardson tableau in $T_{(a,M)}(U)$, afforded by the matrix U. This description, in cases $t = 2, 3$, gives necessary and sufficient conditions for the existence of an unimodular matrix U such that $T_{(a,M)}(U)$ is a matrix realization of a set of Young tableaux, with given shape c/a and weight running over M. If $\mathscr{H}$ is the tableau arising from the sequence of matrices, above, when $a = 0$, it is shown that the words of the tableaux $\mathscr{T}$ and $\mathscr{H}$ are Knuth equivalent. The relationship between this action of the symmetric group and the one described by A. Lascoux and M.P. Schutzenberger [Noncommutative structures in algebra and geometric combinatorics, (Naples, 1978), Quaderni de La Ricerca Scientifica, vol. 109, CNR, Rome, 1981; M. Lothaire, Algebraic Combinatorics on Words, Encyclopedia of Mathematics and its Applications, vol. 90, Cambridge University Press, Cambridge, 2002], on words, is discussed.

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Keywords: Combinatorics on tableaux; Matrix theory; Plactic monoid; Symmetric group

1. Introduction

Let M be the set of all rearrangements of a sequence of t fixed integers in $\{1, \ldots, n\}$. We consider those Young tableaux $\mathscr{T}$, of weight $(m_1, \ldots, m_t)$ in M, arising from a sequence of products of matrices over a local principal ideal domain, with maximal ideal (p),

$$\left(\Delta_a, \Delta_a U(pI_{m_1} \oplus I_{n-m_1}), \Delta_a U \prod_{k=1}^{2}(pI_{m_k} \oplus I_{n-m_k}), \ldots, \right.$$
$$\left. \Delta_a U \prod_{k=1}^{t}(pI_{m_k} \oplus I_{n-m_k})\right),$$

where $\Delta_a = \operatorname{diag}(p^{a_1}, \ldots, p^{a_n})$ is an $n \times n$ diagonal matrix with invariant partition $a = (a_1, \ldots, a_n)$, and U is an $n \times n$ unimodular matrix. When $(m_1, \ldots, m_t)$ is by decreasing order, $\mathscr{T}$ is a Littlewood–Richardson tableau [1–3]. Now, for each partition a and $n \times n$ unimodular matrix U, let $T_{(a,M)}(U)$ be the set of all sequences of matrices, as above, with $(m_1, \ldots, m_t)$ running over M. The symmetric group $\mathscr{S}_t$ acts on M by place permutations of the tuples, and, henceforth, on $T_{(a,M)}(U)$. The action of the symmetric group, on these sequences of matrices, induces an action on the set constituted by the indexing sets of the Young tableaux realized by $T_{(a,M)}(U)$. We describe this action, in cases $t = 2, 3$. The action of $\mathscr{S}_t$ on $T_{(a,M)}(U)$, for $t = 2, 3$, is generated by an explicit decomposition of the indexing sets of the Littlewood–Richardson tableau in $T_{(a,M)}(U)$. This action of the symmetric group has been also described, independently, in [6], in a purely combinatorial way. Here, we shall see a matrix translation of this action.

The paper is divided into six sections. In Section 2, we introduce the combinatorics of Young tableaux and words. Some well-known results of the plactic monoid, important in the sequel, are also discussed. We follow the terminology of [2,3,9], where strict row tableaux are encoded by indexing sets. It is shown the correspondence between words and indexing sets.

Section 3 is divided into three subsections. In Section 3.1, we discuss properties of integral matrices, decompositions of unimodular matrices, and subgroups of unimodular matrices. In Section 3.2, we discuss the notions of matrix realization of an Young tableau $\mathscr{T}$, and of a pair of Young tableaux $(\mathscr{T}, \mathscr{H})$, where $\mathscr{T}$ is of type $(a, (m_1, \ldots, m_t), c)$ and $\mathscr{H}$ is of type $(0, (m_1, \ldots, m_t), b)$ [2–4]. When such a matrix realization exists, $(\mathscr{T}, \mathscr{H})$ is called an admissible pair [3,4]. In this paper, we shall be concerned on admissible pairs, where $\mathscr{H}$ is the tableau $(0, (1^{m_1}), \sum_{i=1}^{2}(1^{m_i}), \ldots, \sum_{i=1}^{t}(1^{m_i}))$. If $m_1 \geqslant \cdots \geqslant m_t$, $(\mathscr{T}, \mathscr{H})$ is an admissible pair if and only if $\mathscr{T}$ is a Littlewood–Richardson tableau [1–3]. In Section 3.3, we introduce the notion of extension of a matrix. A matrix Z is an extension of X, if X is obtained by zero out some entries of Z. This concept turns out to be the key for the matrix description of the aforesaid action of the symmetric group.

In Section 4, we present the main results, Theorems 4.1, 4.5 and 4.7 and their corollaries. Given a tableau $\mathscr{T}$ of type (a, m, c), these theorems, in cases $t = 2, 3$, answer the following questions: (i) Under what conditions does $\mathscr{T}$ belong to $T_{(a,M)}(U)$; (ii) Under what conditions is the pair $(\mathscr{T}, \mathscr{H})$ admissible. The answer to question (i) is equivalent to the description of the action of the symmetric group on $T_{(a,M)}(U)$, discussed above. The answer to question (ii) follows from the answer to question (i), and from the characterization of the elements of the Knuth equivalence class of $\mathscr{H}$, Proposition 4.4 (see also [6]), as shuffles of the rows of the tableau $\mathscr{H}$. $(\mathscr{T}, \mathscr{H})$ is an admissible pair if and only if the words of $\mathscr{T}$ and $\mathscr{H}$ are Knuth equivalent. In remark 3, for $t = 2$, it is shown that, given two unimodular matrices U and V realizing the same LR tableau $\mathscr{T}$, we may have $T_{(a,M)}(U) \neq T_{(a,M)}(V)$. This means that U and V generate different decompositions of the indexing sets of the LR tableau $\mathscr{T}$, and, thereby, give rise to different parentheses matching operations of the corresponding Yamanouchi word over a two-letters alphabet. Theorems 4.5 and 4.7 are proved in Section 5. When $t \geqslant 4$, the rows of $\mathscr{H}$ are not enough to characterize the elements of the Knuth equivalence class of $\mathscr{H}$. For instance, the word $w = 431421$ belongs to the Knuth equivalence class of $\mathscr{H} = 432141$, but it is clear that w is not a shuffle of the rows of $\mathscr{H}$, 4321 and 41. The analysis of the case $t \geqslant 4$ needs a different approach. This will be the content of a subsequent paper.

In the last section, we translate into words over the three-letters alphabet $\{1, 2, 3\}$, the group action generated by the decomposition of the indexing sets of an LR tableau described in Theorem 4.1. This decomposition of the indexing sets is equivalent to a plactic parentheses matching operation satisfying the Moore–Coxeter relations for $\mathscr{S}_3$ on the corresponding Yamanouchi word. We compare it with the one described by Lascoux and Schutzenberger [11,13] on words. Actually, what we get, in the matrix context, is a family of parentheses matching operations on a Yamanouchi word over the alphabet $\{1, 2, 3\}$, compatible with the Knuth or plactic congruence, given by shuffling the output of the Lascoux and Schutzenberger parentheses matching operation on the words 1, 21, 3121 and 321. The output of the Lascoux and Schutzenberger parentheses matching operation on a Yamanouchi word, over the alphabet $\{1, 2, 3\}$, is itself a special shuffle of this kind.

2. Young tableaux and words

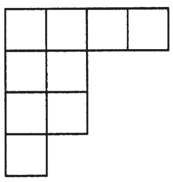
Let $\mathbb{N}$ be the set of nonnegative integers with the usual order "$\geqslant$".

A partition is a sequence of nonnegative integers $a = (a_1, a_2, \ldots)$, all but a finite number of which are nonzero, such that $a_1 \geqslant a_2 \geqslant \cdots$ The number $|a| := \sum_i a_i$ is called the *weight* of a; the maximum value of i for which $a_i > 0$ is called the *length* of a and is denoted by $l(a)$. If $l(a) = |a| = 0$ we have the null partition $a = (0, 0, \ldots)$. If $a_i = 0$, for $i > k$, we shall often write $a = (a_1, \ldots, a_k)$. Sometimes it is convenient to use the notation

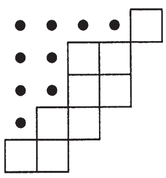
$$a = \left(a_1^{m_1}, a_2^{m_2}, \ldots, a_k^{m_k}\right),$$

where $a_1 > a_2 > \cdots > a_k$ and $a_i^{m_i}$, with $m_i \geqslant 0$, means that a_i appears m_i times as a part of a. We say that a is an *elementary partition* if there is $m > 0$ such that $a = (1^m)$.

Suppose $a = (a_1, \ldots, a_k)$ is a partition of length k with $|a| = n$. The *Young diagram* of a is an array of n boxes, (or dots), having k left-justified rows with row i containing a_i boxes for $1 \leqslant i \leqslant k$. We shall identify a partition with its Young diagram. For example, the Young diagram of $a = (4, 2, 2, 1)$ is



The conjugate partition of a is the partition whose Young diagram is the transpose of the Young diagram of a. For instance, $(4, 3, 1, 1)$ is the conjugate of $a = (4, 2, 2, 1)$. Given two partitions a and c, we write $a \subseteq c$ to mean $a_i \leqslant c_i$, for all i. Graphically, this means that the Young diagram of a is contained in the Young diagram of c. A *skew diagram* c/a is obtained by removing the smaller diagram of a from the diagram of c. For example, if $a = (4, 2, 2, 1)$ and $c = (5, 4, 4, 3, 2)$, the following shows the skew diagram c/a:



We write $|c/a| := |c| - |a|$. A skew-diagram is called a vertical [horizontal] *m-strip*, where $m > 0$, if it has m boxes and at most one box in each row [column].

Let a and c be partitions such that $a \subseteq c$, and $(m_1, \ldots, m_t)$ a sequence of non-negative integers. A *Young tableau* (strictly row) $\mathscr{T}$ of type $(a, (m_1, \ldots, m_t), c)$ is a sequence of partitions

$$\mathscr{T} = (a^0, a^1, \ldots, a^t) \tag{1}$$

such that $a = a^0 \subseteq a^1 \subseteq \cdots \subseteq a^t = c$ and, for each $k = 1, \ldots, t$. The skew-diagram a^k/a^{k-1} is a vertical strip labeled by k, with $m_k = |a^k/a^{k-1}|$. When $a^0 \neq 0$, $\mathscr{T}$ is often called a skew tableau. The *indexing sets* $J_1, \ldots, J_t$ of $\mathscr{T}$ [2,3] are finite subsets of $\mathbb{N}$ given by

$$J_k = \left\{i : a_i^k - a_i^{k-1} \neq 0\right\}, \quad 1 \leqslant k \leqslant t.$$

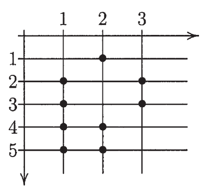
That is, J_k is defined by the row indices of the boxes of c/a labeled by k, $1 \leqslant k \leqslant t$. Notice that $(|J_1|, \ldots, |J_t|) = (m_1, \ldots, m_t)$ ($|J_i|$ denotes the cardinality of J_i). The skew-diagram c/a is called the *shape* of the tableau $\mathscr{T}$ and $(m_1, \ldots, m_t)$ the *weight* of $\mathscr{T}$. For example,

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \boxed{2} \\ \bullet & \bullet & \boxed{1} & \boxed{3} & \\ \bullet & \bullet & \boxed{1} & \boxed{3} & \\ \bullet & \boxed{1} & \boxed{2} & & \\ \boxed{1} & \boxed{2} & & & \end{array} \tag{2}$$

is a (skew) tableau of type $((4, 2, 2, 1), (4, 3, 2), (5, 4, 4, 3, 2))$, with indexing sets $J_1 = \{2, 3, 4, 5\}$, $J_2 = \{1, 4, 5\}$, $J_3 = \{2, 3\}$.

Given $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \ldots, n\}$, and $2^{[n]}$ the power-set of $[n]$.

A sequence $(J_1, \ldots, J_t)$ of subsets of $[n]$ may be represented in a grid of points of $\mathbb{N}^2$, as with matrices, by the set of points $(i, k) \in \mathbb{N}^2$ such that $i \in J_k$, $1 \leqslant k \leqslant t$, where the first coordinate, the row index, increases as one goes downwards, and the second coordinate, the column index, increases as one goes from left to right. For example, the graphical representation of the sequence (J_1, J_2, J_3) defined by the indexing sets of the skew tableau (2), above, is

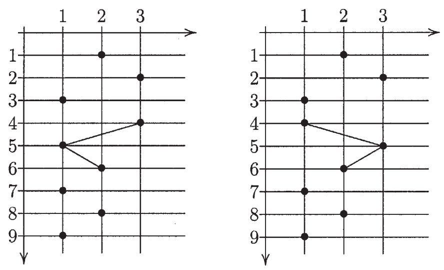


(3)

On its turn, each sequence $(J_1, \ldots, J_t)$ of subsets of $[n]$ gives rise to a word $w(J_1, \ldots, J_t)$ over the alphabet $[t]$, called the *word generated* by $(J_1, \ldots, J_t)$, obtained by reading the grid from top to bottom, along each row, from right to left, by assigning a label i to each dot in column i, for $i = 1, \ldots, t$. The sets $J_1, \ldots, J_t$

are called *indexing sets* of $w(J_1, \ldots, J_t)$. In picture (3), we have $w(J_1, J_2, J_3) = 231312121$. We may now define $w(\mathscr{T})$ the *word of the (skew) tableau* $\mathscr{T}$ (1) as the word generated by the indexing sets of $\mathscr{T}$. That is, $w(\mathscr{T}) = w(J_1, \ldots, J_t)$. In picture (2), the word of $\mathscr{T}$ is $w(\mathscr{T}) = 231312121 = w(J_1, J_2, J_3)$.

Conversely, a word $w = x_1 \ldots x_r$ over the alphabet $[t]$ may be represented in a grid of $\mathbb{N}^2$ as the set of points $(i, x_i) \in \mathbb{N}^2$, $1 \leqslant i \leqslant r$. Putting $F_k = \{i \in [r] : x_i = k\}$, for $k = 1, \ldots, t$, we obtain $w(F_1, \ldots, F_t) = x_1 \ldots x_r$, and $F_1, \ldots, F_t$ are indexing sets of $w = x_1 \ldots x_r$. For example, according to this definition, we have respectively the following graphical representations of the words $w = 231312121$, already considered in picture (3), and $v = 231132121$:



(4)

The sets $F_1 = \{3, 5, 7, 9\}$, $F_2 = \{1, 6, 8\}$ and $F_3 = \{2, 4\}$ are also indexing sets of $w = 231312121$, and therefore $w(J_1, J_2, J_3) = w(F_1, F_2, F_3) = 231312121$, where J_1, J_2, J_3 are the indexing sets of the (skew) tableau (2). The sets $G_1 = \{3, 4, 7, 9\}$, $G_2 = F_2$ and $G_3 = \{2, 5\}$ are indexing sets of $v = 231132121$. Clearly, a word may be generated by different indexing sets. In particular, we may choose always pairwise disjoint indexing sets.

Given a word w over the alphabet $[t]$, we write $|w|_k$, $k \in [t]$, to mean the multiplicity of the letter k in the word w. The sequence $(|w|_1, \ldots, |w|_t)$ is called the *evaluation* (or *weight*) of w, and $|w| = |w|_1 + \cdots + |w|_t$ the *length* of w. Thus if $(J_1, \ldots, J_t)$ are indexing sets of w, the evaluation and the length of w are respectively $(|J_1|, \ldots, |J_t|)$ and $|J_1| + \cdots + |J_t|$. Notice that every skew tableau gives rise to a word, and every word arises from some skew tableau.

A word w is said a *row* if the letters are by strictly decreasing order. Every sequence of indexing sets $p = (X_1, \ldots, X_t)$ of a row word w is such that $X_i = \emptyset$ if the letter i is missing, otherwise, $X_i = \{x_i\}$ and $X_j = \{x_j\}$ with $x_i \geqslant x_j$, whenever $i < j$ are letters of w. Graphically, a row word may be identified with a *polygonal line* p with *line segments of nonnegative slope*. In (4), 321 is a row but neither 312 nor 132 are rows.

A word is said a *tableau* if it is the word of a tableau (1) with $a^0 = 0$. In this case, the word has a factorization into *rows* whose sequence of lengths is the shape of the tableau. For instance, given $m_1 \geqslant \cdots \geqslant m_t$, the word $w([m_1], \ldots, [m_t])$ is the tableau $(t \cdots 2\,1)^{m_t}(t-1 \cdots 2\,1)^{m_{t-1}-m_t} \ldots (2\,1)^{m_2-m_3}\ 1^{m_1-m_2}$, where exponentiation signifies repetition of the same word, with shape the conjugate partition of $(m_1, \ldots, m_t)$. When we mention the rows of a tableau we are referring to those whose sequence of lengths is the shape of the tableau.

Knuth's relation $\equiv$ [10] on words over the alphabet $[t]$ is the equivalence relation generated by the so-called elementary transformations, where x, y, z are letters and u, v are words in $[t]$:

$$uxzyv \equiv uzxyv, \quad x \leqslant y < z, \tag{5}$$

$$uyzxv \equiv uyxzv, \quad x < y \leqslant z. \tag{6}$$

In picture (4), using Knuth relation (5), $w(J_1, J_2, J_3) = w(F_1, F_2, F_3) = 231\ (312)\ 121 \equiv 231\ (132)\ 121 = w(G_1, G_2, G_3)$ (the parentheses indicate the elementary Knuth operation $312 \equiv 132$).The triangular polygonal lines drawn in (4) represent the words 312 and 132 respectively.

In [16], Schensted has described an algorithm, known as *Schensted's insertion algorithm*, which associates to each word w a strictly row tableau $P(w)$. The elementary step consists in the insertion of a letter x into a strictly row tableau $\mathscr{T}$, denoted $P(x.\mathscr{T})$. It takes a positive integer x and a tableau $\mathscr{T}$ and puts x in a new box at the end of the first row if possible, that is, if x is strictly larger than all the entries of the row. If not, it bumps the smallest entry of that row that is larger or equal to x. This bumped entry moves to the next row, going to the end if possible, and bumping an element to the next row, otherwise. The process continues until the bumped entry can go at the end of the next row, or until it becomes the only entry of a new row. Here is an example of the insertion of 3 in a tableau:

$$3 \rightarrow \begin{array}{llll} 1 & 2 & \underline{4} & 5 \\ 1 & 2 & 5 & \\ 2 & & & \end{array} \quad 4 \rightarrow \begin{array}{llll} 1 & 2 & 3 & 5 \\ 1 & 2 & \underline{5} & \\ 2 & & & \end{array} \quad 5 \rightarrow \begin{array}{llll} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & \\ 2 & & & \end{array} \quad \begin{array}{llll} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & \\ 2 & 5 & & \end{array}.$$

For an arbitrary word $w = x_1 \ldots x_k$ in $[t]$, one defines $P(w)$ as the result of inserting x_{k-1} into the unitary tableau $x_k = P(x_k)$, then inserting x_{k-2} into the resulting tableau $P(x_{k-1}.P(x_k))$, and so on. As an example of the general case, the successive steps of the calculation of $P(231312121)$ are

$$\begin{aligned} &1 \rightarrow 1 \quad 2 \rightarrow \begin{array}{ll} 1 & 2 \\ 1 & \end{array} \rightarrow \begin{array}{ll} 1 & 2 \\ 1 & 2 \end{array} \rightarrow \begin{array}{ll} 1 & 2 \\ 1 & 2 \\ 1 & \end{array} \rightarrow \begin{array}{lll} 1 & 2 & 3 \\ 1 & 2 & \\ 1 & & \end{array} \rightarrow \begin{array}{lll} 1 & 2 & 3 \\ 1 & 2 & \\ 1 & & \\ 1 & & \end{array} \\ &\rightarrow \begin{array}{lll} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & & \\ 1 & & \end{array} \rightarrow \begin{array}{lll} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \\ 1 & & \end{array}. \end{aligned} \tag{7}$$

In [8,10,13] is shown that two words w, w' are Knuth equivalent if and only if $P(w) = P(w')$. Therefore, the word 231312121, in (2), (3) and (4), is Knuth equivalent with the tableau 321 321 21 1 in (7).

Definition 2.1. Let $A, B \subseteq [n]$. We write $A \geqslant B$ if there exists an injection $i : B \to A$ such that $b \leqslant i(b)$, for all $b \in B$. We call such an injection a witness for $A \geqslant B$.

Note that if additionally $|A| = |B|$, every witness of $A \geqslant B$ is a bijection. The relation $\geqslant$ defined by $A \geqslant B$ is a partial order in $2^{[n]}$, and we denote it by $\mathscr{P}[n]$. This relation can be characterized in a number of ways as we shall see in the proposition below.

Given a finite set $A \subseteq [n]$, let $\overline{A} := [n]\backslash A$.

Proposition 2.1 [6]. *Given $A, B \subseteq [n]$, the following statements are equivalent*:

(a) $A \geqslant B$.
(b) *There exists an injection $i : B \to A$ such that $b \leqslant i(b)$, for all $b \in B$, and satisfying additionally $i_{|A\cap B} = id_{|A\cap B}$ (id denotes the identity map).*
(c) *For any $k \in \mathbb{N}$, it holds $|\{a \in A : a \geqslant k\}| \geqslant |\{b \in B : b \geqslant k\}|$.*
(d) *If $a = (a_1, a_2, \ldots a_{|A|}, 0, \ldots)$ and $b = (b_1, b_2, \ldots b_{|B|}, 0, \ldots)$ are the decreasing rearrangement of the elements of A and B as embedded into $\mathbb{N}^{\mathbb{N}}$, then $a \geqslant b$ in the componentwise order.*
(e) *There exists $X \subseteq A$ such that $|X| = |B|$ and $X \geqslant B$.*
(f) *$A\backslash Z \geqslant B\backslash Z$, with $Z \subseteq A \cap B$.*

Observe that, when $|A| = |B|$, $A \geqslant B$ if and only if $\overline{B} \geqslant \overline{A}$.

Notice that using (d) of this proposition, $\mathscr{P}[n]$ is clearly a lattice in which the family of all subsets of a given cardinality forms a sublattice. Thus, given $A \geqslant B$ we may define the least upper bound of B in 2^A:

$$\min_B A = \min\left\{X \subseteq A : |X| = |B| \text{ and } X \geqslant B\right\}.$$

Let X be any finite set, and let $\mathscr{S}_X$ denote the set of all bijections on X. In particular, when $X = [n]$ we write $\mathscr{S}_n$ for the symmetric group of order n. Given $A \geqslant B$ with $|A| = |B| = m$, each witness $i : A \to B$, with $i_{|A\cap B} = id_{|A\cap B}$, induces a permutation $\varepsilon \in \mathscr{S}_m$, such that $A\backslash B = \{u_1 > \cdots > u_r\}$, $B\backslash A = \{v_{\varepsilon(1)} > \cdots > v_{\varepsilon(r)}\}$, with $u_j \geqslant i(u_j) = v_j$, $j = 1, \ldots, r$, and $\varepsilon(j) = j$, $j = r+1, \ldots, m$. Notice that if $A = B$, $\varepsilon = id$. Therefore, any witness i can be described completely by the permutation that it induces. In what follows, by a witness of $A \geqslant B$ we mean the permutation $\varepsilon \in \mathscr{S}_m$.

We denote by (uv) the transposition in $\mathscr{S}_n$ of the integers u and v.

Definition 2.2. Given $A, B \subseteq [n]$ with $|A| = |B|$ and $A \geqslant B$, for each witness ε of $A \geqslant B$, as above, we define the permutation $\lambda_{A,B,\varepsilon} = \prod_{k=1}^{r}(u_k v_k)$ in $\mathscr{S}_n$.

When $\varepsilon = id$, we write $\lambda_{A,B}$. If $A = B$, $\lambda_{A,B} = id$. Clearly, $\lambda_{A,B,\varepsilon}(A) = B$, $\lambda_{A,B,\varepsilon}(B) = A$, $(\lambda_{A,B,\varepsilon})_{|A\cap B} = id$, $\lambda^{-1}_{A,B,\varepsilon} = \lambda_{A,B,\varepsilon}$; and $\lambda_{A,B,\varepsilon}\lambda_{C,D,\varrho} = \lambda_{C,D,\varepsilon}\lambda_{A,B,\varrho}$, if $(A \cup B) \cap (C \cup D) = \emptyset$.

Using Proposition 2.1, we may define another relation in $2^{[n]}$.

Definition 2.3 [5]. Let $A, B \subseteq [n]$. We write $A \geqslant_{\text{op}} B$ if $A \geqslant X$, for some $X \subseteq B$ with $|X| = |A|$.

The relation $\geqslant_{\text{op}}$ is a partial order in $2^{[n]}$, and we denote it by $\mathscr{P}^{op}[n]$. Let *op* denote the reverse permutation of $\mathscr{S}_n$. Since $A \geqslant_{\text{op}} B$ if and only if $\text{op}(B) \geqslant \text{op}(A)$, $\mathscr{P}^{op}[n]$ is isomorphic to the dual lattice of $\mathscr{P}[n]$.

A word w over the alphabet $[t]$ is said a *Yamanouchi* word [13] if any right factor v of w satisfies $|v|_1 \geqslant |v|_2 \geqslant \cdots \geqslant |v|_t$. Recalling Proposition 2.1, this is equivalent to say that if $(J_1, \ldots, J_t)$ are indexing sets of w, then every pair (J_i, J_{i+1}), $i = 1, \ldots, t-1$, satisfy condition (c) of that proposition. Henceforth, $w(J_1, \ldots, J_t)$ is a Yamanouchi word if and only if $J_1 \geqslant \cdots \geqslant J_t$. The evaluation of a Yamanouchi word is a partition.

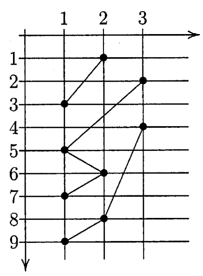
Definition 2.4. Let $u = u_1 \ldots u_r$ and $v = v_1 \ldots v_r$, where $u_1, \ldots, u_r, v_1, \ldots, v_r$ are words over the alphabet $[t]$. The word $\text{sh}(u, v) = u_1 v_1 u_2 v_2 \ldots u_r v_r$, is called a *shuffle* of u and v. That is, $\text{sh}(u, v)$ is obtained by moving u and v through one another.

Let u, v and q be words. We define recursively the shuffle of three (or more words) by $\text{sh}(u, v, q) = \text{sh}(\text{sh}(u, v), q)$.

For instance, the shuffles of 1 and 321 are: $\underline{1}321$, $3\underline{1}21$, $32\underline{1}1 = 321\underline{1}$ (the underlines indicate the position of the word 1 in the shuffle). The word 3211 can be written as a shuffle of 321 and 1 into two different ways. The word 132121 is a shuffle of 321, 21 and 1 but not a shuffle of 3121 and 21. On the other hand, 312211 is both a shuffle of 321, 21, 1, and 3121, 21.

If $(J_1, \ldots, J_t)$ are indexing sets of $\text{sh}(u, v)$ then $J_i = H_i \cup F_i, i = 1, \ldots, t$, where $(H_1, \ldots, H_t)$ and $(F_1, \ldots, F_t)$ are indexing sets of u and v respectively, such that $H_i \cap F_i = \emptyset$. In this case, we say that $(J_1, \ldots, J_t)$ has a *decomposition* into $(H_1, \ldots, H_t)$ and $(F_1, \ldots, F_t)$ and we write $(J_1, \ldots, J_t) = (H_1, \ldots, H_t) \uplus (F_1, \ldots, F_t)$.

The word $w = \underline{2}\,\overline{3}\,\underline{1}\,3\,\overline{1}\,\overline{2}\,\overline{1}\,21$, in (4), is a shuffle of $\overline{3}\,\overline{1}\,\overline{2}\,\overline{1}$, $\underline{2}\,\underline{1}$ and 321 (the overlines and underlines indicate the corresponding shuffle components). Below we exhibit a graphical representation of the word $w = 231312121$ as a shuffle of the words $w(\{3\}, \{1\}) = \underline{21}$, $w(\{9\}, \{8\}, \{4\}) = 321$, and $w(\{5, 7\}, \{6\}, \{2\}) = \overline{3}\,\overline{1}\,\overline{2}\,\overline{1}$. Graphically, w is a union of pairwise disjoint polygonal lines (polygonal lines without overlapping vertexes):



Another way to write $w = 231312121$ as a shuffle of 3121, 21 and 321 is $w = \underline{2}\,3\,\underline{1}\,\overline{3}\,\overline{1}\,2\,1\,\overline{2}\,\overline{1}$ (the overlined and underlined letters indicate respectively the subwords 3121 and 21).

The notion of shuffle allows us to give the following characterization of Yamanouchi word.

Proposition 2.2. *Let w be a word with evaluation $(m_1, \ldots, m_t)$, $m_1 \geqslant \cdots \geqslant m_t$, and indexing sets $(J_1, \ldots, J_t)$. The following conditions are equivalent*:

(a) w is a Yamanouchi word.

(b) $(J_1, \ldots, J_t)$ has a decomposition of the form

$$\begin{array}{llll} A_1^1 & & & \\ A_1^2 & A_2^2 & & \\ \vdots & \vdots & & \\ A_1^t & A_2^t & \ldots & A_t^t, \end{array}$$

where $A_1^k \geqslant A_2^k \geqslant \cdots \geqslant A_k^k$, $|A_1^k| = |A_2^k| = \cdots = |A_k^k| = m_k - m_{k+1}$, $1 \leqslant k \leqslant t$, with $m_{t+1} = 0$, and $A_j^r \cap A_j^s = \emptyset$, $1 \leqslant j < t$, $r \neq s$.

(c) w is a shuffle of the rows of the tableau $w([m_1], \ldots, [m_t])$.

Proof. Let $r_1, \ldots, r_{m_1}$ be the rows of the tableau $w([m_1], \ldots, [m_t])$, by decreasing order of length, and $(l_1, \ldots, l_{m_1})$ be the conjugate partition of $(m_1, \ldots, m_t)$.

(a) $\Leftrightarrow$ (b) By Proposition 2.1(d), $J_1 \geqslant \cdots \geqslant J_t$ if and only if is the union of pairwise disjoint polygonal lines with line segments of nonnegative slope $p_i = (x_1^i \geqslant \cdots \geqslant x_{l_i}^i)$ where $x_k^i \in J_k$, $k = 1, \ldots, l_i$, $1 \leqslant i \leqslant m_1$.

(b) $\Rightarrow$ (c) Suppose $(J_1, \ldots, J_t)$ has a decomposition as displayed in (b). For $1 \leqslant k \leqslant t$, $A_1^k \geqslant A_2^k \geqslant \cdots \geqslant A_k^k$ are indexing sets of a subword of w which is a shuffle of $m_k - m_{k+1}$ row words $k \ldots 2\,1$.

Now suppose that w is a shuffle of the row words $r_1, \ldots, r_{m_1}$. Since $J_1, \ldots, J_t$ are indexing sets of w, each row r_i determines a polygonal line with line segments of nonnegative slope $p_i = (X_1^i \geqslant \cdots \geqslant X_{l_i}^i)$ where $X_k^i = \{x_k^i\} \subseteq J_k$, $k = 1, \ldots, l_i$, $1 \leqslant i \leqslant m_1$. Clearly, $p_1, \ldots, p_{m_1}$ are pairwise disjoint.

Henceforth, $(J_1, \ldots, J_t) = \biguplus_{i=1}^{m_1} p_i$ and $J_1 \geqslant \cdots \geqslant J_t$. $\square$

On the other hand

Proposition 2.3 (13, Lemma 5.4.7). *The set of Yamanouchi words with evaluation $(m_1, \ldots, m_t)$, forms a single Knuth equivalence class, whose representative word is the tableau $w([m_1], \ldots, [m_t])$.*

From these two propositions, we find that Knuth operations on Yamanouchi words of evaluation $(m_1, \ldots, m_t)$ are equivalent to shuffle the rows of the tableau $w([m_1],$

$\ldots, [m_t])$. For instance, $w = \underline{2}\,\overline{3}\,\underline{1}\,3\,\overline{1}\,\overline{2}\,\overline{1}\,2\,1 \equiv \underline{2}\,\overline{1}\,\underline{1}\,3\,\overline{3}\,\overline{2}\,\overline{1}\,2\,1 \equiv \underline{2}\,\overline{3}\,\underline{1}\,3\,\overline{2}\,\overline{1}\,\overline{1}\,2\,1 \equiv$ (7) which are shuffles of $\overline{3}\,\overline{2}\,\overline{1}$, $\overline{1}$, 321, and $\underline{2}\,\underline{1}$.

Indeed not every Knuth class satisfy this property. There are two reasons: either a shuffle of the rows of the tableau in the Knuth class can not be performed by Knuth operations, and we stay out of the Knuth class, or we stay in the Knuth class but there are Knuth operations which can not be performed by a shuffle of the rows of the tableau in the Knuth class. For example, in the first case, the tableau $5321\,421\,52 \not\equiv 54321\,21\,52 = \mathrm{sh}(5321, 421, 52)$. In the second case, the Knuth operation $412 \equiv 142$ on a Yamanouchi word over the alphabet [4] always implies a shuffle of the row words 4321, 21 and 1 but, on the other hand, considering the word $43\underline{4}\,\underline{1}21 \equiv 4321\,41$, a shuffle of the rows of the tableau 4321 41, the same Knuth operation on this word can not be performed by a shuffle of the row words 4321, 41, since $43\underline{4}\,\underline{1}21 = \mathrm{sh}(4321, 41) \equiv 43\underline{1}\,\underline{4}21 \neq \mathrm{sh}(4321, 41)$.

The dual word of $w = x_1 \cdots x_r$ in the alphabet $[t]$ is $w_{\mathrm{op}} := \mathrm{op}(x_r) \cdots \mathrm{op}(x_1)$, a word in the alphabet $[t]$, with $\mathrm{op}(i) = t - i + 1$ the reverse permutation of $\mathscr{S}_t$. Clearly, given $J_1, \ldots, J_t \subseteq [n]$, $J_1, \ldots, J_t$ are indexing sets of w if and only if $\mathrm{op}(J_t), \ldots, \mathrm{op}(J_1)$, with $op \in \mathscr{S}_n$, are indexing sets of w_{op}.

A word over the alphabet $[t]$ is said a *dual Yamanouchi* word if it is the dual of some Yamanouchi word over $[t]$. Therefore, a word w with indexing sets $J_1, \ldots, J_t$ is a dual Yamanouchi word if and only if $J_1 \geqslant_{\mathrm{op}} \cdots \geqslant_{\mathrm{op}} J_t$. Attending to the characterizations of Yamanouchi words given above, we also find that

Corollary 2.4. *Let w be a word with evaluation $(m_1, \ldots, m_t)$, $m_1 \leqslant \cdots \leqslant m_t$, and indexing sets $(J_1, \ldots, J_t)$. The following conditions are equivalent*:

(a) *w is a dual Yamanouchi word.*
(b) *$(J_1, \ldots, J_t)$ has a decomposition of the form*

$$\begin{array}{cccc} & & & A_t^1 \\ & & A_{t-1}^2 & A_t^2 \\ & \vdots & \vdots & \\ A_1^t & \ldots & A_{t-1}^t & A_t^t, \end{array}$$

where $A_{t-k+1}^k \geqslant \cdots \geqslant A_t^k$, $|A_{t-k+1}^k| = \cdots = |A_t^k| = m_{t-k+1} - m_{t-k}$, $1 \leqslant k \leqslant t$, with $m_0 = 0$, and $A_j^r \cap A_j^s = \emptyset$, $1 < j \leqslant t$, $r \neq s$.
(c) *w is a shuffle of the rows of the tableau $w([m_1], \ldots, [m_t])$.*

Recalling the Knuth relations (5) and (6), since $x \geqslant y$ if and only if $\mathrm{op}(y) \geqslant \mathrm{op}(x)$, we find that $xzy \equiv zxy$, with $x \leqslant y < z$ if and only if $\mathrm{op}(y)\mathrm{op}(z)\mathrm{op}(x) \equiv \mathrm{op}(y)\mathrm{op}(x)\mathrm{op}(z)$, with $\mathrm{op}(z) < \mathrm{op}(y) \leqslant \mathrm{op}(x)$. Therefore, we have $w \equiv w'$ if and only if $w_{op} \equiv w'_{op}$, which allows us to obtain the following characterization of dual Yamanouchi words:

Corollary 2.5. *The set of dual Yamanouchi words with evaluation* $(m_1, \ldots, m_t)$, $m_1 \leqslant \ldots \leqslant m_t$, *forms a single Knuth equivalence class, whose representative word is the tableau* $w([m_1], \ldots, [m_t])$.

Thus, a word w with evaluation $(m_1, \ldots, m_t)$, $m_1 \leqslant \cdots \leqslant m_t$, is a dual Yamanouchi word if and only if it is Knuth equivalent to $w([m_1], \ldots, [m_t])$.

With the relation $\geqslant$ and $\geqslant_{\mathrm{op}}$ in $2^{[n]}$, we may give the following definition of Littlewood–Richardson tableau [12] and opposite Littlewood–Richardson tableau.

Definition 2.5 ([2,3,5]). Let $\mathscr{T}$ be a Young tableau of type $(a, (m_1, \ldots, m_t), c)$ with indexing sets $J_1, \ldots, J_t$. We say that:

(I) $\mathscr{T}$ is a Littlewood–Richardson (LR for short) tableau if $J_1 \geqslant \cdots \geqslant J_t$.
(II) $\mathscr{T}$ is an opposite Littlewood–Richardson (LR_{op} for short) tableau if $J_1 \geqslant_{\mathrm{op}} \ldots \geqslant_{\mathrm{op}} J_t$.

Equivalently, $\mathscr{T}$ is an LR ($\mathrm{LR_{op}}$) tableau if and only if $w(J_1, \ldots, J_t)$ is a (dual) Yamanouchi word. In Section 5, we shall look at an $\mathrm{LR_{op}}$ tableau and a dual Yamanouchi word under the point of view of an action of the symmetric group.

3. Matrix realizations of Young tableaux

3.1. Smith normal form and subgroups of unimodular matrices

Let $\mathscr{R}_p$ be a local principal ideal domain with maximal ideal (p). In this paper, all matrices are square and nonsingular, with entries over $\mathscr{R}_p$. Let $\mathscr{U}_n$ be the group of $n \times n$ unimodular matrices. We denote by E_{ij} the $n \times n$ matrix having 1 in position (i, j) and 0's elsewhere, and define the elementary unimodular matrices $T_{ij}(x)$ as follows:

$$\begin{aligned} T_{ij}(x) &= I + xE_{ij}, && \text{where } i \neq j \text{ and } x \in \mathscr{R}_p; \\ T_{ii}(v) &= I + (v-1)E_{ii}, && \text{where } v \text{ is a unit of } \mathscr{R}_p. \end{aligned}$$

It is obvious, that $E_{ij}E_{rs} = \delta_{jr}E_{is}$, where δ_{jr} denotes the Kronecker symbol, that is, $\delta_{jr} = 1$ if $j = r$, and equals 0 otherwise. Therefore, if $i \neq j$ and $r \neq s$, we find that

$$T_{ij}(x)T_{rs}(y) = I + xE_{ij} + yE_{rs} + xy\delta_{jr}E_{is}. \tag{8}$$

In particular, $T_{ij}(x)T_{ij}(y) = T_{ij}(x + y)$, if $i \neq j$, and the elementary matrices $T_{ij}(x)$ and $T_{rs}(y)$ commute, whenever $i \neq s$ and $j \neq r$.

If $\sigma \in \mathscr{S}_n$, we denote by P_σ the permutation matrix having $\delta_{i\sigma(j)}$ in position (i, j). Note that if $[n] = \{i_1, \ldots, i_n\} = \{j_1, \ldots, j_n\}$, then $\sum_{k=1}^{n} E_{i_k j_k} = P_\sigma$, where σ is the permutation defined by $\sigma(j_k) = i_k$, for $k = 1, \ldots, n$.

Lemma 3.1. *Let* $i_k, j_k \in [n]$, *for* $k = 1, \ldots, r$, *such that* $\{i_1, \ldots, i_r\} \cap \{j_1, \ldots, j_r\} = \emptyset$. *Then, if* $\xi = \prod_{k=1}^{r}(i_k j_k)$,

$$\left(\prod_{k=1}^{r} T_{j_k j_k}(-1)\right)\left(I - \sum_{k=1}^{r} E_{j_k i_k}\right)\left(I + \sum_{k=1}^{r} E_{i_k j_k}\right)\left(I - \sum_{k=1}^{r} E_{j_k i_k}\right) = P_\xi. \tag{9}$$

Proof. Attending to (8) and since $\{i_1, \ldots, i_r\} \cap \{j_1, \ldots, j_r\} = \emptyset$, a simple induction on r shows that $\prod_{k=1}^{r} T_{i_k j_k}(1) = I + \sum_{k=1}^{r} E_{i_k j_k}$. Therefore, we may write the first member of (9) as

$$\begin{aligned}
&\prod_{k=1}^{r}\left[T_{j_k j_k}(-1) T_{j_k i_k}(-1) T_{i_k j_k}(1) T_{j_k i_k}(-1)\right] \\
&\quad = \prod_{k=1}^{r}\left[T_{j_k j_k}(-1)\left(\sum_{s \neq i_k, j_k} E_{ss} + E_{i_k j_k} - E_{j_k i_k}\right)\right] \\
&\quad = \prod_{k=1}^{r} P_{(i_k j_k)} = P_\xi. \qquad \square
\end{aligned}$$

Given $n \times n$ matrices A and B, we say that B is *left equivalent* to A (written $B \sim_L A$) if $B = UA$ for some unimodular matrix U; B is *right equivalent* to A (written $B \sim_R A$) if $B = AV$ for some unimodular matrix V; and B is *equivalent* to A (written $B \sim A$) if $B = UAV$ for some unimodular matrices U, V. The relations $\sim_L$, $\sim_R$ and $\sim$ are equivalence relations in the set of all $n \times n$ matrices over $\mathscr{R}_p$.

Let A be an $n \times n$ nonsingular matrix. By the Smith normal form theorem (see [7,15]), there exist nonnegative integers $a_1, \ldots, a_n$ with $a_1 \geqslant \cdots \geqslant a_n$ such that A is equivalent to

$$\mathrm{diag}(p^{a_1}, \ldots, p^{a_n}).$$

The sequence $a = (a_1, \ldots, a_n)$ by decreasing order, of the exponents of the p-powers in the Smith normal form of A, is a partition of length $\leqslant n$, uniquely determined by the matrix A. We call a the *invariant partition* of A. More generally, if we are given a sequence of nonnegative integers $e_1, \ldots, e_n$, the following notation for p-powered diagonal matrices will be used:

$$\mathrm{diag}_p(e_1, \ldots, e_n) := \mathrm{diag}(p^{e_1}, \ldots, p^{e_n}).$$

Given a partition a of length $\leqslant n$, let $\Delta_a := \operatorname{diag}_p(a)$. If $a = 0$ is the null partition, then $\Delta_0 = I$. If $F \subseteq [n]$, let $D_F := \operatorname{diag}_p(\chi^F)$, where χ^F is the characteristic function of F, that is, $\chi^F(i) = 1$ if $i \in F$, and equals 0 if $i \notin F$.

Given a sequence of nonnegative integers $m = (m_1, \ldots, m_t)$ and $\sigma \in \mathscr{S}_t$, let $\sigma m := (m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(t)})$. That is, $\sigma m = P_\sigma[m_1 \cdots m_t]^{\mathrm{T}}$. It is a simple exercise to prove that

$$P_\sigma \Delta_a = \Delta_{\sigma a} P_\sigma, \quad P_\sigma^{-1} = P_\sigma^T = P_{\sigma^{-1}}, \quad \text{and} \tag{10}$$

$$P_\sigma^T \Delta_a P_\sigma = \Delta_{\sigma^{-1} a} = \operatorname{diag}_p(a_{\sigma(1)}, \ldots, a_{\sigma(n)}). \tag{11}$$

Let $(m_1, \ldots, m_t)$ be a sequence of t integers in $[n]$, and define

$$M_t := \left\{ m \in \mathbb{Z}^t : m \text{ is a rearrangement of } (m_1, \ldots, m_t) \right\}. \tag{12}$$

Note that there exists $\sigma \in \mathscr{S}_t$ such that $\sigma^{-1}(m_1, \ldots, m_t)$ is the only partition of M_t. The symmetric group $\mathscr{S}_t$ acts on M_t by place permutations of the t-uples of M_t. For each permutation $\sigma \in \mathscr{S}_t$, the map $\phi(\sigma) : M_t \to M_t$ defined by $\phi(\sigma)(m) = \sigma m$ is a bijection. Thus, the map $\phi : \mathscr{S}_t \to \mathscr{S}_{M_t}$ defined by $\phi(\sigma)(m) = \sigma m$, for $\sigma \in \mathscr{S}_t$, is a group action on M_t.

Definition 3.1. Given $F \subseteq [n]$, let $\mathscr{M}(F)$ be the set of $n \times n$ matrices of the form $I + X$, where $X = (x_{ij})$ satisfy the condition: $x_{ij} \neq 0$ only if $i \in F$ and $j \notin F$.

Note that if $m = |F|$ and $\omega \in \mathscr{S}_n$ is such that $F = \{\omega(1), \ldots, \omega(m)\} = \omega([m])$, then $P_\omega^{\mathrm{T}} \mathscr{M}(F) P_\omega = \mathscr{M}([m])$. Clearly, $\mathscr{M}([m])$ is a subgroup of $\mathscr{U}_n$ and, therefore, $\mathscr{M}(F)$ as well. We also consider $\mathscr{M}_p(F) := \{I + pX : I + X \in \mathscr{M}(F)\}$, a subgroup of $\mathscr{M}(F)$.

Notice that $[\mathscr{M}(F)]^{\mathrm{T}} = \mathscr{M}(\overline{F})$ and $\mathscr{M}(\emptyset) = \{I\} = \mathscr{M}([n])$.

Given $F, G \subseteq [n]$, we define

$$\mathscr{M}(F, G) := \mathscr{M}(F) \cap \mathscr{M}(G)$$

and

$$\mathscr{M}_p(F, G) := \left\{ I + pX : I + X \in \mathscr{M}(F, G) \right\}.$$

Clearly, $\mathscr{M}(F, G)$ is a subgroup of $\mathscr{M}(F \cap G)$, $\mathscr{M}(F)$, and $\mathscr{M}(G)$. Notice that $\mathscr{M}(F, F) = \mathscr{M}(F)$ and $\mathscr{M}_p(F, F) = \mathscr{M}_p(F)$. We have $[\mathscr{M}(F, G)]^{\mathrm{T}} = \mathscr{M}(\overline{F}, \overline{G}) = \mathscr{M}(\overline{F}) \cap \mathscr{M}(\overline{G})$; and $\mathscr{M}(\emptyset, G) = \{I\} = \mathscr{M}(F, [n])$.

Lemma 3.2. *Let $F, G, H \subseteq [n]$ such that $F \subseteq G$ and $H \subseteq G \backslash F$. Then*:

(i) $\mathscr{M}(F, G)\mathscr{M}(H) = \mathscr{M}(H)\mathscr{M}(F, G)$;
(ii) $\mathscr{M}(\overline{F}, \overline{G}) D_F = D_F \mathscr{M}_p(\overline{F}, \overline{G})$.

Proof. It is enough to prove the result when $F = [r]$ and $G = [s]$, with $0 \leqslant r \leqslant s$. $\square$

In the conditions of the lemma above, we also have $[\mathscr{M}(F,G)]^{\mathrm{T}}\mathscr{M}(H)=\mathscr{M}(H)$ $[\mathscr{M}(F,G)]^{\mathrm{T}}$, since $H\subseteq G\backslash F$ if and only if $H\subseteq\overline{F}\backslash\overline{G}$, and $\mathscr{M}_p(F,G)D_F=D_F\mathscr{M}(F,G)$.

Given $F\subseteq[n]$, let

$$\mathscr{U}(F):=\left\{I+(x_{ij})\in\mathscr{U}_n : x_{ij}\neq 0 \text{ only if } i,j\in F\right\}.$$

If $m=|F|$ and $\omega\in\mathscr{S}_n$ is such that $F=\{\omega(1),\ldots,\omega(m)\}=\omega([m])$, then $P_\omega^{\mathrm{T}}\mathscr{U}(F)P_\omega=\mathscr{U}([m])$. Note that $\mathscr{U}([n])=\mathscr{U}_n$. Clearly, $\mathscr{U}(F)$ is a subgroup of $\mathscr{U}_n$.

Lemma 3.3. *Let $F,G,H\subseteq[n]$ such that $F\subseteq G$. Then*:

(i) $\mathscr{U}(F)\mathscr{M}(F,G)=\mathscr{M}(F,G)\mathscr{U}(F)$;
(ii) $\mathscr{U}(F)\mathscr{M}(H)=\mathscr{M}(H)\mathscr{U}(F)$, *if* $H\subseteq\overline{F}$;
(iii) $(\mathscr{U}(F)\mathscr{M}(F))\left(\mathscr{M}_p(H)\mathscr{U}(H)\right)=\left(\mathscr{M}_p(H)\mathscr{U}(H)\right)(\mathscr{U}(F)\mathscr{M}(F))$, *if* $H\subseteq\overline{F}$;
(iv) $\mathscr{M}(\overline{H},\overline{G})\mathscr{M}(F\backslash H,G)\subseteq\mathscr{U}(F)\mathscr{M}(F\backslash H,G)\mathscr{M}(\overline{H},\overline{G})$, *if* $H\subseteq F$.

Proof. For (iii), notice that, given an $n\times n$ matrix U, $\det(U+pX)=\det(U)(\mathrm{mod}\ p)$, for every $n\times n$ matrix X. Thus, if $U\in\mathscr{U}_n$, $U+pX$ is also unimodular. $\square$

Observe that for $x\in\mathscr{R}_p$, $\Delta_{\sigma^{-1}a}T_{ij}(x)\sim_L\Delta_{\sigma^{-1}a}$, whenever $\sigma(j)\geqslant\sigma(i)$, $T_{ij}(p)D_F\sim_R D_F$, and $T_{ij}(x)D_F\sim_R D_F$, if $i\notin F$.

Theorem 3.4. *Let $U\in\mathscr{U}_n$. Then, there exists $\sigma\in\mathscr{S}_n$ such that $U=TP_\sigma R$, where T is a $n\times n$ upper triangular matrix, having* 1*'s along the main diagonal, and R is a $n\times n$ unimodular matrix, with units along the main diagonal, and multiples of p above it.*

Proof. Let $U=[u_{ij}]$. Noticing that every row of an unimodular matrix has a unit, we define

$$j_n:=\max_{1\leqslant j\leqslant n}\left\{j : u_{nj} \text{ is a unit}\right\}.$$

Multiplying U, on the left, by suitable elementary matrices $T_{kn}(x)$, $k<n$, we may use u_{nj_n} as a pivot to eliminate all nonzero elements of column j_n above row n. Observe that all these matrices are upper triangular with 1's along the main diagonal. Denote the product of these elementary matrices by T_n.

By columns operations, we may use u_{nj_n} to eliminate all nonzero elements of row n to the left and right of u_{nj_n}. To eliminate the elements to the left of u_{nj_n}, we use lower triangular matrices with 1's along the main diagonal, and to eliminate the elements to the right, we use upper triangular matrices whose nondiagonal entries are multiples of p. Then, multiplying on the right by a suitable diagonal matrix, we divide column j_n by $u_{nj_n}^{-1}$. We denote the product of this elementary matrices by R_n.

The resulting matrix $U_n:=T_nUR_n$ has all entries of row n and column j_n zero, except the entry (n,j_n), which is 1.

The process is now repeated with row $n-1$ of U_n, obtaining $U_{n-1} := T_{n-1}T_nUR_nR_{n-1}$ with all entries of rows $n, n-1$ and columns j_n, j_{n-1} zero, except the entries (n, j_n) and $(n-1, j_{n-1})$ which are 1.

Continuing the process above, we obtain $T_1 \cdots T_nUR_n \cdots R_1 = E_{1j_1} + \cdots + E_{nj_n}$, with $\{j_1, \ldots, j_n\} = [n]$. Define $\sigma \in \mathscr{S}_n$ by $\sigma(j_i) = i$, $i = 1, \ldots, n$. Then $P_\sigma = E_{1j_1} + \cdots + E_{nj_n}$ and $U = TP_\sigma R$, where $T = T_n^{-1} \cdots T_1^{-1}$ and $R = R_1^{-1} \cdots R_n^{-1}$ are as requested. $\square$

Theorem 3.5. *Let $U \in \mathscr{U}_n$. Then, there exists $\sigma \in \mathscr{S}_n$ such that $U = TP_\sigma QL$, where T is an $n \times n$ upper triangular matrix, with 1's along the main diagonal, Q is an $n \times n$ upper triangular matrix, with 1's along the main diagonal, and multiples of p above it, and L is an $n \times n$ lower triangular matrix, with units along the main diagonal.*

Proof. Given an unimodular matrix U, by Theorem 3.4, there exists $\sigma \in \mathscr{S}_n$ such that $U = TP_\sigma R$, where T is an $n \times n$ upper triangular matrix, with $1's$ along its main diagonal, and R is an unimodular matrix, with units along the main diagonal, and multiples of p above it.

Attending to the form of matrix R, the application of Theorem 3.4 to R gives $R = T'IR'$, where T' is upper triangular, with 1's in the main diagonal, and multiples of p above it, and R' is lower triangular matrix, with units along the main diagonal. So let $Q := T'$ and $L := R'$. $\square$

Remark 1. Notice that the decomposition given in this theorem is not unique. For instance, let

$$A = \begin{bmatrix} 1+p & 1 \\ 1 & 0 \end{bmatrix} \in \mathscr{U}_2.$$

We get the decompositions $A = TP_{(12)}QL$, with $T = T_{12}(p)$, $Q = I$ and $L = T_{21}(1)$, and also $A = T'P_{(12)}QL'$, with $T = T_{12}(1+p)$, and $Q = L' = I$.

3.2. Matrix realizations of Young tableaux

Now, we analyze products of matrices of the form $\Delta_a UD_{[m]}$, where $1 \leqslant m \leqslant n$ and $U \in \mathscr{U}_n$. By the previous theorem, we may write $U = TP_\sigma QL$. Since T is upper triangular with 1's along the main diagonal, and L is lower triangular with units along the main diagonal, we have

$$\Delta_a UD_b \sim \Delta_a P_\sigma QD_b,$$

for any partition b of length $\leqslant n$. Thus, without loss of generality, we may assume that $U = P_\sigma Q$, where Q is upper triangular with 1's along the main diagonal and multiples of p above it, and $\sigma \in \mathscr{S}_n$.

Lemma 3.6. *Let a be a partition of length $\leqslant n$, and F a subset of $\{1, \dots, n\}$. Then, there exists a permutation $\sigma \in \mathscr{S}_n$ such that $\sigma = \sigma^{-1}$, $a + \chi^{\sigma(F)}$ is a partition, $F \geqslant \sigma(F)$ and $\sigma(a) = a$. In particular, if $a = (a_1, \dots, a_n)$ is such that $a_1 > \cdots > a_n$, $a + \chi^F$ is always a partition.*

Proof. Straightforward. $\square$

In order to avoid cumbersome notation, we write $\sigma[m] := \sigma([m])$.

Theorem 3.7. *Let $U \in \mathscr{U}_n$, and $1 \leqslant m \leqslant n$. Given a partition a of length $\leqslant n$, there exists $\sigma \in \mathscr{S}_n$ such that $\Delta_a U D_{[m]} \sim \operatorname{diag}_p(a + \chi^{\sigma[m]})$, where $a + \chi^{\sigma[m]}$ is a partition.*

Proof. Let $U = P_\sigma Q$, with $\sigma \in \mathscr{S}_n$ and Q an upper triangular matrix, with 1's along the main diagonal, and multiples of p above it. We may write

$$Q = \begin{bmatrix} B_1 & pB_2 \\ 0 & B_3 \end{bmatrix},$$

where B_1 and B_3 are, respectively, $m \times m$ and $(n-m) \times (n-m)$ upper triangular matrices, with 1's along its main diagonal, and multiples of p above it. Thus, we have

$$\Delta_a P_\sigma Q D_{[m]} = \Delta_a P_\sigma D_{[m]} Q', \quad \text{where } Q' = \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix}. \tag{13}$$

Therefore, $\Delta_a P_\sigma Q D_{[m]} \sim_R \Delta_a P_\sigma D_{[m]} \sim_R \Delta_a D_{\sigma[m]} = \operatorname{diag}_p(a + \chi^{\sigma[m]})$.

If $a + \chi^{\sigma[m]}$ is not a partition, then by previous lemma and conditions (10), there exists a permutation μ such that $P_\mu \Delta_a P_\mu = \Delta_a$ and $a + \chi^{\mu\sigma[m]}$ is a partition. Hence, $\Delta_a U D_{[m]} \sim \operatorname{diag}_p(a + \chi^{\mu\sigma[m]})$. $\square$

From this proof, it follows

Corollary 3.8. *Let $U \in \mathscr{U}_n$ and $1 \leqslant m' \leqslant m \leqslant n$. Let a^1 be the invariant partition of $\Delta_a U D_{[m]}$, and a' the invariant partition of $\Delta_a U D_{[m']}$. Then, $a' \subseteq a^1$ and a^1/a' is a vertical strip.*

Given a sequence of $n \times n$ nonsingular matrices $A_0, B_1, \dots, B_t$, where A_0 has invariant partition a, and B_r has elementary invariant partition (1^{m_r}), for $r = 1, \dots, t$, it holds

$$A_0 B_1 \dots B_k \sim \Delta_a U_1 D_{[m_1]} U_2 D_{[m_2]} \dots U_k D_{[m_k]}, \quad k = 1, \dots, t, \tag{14}$$

for some $n \times n$ unimodular matrices $U_1, \dots, U_t$. Therefore, by the application of the previous theorem, there exist $\sigma_1, \dots, \sigma_t \in \mathscr{S}_n$ such that (14) is equivalent to the diagonal matrix

$$\Delta_a D_{\sigma_1[m_1]} \cdots D_{\sigma_k[m_k]} = \operatorname{diag}_p\big(a + \chi^{\sigma_1[m_1]} + \cdots + \chi^{\sigma_k[m_k]}\big), \quad k = 1, \dots, t.$$

This leads us to the notion of matrix realization of a Young tableau.

Definition 3.2. Let $\mathscr{T} = (a^0, a^1, \ldots, a^t)$ be a Young tableau of type $(a, (m_1, \ldots, m_t), c)$, with $l(c) \leqslant n$. A sequence of $n \times n$ nonsingular matrices $A_0, B_1, \ldots, B_t$ is a matrix realization of $\mathscr{T}$ (or realizes $\mathscr{T}$) if:

(I) For each $r \in \{1, \ldots, t\}$, the matrix B_r has invariant partition $(1^{m_r}, 0^{n-m_r})$.
(II) For each $r \in \{0, 1, \ldots, t\}$, the matrix $A_r := A_0 B_1 \cdots B_r$ has invariant partition a^r.

Observe that, according to Theorem 3.7, given a sequence of $n \times n$ nonsingular matrices $A_0, B_1, \ldots, B_t$, where A_0 has invariant partition a, and B_r has elementary invariant partition $(1^{m_r}, 0^{n-m_r})$, $r = 1, \ldots, t$, $A_0, B_1, \ldots, B_t$ is a matrix realization of one and only one Young tableau of type $(a, (m_1, \ldots, m_t), c)$, where c is the invariant partition of $A_0 B_1 \ldots B_t$. In particular, $I, B_1, \ldots, B_t$ is a matrix realization of a Young tableau of type $(0, (m_1, \ldots, m_t), b)$. Thus, it is natural to give the following definition.

Definition 3.3. Let $\mathscr{T} = (a^0, a^1, \ldots, a^t)$ and $\mathscr{H} = (0, b^1, \ldots, b^t)$ be Young tableaux of types $(a, (m_1, \ldots, m_t), c)$ and $(0, (m_1, \ldots, m_t), b)$, respectively, where $l(c) \leqslant n$. We say that a sequence of $n \times n$ nonsingular matrices $A_0, B_1, \ldots, B_t$ is a matrix realization of the pair of Young tableaux $(\mathscr{T}, \mathscr{H})$ (or realizes $(\mathscr{T}, \mathscr{H})$) if:

(I) For each $r \in \{1, \ldots, t\}$, the matrix B_r has invariant partition $(1^{m_r}, 0^{n-m_r})$.
(II) For each $r \in \{0, 1, \ldots, t\}$, the matrix $A_r := A_0 B_1 \ldots B_r$ has invariant partition a^r.
(III) For each $r \in \{1, \ldots, t\}$, the matrix $B_1 \ldots B_r$ has invariant partition b^r.

$(\mathscr{T}, \mathscr{H})$ is called an admissible pair of tableaux.

Clearly $\mathscr{H} = \left(0, (1^{m_1}), \sum_{i=1}^{2}(1^{m_i}), \ldots, \sum_{i=1}^{t}(1^{m_i})\right)$ is the only Young tableau of type $(0, (m_1, \ldots, m_t), \sum_{i=1}^{t}(1^{m_i}))$, and its indexing sets are $[m_1], \ldots, [m_t]$. For the remainder of this paper, we shall consider pairs of Young tableaux $(\mathscr{T}, \mathscr{H})$, where $\mathscr{H}$ is this tableau. Thus, in order to verify property (III), it is sufficient to show that $B_1 \cdots B_t$ has invariant partition $(1^{m_1}) + \cdots + (1^{m_t})$.

Given a matrix realization $A_0, B_1, \ldots, B_t$ of a pair of Young tableaux $(\mathscr{T}, \mathscr{H})$, there are, in general, many sequences of matrices $S_1, \ldots, S_t$ realizing $\mathscr{H}$ and such that $B_1 \cdots B_t = S_1 \cdots S_t$. When $m_1 \geqslant \cdots \geqslant m_t$, it was proved in [2] that $A_0, S_1, \ldots, S_t$ is also a matrix realization of $(\mathscr{T}, \mathscr{H})$. The next theorem generalizes this result to any sequence $(m_1, \ldots, m_t)$.

Proposition 3.9 (Hermite normal form). *Given an $n \times n$ matrix A, there exists a matrix $U \in \mathscr{U}_n$ such that AU is lower triangular.*

Proof. See [15]. □

Theorem 3.10. *Let $A_0, B_1, \ldots, B_t$ be a matrix realization of the pair $(\mathscr{T}, \mathscr{H})$. Moreover, assume that we are given $n \times n$ matrices $S_1, \ldots, S_t$ such that $I, S_1, \ldots, S_t$ realizes $\mathscr{H}$ and $B_1 \cdots B_t = S_1 \cdots S_t$. Then $A_0, S_1, \ldots, S_t$ is also a matrix realization of $(\mathscr{T}, \mathscr{H})$.*

Proof. We may assume without loss of generality that $B = B_1 \ldots B_t = S_1 \ldots S_t$ is in Smith normal form $B = \operatorname{diag}_p((1^{m_1}) + \cdots + (1^{m_t}))$. We claim that there exist unimodular matrices $W_0, \ldots, W_t$ such that $W_0 = W_t = I$ and

$$W_{i-1}^{-1} B_i W_i \text{ is the Smith normal form of } B_i. \tag{15}$$

By the Hermite normal form theorem, there exist unimodular matrices $V_1, \ldots, V_{t-1}$ such that $B_1 V_1, V_1^{-1} B_2 V_2, \ldots, V_{t-2}^{-1} B_{t-1} V_{t-1}$ are lower triangular. It follows that $V_{t-1}^{-1} B_t$ is lower triangular as well. So, we may assume that each B_i is lower triangular and that its diagonal $D_i = \operatorname{diag}(B_i)$ has powers of p along the main diagonal. Thus, D_i contains m_i elements equal to p and the others equal to 1. As $D_1 \ldots D_t = \operatorname{diag}_p((1^{m_1}) + \cdots + (1^{m_t}))$, we find that D_i is the Smith normal form of B_i, for $i = 1, \ldots, t$. Therefore we may find lower triangular unimodular matrices $T_1, \ldots, T_{t-1}$ in such a way that $B_1 T_1 = D_1$, $T_1^{-1} B_2 T_2 = D_2, \ldots, T_{t-2}^{-1} B_{t-1} T_{t-1} = D_{t-1}$. This forces $T_{t-1}^{-1} B_t = D_t$. Our claim (15) is proved.

We may apply the same argument to the S_i's. Therefore $A_0 B_1 \cdots B_r$ and $A_0 S_1 \cdots S_r$ are right equivalent, for $r = 1, \ldots, t$. □

Let $I, B_1, \ldots, B_t$ be a matrix realization of $\mathscr{H}$. Since $B_1 \cdots B_t \sim_R U D_{[m_1]} \cdots D_{[m_t]}$ for some $n \times n$ unimodular matrix U, and $I, U D_{[m_1]}, \ldots, D_{[m_t]}$ is also a matrix realization of $\mathscr{H}$, it follows from previous theorem:

Corollary 3.11. *The following conditions are equivalent*:

(a) *$(\mathscr{T}, \mathscr{H})$ is an admissible pair.*
(b) *There exists $U \in \mathscr{U}_n$ such that $\Delta_a, U D_{[m_1]}, \ldots, D_{[m_t]}$ realizes $(\mathscr{T}, \mathscr{H})$.*
(c) *There exists $U \in \mathscr{U}_n$ such that $\Delta_a, U D_{[m_1]}, \ldots, D_{[m_t]}$ realizes $\mathscr{T}$.*

Therefore, when we are referring to a matrix realization of $(\mathscr{T}, \mathscr{H})$ we may assume, without loss of generality, that it is of the form Δ_a, $U D_{[m_1]}, \ldots, D_{[m_t]}$, for some $U \in \mathscr{U}_n$. Thus, often, we shall say that U realizes $\mathscr{T}$.

Next, we analyze the invariant partitions associated with product of matrices $\Delta_a U D_{[m_1]} D_{[m_2]}$, where $U \in \mathscr{U}_n$, and $m_1, m_2 \in [n]$.

Proposition 3.12. *Let $U \in \mathscr{U}_n$ and $m_1, m_2 \in [n]$. Then, there exist $\sigma \in \mathscr{S}_n$ and $I + X \in \mathscr{M}([m_1], [m_2])$, such that $\Delta_a U D_{[m_1]} D_{[m_2]} \sim \Delta_a P_\sigma D_{[m_1]} (I + X) D_{[m_2]} \sim \Delta_a U D_{[m_2]} D_{[m_1]}$ for every partition a of length $\leqslant n$.*

Proof. In view of the proof of Theorem 3.7, we may write

$$\Delta_a U D_{[m_1]} D_{[m_2]} \sim \Delta_a P_\sigma D_{[m_1]} Q' D_{[m_2]}$$

where Q' is as in (13). Without loss of generality, assume $m_1 \geqslant m_2$. We may write the matrix

$$Q' = \begin{bmatrix} A_1 & pA_2 & A_3 \\ 0 & A_4 & A_5 \\ 0 & 0 & A_6 \end{bmatrix},$$

where A_1, $m_2 \times m_2$, A_4, $(m_1 - m_2) \times (m_1 - m_2)$, and A_6, $(n - m_1) \times (n - m_1)$, are upper triangular matrices with 1's along its main diagonal and multiples of p above it. Hence,

$$Q' = \underbrace{\begin{bmatrix} I_{m_2} & 0 & X_1 \\ 0 & I_{m_1-m_2} & 0 \\ 0 & 0 & I_{n-m_1} \end{bmatrix}}_{I+X} \underbrace{\begin{bmatrix} A_1 & pA_2 & 0 \\ 0 & A_4 & A_5 \\ 0 & 0 & A_6 \end{bmatrix}}_{Q''},$$

where $X_1 = A_3 A_6^{-1}$, $I + X \in \mathscr{M}([m_1], [m_2])$ and Q'' is unimodular. Therefore,

$$\begin{aligned} \Delta_a P_\sigma D_{[m_1]} Q' D_{[m_2]} &= \Delta_a P_\sigma D_{[m_1]} (I + X) Q'' D_{[m_2]} \\ &\sim_R \Delta_a P_\sigma D_{[m_1]} (I + X) D_{[m_2]} \\ &= \Delta_a P_\sigma D_{[m_2]} D_{[m_1] \backslash [m_2]} (I + X) D_{[m_2]} \\ &= \Delta_a P_\sigma D_{[m_2]} (I + X) D_{[m_1]}. \qquad \square \end{aligned}$$

According to this proposition, it is enough to consider products of matrices $\Delta_a P_\sigma (I + X) D_F$, with $I + X \in \mathscr{M}(F)$.

Definition 3.4. Given $\sigma \in \mathscr{S}_n$, let $\{i_1, \ldots, i_n\} = [n]$ such that $[n] = \{\sigma(i_1) > \cdots > \sigma(i_n)\}$. We define $\hat{\sigma} \in \mathscr{S}_n$ by $\hat{\sigma}(i_k) = k$, for $k = 1, \ldots, n$.

We have $\sigma(i) \geqslant \sigma(j)$ if and only if $\hat{\sigma}(j) \geqslant \hat{\sigma}(i)$. Thus, given $A, B \subseteq [n]$ with $|A| = |B|$, we find that $\sigma(A) \geqslant \sigma(B)$ if and only if $\hat{\sigma}(B) \geqslant \hat{\sigma}(A)$.

Lemma 3.13 [3]. *Let $F \subseteq [n]$, $I + X \in \mathscr{M}(F)$ and $\sigma \in \mathscr{S}_n$. Then, there exist $\{i_1, \ldots, i_r\} \subseteq F$ and $\{j_1, \ldots, j_r\} \subseteq [n] \backslash F$, with $\sigma(i_s) > \sigma(j_s)$, for $s = 1, \ldots, r$, and $\sigma(i_1) > \cdots > \sigma(i_r)$, such that*

(i) $\Delta_a P_\sigma (I + X) D_F \sim \Delta_a P_\sigma (I + \sum_{k=1}^{r} E_{i_k j_k}) D_F$,
(ii) $\Delta_a P_{\hat{\sigma}} (I + X^{\mathrm{T}}) D_{\overline{F}} \sim \Delta_a P_{\hat{\sigma}} (I + \sum_{k=1}^{r} E_{j_k i_k}) D_{\overline{F}}$,

for every partition a of length $\leqslant n$.

Proof. Fix a partition a, arbitrarily, of length $\leqslant n$. Recall that $P_\sigma^{\mathrm{T}} \Delta_a P_\sigma = \mathrm{diag}_p(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. Without loss of generality, we may assume that all nonzero elements of X are units. Let x_{ij} be the unit in row $i \in F$, and column $j \notin F$ of X. If $\sigma(j) \geqslant \sigma(i)$ we use 1, in position (j, j) of $I + X$, as a pivot to zero out x_{ij} by row operations. Therefore, we may assume that $I + X \in \mathscr{M}(F)$ satisfy $x_{ij} \neq 0$ only if x_{ij} is a unit and $\sigma(i) > \sigma(j)$.

If $X = 0$, then $\Delta_a P_\sigma (I + X) D_F = \Delta_a P_\sigma D_F$.

If $X \neq 0$, let

$$\sigma(i_1) = \max\left\{\sigma(i) : i \in F \text{ and } \exists j : x_{ij} \neq 0\right\}$$

and

$$\sigma(j_1) = \min\left\{\sigma(j) : j \notin F \text{ and } x_{i_1 j} \neq 0\right\}.$$

Clearly, $\sigma(i_1) > \sigma(j_1)$. Also, if $i \in F$ and $x_{ij} \neq 0$, we have $\sigma(i_1) \geqslant \sigma(i)$. Then, we use the unit in position (i_1, j_1), say z_1, as a pivot to zero out the remaining entries of row i_1 and afterwards the remaining entries of column j_1 in X. Note that $i_1 \in F$ and $j_1 \notin F$.

Therefore, $(I + X)D_F \sim_R T(I + X_1 + z_1 E_{i_1 j_1})D_F$, where z_1 is a unit, T is a product of elementary matrices $T_{ii_1}(x)$ such that $\sigma(i_1) > \sigma(i)$, $I + X_1 \in \mathscr{M}(F)$, and $X_1 = (x^1_{ij})$ has row i_1 and column j_1 null, and $x^1_{ij} \neq 0$ only if x^1_{ij} is a unit and $\sigma(i) > \sigma(j)$.

If $X_1 = 0$, the reduction process is finished. If not, we repeat the above process with the matrix X_1. Eventually, after a finite number of steps, we obtain

$$(I + X)D_F \sim_R T'(I + z_1 E_{i_1 j_1} + \cdots + z_r E_{i_r j_r})D_F,$$

where $z_1, \ldots, z_r$ are units, $i_1, \ldots, i_r$ are distinct elements of F, and $j_1, \ldots, j_r$ are distinct elements of $\{1, \ldots, n\}\backslash F$ such that $\sigma(i_s) > \sigma(j_s)$, for $s = 1, \ldots, r$, and $\sigma(i_1) > \cdots > \sigma(i_r)$, and T' is a product of elementary matrices $T_{ki}(x)$ such that $\sigma(i) > \sigma(k)$.

Let $Y := \operatorname{diag}(y_1, \ldots, y_n)$, where $y_s = z_s^{-1}$ if $s \in \{i_1, \ldots, i_r\}$, and $y_s = 1$ if $s \notin \{i_1, \ldots, i_r\}$. Then

$$Y^{-1}(I + E_{i_1 j_1} + \cdots + E_{i_r j_r})Y = I + z_1 E_{i_1 j_1} + \cdots + z_r E_{i_r j_r},$$

and we may write

$$\Delta_a P_\sigma (I + X) D_F \sim_R \Delta_a P_\sigma T' Y (I + E_{i_1 j_1} + \cdots + E_{i_r j_r}) D_F. \tag{16}$$

Since T' is a product of elementary matrices $T_{ki}(x)$ with $\sigma(i) > \sigma(k)$, using row operations, we find that $\Delta_a P_\sigma T' \sim_L \Delta_a P_\sigma$. Therefore,

$$(16) \sim_L \Delta_a P_\sigma (I + E_{i_1 j_1} + \cdots + E_{i_r j_r}) D_F.$$

Finally, recalling that $I + X^{\mathrm{T}} \in \mathscr{M}(\overline{F})$, and that $\sigma(i) \geqslant \sigma(j)$ if and only if $\hat{\sigma}(j) \geqslant \hat{\sigma}(i)$, we may repeat on $\Delta_a P_{\hat{\sigma}} (I + X)^{\mathrm{T}} D_{\overline{F}}$ the operations performed on $\Delta_a P_\sigma (I + X) D_F$, to get equation (i). In this way, we obtain equation (ii). $\square$

Notice that in this lemma, $\xi = (i_1 j_1) \cdots (i_r j_r) \in \mathscr{S}_n$ satisfy $\sigma(F) \geqslant \sigma\xi(F)$. This leads us to the following definition.

Definition 3.5. Let $F, J \subseteq [n]$ and $\sigma \in \mathscr{S}_n$ such that $|F| = |J| = m$ and $\sigma(F) \geqslant J$. Let $\varepsilon \in \mathscr{S}_m$ be a witness of $\sigma(F) \geqslant J$. We define the $n \times n$ matrix $S(\sigma(F), J, \sigma, \varepsilon)$, whose entry s_{ij} satisfy

$$s_{ij} = \begin{cases} 1 & \text{if } \sigma(i) \in \sigma(F)\backslash J \text{ and } \lambda_{\sigma(F),J,\varepsilon}\sigma(i) = \sigma(j), \\ 0 & \text{otherwise.} \end{cases}$$

When $\varepsilon = id$, we write $S(\sigma(F), J, \sigma)$.

Clearly, $I + S(\sigma(F), J, \sigma, \varepsilon) \in \mathscr{M}(F)$, and if $J = \sigma(F)$, $S(\sigma(F), J, \sigma) = 0$. Notice that for each witness $\varepsilon \in \mathscr{S}_m$ of $\sigma(F) \geqslant J$, in the conditions of definition 2.2, there exist $\{i_1, \ldots, i_r\} \subseteq F$ with $\sigma(i_1) > \cdots > \sigma(i_r)$, and $\{j_1, \ldots, j_r\} \subseteq [n]\backslash F$ with $\sigma(i_s) > \sigma(j_s)$, for $s = 1, \ldots, r$, and $\sigma(j_{\varepsilon(1)}) > \cdots > \sigma(j_{\varepsilon(r)})$, such that $\sigma^{-1}\lambda_{\sigma(F),J,\varepsilon}\sigma = (i_1 j_1) \cdots (i_r j_r)$. Therefore, $S(\sigma(F), J, \sigma, \varepsilon) = \sum_{k=1}^{r} E_{i_k j_k}$.

Lemma 3.14. *In the conditions of the definition above, put $S_\varepsilon = S(\sigma(F), J, \sigma, \varepsilon)$. Then, we have always*

$$\begin{aligned} \Delta_a P_\sigma (I + S_\varepsilon) D_F &\sim \Delta_a P_\sigma P_{(\sigma^{-1}\lambda_{\sigma(F),J,\varepsilon}\sigma)} D_F \\ &\sim_R \operatorname{diag}_p(a + \chi^J), \end{aligned}$$

for every partition a of length $\leqslant n$. In other words, the invariant partition of $\Delta_a P_\sigma (I + S_\varepsilon) D_F$ does not depend on the witness ε of $\sigma(F) \geqslant J$.

Proof. Fix an arbitrary partition a. Recall that $I + S_\varepsilon \in \mathscr{M}(F)$. Thus, we have

$$\begin{aligned} \Delta_a P_\sigma (I + S_\varepsilon) D_F &\sim_R \Delta_a P_\sigma (I + S_\varepsilon) D_F (I - p S_\varepsilon^{\mathrm{T}}) &\quad (17)\\ &= \Delta_a P_\sigma (I + S_\varepsilon)(I - S_\varepsilon^{\mathrm{T}}) D_F. &\quad (18) \end{aligned}$$

Consider now the permutation $\sigma^{-1}\lambda_{\sigma(F),J,\varepsilon}\sigma = (i_1 j_1) \cdots (i_r j_r)$, and note that, by Lemma 3.1, we have

$$P_{(\sigma^{-1}\lambda_{J,\sigma(F),\varepsilon}\sigma)} = Z(I - S_\varepsilon^{\mathrm{T}})(I + S_\varepsilon)(I - S_\varepsilon^{\mathrm{T}}),$$

where $Z = \prod_{k=1}^{r} T_{j_k j_k}(-1)$. Since $\sigma(i_s) > \sigma(j_s)$, $s = 1, \ldots, r$, we may use row operations to zero out all nonzero elements of $S_\varepsilon^{\mathrm{T}}$, and obtain

$$\Delta_a P_\sigma Z\big(I - S_\varepsilon^{\mathrm{T}}\big) \sim_L \Delta_a P_\sigma.$$

Therefore, we have

$$\begin{aligned} \Delta_a P_\sigma P_{(\sigma^{-1}\lambda_{\sigma(F),J,\varepsilon}\sigma)} D_F &= \Delta_a P_\sigma Z(I - S_\varepsilon^{\mathrm{T}})(I + S_\varepsilon)(I - S_\varepsilon^{\mathrm{T}}) D_F \\ &\sim_L \Delta_a P_\sigma (I + S_\varepsilon)(I - S_\varepsilon^{\mathrm{T}}) D_F. &\quad (19) \end{aligned}$$

By (17) and (19) we find that

$$\Delta_a P_\sigma(I+X)D_F \sim \Delta_a P_{(\lambda_{\sigma(F),J,\varepsilon}\sigma)}D_F \sim_R \operatorname{diag}_p(a+\chi^J). \qquad \square$$

Theorem 3.15. *Given $F \subseteq [n]$, $I+X \in \mathscr{M}(F)$, and $\sigma \in \mathscr{S}_n$, there exists $J \subseteq [n]$ with $|J| = |F|$ and $\sigma(F) \geqslant J$, such that, by putting $S = S(\sigma(F), J, \sigma)$,*

$$\begin{aligned}\Delta_a P_\sigma(I+X)D_F &\sim \Delta_a P_\sigma(I+S)D_F\\ &\sim \Delta_a P_\sigma P_{(\sigma^{-1}\lambda_{\sigma(F),J}\sigma)}D_F\\ &\sim_R \operatorname{diag}_p(a+\chi^J),\end{aligned}$$

for every partition a of length $\leqslant n$.

Proof. Fix a partition a. Let $m := |F|$. By Lemma 3.13, there exist $\{i_1, \ldots, i_r\} \subseteq F$ and $\{j_1, \ldots, j_r\} \subseteq [n]\backslash F$ with $\sigma(i_s) > \sigma(j_s)$, for $s = 1, \ldots, r$, and $\sigma(i_1) > \cdots > \sigma(i_r)$, such that

$$\Delta_a P_\sigma(I+X)D_F \sim \Delta_a P_\sigma(I+E_{i_1 j_1}+\cdots+E_{i_r j_r})D_F.$$

Let $J := \left[\sigma(F)\backslash\{\sigma(i_1), \ldots, \sigma(i_r)\}\right] \cup \{\sigma(j_1), \ldots, \sigma(j_r)\}$. Clearly, $\sigma(F) \geqslant J$, and the permutation $\varepsilon \in \mathscr{S}_m$ such that $\sigma(j_{\varepsilon(1)}) > \cdots > \sigma(j_{\varepsilon(r)})$ is a witness of $\sigma(F) \geqslant J$. Thus, $\lambda_{\sigma(F),J,\varepsilon} = (\sigma(i_1)\sigma(j_1))\ldots(\sigma(i_r)\sigma(j_r))$, and, by definition of $S_\varepsilon = S(\sigma(F), J, \sigma, \varepsilon)$, we obtain $I + S_\varepsilon = I + E_{i_1 j_1} + \cdots + E_{i_r j_r}$. Therefore,

$$\Delta_a P_\sigma(I+X)D_F \sim \Delta_a P_\sigma(I+S_\varepsilon)D_F \sim \operatorname{diag}_p(a+\chi^J).$$

From previous lemma, we may choose $\varepsilon = id$, hence

$$\begin{aligned}\Delta_a P_\sigma(I+X)D_F &\sim \Delta_a P_\sigma(I+S_\varepsilon)D_F \sim \Delta_a P_\sigma(I+S)D_F\\ &\sim \operatorname{diag}_p(a+\chi^J). \qquad \square\end{aligned}$$

Observe that if $a+\chi^J$ is not a partition then, by Lemma 3.6, there exists a permutation μ such that $\Delta_a D_J \sim \operatorname{diag}_p(a+\chi^{\mu(J)})$ and $\sigma(F) \geqslant J \geqslant \mu(J)$. Therefore, we obtain:

Corollary 3.16. *In the conditions of the theorem above, given a partition a, let $a+\chi^J$ be the invariant partition of $\Delta_a P_\sigma(I+X)D_F$. If a' is a partition of length $\leqslant n$ such that either a'/a or a/a' is a vertical strip, then there exists $\mu \in \mathscr{S}_n$ such that the invariant partition of $\Delta_{a'} P_\sigma(I+X)D_F$ is given by $a'+\chi^{\mu(J)}$, where $J \geqslant \mu(J)$.*

This corollary will be useful in the following section. As an application of the previous theorem, we shall characterize the tableaux realized by a sequence of matrices of the form $\Delta_a, UD_{[m_1]}, \ldots, D_{[m_t]}$, where $n \geqslant m_1 \geqslant \cdots \geqslant m_t \geqslant 1$.

Proposition 3.17 [3]. *Let* $U \in \mathscr{U}_n$ *and* $n \geqslant m_1 \geqslant m_2 \geqslant 1$. *Then* $\Delta_a, UD_{[m_1]}, D_{[m_2]}$ *realizes an LR tableau of weight* (m_1, m_2).

Proof. By Lemma 3.6 and Proposition 3.12, there exists $\sigma \in \mathscr{S}_n$ such that

$$\Delta_a UD_{[m_1]}D_{[m_2]} \sim \Delta_a P_\sigma D_{[m_1]}(I+X)D_{[m_2]} = \Delta_a D_{\sigma[m_1]}P_\sigma(I+X)D_{[m_2]} \tag{20}$$

with $I+X \in \mathscr{M}([m_1],[m_2])$ and $a+\chi^{\sigma[m_1]}$ a partition. Let $J_1 := \sigma[m_1]$.

By Theorem 3.15, there exists $J_2 \subseteq [n]$ with $\sigma[m_2] \geqslant J_2$ and $|J_2| = m_2$, such that

$$(20) \sim \mathrm{diag}_p(a+\chi^{J_1}+\chi^{J_2}),$$

with $a+\chi^{J_1}+\chi^{J_2}$ is a partition.

Finally, note that $J_1 = \sigma[m_1] \geqslant \sigma[m_2] \geqslant J_2$. Then $\Delta_a, UD_{[m_1]}, D_{[m_2]}$ is a matrix realization of the LR tableau $\mathscr{T} = (a, a+\chi^{J_1}, a+\chi^{J_1}+\chi^{J_2})$. $\square$

Next result generalizes the proposition above.

Theorem 3.18 [3]. *Let* $U \in \mathscr{U}_n$ *and* $n \geqslant m_1 \geqslant \cdots \geqslant m_t \geqslant 1$. *Then* $\Delta_a, UD_{[m_1]}, \ldots, D_{[m_t]}$ *realizes an LR tableau of weight* $(m_1, \ldots, m_t)$.

Proof. By induction on t. For $t = 1$ there exists a permutation $\sigma \in \mathscr{S}_n$ such that $\Delta_a UD_{[m_1]} \sim \mathrm{diag}_p(a+\chi^{\sigma[m_1]})$ where $a+\chi^{\sigma[m_1]}$ is a partition. Therefore, Δ_a, $UD_{[m_1]}$ realizes the tableau $\mathscr{T} = (a, a+\chi^{\sigma[m_1]})$, which is an LR tableau. The case $t = 2$ was proved in previous lemma.

Let $t > 2$. By induction, the sequence $\Delta_a U, D_{[m_1]}, \ldots, D_{[m_{t-1}]}$ is a matrix realization of an LR tableau with indexing sets $J_1 \geqslant \cdots \geqslant J_{t-1}$. Therefore, there exists an $n \times n$ unimodular matrix V such that

$$\Delta_a UD_{[m_1]} \cdots D_{[m_{t-1}]}D_{[m_t]} \sim_L \Delta^1 VD_{[m_{t-1}]}D_{[m_t]},$$

where $\Delta^1 = \mathrm{diag}_p(a+\chi^{J_1}+\cdots+\chi^{J_{t-2}})$.

By the previous lemma, $\Delta^1 VD_{[m_{t-1}]}D_{[m_t]}$ realizes an LR tableau $\mathscr{T}'$ with indexing sets $J_{t-1} \geqslant J_t$. Therefore, $\Delta_a UD_{[m_1]} \ldots D_{[m_t]}$ realizes the LR tableau $\mathscr{T} = (a, a^1, \ldots, a^t)$, with $a^i = a+\chi^{J_1}+\cdots+\chi^{J_i}$, for $i = 1, \ldots, t$. $\square$

In view of this result, we conclude that a pair of Young tableaux $(\mathscr{T}, \mathscr{H})$ of weight $(m_1, \ldots, m_t)$, where $m_1 \geqslant \cdots \geqslant m_t$, is an admissible pair only if $\mathscr{T}$ is an LR tableau. In [2,3] was also proved that $(\mathscr{T}, \mathscr{H})$ is an admissible pair if $\mathscr{T}$ is an LR tableau. We shall recover the "if" part in the last section for $t = 2, 3$. In [1], using a different characterization of LR tableau, the "if" part was proved as well.

3.3. Matrix extensions

Let X be an $n \times n$ matrix, and denote by $R(X)$ the set of the indices of the nonnull rows of X, and by $C(X)$ the set of the indices of the nonnull columns of X. Given an $n \times n$ matrix Z, we say that Z is an *extension* of X if there exists an $n \times n$ matrix $X' = (x'_{ij})$ with $x'_{ij} \neq 0$ only if $x_{ij} = 0$ such that $Z = X + X'$. When $Z = X + X'$ is an extension of X such that $C(X) \cap C(X') = \emptyset$ [$R(X) \cap R(X') = \emptyset$], we say that Z is a *column* [*row*] *extension* of X.

Let $F \subseteq [n]$, $\sigma \in \mathscr{S}_n$ and $I + X \in \mathscr{M}(F)$. By the application of Theorem 3.15 and Lemma 3.6, we conclude that, for every partition a, there exists $J \subseteq [n]$ such that the invariant partition of the product of matrices

$$\Delta_a P_\sigma (I + X) D_F \tag{21}$$

is $a + \chi^J$. In the following results, using Lemma 3.13, we analyze the relationship between the invariant partition of the product (21) and the product $\Delta_a P_\sigma (I + Z) D_F$, with $I + Z \in \mathscr{M}(F)$ and Z an extension of X. We start with the case where Z is a column extension of X.

Lemma 3.19. *Let $F \subseteq [n]$, $\{i_1, \ldots, i_r\} \subseteq F$, $\{j_0, j_1, \ldots, j_r\} \subseteq [n]\backslash F$ and $\sigma \in \mathscr{S}_n$ such that $\sigma(i_k) > \sigma(j_k)$, $k = 1, \ldots, r$. Consider a matrix X' such that $C(X') = \{j_0\}$ and $R(X') \subseteq F$. Then, there exist $\{v_1, \ldots, v_s\} \subseteq F$ and $\{f_1, \ldots, f_s\} \subseteq \{j_0, j_1, \ldots, j_r\}$, with $\sigma(v_k) > \sigma(f_k)$, $k = 1, \ldots, s$, and $\sigma\xi(F) \geqslant \sigma\xi'(F)$, where $\xi = (i_1 j_1) \cdots (i_r j_r)$ and $\xi' = (v_1 f_1) \cdots (v_s f_s)$, such that*

$$\Delta_a P_\sigma \left(I + \sum_{k=1}^{r} E_{i_k j_k} \right) D_F \sim \operatorname{diag}_p(a + \chi^{\sigma\xi(F)})$$

and

$$\begin{aligned} \Delta_a P_\sigma \left(I + \sum_{k=1}^{r} E_{i_k j_k} + X' \right) D_F &\sim \Delta_a P_\sigma \left(I + \sum_{k=1}^{s} E_{v_k f_k} \right) D_F \\ &\sim \operatorname{diag}_p(a + \chi^{\sigma\xi'(F)}), \end{aligned}$$

for every partition a of length $\leqslant n$.

Proof. Fix a partition a. The proof will be handle by induction on the number m of nonzero entries of X'. Let $X = \sum_{k=1}^{r} E_{i_k j_k}$ and notice that, by Theorem 3.15, we have $\Delta_a P_\sigma (I + X) D_F \sim \operatorname{diag}_p(a + \chi^{\sigma\xi(F)})$, where $\xi = (i_1 j_1) \cdots (i_r j_r)$.

Without loss of generality, we may assume that all nonzero entries of $X' = (x_{ij})$ are units, and that $x_{ij_0} \neq 0$ only if $\sigma(i) > \sigma(j_0)$, with $i \in F$ and $j_0 \notin F$.

Suppose that $m = 1$, that is, $X' = z_0 E_{i_0 j_0}$ for some unit z_0. Clearly, $\sigma(i_0) > \sigma(j_0)$. If $i_0 \notin R(X)$, then by Theorem 3.15, we have

$$\Delta_a P_\sigma(I + X + X')D_F = \Delta_a P_\sigma \left(I + \sum_{k=0}^{r} E_{i_k j_k} \right) D_F \sim \operatorname{diag}_p(a + \chi^{\sigma\xi'(F)}),$$

where $\xi' = (i_0 j_0)(i_1 j_1) \cdots (i_r j_r)$ satisfy $\sigma\xi(F) \geqslant \sigma\xi'(F)$. If $i_0 \in R(X)$, without loss of generality, we may assume that $i_0 = i_1$. Now, either we have $\sigma(j_0) > \sigma(j_1)$ or $\sigma(j_1) > \sigma(j_0)$.

If $\sigma(j_0) > \sigma(j_1)$, since $j_1, j_0 \notin F$, we may eliminate z_0 by column operations, using the unit in position (i_1, j_1) as a pivot, obtaining $\Delta_a P_\sigma(I + X + X')D_F \sim \Delta_a P_\sigma(I + X)D_F$. Clearly, $\xi = \xi'$.

If $\sigma(j_1) > \sigma(j_0)$, since $j_1, j_0 \notin F$, we use z_0 as a pivot to eliminate, by column operations, the unit in position (i_1, j_1). Thus, by Theorem 3.15, we find that

$$\begin{aligned}\Delta_a P_\sigma(I + X + X')D_F &\sim \Delta_a P_\sigma \left(I + E_{i_1 j_0} + \sum_{k=2}^{r} E_{i_k j_k} \right) D_F \\ &\sim \operatorname{diag}_p(a + \chi^{\sigma\xi'(F)}),\end{aligned}$$

where $\xi' := (i_1 j_0)(i_2 j_2) \cdots (i_r j_r)$ satisfy $\sigma\xi(F) \geqslant \sigma\xi'(F)$, since $\sigma(j_1) > \sigma(j_0)$.

Now, suppose $m > 1$. Let $X' = (x_{ij})$, and denote by z_0 the unit in position (i_0, j_0) of X', where $\sigma(i_0) := max\{\sigma(i) : i \in F \text{ and } x_{i,j_0} \neq 0\}$. If $i_0 \notin R(X)$, then we may use z_0 to eliminate, by row operations, all entries of column j_0 of X', obtaining

$$\Delta_a P_\sigma(I + X + X')D_F \sim \Delta_a P_\sigma \left(I + \sum_{k=0}^{r} E_{i_k j_k} \right) D_F \sim \operatorname{diag}_p(a + \chi^{\sigma\xi'(F)}),$$

where $\xi' = (i_0 j_0)(i_1 j_1) \cdots (i_r j_r)$ satisfy $\sigma\xi(F) \geqslant \sigma\xi'(F)$.

Assume now that $i_0 = i_1 \in R(X)$. If $\sigma(j_0) > \sigma(j_1)$, we use the unit in position (i_1, j_1), as a pivot, to eliminate z_0 by column operations. Thus, for every partition a, we have

$$\Delta_a P_\sigma(I + X + X')D_F \sim \Delta_a P_\sigma(I + X + X'')D_F, \tag{22}$$

where X'' has $m - 1$ nonzero entries in column j_0, and zero elsewhere. By the inductive step and Theorem 3.15, there exist $\{v_1, \ldots, v_s\} \subseteq F$ and $\{f_1, \ldots, f_s\} \subseteq \{j_0, j_1, \ldots, j_r\}$ with $\sigma(v_k) > \sigma(f_k)$, $k = 1, \ldots, s$, and $\sigma\xi(F) \geqslant \sigma\xi'(F)$, such that

$$(22) \sim \Delta_a P_\sigma \left(I + \sum_{k=1}^{s} E_{v_k f_k} \right) D_F \sim \operatorname{diag}_p(a + \chi^{\sigma\xi'(F)}),$$

where $\xi' = (v_1 f_1) \cdots (v_s f_s)$.

If $\sigma(j_1) > \sigma(j_0)$, we use z_0 to zero out, by column operations, the unit in position (i_1, j_1), and all entries of column j_0 of X', by row operations. Therefore, we obtain

$$\Delta_a P_\sigma(I + X + X')D_F \sim \Delta_a P_\sigma \left(I + E_{i_1 j_0} + \sum_{k=2}^{r} E_{i_k j_k} + X'' \right) D_F, \tag{23}$$

where X'' has $m - 1$ nonzero entries in column j_1, and zero elsewhere. Notice that, by Theorem 3.15, $\Delta_a P_\sigma(I + E_{i_1 j_0} + \sum_{k=2}^{r} E_{i_k j_k})D_F \sim \operatorname{diag}_p(a + \chi^{\sigma\xi'(F)})$, where $\xi' := (i_1 j_0)(i_2 j_2) \cdots (i_r j_r)$ satisfy $\sigma\xi(F) \geqslant \sigma\xi'(F)$. Then, by the inductive step, there exist $\{v_1, \ldots, v_s\} \subseteq F$, and $\{f_1, \ldots, f_s\} \subseteq \{j_0, j_1, \ldots, j_r\}$ with $\sigma(v_k) > \sigma(f_k)$, $k = 1, \ldots, s$, such that $\sigma\xi(F) \geqslant \sigma\xi'(F) \geqslant \sigma\xi''(F)$ and

$$(23) \sim \Delta_a P_\sigma \left(I + \sum_{k=1}^{s} E_{v_k f_k} \right) D_F \sim \operatorname{diag}_p(a + \chi^{\sigma\xi''(F)}),$$

where $\xi'' = (v_1 f_1) \cdots (v_s f_s)$. $\square$

Theorem 3.20. *Let $F \subseteq [n]$ and $\sigma \in \mathscr{S}_n$. Let $I + X, I + Z \in \mathscr{M}(F)$ such that Z is a column extension of X. Then, there exist $J, J' \subseteq [n]$ with $J \geqslant J'$ satisfying*

$$\Delta_a P_\sigma (I + X) D_F \sim \operatorname{diag}_p(a + \chi^J),$$

$$\Delta_a P_\sigma (I + Z) D_F \sim \operatorname{diag}_p(a + \chi^{J'}),$$

for every partition a of length $\leqslant n$.

Proof. Fix a partition a, arbitrarily. Since Z is a column extension of X, we have $Z = X + X'$ such that $C(X) \cap C(X') = \emptyset$. Without loss of generality, we may assume that all nonzero entries of X and X' are units. As in Lemma 3.13, using row operations, let us zero out the elements x_{ij} of X and x'_{ij} of X' such that $\sigma(j) > \sigma(i)$.

Using the decomposition, of Lemma 3.13, on matrix $I + X$, there exist $\{i_1, \ldots, i_r\} \subseteq [n]$ and $\{j_1, \ldots, j_r\} \subseteq [n] \backslash F$ such that $\sigma(i_k) > \sigma(j_k)$, $k = 1, \ldots, r$, $\sigma(i_1) > \cdots > \sigma(i_r)$, and

$$\Delta_a P_\sigma (I + X) D_F \sim \Delta_a P_\sigma \left(I + \sum_{k=1}^{r} E_{i_k j_k} \right) D_F. \tag{24}$$

By Theorem 3.15, we find that (24) $\sim \operatorname{diag}_p(a + \chi^J)$, where $\sigma(F) \geqslant J = \sigma\xi(F)$ with $\xi = (i_1 j_1) \cdots (i_r j_r)$. We may repeat on $I + X + X'$ the operations just performed on $I + X$ to get (24). So we have

$$\Delta_a P_\sigma (I + X + X') D_F \sim \Delta_a P_\sigma \left(I + \sum_{k=1}^{r} E_{i_k j_k} + Y \right) D_F, \tag{25}$$

where the matrix Y satisfy $R(Y) \subseteq F$ and $C(Y) \cap C(X) = \emptyset$.

We will prove, by induction on the number $m := |C(Y)|$, the existence of a set $J' \subseteq [n]$ such that $J \geqslant J'$ and (25) $\sim \operatorname{diag}_p(a + \chi^{J'})$.

When $m = 1$, the result was proved in Proposition 3.19. Suppose now $m > 1$. Let $j_0 \in C(Y)$ and consider the matrix Y' obtained from Y by replacing all nonzero entries, outside column j_0, by zero. Again, by Proposition 3.19, there exist $\{v_1, \ldots, v_s\} \subseteq F$ and $\{f_1, \ldots, f_s\} \subseteq \{j_0, j_1, \ldots, j_r\}$ with $\sigma(v_k) > \sigma(f_k)$, $k = 1, \ldots, s$, and $J \geqslant \sigma\xi'(F)$, such that

$$\Delta_a P_\sigma \left(I + \sum_{k=1}^{r} E_{i_k j_k} + Y' \right) D_F \sim \Delta_a P_\sigma \left(I + \sum_{k=1}^{s} E_{v_k f_k} \right) D_F$$
$$\sim \mathrm{diag}_p(a + \chi^{\sigma\xi'(F)}), \tag{26}$$

where $\xi' = (v_1 f_1) \cdots (v_s f_s)$. We may repeat on $I + \sum_{k=1}^{r} E_{i_k j_k} + Y$ the operations just performed on $I + \sum_{k=1}^{r} E_{i_k j_k} + Y'$ to get (26). Therefore, we obtain

$$(25) \sim \Delta_a P_\sigma \left(I + \sum_{k=1}^{s} E_{v_k f_k} + Y'' \right) D_F, \tag{27}$$

where Y'' satisfy $C(Y'') \cap \{f_1, \ldots, f_s\} = \emptyset$ and $|C(Y'')| = m - 1$. Applying the inductive step to equations (26) and (27), there exists $J' \subseteq [n]$ such that (27) $\sim \mathrm{diag}_p(a + \chi^{J'})$ and $J \geqslant \sigma\xi'(F) \geqslant J'$. $\square$

Next, we prove the analogous of the theorem above, in the case, of a row extension of X.

Theorem 3.21. *Let $F \subseteq [n]$ and $\sigma \in \mathscr{S}_n$. Let $I + X, I + Z \in \mathscr{M}(F)$ such that Z is a row extension of X. Then, there exist $J, J' \subseteq [n]$ with $J \geqslant J'$ satisfying*

$$\Delta_a P_\sigma (I + X) D_F \sim \mathrm{diag}_p(a + \chi^J),$$

$$\Delta_a P_\sigma (I + Z) D_F \sim \mathrm{diag}_p(a + \chi^{J'}),$$

for every partition a of length $\leqslant n$.

Proof. Let a be an arbitrarily partition. Since Z is a row extension of X, we must have $Z = X + X'$, where $R(X) \cap R(X') = \emptyset$. Note that $I + X^{\mathrm{T}}, I + X + X'^{T} \in \mathscr{M}(\overline{F})$ with $C(X'^{T}) \cap C(X^{\mathrm{T}}) = \emptyset$. In view of the proof of Theorem 3.20, there exist $\xi, \xi' \in \mathscr{S}_n$ such that

$$\Delta_a P_{\hat\sigma} (I + X^{\mathrm{T}}) D_{\overline{F}} \sim \Delta_a P_{\hat\sigma} P_\xi D_{\overline{F}} \sim \mathrm{diag}_p(a + \chi^{\hat\sigma\xi(\overline{F})})$$

and

$$\Delta_a P_{\hat\sigma} (I + X^{\mathrm{T}} + X'^{T}) D_{\overline{F}} \sim \Delta_a P_{\hat\sigma} P_{\xi'} D_{\overline{F}} \sim \mathrm{diag}_p(a + \chi^{\hat\sigma\xi'(\overline{F})}),$$

with $\hat\sigma\xi(\overline{F}) \geqslant \hat\sigma\xi'(\overline{F})$. Thus, we have $\hat\sigma\xi'(F) \geqslant \hat\sigma\xi(F)$, and, by the definition of $\hat\sigma$ (Definition 3.4), we find that $\sigma\xi(F) \geqslant \sigma\xi'(F)$. Finally, recall from (i) and (ii) of Lemma 3.13, that the permutations ξ, ξ' are such that

$$\Delta_a P_\sigma (I + X) D_F \sim \Delta_a P_\sigma P_\xi D_F \sim \mathrm{diag}_p(a + \chi^{\sigma\xi(F)}),$$

and

$$\Delta_a P_\sigma (I + X + X') D_F \sim \Delta_a P_\sigma P_{\xi'} D_F \sim \mathrm{diag}_p(a + \chi^{\sigma\xi'(F)}). \qquad \square$$

Next theorem states the relationship between the invariant partition of the product of matrices $\Delta_a P_\sigma(I+X)D_F$ and $\Delta_a P_\sigma(I+Z)D_F$, when Z is an extension of X and $I+X, I+Z \in \mathscr{M}(F)$.

Theorem 3.22. *Let $F \subseteq [n]$ and $\sigma \in \mathscr{S}_n$. Let $I+X, I+Z \in \mathscr{M}(F)$ such that Z is an extension of X. Then, there exist $J, J' \subseteq [n]$ with $J \geqslant J'$ satisfying*

$$\Delta_a P_\sigma(I+X)D_F \sim \operatorname{diag}_p(a+\chi^J),$$

$$\Delta_a P_\sigma(I+Z)D_F \sim \operatorname{diag}_p(a+\chi^{J'}),$$

for every partition a of length $\leqslant n$.

Proof. Fix a partition a. Since Z is an extension of X, there exists an $n \times n$ matrix X' such that $Z = X + X'$. Let Y be the matrix obtained from X' by replacing all entries x'_{ij} with $i \notin R(X)$ by zero. Thus, $I+X+Y \in \mathscr{M}(F)$ and $C(Y) \cap C(X) = \emptyset$. By Theorem 3.20, there exist $J, \widehat{J} \subseteq [n]$ such that $J \geqslant \widehat{J}$,

$$\Delta_a P_\sigma(I+X)D_F \sim \operatorname{diag}_p(a+\chi^J)$$

and

$$\Delta_a P_\sigma(I+X+Y)D_F \sim \operatorname{diag}_p(a+\chi^{\widehat{J}}).$$

Let $Y' := X' - Y$ and notice that $R(Y') \cap R(X+Y) = \emptyset$. Therefore, by Theorem 3.21, there exists $J' \subseteq [n]$ with $J \geqslant \widehat{J} \geqslant J'$ such that

$$\Delta_a P_\sigma(I+X+Y+Y')D_F \sim \operatorname{diag}_p(a+\chi^{J'}). \qquad \square$$

Notice that if, in the theorem above, either $a+\chi^J$ or $a+\chi^{J'}$ is not a partition then, by Lemma 3.6, there exist permutations $\mu, \mu' \in \mathscr{S}_n$ such that $\operatorname{diag}_p(a+\chi^J) \sim_L \operatorname{diag}_p(a+\chi^{\mu(J)})$ and $\operatorname{diag}_p(a+\chi^{J'}) \sim_L \operatorname{diag}_p(a+\chi^{\mu'(J')})$, with $a+\chi^{\mu(J)}$ and $a+\chi^{\mu'(J')}$ partitions, and satisfying $J \geqslant \mu(J)$, $J' \geqslant \mu'(J')$, and $\mu(J) \geqslant \mu'(J')$. Therefore, without loss of generality, we may assume that the sets J, J' are such that $a+\chi^J$ and $a+\chi^{J'}$ are partitions.

Corollary 3.23. *Let $U \in \mathscr{U}_n$ and $1 \leqslant m_3 \leqslant m_2 \leqslant m_1 \leqslant n$.*

(i) *If J_1, J_2 and F_1, F_2 are the indexing sets of $\Delta_a U D_{[m_1]} D_{[m_3]}$ and $\Delta_a U D_{[m_2]} D_{[m_3]}$, respectively, then $J_2 \geqslant F_2$.*
(ii) *If J_1, J_2 and F_1, F_2 are the indexing sets of $\Delta_a U D_{[m_3]} D_{[m_1]}$ and $\Delta_a U D_{[m_2]} D_{[m_1]}$, respectively, then $J_2 \geqslant F_2$.*

Proof. We may assume $U = P_\sigma Q$, where $\sigma \in \mathscr{S}_n$ and Q is an upper triangular matrix, with 1's along the main diagonal, and multiples of p above it. Without loss of generality, assume that $a^i := a + \chi^{\sigma[m_i]}$ is a partition, $i = 1, 2, 3$.

(i) By Proposition 3.12, we may write

$$\Delta_a P_\sigma Q D_{[m_1]} D_{[m_3]} \sim \mathrm{diag}_p(a^1) P_\sigma (I+X) D_{[m_3]} \sim \mathrm{diag}_p(a^1 + \chi^{J_2}),$$

where $I + X \in \mathscr{M}([m_1],[m_3])$, and

$$\Delta_a P_\sigma Q D_{[m_2]} D_{[m_3]} \sim \mathrm{diag}_p(a^2) P_\sigma (I+Y+Y') D_{[m_3]} \sim \mathrm{diag}_p(a^2 + \chi^{F_2}), \tag{28}$$

where $I+Y+Y' \in \mathscr{M}([m_2],[m_3])$ satisfy $R(Y') = R(X)$, $C(Y') = C(X)$, and $y_{ij} = 0$, $y'_{ij} = x_{ij} + \dot{p}$ for all $(i,j) \in R(X) \times C(X)$, where $X = (x_{ij})$, $Y = (y_{ij})$ and $Y' = (y'_{ij})$. By column operations, we may eliminate all multiples of p in $I + Y + Y'$ and obtain

$$(28) \sim_R \mathrm{diag}_p(a^2) P_\sigma (I+Y+X) D_{[m_3]}.$$

By Corollary 3.8, a^1/a^2 is a vertical strip. Then, by Corollary 3.16, the invariant partition $a^2 + \chi^J$ of $\mathrm{diag}_p(a^2) P_\sigma (I+X) D_{[m_3]}$ satisfy $J_2 \geqslant J$. Applying now theorem 3.22, we have $J \geqslant F_2$.

(ii) Easy calculations, following the proof of Proposition 3.12, give

$$\Delta_a U D_{[m_3]} D_{[m_1]} \sim \mathrm{diag}_p(a^3) P_\sigma (I+X) D_{[m_1]} \sim \mathrm{diag}_p(a^3 + \chi^{J_2}),$$

where $I + X \in \mathscr{M}([m_3],[m_1])$, and

$$\Delta_a U D_{[m_2]} D_{[m_1]} \sim \mathrm{diag}_p(a^2) P_\sigma (I+X+X') D_{[m_1]} \sim \mathrm{diag}_p(a^2 + \chi^{F_2}),$$

where $I + X + X' \in \mathscr{M}([m_2],[m_1])$ satisfy $R(X) \cap R(X') = \emptyset$.

Again, by Corollary 3.8, a^2/a^3 is a vertical strip. Then, by Corollary 3.16, the invariant partition $a^2 + \chi^J$ of $\mathrm{diag}_p(a^2) P_\sigma (I+X) D_{[m_1]}$ satisfy $J_2 \geqslant J$. Finally, by Theorem 3.22, we have $J \geqslant F_2$. $\square$

4. The main results

Let $t \geqslant 2$ and consider the transpositions of consecutive positive integers $s_i = (i\,i+1)$, $1 \leqslant i \leqslant t-1$. Denote the identity by s_0. The symmetric group $\mathscr{S}_t$, $t \geqslant 2$, is generated by these $t-1$ transpositions which satisfy the Moore–Coxeter relations: $s_i^2 = s_0$, $s_i s_j = s_j s_i$, if $|i-j| \neq 1$, and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $1 \leqslant i \leqslant t-1$.

The elements of $\mathscr{S}_t$, $t \geqslant 2$, can be written as words in the alphabet $\{s_1, \ldots, s_{t-1}\}$. We define $\mathscr{S}_t$ recursively:

$$\mathscr{S}_1 = \{s_0\},$$

$$\mathscr{S}_t = \left\{ \begin{array}{c} \omega \\ s_{t-1}\omega \\ s_{t-2}s_{t-1}\omega \\ \vdots \\ s_1 s_2 \ldots s_{t-1}\omega \end{array}, \omega \in \mathscr{S}_{t-1} \right\} \quad \text{if } t \geqslant 2.$$

We call to these presentations of the elements of $\mathscr{S}_t$, canonical words. For example, if $t = 2$ we have $\mathscr{S}_2 = \{s_0, s_1\}$, and if $t = 3$ we have $\mathscr{S}_3 = \{s_0, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$.

Given $m = (m_1, \dots, m_t) \in M_t$ (12), we let D_m denote the sequence of diagonal matrices

$$D_m := \left(I, D_{[m_1]}, D_{[m_1]}D_{[m_2]}, \dots, D_{[m_1]}D_{[m_2]}\dots D_{[m_t]}\right),$$

and define the set of all these sequences, with m running over M_t,

$$T_{M_t} := \{D_m : m \in M_t\}.$$

Let $\sigma \in \mathscr{S}_t$ such that $\sigma^{-1}m$ is the partition of M_t. The sequence D_m realizes the unique tableau $\mathscr{H}_\sigma = (0, (1^{m_1}), \sum_{i=1}^{2}(1^{m_i}), \dots, \sum_{i=1}^{t}(1^{m_i}))$ of type $(0, (m_1, \dots, m_t), \sum_{i=1}^{t}(1^{m_i}))$. We may identify T_{M_t} with the set $\{\mathscr{H}_\sigma : \sigma \in \mathscr{S}_t\}$, the set of tableaux of shape the conjugate partition of M_t and words $w([m_1], \dots, [m_t])$, with m running over M_t.

The symmetric group $\mathscr{S}_t$ acts on M_t by place permutations of the tuples. The map $\psi : \mathscr{S}_t \to \mathscr{S}_{T_{M_t}}$ defined by $\psi(s_i)(D_m) = D_{s_i m}$, for $0 \leqslant i \leqslant t-1$ and $m \in M_t$, is a group action on T_{M_t}. The map $\Theta_i(\mathscr{H}_\sigma) = \mathscr{H}_{s_i\sigma}$, $1 \leqslant i \leqslant t-1$, defines an action of the symmetric group $\mathscr{S}_t$ on $\{\mathscr{H}_\sigma : \sigma \in \mathscr{S}_t\}$.

For example, if $m = (4, 3)$ the tableaux realized by $T_{M_2} = \{D_m, D_{s_1 m}\}$ are

$$\mathscr{H}_{s_0} = \begin{matrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & \end{matrix}, \quad \mathscr{H}_{s_1} = \begin{matrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 2 & \end{matrix}, \tag{29}$$

and, if $m = (4, 3, 2)$, the tableaux realized by

$$T_{M_3} = \left\{D_m, D_{s_1 m}, D_{s_2 m}, D_{s_1 s_2 m}, D_{s_2 s_1 m}, D_{s_1 s_2 s_1 m}\right\}$$

are

$$\mathscr{H}_{s_0} = \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \\ 1 & & \end{matrix}, \quad \mathscr{H}_{s_1} = \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \\ 2 & & \end{matrix}, \quad \mathscr{H}_{s_2 s_1} = \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 3 & \\ 3 & & \end{matrix},$$

$$\mathscr{H}_{s_2} = \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 3 & \\ 1 & & \end{matrix}, \quad \mathscr{H}_{s_1 s_2} = \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & \\ 2 & & \end{matrix}, \quad \mathscr{H}_{s_2 s_1 s_2} = \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & \\ 3 & & \end{matrix}.$$

We may write $T_{M_2} = \{\mathscr{H}_{s_0}, \mathscr{H}_{s_1}\}$, and

$$T_{M_3} = \left\{\mathscr{H}_{s_0}, \mathscr{H}_{s_1}, \mathscr{H}_{s_2 s_1}, \mathscr{H}_{s_2}, \mathscr{H}_{s_1 s_2}, \mathscr{H}_{s_1 s_2 s_1}\right\}.$$

Now, fix a partition $a = (a_1, \ldots, a_n)$ and $U \in \mathscr{U}_n$. For each $m = (m_1, \ldots, m_t) \in M_t$, let

$$\Delta_a U D_m := \Big(\Delta_a, \Delta_a U D_{[m_1]}, \Delta_a U D_{[m_1]} D_{[m_2]}, \ldots, \Delta_a U D_{[m_1]} D_{[m_2]} \ldots D_{[m_t]}\Big),$$

and define

$$T_{(a,M_t)}(U) := \big\{\Delta_a U D_m : m \in M_t\big\}.$$

Clearly the symmetric group $\mathscr{S}_t$ also acts on $T_{(a,M_t)}(U)$ by putting

$$\psi(s_i)(\Delta_a U D_m) = \Delta_a U D_{s_i m}, \quad 0 \leqslant i \leqslant t-1.$$

For each $m \in M_t$, $\Delta_a U D_m$ realizes a pair of Young tableaux $(\mathscr{T}, \mathscr{H}_\sigma)$ with weight m, where $\sigma^{-1} m$ is the partition of M_t. According to Corollary 3.11, we replace the notation $(\mathscr{T}, \mathscr{H}_\sigma)$ by $\mathscr{T}_\sigma$, $\sigma \in \mathscr{S}_t$. Thus, we may identify $T_{(a,M_t)}(U)$ with $\{\mathscr{T}_\sigma, \sigma \in \mathscr{S}_t : \exists m \in M_t, \Delta_a U D_m \text{ realizes } \mathscr{T}_\sigma\}$. We shall characterize this set in cases $t = 2, 3$. In order to do this, we need to introduce the following definitions.

Definition 4.1 [6]. Let $F_1 \geqslant F_2$ and $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma) \in (2^{[n]})^2 : \sigma \in \langle s_1 \rangle\}$. We say that $\mathbb{F}$ is generated by (F_1, F_2), if $(F_1^{s_0}, F_2^{s_0}) = (F_1, F_2)$, and the following relations are satisfied:

$$\text{(i)}\ F_1^{s_1} \subseteq F_1, \qquad \text{(ii)}\ F_1^{s_1} \geqslant F_2,\ \ |F_2| = |F_1^{s_1}|, \tag{30}$$

$$\text{(iii)}\ F_1 \cap F_2 \subseteq F_1^{s_1}, \quad \text{(iv)}\ F_2^{s_1} = F_2 \cup (F_1 \backslash F_1^{s_1}). \tag{31}$$

Recalling Definition 2.3, we have $F_1^{s_1} \geqslant_{\text{op}} F_2^{s_1}$. Let $\Theta(F_1^\sigma, F_2^\sigma) = (F_1^{s_1\sigma}, F_2^{s_1\sigma})$. Then, $\Theta^2 = id$, and the symmetric group $\mathscr{S}_2$ acts on any set generated by (F_1, F_2).

Given sets $F_1 \geqslant F_2$, there exists always a set generated by (F_1, F_2). For instance, $F_1^{s_1} := \min_{F_2} F_1$ and $F_2^{s_1} := F_2 \cup (F_1 \backslash F_1^{s_1})$ satisfy (30). In this case, we say that the set $\mathbb{F}$ is $*$-generated by (F_1, F_2) [6].

Definition 4.2 [6]. Given $F_1 \geqslant F_2 \geqslant F_3$ and $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma, F_3^\sigma) \in (2^{[n]})^3 : \sigma \in \langle s_1, s_2 \rangle\}$, with $(F_1^{s_0}, F_2^{s_0}, F_3^{s_0}) = (F_1, F_2, F_3)$, we say that $\mathbb{F}$ is generated by (F_1, F_2, F_3) if

(I)(a) $F_3^{s_1} = F_3$ and $\{(F_1^\sigma, F_2^\sigma) : \sigma \in \langle s_1 \rangle\}$ is generated by (F_1, F_2).
(b) $F_1^{s_2} = F_1$ and $\{(F_2^\sigma, F_3^\sigma) : \sigma \in \langle s_2 \rangle\}$ is generated by (F_2, F_3).
(II)(a) $F_1^{s_2 s_1} = F_1^{s_1}$ and $\{(F_2^{\sigma s_1}, F_3^{\sigma s_1}) : \sigma \in \langle s_2 \rangle\}$ is generated by $(F_2^{s_1}, F_3^{s_1})$ with $F_2^{s_2} \geqslant F_2^{s_2 s_1}$.
(b) $F_3^{s_1 s_2} = F_3^{s_2}$ and $\{(F_1^{\sigma s_2}, F_2^{\sigma s_2}) : \sigma \in \langle s_1 \rangle\}$ is generated by $(F_1^{s_2}, F_2^{s_2})$ with $F_2^{s_1 s_2} \geqslant F_2^{s_1}$.
(III)(a) $F_3^{s_1 s_2 s_1} = F_3^{s_2 s_1}$, $\{(F_1^{\sigma s_2 s_1}, F_2^{\sigma s_2 s_1}) : \sigma \in \langle s_1 \rangle\}$ is generated by $(F_1^{s_2 s_1}, F_2^{s_2 s_1})$, and $F_1^{s_1 s_2 s_1} = F_1^{s_1 s_2}$.
(b) $\{(F_2^{s_1 s_2}, F_3^{s_1 s_2}), (F_2^{s_1 s_2 s_1}, F_3^{s_1 s_2 s_1})\}$ is generated by $(F_2^{s_1 s_2}, F_3^{s_1 s_2})$.

In [6] it has been shown directly that if we are given sets $F_1 \geqslant F_2 \geqslant F_3$ in $[n]$, there exists always the set $\mathbb{F}*$-generated by (F_1, F_2, F_3). Here, in section 5, Theorem 4.7, we shall see a matrix interpretation of the generation of a set $\mathbb{F}$ based on the following facts: in [2] it has been proved that given an LR tableau $\mathscr{T}$ of type (a, m, c), there exists always an unimodular matrix U such that $\Delta_a U D_m$ realizes $\mathscr{T}$, on the other hand the symmetric group acts on $T_{a,M_3}(U)$ which leads to a such set $\mathbb{F}$. In the next theorem, the elements of a set $\mathbb{F}$, generated by (F_1, F_2, F_3), are given explicitly.

Theorem 4.1. *Let $F_1 \geqslant F_2 \geqslant F_3$. The following assertions are equivalent*:

(a) $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma, F_3^\sigma) : \sigma \in \langle s_1, s_2\rangle\}$ *is generated by* (F_1, F_2, F_3).

(b) *The sequence $F_1 \geqslant F_2 \geqslant F_3$ has a decomposition* $F_1 = \bigcup_{j=1}^5 A_1^j$, $F_2 = \bigcup_{j=3}^5 A_2^j$, $F_3 = A_3^5 \cup A_3^2$,

$$F_1, F_2, F_3 = \begin{matrix} A_1^1 & & \\ A_1^2 & & A_3^2 \\ A_1^3 & A_2^3 & \\ A_1^4 & A_2^4 & \\ A_1^5 & A_2^5 & A_3^5 \end{matrix} \tag{32}$$

satisfying:

1. $A_1^4 \geqslant A_2^4 \geqslant A_1^2 \geqslant A_3^2$, *with* $|A_1^4| = |A_2^4| = |A_1^2| = |A_3^2|$,
   $A_1^5 \geqslant A_2^5 \geqslant A_3^5$, *with* $|A_1^5| = |A_2^5| = |A_3^5|$,
   $A_1^3 \geqslant A_2^3$, *with* $|A_1^3| = |A_2^3|$,
2. $A_1^i \cap A_1^j = \emptyset$, *if* $i \neq j$,
   $A_2^i \cap A_2^j = \emptyset$, *if* $i \neq j$,
   $A_3^2 \cap A_3^5 = \emptyset$,
3. $F_1 \cap A_2^5 \subseteq A_1^5$,
   $(F_1 \backslash A_1^5) \cap A_2^4 \subseteq A_1^4$,
   $[F_1 \backslash (A_1^5 \cup A_1^4)] \cap A_2^3 \subseteq A_1^3$,
   $[F_2 \cup (A_1^2 \cup A_1^1)] \cap A_3^2 \subseteq A_1^2$, *and*
   $[F_2 \cup (A_1^2 \cup A_1^1)] \cap A_3^5 \subseteq A_2^5$,

such that the sets $F_1^\sigma, F_2^\sigma, F_3^\sigma$, with $\sigma \in \{s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$, are obtained from F_1, F_2, F_3 as follows:

$$F_1^{s_1}, F_2^{s_1}, F_3^{s_1} = \begin{matrix} & A_1^1 & \\ & A_1^2 & A_3^2 \\ A_1^3 & A_2^3 & \\ A_1^4 & A_2^4 & \\ A_1^5 & A_2^5 & A_3^5 \end{matrix}, \quad F_1^{s_2s_1}, F_2^{s_2s_1}, F_3^{s_2s_1} = \begin{matrix} & & A_1^1 \\ & A_1^2 & A_3^2 \\ A_1^3 & & A_2^3, \\ A_1^4 & & A_2^4 \\ A_1^5 & A_2^5 & A_3^5 \end{matrix}$$

$$F_1^{s_1s_2s_1}, F_2^{s_1s_2s_1}, F_3^{s_1s_2s_1} = \begin{array}{ccc} & & A_1^1 \\ & A_1^2 & A_3^2 \\ & A_1^3 & A_2^3, \\ A_1^4 & & A_2^4 \\ A_1^5 & A_2^5 & A_3^5 \end{array}$$

$$F_1^{s_2}, F_2^{s_2}, F_3^{s_2} = \begin{array}{ccc} A_1^1 & & \\ A_1^2 & & A_3^2 \\ A_1^3 & & A_2^3, \\ A_1^4 & A_2^4 & \\ A_1^5 & A_2^5 & A_3^5 \end{array} \quad F_1^{s_1s_2}, F_2^{s_1s_2}, F_3^{s_1s_2} = \begin{array}{ccc} & A_1^1 & \\ & A_1^2 & A_3^2 \\ & A_1^3 & A_2^3. \\ A_1^4 & A_2^4 & \\ A_1^5 & A_2^5 & A_3^5 \end{array}$$

Proof. (a) $\Rightarrow$ (b) See the proof of the “only if” part of Theorem 4.7.
(b) $\Rightarrow$ (a) Obvious. □

Remark 2. In the previous theorem, if J_1, J_2 and J_3 are pairwise disjoint, condition 3 vanishes and, in that case, we may consider the decomposition (32) with $A_1^2 = A_1^4 = A_2^4 = A_3^2 = \emptyset$.

Corollary 4.2. *Let $F_1 \geqslant F_2 \geqslant F_3$ and $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma, F_3^\sigma) : \sigma \in \langle s_1, s_2\rangle\}$ generated by (F_1, F_2, F_3). For $i = 1, 2$, let $\Theta_i : \mathbb{F} \to \mathbb{F}$ defined by*

$$\Theta_i\left(F_1^\sigma, F_2^\sigma, F_3^\sigma\right) = \left(F_1^{s_i\sigma}, F_2^{s_i\sigma}, F_3^{s_i\sigma}\right), \quad \sigma \in \langle s_1, s_2\rangle.$$

Then, ${\Theta_i}^2 = id$, $i = 1, 2$, and $\Theta_1\Theta_2\Theta_1 = \Theta_2\Theta_1\Theta_2$. That is, the symmetric group $\mathscr{S}_3$ acts on the set $\mathbb{F}$.

Proof. Follows from Theorem 4.1. □

In what follows we put $m = (m_1, \ldots, m_t)$ for the partition in M_t, $t = 2, 3$. We may now define σ-Yamanouchi word for $\sigma \in \mathscr{S}_t$, $t = 2, 3$.

Definition 4.3. Let $t = 2, 3$ and $\sigma \in \mathscr{S}_t$. Let w be a word over the alphabet $[t]$ with evaluation σm. We say that w is a σ-Yamanouchi word if $w \equiv \mathscr{H}_\sigma$.

In [6], Definition 4.4, we have introduced this concept using the indexing sets of the word. We will see that these two definitions do coincide.

Proposition 4.3. *Let $\sigma \in \mathscr{S}_2$ and w a word over the alphabet* [2], *with evaluation σm and indexing sets (F_1, F_2). The following conditions are equivalent*:

(a) *w is a σ-Yamanouchi word.*
(b) *w is a shuffle of the rows of $\mathscr{H}_\sigma$.*

(c) (F_1, F_2) *has a decomposition either of the form*

$$\begin{array}{ll} A_1^1 & \\ A_1^2 & A_2^2 \end{array} \text{ if } \sigma = s_0 \quad or \quad \begin{array}{ll} & A_2^1 \\ A_1^2 & A_2^2 \end{array} \text{ if } \sigma = s_1,$$

where $A_2^1 \geqslant A_2^2$ *with* $|A_2^1| = |A_2^2| = m_2$, *and* $A_1^1 \cap A_1^2 = A_2^2 \cap A_2^1 = \emptyset$.

(d) (F_1, F_2) *belongs to a set* $\mathbb{F}$ *generated by some* $J_1 \geqslant J_2$.

Proof. (a) $\Leftrightarrow$ (b) follows from Proposition 2.3 and Corollary 2.5.

(b) $\Leftrightarrow$ (c). Notice that $\mathscr{H}_{s_0} = w([m_1], [m_2])$ and $\mathscr{H}_{s_1} = w([m_2], [m_1])$. Clearly, $w(F_1, F_2)$ is a Yamanouchi word, when $\sigma = id$, and $w(F_1, F_2)$ is a dual Yamanouchi word, when $\sigma = s_1$. The result follows from Proposition 2.2 and Corollary 2.4.

$(c) \Leftrightarrow (d)$ follows from Definition 4.1. $\square$

Proposition 4.4. *Let* $\sigma \in \mathscr{S}_3$ *and* w *a word over the alphabet* $[3]$, *with evaluation* σm *and indexing sets* (F_1, F_2, F_3). *The following conditions are equivalent*:

(a) w *is a* σ*-Yamanouchi word.*
(b) w *is a shuffle of the rows of* $\mathscr{H}_\sigma$.
(c) (F_1, F_2, F_3) *has a decomposition according to*

$$\begin{array}{lll} A_1^1 & & \\ A_1^2 & A_2^2 & \\ A_1^3 & A_2^3 & A_3^3 \end{array} \text{ if } \sigma = s_0, \quad \begin{array}{lll} & A_2^1 & \\ A_1^2 & A_2^2 & \\ A_1^3 & A_2^3 & A_3^3 \end{array} \text{ if } \sigma = s_1,$$

$$\begin{array}{lll} & & A_3^1 \\ A_1^2 & & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{array} \text{ if } \sigma = s_2 s_1, \quad \begin{array}{lll} & & A_3^1 \\ & A_2^2 & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{array} \text{ if } \sigma = s_1 s_2 s_1,$$

$$\begin{array}{lll} A_1^1 & & \\ A_1^2 & & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{array} \text{ if } \sigma = s_2, \quad \begin{array}{lll} & A_2^1 & \\ & A_2^2 & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{array} \text{ if } \sigma = s_1 s_2,$$

where $A_1^3 \geqslant A_2^3 \geqslant A_3^3$, *with* $|A_1^3| = |A_2^3| = |A_3^3| = |m_3|$; $A_i^r \cap A_i^s = \emptyset$, *for* $r \neq s$, $i = 1, 2, 3$, *and* $A_1^2 \geqslant A_2^2$, $A_1^2 \geqslant A_3^2$, $A_2^2 \geqslant A_3^2$, *with* $|A_1^2| = |A_2^2| = |A_3^2| = |m_2| - |m_3|$.

(d) (F_1, F_2, F_3) *belongs to a set* $\mathbb{F}$ *generated by some* $J_1 \geqslant J_2 \geqslant J_3$.

Proof. (a) $\Leftrightarrow$ (b) Let σ in $\mathscr{S}_3$. A careful analysis of the Schensted's insertion algorithm, Section 2, shows that when we apply this algorithm to a shuffle of the rows of $\mathscr{H}_\sigma$, we get $\mathscr{H}_\sigma$. So, if w is a shuffle of the rows of $\mathscr{H}_\sigma$, $w \equiv \mathscr{H}_\sigma$.

Notice that the tableau $\mathscr{H}_\sigma$ is respectively $(321)^{m_3}(21)^{m_2-m_3}1^{m_1-m_2}$, if $\sigma = s_0$; $(321)^{m_3}(21)^{m_2-m_3}2^{m_1-m_2}$, if $\sigma = s_1$; $(321)^{m_3}(31)^{m_2-m_3}3^{m_1-m_2}$, if $\sigma = s_2 s_1$;

$(321)^{m_3}(32)^{m_2-m_3}3^{m_1-m_2}$ if $\sigma=s_1s_2s_1$; $(321)^{m_3}(31)^{m_2-m_3}1^{m_1-m_2}$ if $\sigma=s_2$; and $(321)^{m_3}(32)^{m_2-m_3}2^{m_1-m_2}$ if $\sigma=s_1s_2$. Therefore, if w is a shuffle of the rows of $\mathscr{H}_\sigma$, when applying, to w, the elementary Knuth transformations $xyx\equiv yxx$, and $yxy\equiv yyx$, with $1\leqslant x<y\leqslant 3$, we do still obtain a word of the same form. In the case of the Knuth transformations $132\equiv 312$ and $231\equiv 213$, notice that 31 is a row of the tableau $\mathscr{H}_\sigma$ only when $\sigma=s_2s_1$. In this case, w is a shuffle of m_1-m_2 rows 321, m_2-m_3 rows 31 and m_3 rows 3. Thus the letter 2 appears only as a letter of the row 321. So, $w\equiv\mathscr{H}_\sigma$ implies that w is a shuffle of the rows of $\mathscr{H}_\sigma$. It is now easy to conclude that a Knuth class containing a word which is a shuffle of the rows of $\mathscr{H}_\sigma$, only contains words which are shuffles of those rows, and the representative tableau of this Knuth class is $\mathscr{H}_\sigma$.

(b) $\Leftrightarrow$ (c) Notice that $w(A_1^3,A_2^3,A_3^3)$ is a shuffle of m_3 rows 321, $w(A_1^2,A_2^2)$ is a shuffle of m_2-m_3 rows 21, $w(A_1^2,A_3^2)$ is a shuffle of m_2-m_3 rows 31, $w(A_3^2,A_2^2)$ a shuffle of m_2-m_3 rows 32, $w(A_1^1)$ is a shuffle of m_1-m_2 rows 1, $w(A_2^1)$ is a shuffle of m_2-m_3 rows 2, and $w(A_3^1)$ is a shuffle of m_2-m_3 rows 3.

(c) $\Rightarrow$ (d) If F_1,F_2,F_3 are pairwise disjoint then condition 3 of Theorem 4.1 vanishes and we may consider $A_3^2=\emptyset$. Otherwise, it has been shown, in [6], the existence of a set $\mathbb{F}$ $*$-generated by a sequence $J_1\geqslant J_2\geqslant J_3$, containing (F_1,F_2,F_3). Furthermore, if (F_1,F_2,F_3) are the indexing sets of some tableau $\mathscr{T}$ of type $(a,\sigma m,c)$, then $J_1\geqslant J_2\geqslant J_3$ are the indexing sets of an LR tableau of type (a,m,c).

(d) $\Rightarrow$ (c) From Theorem 4.1 it is clear that (F_1,F_2,F_3) has a decomposition of one of these forms. □

We are now in conditions to state the two main theorems of this paper. Let $t=2,3$. Let c be the invariant partition of $\Delta_a UD_m$. Given a Young tableau $\mathscr{T}$ of type $(a,\sigma m,c)$, $\sigma\in\mathscr{S}_t$, the theorems, below, show that $\mathscr{T}\in T_{(a,M_t)}(U)$ if and only if the indexing sets of $\mathscr{T}$ belong to some set $\mathbb{F}$ generated by the indexing sets of the LR tableau in $T_{(a,M_t)}(U)$.

Theorem 4.5. *Let $\mathscr{T}$ and $\mathscr{T}_{s_1}$ be Young tableaux, respectively, with indexing sets J_1,J_2,F_1,F_2, and types $(a,m,c),(a,s_1m,c)$, where $l(c)\leqslant n$. Then, there exists an $n\times n$ unimodular matrix U such that $T_{(a,M_2)}(U)=\{\mathscr{T},\mathscr{T}_{s_1}\}$ if and only if $\{(J_1,J_2),(F_1,F_2)\}$ is generated by $J_1\geqslant J_2$.*

This theorem has been stated in [4], without proof, using a different language.

Corollary 4.6. *Let $\sigma\in\mathscr{S}_2$. Let $\mathscr{T}$ be a Young tableau of type $(a,\sigma m,c)$. Then, $(\mathscr{T},\mathscr{H}_\sigma)$ is an admissible pair if and only if $w(\mathscr{T})\equiv\mathscr{H}_\sigma$.*

Proof. Let F_1,F_2 be the indexing sets of $\mathscr{T}$. From [2] and [5], $(\mathscr{T},\mathscr{H}_\sigma)$ is an admissible pair if and only if $w(F_1,F_2)$ is a Yamanouchi word, when $\sigma=id$, and $w(F_1,F_2)$ is a dual Yamanouchi word, when $\sigma=s_1$. Therefore, the result follows from Proposition 4.3. □

Theorem 4.7. *For each $\sigma \in \mathscr{S}_3$, let $\mathscr{T}_\sigma$ be a Young tableau of type $(a, \sigma m, c)$, with indexing sets $F_1^\sigma, F_2^\sigma, F_3^\sigma$, and $l(c) \leqslant n$. Then, there exists an $n \times n$ unimodular matrix U such that $T_{(a,M_3)}(U) = \{\mathscr{T}_\sigma, \sigma \in \mathscr{S}_3\}$ if and only if the set $\{(F_1^\sigma, F_2^\sigma, F_3^\sigma) : \sigma \in \mathscr{S}_3\}$ is generated by $F_1^{s_0} \geqslant F_2^{s_0} \geqslant F_3^{s_0}$.*

Corollary 4.8. *Let $\sigma \in \mathscr{S}_3$. Let $\mathscr{T}$ be a Young tableau of type $(a, \sigma m, c)$. Then, $(\mathscr{T}, \mathscr{H}_\sigma)$ is an admissible pair if and only if $w(\mathscr{T}) \equiv \mathscr{H}_\sigma$.*

Proof. Let F_1, F_2, F_3 be the indexing sets of $\mathscr{T}$. If $(\mathscr{T}, \mathscr{H}_\sigma)$ is an admissible pair, there exists an unimodular matrix U such that $\Delta_a U D_{\sigma m}$ realizes $(\mathscr{T}, \mathscr{H}_\sigma)$. Therefore, by previous theorem, $\mathscr{T}$ is an element of $T_{(a,M_3)}(U)$ and, by Proposition 4.4, we have $w(\mathscr{T}) \equiv \mathscr{H}_\sigma$.

Conversely, if $w(\mathscr{T}) \equiv \mathscr{H}_\sigma$, by Proposition 4.4, there exists a set $\mathbb{F}$ generated by a sequence $J_1 \geqslant J_2 \geqslant J_3$ which contains (F_1, F_2, F_3). By previous theorem, $(\mathscr{T}, \mathscr{H}_\sigma)$ is an admissible pair. □

5. Proof of the main results

We start this section with an auxiliary result in which we analyze the structure of some $n \times n$ matrices.

Lemma 5.1 [14]. *Let $0 \leqslant m_3 \leqslant m_2 \leqslant m_1 \leqslant n$. Let $J_1 = \bigcup_{k=3}^{1} A_1^k$, $J_2 = \bigcup_{k=3}^{2} A_2^k$ be subsets of $[n]$, with $J_1 \geqslant J_2$, and $\sigma, \theta \in \mathscr{S}_n$ such that*

1. $A_i^k \cap A_i^j = \emptyset$, *for* $i = 1, 2$, $k \neq j$,
   $|J_1| = m_1$, $|A_1^2| = |A_2^2| = m_2 - m_3$, $|A_1^3| = |A_2^3| = m_3$ *with* $A_1^k \geqslant A_2^k$, *for* $k = 2, 3$,
2. $J_1 \cap A_2^3 \subseteq A_1^3$,
   $(J_1 \backslash A_1^3) \cap A_2^2 \subseteq A_1^2$,
3. $\sigma[m_3] = A_1^3$, $\sigma([m_k] \backslash [m_{k+1}]) = A_1^k$, *for* $k = 1, 2$, *and* $\theta = \lambda_{A_1^3 A_2^3}$.

Then,

(I) $I + S(A_1^2, A_2^2, \theta\sigma) \in \mathscr{M}([m_2] \backslash [m_3], [m_1])$;

(II) *if* $|A_1^2| = |A_2^3|$, *the matrix* $S(A_1^2, A_2^3, \theta\sigma)$ *has nonzero entries in position* (i, j) *only if* $i \in \sigma^{-1}(A_2^3)$ *and* $j \in \sigma^{-1}(A_1^2)$.

Proof. (I) By definition of $S(A_1^2, A_2^2, \theta\sigma) = (s_{ij})$, if $s_{ij} = 1$ we must have $\theta\sigma(i) \in A_1^2 \backslash A_2^2$ and $\theta\sigma(j) \in A_2^2 \backslash A_1^2$. It follows that $i \in \sigma^{-1}\theta^{-1}(A_1^2) = \sigma^{-1}(A_1^2) = [m_2] \backslash [m_3]$.

Suppose $j \in [m_1]$. Then $\theta\sigma(j) \in \theta\sigma[m_1] = A_2^3 \cup A_1^2 \cup A_1^1$. Since $\theta\sigma(j) \in A_2^2$ and the sets A_2^2 and A_2^3 are disjoint, we find that $\theta\sigma(j) \in (A_1^1 \cup A_1^2) \cap A_2^2 \subseteq A_1^2$, which is a contradiction.

Therefore, $I + S(A_1^2, A_2^2, \theta\sigma) \in \mathscr{M}([m_2]\backslash[m_3], [m_1])$.

(II) Again by definition of $S(A_1^2, A_2^3, \theta\sigma)$, we have $\theta\sigma(i) \in A_2^3\backslash A_1^2 = A_2^3$ and $\theta\sigma(j) \in A_1^2\backslash A_2^3 = A_1^2$, since the sets A_1^2 and A_2^3 are disjoint. □

5.1. The case $t = 2$

Proof of Theorem 4.5 [4,14]. *The "only if" part*. Let $\mathscr{T}$ and $\mathscr{T}_{s_1}$ be tableaux, respectively, of type (a, m, c), with indexing sets J_1, J_2, and of type $(a, s_1 m, c)$, with indexing sets F_1, F_2, with $l(c) \leqslant n$. Suppose there exists an $n \times n$ unimodular matrix U such that $T_{(a,M_2)}(U) = \{\mathscr{T}, \mathscr{T}_{s_1}\}$. We will prove that conditions (i), (ii), (iii) and (iv) of Definition 4.1 are fulfilled.

Assume $U = P_\sigma Q$, where $\sigma \in \mathscr{S}_n$ and Q is a upper triangular matrix, with 1's along its main diagonal, and multiples of p above it.

By Proposition 3.12, we find that $\Delta_a P_\sigma Q D_{[m_1]} D_{[m_2]} \sim \Delta_a P_\sigma D_{[m_1]}(I + X) D_{[m_2]}$, with $I + X \in \mathscr{M}([m_1], [m_2])$. Therefore,

$$\Delta_a, P_\sigma D_{[m_2]} D_{([m_1]\backslash[m_2])}, (I + X) D_{[m_2]} \quad \text{realizes} \quad (\mathscr{T}, \mathscr{H}), \tag{33}$$

$$\Delta_a, P_\sigma D_{[m_2]}, (I + X) D_{([m_1]\backslash[m_2])} D_{[m_2]} \quad \text{realizes} \quad (\mathscr{T}_{s_1}, \mathscr{H}_{s_1}). \tag{34}$$

Recalling the type and the indexing sets of $\mathscr{T}_{s_1}$, we find that $\Delta_a P_\sigma D_{[m_2]}$ has invariant partition $a + \chi^{F_1}$, and is equivalent to $\operatorname{diag}_p(a + \chi^{\sigma[m_2]})$. It follows, by Lemma 3.6, that there exists a permutation $\theta = \theta^{-1}$ such that

$$\Delta_a = \Delta_{\theta a} = P_\theta^{\mathrm{T}} \Delta_a P_\theta \quad \text{and} \quad \theta\sigma[m_2] = F_1. \tag{35}$$

Now, we have

$$\begin{aligned}
\operatorname{diag}_p(a + \chi^{J_1}) &\sim \Delta_a P_\sigma D_{[m_1]} \text{ by hypothesis on } \mathscr{T} \\
&= P_\theta^{\mathrm{T}} \Delta_a P_{\theta\sigma} D_{[m_2]} D_{([m_1]-[m_2])} \text{ by (35)} \\
&\sim \Delta_a D_{\theta\sigma[m_2]} P_{\theta\sigma} D_{([m_1]-[m_2])} \\
&\sim \operatorname{diag}_p(a + \chi^{F_1} + \chi^{\theta\sigma([m_1]-[m_2])}).
\end{aligned}$$

Note that $F_1 \cap \theta\sigma([m_1]\backslash[m_2]) = \emptyset$. Hence, there exists an $\alpha \in \mathscr{S}_n$ such that, with $\theta' = \alpha\theta\sigma$, it follows $J_1 = F_1 \cup \theta'([m_1]\backslash[m_2])$. In particular, (i) follows.

By hypothesis,

$$c - a := (c_1 - a_1, \ldots, c_n - a_n) = \chi^{J_1} + \chi^{J_2} = \chi^{F_1} + \chi^{F_2}. \tag{36}$$

Hence, subtracting χ^{F_1} on both sides of (36), and using (i), we find (iv). Furthermore, (36) also shows us that $J_1 \cap J_2 = F_1 \cap F_2$. So, necessarily (ii) follows. Finally, note that, by Theorem 3.15, there exist $J \subseteq [n]$ with $|J| = m_2$ and $\theta\sigma[m_2] \geqslant J$ such that

$$\begin{aligned}\operatorname{diag}_p(a+\chi^{J_1}+\chi^{J_2}) &\sim \Delta_a P_\sigma D_{[m_1]}(I+X)D_{[m_2]}\\ &\sim \operatorname{diag}_p(a+\chi^{J_1})P_{\theta\sigma}(I+X)D_{[m_2]}\\ &\sim \operatorname{diag}_p(a+\chi^{J_1}+\chi^{J}),\end{aligned} \tag{37}$$

with $F_1=\theta\sigma[m_2]\geqslant J$. If $J\neq J_2$ then, by Lemma 3.6, there exists a permutation $\mu=\mu^{-1}$ such that $\mu(a+\chi^{J_1})=a+\chi^{J_1}$ and $J\geqslant\mu(J)=J_2$. Thus, $F_1\geqslant J_2$ and (30) is satisfied. Therefore, $\{(J_1,J_2),(F_1,F_2)\}$ is generated by $J_1\geqslant J_2$.

The "if" part. Given J_1, J_2 and $F_1, F_2\subseteq[n]$ satisfying (i), (ii), (iii), and (iv) of Definition 4.1 with $|J_1|=|F_2|=m_1$, $|J_2|=|F_1|=m_2$, let $\sigma_1\in\mathscr{S}_n$ be a permutation such that $\sigma_1[m_2]=F_1$ and $\sigma_1([m_1]\backslash[m_2])=J_1\backslash F_1$.

Since $F_1\geqslant J_2$, we may consider the permutations $\theta_2=\lambda_{F_1J_2}$, $\sigma_2=\theta_2\sigma_1$, and the matrix $S=S(F_1,J_2,\sigma_1)$, which, by Lemma 5.1, belongs to $\mathscr{M}([m_1],[m_2])$.

Consider the sequence

$$\Delta_a,\ P_{\sigma_1}D_{[m_1]},\ (I+S)(I-S^{\mathrm{T}})D_{[m_2]}. \tag{38}$$

In view of the proof of Theorem 3.15, we have

$$\begin{aligned}(38) &= \Delta_a D_{\sigma_1[m_1]}P_{\sigma_1}(I+S)(I-S^{\mathrm{T}})D_{[m_2]}\\ &\sim_L\ \Delta_a D_{J_1}P_{\sigma_1}P_{\sigma_1^{-1}\theta_2\sigma_1}D_{[m_2]}\\ &\sim_R\ \Delta_a D_{J_1}D_{\theta_2\sigma_1[m_2]}\\ &= \operatorname{diag}_p(a+\chi^{J_1}+\chi^{J_2}).\end{aligned}$$

On the other hand, since $I+S\in\mathscr{M}([m_1],[m_2])$, we may write

$$\begin{aligned}(38) &= \Delta_a P_{\sigma_1}D_{[m_2]}D_{([m_1]\backslash[m_2])}(I+S)(I-S^{\mathrm{T}})D_{[m_2]}\\ &= \Delta_a P_{\sigma_1}D_{[m_2]}(I+S)(I-S^{\mathrm{T}})D_{([m_1]\backslash[m_2])}D_{[m_2]}.\end{aligned} \tag{39}$$

Thus, again, by Theorem 3.15, we have

$$\begin{aligned}(38) &= \Delta_a D_{\sigma_1[m_2]}P_{\sigma_1}(I+S)(I-S^{\mathrm{T}})D_{[m_1]}\\ &\sim_L\ \Delta_a D_{F_1}P_{\sigma_1}P_{\sigma_1^{-1}\theta_2\sigma_1}D_{[m_1]}\\ &\sim_R\ \Delta_a D_{F_1}D_{\theta_2\sigma_1[m_1]}\\ &= \operatorname{diag}_p(a+\chi^{F_1}+\chi^{F_2}).\end{aligned}$$

Finally, note, that by Lemma 3.2(ii), we have

$$(38)\sim_R\ \Delta_a P_{\sigma_1}D_{[m_1]}(I+S)D_{[m_2]}=\Delta P_{\sigma_1}(I+pS)D_{[m_1]}D_{[m_2]}.$$

Therefore, the matrix $U:=P_{\sigma_1}(I+pS)$ is such that $\Delta_a UD_m$ and $\Delta_a UD_{s_1m}$ realizes, respectively, $(\mathscr{T},\mathscr{H})$ and $(\mathscr{T}_{s_1},\mathscr{H}_{s_1})$. That is, $\{\mathscr{T},\mathscr{T}_{s_1}\}=T_{(a,M_2)}(U)$. $\square$

In view of the theorem above, the indexing sets of $\mathscr{T}_{s_1}$ satisfy $F_1 \geqslant_{\text{op}} F_2$. As a consequence of this result, we obtain, below, necessary conditions for the admissibility of a pair $(\mathscr{T}, \mathscr{H})$, with $t \geqslant 2$. As we shall see, in the case $t = 3$, these conditions are not, in general, sufficient.

Theorem 5.2. *Let $(m_1, \ldots, m_t) \in M_t$, with $t \geqslant 2$, and let $\mathscr{T}$ be a Young tableau of type $(a, (m_1, \ldots, m_t), c)$, with indexing sets $F_1, \ldots, F_t$. Suppose $(\mathscr{T}, \mathscr{H})$ is an admissible pair. Then we have:*

1. *If $m_i \geqslant m_{i+1}$, $F_i \geqslant F_{i+1}$.*
2. *If $m_i \leqslant m_{i+1}$, $F_i \geqslant_{\text{op}} F_{i+1}$.*

Proof. By hypothesis, there exists $U \in \mathscr{U}_n$ such that $\Delta_a, UD_{[m_1]}, \ldots, D_{[m_t]}$ is a matrix realization of $(\mathscr{T}, \mathscr{H})$.

Thus, $\Delta_a, UD_{[m_1]}, \ldots, D_{[m_{i-1}]} \sim_R \Delta_1 V$, where $\Delta_1 = \operatorname{diag}_p(a + \chi^{F_1} + \cdots + \chi^{F_{i-1}})$ and V is a unimodular matrix.

If we denote by a' the partition $a + \chi^{F_1} + \cdots + \chi^{F_{i-1}}$, we have

$$\Delta_1 V D_{[m_i]} D_{[m_{i+1}]} \sim \operatorname{diag}_p(a' + \chi^{F_i} + \chi^{F_{i+1}}).$$

Now, if $m_i \geqslant m_{i+1}$ then Theorem 3.17 says that the sequence $\Delta_1, VD_{[m_i]}, D_{[m_{i+1}]}$ realizes a pair $(\mathscr{T}', \mathscr{H}')$, where $\mathscr{T}'$ is an LR-tableau with indexing sets J_i, J_{i+1}, and $\mathscr{H}' = (0, (1^{m_i}), (1^{m_i}) + (1^{m_{i+1}}))$. Therefore, $J_i \geqslant J_{i+1}$.

If $m_i < m_{i+1}$. The sequence $\Delta_1, VD_{[m_i]}, D_{[m_{i+1}]}$ realizes a pair of tableaux $(\mathscr{T}'', \mathscr{H}'')$, where $\mathscr{T}''$ is a tableau with indexing sets F_i, F_{i+1}, and $\mathscr{H}'' = (0, (1^{m_i}), (1^{m_i}) + (1^{m_{i+1}}))$. Since $\Delta_1, VD_{[m_{i+1}]}, D_{[m_i]}$ is a matrix realization of a pair $(\mathscr{F}, \widetilde{\mathscr{H}})$, where $\mathscr{F}$ is an LR tableau, and $\widetilde{\mathscr{H}} = (0, (1^{m_{i+1}}), (1^{m_{i+1}}) + (1^{m_i}))$, by Theorem 4.5, we have $F_i \geqslant_{\text{op}} F_{i+1}$. $\square$

Remark 3. In general, an LR tableau may be realized by more than one unimodular matrix U. For example, let $\mathscr{T}$ be the LR tableau $(a, a + \chi^{J_1}, a + \chi^{J_1} + \chi^{J_2})$, where $a = (3, 2, 0, 0)$, $J_1 = \{4, 3, 2\}$ and $J_2 = \{1\}$, and consider the matrices $U = P_{(14)}(I + pE_{14})$ and $U' = P_{(14)}(I + E_{12})$. Let $\sigma = (14) \in \mathscr{S}_4$, $m_1 = 3$ and $m_2 = 1$, and note that, by Proposition 3.12, since $\sigma[m_1] = J_1$, we may write

$$\Delta_a P_{(14)}(I + pE_{14}) D_{[m_1]} D_{[m_2]} = \operatorname{diag}_p(a + \chi^{J_1}) P_{(14)}(I + E_{14}) D_{[m_2]} \tag{40}$$

and

$$\Delta_a P_{(14)}(I + E_{12}) D_{[m_1]} D_{[m_2]} = \operatorname{diag}_p(a + \chi^{J_1}) P_{(14)}(I + E_{12}) D_{[m_2]}. \tag{41}$$

Now, Theorem 3.15 and Lemma 3.6 give

$$(40) \sim_L \operatorname{diag}_p(a + \chi^{J_1}) P_{(14)} P_{(14)} D_{[m_2]} = \operatorname{diag}_p(a + \chi^{J_1} + \chi^{J_2})$$

and

$$(41) \sim_L \operatorname{diag}_p(a + \chi^{J_1}) P_{(14)} P_{(12)} D_{[m_2]} \sim \operatorname{diag}_p(a + \chi^{J_1} + \chi^{J_2}).$$

Therefore, both matrices U and U' realize $\mathscr{T}$. On the other hand, applying the procedure used above, we may show that $\Delta U D_{[m_2]} D_{[m_1]} \sim \mathrm{diag}_p(a + \chi^{F_1} + \chi^{F_2})$ and $\Delta U' D_{[m_2]} D_{[m_1]} \sim \mathrm{diag}_p(a + \chi^{F_1'} + \chi^{F_2'})$, where $F_1 = \{3\}$, $F_2 = \{4, 2, 1\}$, and $F_1' = \{2\}$, $F_2' = \{4, 3, 1\}$. That is, matrix U gives rise to the set $\{(J_1, J_2), (F_1, F_2)\}$ generated by (J_1, J_2), while matrix U' gives rise to the set $\{(J_1, J_2), (F_1', F_2')\}$ generated by (J_1, J_2) as well, with $F_1' = \min_{J_2} J_1$ and $F_2' = J_2 \cup (J_1 \backslash F_1')$. This is consistent with Definition 4.1, given sets $J_1 \geqslant J_2$, there is, in general, more that one set generated by the sequence (J_1, J_2).

5.2. *The case $t = 3$*

Proof of Theorem 4.7. *The "only if" part*. Let $\sigma \in \mathscr{S}_3$, and suppose there exists an unimodular matrix U such that $\Delta_a U D_{\sigma m}$ realizes $(\mathscr{T}_\sigma, \mathscr{H}_\sigma)$, where the tableau $\mathscr{T}_\sigma$ has indexing sets $F_1^\sigma, F_2^\sigma, F_3^\sigma$. For simplicity, we shall often say that $F_1^\sigma, F_2^\sigma, F_3^\sigma$ are the indexing sets of $\Delta_a U D_{\sigma m}$. We observe, as we shall see through the proof, that in proving that $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma, F_3^\sigma) : \sigma \in \mathscr{S}_3\}$ is generated by the sequence $(F_i^{s_0})_{i=1}^3$, we also prove that (a) $\Rightarrow$ (b) in Theorem 4.1.

By Theorem 5.2, the indexing sets of $\Delta_a U D_{(m_1,m_2,m_3)}$ satisfy $F_1^{s_0} \geqslant F_2^{s_0} \geqslant F_3^{s_0}$, and the indexing sets of $\Delta_a U D_{(m_2,m_1,m_3)}$ satisfy $F_1^{s_1} \geqslant_{\mathrm{op}} F_2^{s_1} \geqslant F_3^{s_1}$, with $F_3^{s_0} = F_3^{s_1}$. Applying Theorem 4.5 to the set $\{\Delta_a U D_{(m_1,m_2)}, \Delta_a U D_{(m_2,m_1)}\}$, we find that $|F_1^{s_1}| = |F_2^{s_0}|$,

$$\begin{aligned} &F_1^{s_1} \subseteq F_1^{s_0}, \quad F_1^{s_0} \cap F_2^{s_0} \subseteq F_1^{s_1}, \quad F_1^{s_1} \geqslant F_2^{s_0} \quad \text{and} \\ &F_2^{s_1} = F_2^{s_1} \cup (F_1^{s_0} \backslash F_1^{s_1}). \end{aligned} \tag{42}$$

There exists an $n \times n$ unimodular matrix V such that $\Delta_a U D_{[m_1]} \sim_L \Delta_{a'} V$, where $\Delta_{a'} = \mathrm{diag}_p(a + \chi^{F_1^{s_0}})$. Recalling Theorem 5.2, the indexing sets of $\Delta_{a'} V D_{(m_2,m_3)}$ and $\Delta_a U D_{(m_3,m_2)}$ are $F_2^{s_0} \geqslant F_3^{s_0}$ and $F_2^{s_2} \geqslant_{\mathrm{op}} F_3^{s_2}$, respectively. Thus, applying again Theorem 4.5, it follows that $|F_2^{s_2}| = |F_3^{s_0}|$,

$$\begin{aligned} &F_2^{s_2} \subseteq F_2^{s_0}, \quad F_2^{s_0} \cap F_3^{s_0} \subseteq F_2^{s_2}, \quad F_2^{s_2} \geqslant F_3^{s_0} \quad \text{and} \\ &F_3^{s_2} = F_3^{s_0} \cup (F_2^{s_0} \backslash F_2^{s_2}). \end{aligned} \tag{43}$$

We have $\Delta_a U D_{[m_2]} \sim_L \Delta_{a''} V'$, for some unimodular matrix V', with $\Delta_{a''} = \mathrm{diag}_p(a + \chi^{F_1^{s_1}})$. Since the indexing sets of $\Delta_a U D_{(m_2,m_1,m_3)}$ and $\Delta_a U D_{(m_2,m_3,m_1)}$ are $F_1^{s_1}, F_2^{s_1}, F_3^{s_0}$ and $F_1^{s_1}, F_2^{s_2 s_1}, F_3^{s_2 s_1}$, respectively, recalling Theorem 5.2, we find that $\Delta_{a'} V D_{(m_1,m_3)}$ has indexing sets $F_2^{s_1} \geqslant F_3^{s_0}$ and $\Delta_{a'} V D_{(m_3,m_1)}$ has indexing sets $F_2^{s_2 s_1} \geqslant_{\mathrm{op}} F_3^{s_2 s_1}$. Again by Theorem 4.5, it follows that $|F_2^{s_1 s_2}| = |F_3^{s_0}|$,

$$\begin{aligned} &F_2^{s_2 s_1} \subseteq F_2^{s_1}, \quad F_2^{s_1} \cap F_3^{s_0} \subseteq F_2^{s_2 s_1}, \quad F_2^{s_2 s_1} \geqslant F_3^{s_0} \quad \text{and} \\ &F_3^{s_2 s_1} = F_3^{s_0} \cup (F_2^{s_1} \backslash F_2^{s_2 s_1}). \end{aligned} \tag{44}$$

By (42) and (44), we have $F_2^{s_2s_1} \subseteq F_2^{s_1} = F_2^{s_0} \cup (F_1^{s_0} \backslash F_1^{s_1})$, so we may write

$$F_2^{s_2s_1} = A_2^5 \cup A_1^2, \tag{45}$$

where $A_2^5 \subseteq F_2^{s_1}$ and $A_1^2 \subseteq F_1^{s_0} \backslash F_1^{s_1}$. Let $A_1^1 := (F_1^{s_0} \backslash F_1^{s_1}) \backslash A_1^2$.

From (45), and since $F_2^{s_2s_1} \geqslant F_3^{s_0}$ and $|F_2^{s_2s_1}| = |F_3^{s_0}|$, we can factorize $F_3^{s_0}$ as

$$F_3^{s_0} = A_3^5 \cup A_3^2,$$

where $A_2^5 \geqslant A_3^5$, $A_1^2 \geqslant A_3^2$ are such that $|A_2^5| = |A_3^5|$, $|A_1^2| = |A_3^2|$, $F_2^{s_1} \cap A_3^5 \subseteq A_2^5$, and $F_2^{s_1} \cap A_3^2 \subseteq A_1^2$.

Recall again Theorem 5.2, and consider $\Delta_a UD_{(m_2,m_3,m_1)}$ and $\Delta_a UD_{(m_3,m_2,m_1)}$, which have indexing sets

$$F_1^{s_1} \geqslant F_2^{s_2s_1} \geqslant_{\text{op}} F_3^{s_2s_1} \quad \text{and} \quad F_1^{s_1s_2s_1} \geqslant_{\text{op}} F_2^{s_1s_2s_1} \geqslant_{\text{op}} F_3^{s_2s_1}, \tag{46}$$

respectively. The application of Theorem 4.5 to the set $\{\Delta_a UD_{(m_2,m_3)}, \Delta_a UD_{(m_3,m_2)}\}$ gives

$$\begin{aligned} &F_1^{s_1s_2s_1} \subseteq F_1^{s_1}, \quad F_1^{s_1} \cap F_2^{s_2s_1} \subseteq F_1^{s_1s_2s_1}, \quad F_1^{s_1s_2s_1} \geqslant F_2^{s_2s_1} \quad \text{and} \\ &F_2^{s_1s_2s_1} = F_2^{s_2s_1} \cup (F_1^{s_1} \backslash F_1^{s_1s_2s_1}). \end{aligned} \tag{47}$$

Since $F_1^{s_1s_2s_1} \geqslant F_2^{s_2s_1} = A_2^5 \cup A_1^2$, and $|F_1^{s_1s_2s_1}| = |F_2^{s_2s_1}|$, define

$$A_1^5 := \min\left\{X \subseteq F_1^{s_1s_2s_1} : |X| = |A_2^5| \text{ and } X \geqslant A_2^5\right\},$$

and

$$A_1^4 := F_1^{s_1s_2s_1} \backslash A_1^5.$$

Since $F_1^{s_1s_2s_1} \subseteq F_1^{s_1}$, let $A_1^3 := F_1^{s_1} \backslash F_1^{s_1s_2s_1}$. Then we obtain $F_1^{s_1s_2s_1} = A_1^5 \cup A_1^4$ and $F_2^{s_1s_2s_1} = A_2^5 \cup A_1^2 \cup A_1^3$. From the inequality $F_1^{s_1s_2s_1} \geqslant F_2^{s_2s_1}$ and the definition of A_1^5, it follows that

$$A_1^4 > A_1^2 \geqslant A_3^2 \quad \text{and} \quad A_1^5 \geqslant A_2^5 \geqslant A_3^5.$$

Also, from (42), (44) and (47), we obtain $F_1 \cap A_2^5 \subseteq A_1^5$.

Observe that $\Delta_a UD_{(m_1,m_3)}$ has indexing sets $F_1^{s_0} \geqslant F_2^{s_2}$, and from (46), $\Delta_a UD_{(m_2,m_3)}$ has indexing sets $F_1^{s_1} \geqslant F_2^{s_2s_1}$. Then, by Corollary 3.23(i), we must have

$$F_2^{s_2} \geqslant F_2^{s_2s_1}. \tag{48}$$

Since the tableaux $\Delta_a UD_{(m_1,m_3,m_2)}$ and $\Delta_a UD_{(m_3,m_1,m_2)}$ have indexing sets

$$F_1^{s_2} = F_1^{s_0}, F_2^{s_2}, F_3^{s_2} \quad \text{and} \quad F_1^{s_1s_2}, F_2^{s_1s_2}, F_3^{s_2},$$

Theorem 4.5 applied to $\{\Delta_a UD_{(m_1,m_3)}, \Delta_a UD_{(m_3,m_1)}\}$ gives

$$\begin{aligned} &F_1^{s_1s_2} \subseteq F_1^{s_0}, \quad F_1^{s_0} \cap F_2^{s_1} \subseteq F_1^{s_1s_2}, \quad F_1^{s_1s_2} \geqslant F_2^{s_2} \quad \text{and} \\ &F_2^{s_1s_2} = F_2^{s_0} \cup (F_1^{s_0} \backslash F_1^{s_1s_2}). \end{aligned} \tag{49}$$

Observe that $\Delta_a UD_{(m_3,m_1)}$ has indexing sets $F_1^{s_1s_2} \geqslant_{\text{op}} F_2^{s_1s_2}$, and $\Delta_a UD_{(m_2,m_1)}$ has indexing sets $F_1^{s_1} \geqslant_{\text{op}} F_2^{s_1}$. Then, by Corollary 3.23(ii), we must have

$$F_2^{s_1s_2} \geqslant F_2^{s_1}. \tag{50}$$

Finally, consider the tableaux $\Delta_a U D_{(m_3,m_1,m_2)}$ and $\Delta_a U D_{(m_3,m_2,m_1)}$, which have, respectively, indexing sets

$$F_1^{s_1s_2}, F_2^{s_1s_2}, F_3^{s_2} \quad \text{and} \quad F_1^{s_1s_2s_1}, F_2^{s_1s_2s_1}, F_3^{s_2s_1},$$

with $F_1^{s_1s_2} = F_1^{s_1s_2s_1}$. There exists an unimodular matrix V'' such that $\Delta_a U D_{[m_3]} \sim_L \Delta_{a'''} V''$. Then, the application of Theorem 4.5 to the set $\{\Delta_{a'''} V'' D_{(m_1,m_2)}, \Delta_{a'''} V'' D_{(m_2,m_1)}\}$ gives

$$\begin{aligned}&F_2^{s_1s_2s_1} \subseteq F_2^{s_1s_2}, \quad F_2^{s_1s_2} \cap F_3^{s_2} \subseteq F_2^{s_1s_2s_1}, \quad F_2^{s_1s_2s_1} \geqslant F_3^{s_2} \quad \text{and}\\ &F_3^{s_2s_1} = F_3^{s_2} \cup (F_2^{s_1s_2} \backslash F_2^{s_1s_2s_1}).\end{aligned} \tag{51}$$

From (43) and the inclusion $A_2^5 \cup A_1^2 \cup A_1^3 = F_2^{s_1s_2s_1} \subseteq F_2^{s_1s_2} = F_2^{s_2} \cup A_1^1 \cup A_1^2 \cup A_1^3$, it follows that

$$A_2^5 \subseteq F_2^{s_2} \cup A_1^1.$$

But the sets A_2^5 and A_1^1 are disjoint. Therefore $A_2^5 \subseteq F_2^{s_2}$. Let $A_2^4 := F_2^{s_2} \backslash A_2^5$ and $A_2^3 := F_2^{s_0} \backslash F_2^{s_2}$. Since $|F_2^{s_2}| = |F_1^{s_1s_2}|$, we also have $|A_1^4| = |A_2^4|$, $|A_1^3| = |A_2^3|$, $(F_1^{s_0} \backslash A_1^5) \cap A_2^4 \subseteq A_1^4$ and $\left(F_1^{s_0} \backslash (A_1^5 \cup A_1^4)\right) \cap A_2^3 \subseteq A_1^3$. Moreover, from the inequality $F_1^{s_1s_2} \geqslant F_2^{s_2}$, we obtain $A_1^4 \geqslant A_2^4$. From the inequalities (48) and (50), we find that $A_2^4 \geqslant A_1^2$ and $A_1^3 \geqslant A_2^3$.

Thus, the sequence $(F_1^{s_0}, F_2^{s_0}, F_3^{s_0})$ satisfy (b) of Theorem 4.1, and, therefore, $\mathbb{F}$ is generated by $F_1^{s_0} \geqslant F_2^{s_0} \geqslant F_3^{s_0}$.

The "if" part. Suppose the set $F = \{(F_i^{\sigma})_{i=1}^3 : \sigma \in \mathscr{S}_3\}$ is generated by $(F_i^{s_0})_{i=1}^3$. Then, there exists a decomposition of $(F_i^{s_0})_{i=1}^3$ satisfying (b) of Theorem 4.1. We will prove the existence of a unimodular matrix U such that $\{\mathscr{T}_\sigma : \sigma \in \mathscr{S}_3\} = T_{(a,M_3)}(U)$.

Let $m_3' := |A_1^5|$ and $m_1' := |F_1^{s_0} \backslash A_1^1|$. Let σ_1 be a permutation in $\mathscr{S}_n$ such that

$$\begin{aligned}\sigma_1([m_3']) &= A_1^5,\\ \sigma_1([m_3] \backslash [m_3']) &= A_1^4,\\ \sigma_1([m_2] \backslash [m_3]) &= A_1^3,\\ \sigma_1([m_1'] \backslash [m_2]) &= A_1^2,\\ \sigma_1([m_1] \backslash [m_1']) &= A_1^1,\end{aligned}$$

and consider the following permutations:

$$\begin{aligned}&\theta_{25} = \lambda_{A_1^5, A_2^5}, \quad \theta_{35} = \lambda_{A_2^5, A_3^5},\\ &\theta_{24} = \lambda_{A_1^4, A_2^4}, \quad \theta_{32} = \lambda_{A_1^2, A_2^3},\\ &\theta_{23} = \lambda_{A_1^3, A_2^3}, \quad \theta_{12} = \lambda_{A_2^4, A_1^2}.\end{aligned}$$

Let $\sigma_2 := \theta_{23}\theta_{24}\theta_{25}\sigma_1$ and $\sigma_3 := \theta_{32}\theta_{35}\theta_{12}\sigma_2$. Note that since $(A_1^2 \cup A_2^4) \cap (A_1^3 \cup A_2^3) = \emptyset$, the permutations θ_{23} and θ_{12} commute. Consider the following matrices:

$$S_{25} = S(A_1^5, A_2^5, \sigma_1), \qquad S_{12} = S(A_2^4, A_1^2, \sigma_2),$$
$$S_{24} = S(A_1^4, A_2^4, \theta_{25}\sigma_1), \qquad S_{35} = S(A_2^5, A_3^5, \theta_{12}\sigma_2),$$
$$S_{23} = S(A_1^3, A_2^3, \theta_{24}\theta_{25}\sigma_1), \qquad S_{32} = S(A_1^2, A_3^2, \theta_{35}\theta_{12}\sigma_2).$$

Notice that by Lemma 5.1(II), the entry (i, j) of S_{12} is nonnull only if $i \in [m_3]\backslash[m_3']$ and $j \in [m_1']\backslash[m_2]$. Again, by Lemma 5.1(I), we have

$$\begin{aligned} &I + S_{25}, I + S_{35} \in \mathscr{M}([m_3'], [m_1]),\\ &I + S_{24}, I + S_{32} \in \mathscr{M}([m_3]\backslash[m_3'], [m_1]),\\ &\qquad I + S_{23} \in \mathscr{M}([m_2]\backslash[m_3], [m_1]). \end{aligned} \tag{52}$$

Let $\overline{S}_{ij} := (I + S_{ij})(I - S_{ij}^{\mathrm{T}})$, and consider the following product of matrices:

$$\Delta_a P_{\sigma_1} D_{[m_1]} \overline{S}_{25}\overline{S}_{24}\overline{S}_{23} D_{[m_2]} \overline{S}_{12}\overline{S}_{35}\overline{S}_{32} D_{[m_3]}. \tag{53}$$

Recall (52). Since $D_{([m_1]\backslash[m_2])}$ commute with $\overline{S}_{25}\overline{S}_{24}\overline{S}_{23}$, we may write

$$\begin{aligned} (53) &= \Delta_a P_{\sigma_1} D_{[m_2]} \overline{S}_{25}\overline{S}_{24}\overline{S}_{23} D_{([m_1]\backslash[m_2])} D_{[m_2]} \overline{S}_{12}\overline{S}_{35}\overline{S}_{32} D_{[m_3]}\\ &= \Delta_a P_{\sigma_1} D_{[m_2]} \overline{S}_{25}\overline{S}_{24}\overline{S}_{23} D_{[m_1]} \overline{S}_{12}\overline{S}_{35}\overline{S}_{32} D_{[m_3]}. \end{aligned} \tag{54}$$

Matrices $D_{[m_1]}$ and $D_{([m_1]\backslash[m_3])}$ commute with $\overline{S}_{12}$ and $\overline{S}_{35}\overline{S}_{32}$, respectively. Thus, we have

$$(54) = \Delta_a P_{\sigma_1} D_{[m_2]} \overline{S}_{25}\overline{S}_{24}\overline{S}_{23}\overline{S}_{12} D_{[m_3]} \overline{S}_{35}\overline{S}_{32} D_{[m_1]}. \tag{55}$$

Note that $\overline{S}_{12}\overline{S}_{23} = \overline{S}_{23}\overline{S}_{12}$, and the diagonal matrices $D_{([m_2]\backslash[m_3])}$ and $D_{[m_3]}$ commute with $\overline{S}_{25}\overline{S}_{24}\overline{S}_{12}$ and $\overline{S}_{23}$, respectively. So,

$$(55) = \Delta_a P_{\sigma_1} D_{[m_3]} \overline{S}_{25}\overline{S}_{24}\overline{S}_{12} D_{[m_2]} \overline{S}_{23}\overline{S}_{35}\overline{S}_{32} D_{[m_1]}. \tag{56}$$

Consider again (53), and observe that the diagonal matrices $D_{([m_2]\backslash[m_3])}$ and $D_{[m_3]}$ commute with $\overline{S}_{12}\overline{S}_{35}\overline{S}_{32}$ and $\overline{S}_{23}$, respectively. So, we get

$$(53) = \Delta_a P_{\sigma_1} D_{[m_1]} \overline{S}_{25}\overline{S}_{24} D_{[m_3]} \overline{S}_{23}\overline{S}_{12}\overline{S}_{35}\overline{S}_{32} D_{[m_2]}. \tag{57}$$

Finally, note that $D_{([m_1]\backslash[m_3])}$ commute with $\overline{S}_{25}\overline{S}_{24}$. Therefore,

$$(57) = \Delta_a P_{\sigma_1} D_{[m_3]} \overline{S}_{25}\overline{S}_{24} D_{[m_1]} \overline{S}_{23}\overline{S}_{12}\overline{S}_{35}\overline{S}_{32} D_{[m_2]}. \tag{58}$$

We will show that (53), (54), (55), (56), (57) and (58) are, respectively, matrix realizations of the pair of Young tableaux $(\mathscr{T}_\sigma, \mathscr{H}_\sigma)$, for $\sigma = s_0, s_1, s_2s_1, s_1s_2s_1, s_2, s_1s_2$. Consider the sequence (53). Recalling Lemma 3.14, we may write

$$\Delta_a P_{\sigma_1} D_{[m_1]} = \Delta_a D_{\sigma_1[m_1]} P_{\sigma_1} \sim_R \Delta_a D_{\sigma_1[m_1]},$$

$$\begin{aligned} \Delta_a P_{\sigma_1} D_{[m_1]} \overline{S}_{25}\overline{S}_{24}\overline{S}_{23} D_{[m_2]} &= \Delta_a D_{\sigma_1[m_1]} P_{\sigma_1} \overline{S}_{25}\overline{S}_{24}\overline{S}_{23} D_{[m_2]}\\ &\sim_L \Delta_a D_{\sigma_1[m_1]} P_{\sigma_2} D_{[m_2]}\\ &= \Delta_a D_{\sigma_1[m_1]} D_{\sigma_2[m_2]} P_{\sigma_2}\\ &\sim_R \Delta_a D_{\sigma_1[m_1]} D_{\sigma_2[m_2]}, \end{aligned}$$

and

$$\begin{aligned}(53) &= \Delta_a D_{\sigma_1[m_1]} D_{\sigma_2[m_2]} P_{\sigma_2} \overline{S}_{12} \overline{S}_{35} \overline{S}_{32} D_{[m_3]} \\ &\sim_L \Delta_a D_{\sigma_1[m_1]} D_{\sigma_2[m_2]} P_{\sigma_3} D_{[m_3]} \\ &\sim_R \Delta_a D_{\sigma_1[m_1]} D_{\sigma_2[m_2]} D_{\sigma_3[m_3]}.\end{aligned}$$

Since $\sigma_i[m_i] = F_i^{s_0}$ for $i = 1, 2, 3$, we obtain (53) $\sim \Delta_a D_{F_1^{s_0}} D_{F_2^{s_0}} D_{F_3^{s_0}}$. By a similar process, we find that

$$(54) \sim \Delta_a D_{\sigma_1[m_2]} D_{\sigma_2[m_1]} D_{\sigma_3[m_3]} = \Delta_a D_{F_1^{s_1}} D_{F_2^{s_1}} D_{F_3^{s_1}},$$

$$(55) \sim \Delta_a D_{\sigma_1[m_2]} D_{\theta_{12}\sigma_2[m_3]} D_{\sigma_3[m_1]} = \Delta_a D_{F_1^{s_2 s_1}} D_{F_2^{s_2 s_1}} D_{F_3^{s_2 s_1}},$$

$$(56) \sim \Delta_a D_{\sigma_1[m_3]} D_{\theta_{12}\theta_{24}\theta_{25}\sigma_1[m_2]} D_{\sigma_3[m_1]} = \Delta_a D_{F_1^{s_1 s_2 s_1}} D_{F_2^{s_1 s_2 s_1}} D_{F_3^{s_1 s_2 s_1}},$$

$$(57) \sim \Delta_a D_{\sigma_1[m_1]} D_{\theta_{24}\theta_{25}\sigma_1[m_3]} D_{\sigma_3[m_2]} = \Delta_a D_{F_1^{s_2}} D_{F_2^{s_2}} D_{F_3^{s_2}},$$

$$(58) \sim \Delta_a D_{\sigma_1[m_3]} D_{\theta_{24}\theta_{25}\sigma_1[m_1]} D_{\sigma_3[m_2]} = \Delta_a D_{F_1^{s_1 s_2}} D_{F_2^{s_1 s_2}} D_{F_3^{s_1 s_2}}.$$

By Theorem 3.10, it remains to prove the existence of an unimodular matrix U such that

$$P_{\sigma_1} D_{[m_1]} \overline{S}_{25} \overline{S}_{24} \overline{S}_{23} D_{[m_2]} \overline{S}_{12} \overline{S}_{35} \overline{S}_{32} D_{[m_3]} \sim_R U D_{[m_1]} D_{[m_2]} D_{[m_3]}.$$

We start by noticing that, attending to (52) and to Lemma 3.3(iv), we may write

$$\prod_{k=5}^{3} \overline{S}_{2k} = A_2 B_2 \quad \text{and} \quad \prod_{k=5,2} \overline{S}_{3k} = A_3 B_3,$$

where $A_i \in \mathscr{U}([m_i])\mathscr{M}([m_i], [m_1])$ and $B_i \in \mathscr{M}(\overline{[m_i]}, \overline{[m_1]})$, $i = 2, 3$. Thus, by Lemma 3.2(ii), we have

$$\begin{aligned}(53) &= \Delta_a P_{\sigma_1} D_{[m_1]} A_2 B_2 D_{[m_2]} (I + S_{12})(I - S_{12}^{\mathrm{T}}) A_3 B_3 D_{[m_3]} \\ &\sim_R \Delta_a P_{\sigma_1} A_2' D_{[m_1]} D_{[m_2]} B_2' (I + S_{12})(I - S_{12}^{\mathrm{T}}) A_3 D_{[m_3]},\end{aligned} \tag{59}$$

where $A_2' \in \mathscr{U}([m_2])\mathscr{M}_p([m_2], [m_1])$ and $B_2' \in \mathscr{M}_p(\overline{[m_2]}, \overline{[m_1]}) \subseteq \mathscr{U}(\overline{[m_1]})\mathscr{M}_p(\overline{[m_1]})$.

Next, note that

$$I + S_{12} \in \mathscr{U}([m_3])\mathscr{M}([m_3]).$$

Then, by Lemma 3.3(iii), there exist $C, C' \in \mathscr{U}([m_3])\mathscr{M}([m_3])$ and $B_2'' \in \mathscr{U}(\overline{[m_1]})\mathscr{M}_p(\overline{[m_1]})$ such that

$$D_{[m_1]} D_{[m_2]} B_2' (I + S_{12}) = D_{[m_1]} D_{[m_2]} C B_2'' = C' D_{[m_1]} D_{[m_2]} B_2''.$$

Attending to the structure of S_{12}^{T}, we have $I - S_{12}^{\mathrm{T}} \in \mathscr{M}([m_1'] \backslash [m_2])$. Thus, by Lemmas 3.3(ii) and 3.2(i), we may write

$$(59) = \Delta_a P_{\sigma_1} A_2' C' D_{[m_1]} D_{[m_2]} B_2'' A_3' F D_{[m_3]}, \tag{60}$$

for some matrices $F \in \mathscr{M}([m_1']\backslash[m_2])$ and $A_3' \in \mathscr{U}([m_3])\mathscr{M}([m_3],[m_1]) \subseteq \mathscr{U}([m_3])\mathscr{M}([m_3])$. Finally, again by Lemmas 3.3(iii) and 3.2(ii), we obtain

$$(60) \sim_R \Delta_a P_{\sigma_1} A_2' C' A_3'' D_{[m_1]} D_{[m_2]} D_{[m_3]},$$

for some $A_3''' \in \mathscr{U}([m_3])\mathscr{M}([m_3])$. Therefore, the matrix $U := P_{\sigma_1} A_2' C' A_3''$ is unimodular and satisfy $\{\mathscr{T}_\sigma : \sigma \in \mathscr{S}_3\} = T_{(a,M_3)}(U)$. $\square$

6. Final remarks and examples

In this section we translate into words the action of the symmetric group $\mathscr{S}_3$ described in Theorems 4.1 and 4.7, and relate it with the action of the symmetric group generated by the parentheses matching operation on words as described by Lascoux and Schutzenberger in [11,13]. Actually, from the matrix context we get a family of parentheses matching operations on a Yamanouchi word over the alphabet $\{1, 2, 3\}$, compatible with the Knuth equivalence, given by shuffling the output of the Lascoux and Schutzenberger parentheses matching operation on words 1, 21, 3121 and 321. The output of the Lascoux and Schutzenberger parentheses matching operation on a Yamanouchi word, over the alphabet $\{1, 2, 3\}$, is itself a special shuffle of this kind.

A parentheses matching operation θ_i, $1 \leqslant i \leqslant t-1$, on a word w over the alphabet $[t]$ consists of a longest matching between letters $i+1$ and letters i to their right, by putting a left parenthesis on the left of each letter $i+1$, and a right parenthesis on the right of each letter i, such that the unmatched right and left parentheses indicate a subword of the form $i^s(i+1)^r$ which will be replaced in w with $i^r(i+1)^s$. For each $i \in \{1, \ldots, t-1\}$, the nonnegative integers r and s are uniquely determined.

Lascoux and Schutzenberger have introduced involutions θ_i^*, for $i = 1, \ldots, t-1$, to describe the following parentheses matching operation on words over the alphabet $[t]$. Let w be a word over the alphabet $[t]$. To compute $\theta_i^*(w)$, first extract from w the subword w' containing the letters i and $i+1$ only. Second, bracket every factor $i+1i$ of w'. The letters which are not bracketed constitute a subword w_1' of w'. Then bracket every factor $i+1i$ of w_1'. There remains a subword w_2'. Continue this procedure until it stops, giving a word w_k' of type $i^r(i+1)^s$. Then, replace it with the word $i^s(i+1)^r$ and, after this, recover all the removed letters of w, including the ones different from i and $i+1$.

The operations θ_i^* are compatible with the plactic or Knuth equivalence $\equiv$ [11,13].

For example, let $w = 231312121$ be a Yamanouchi word over the alphabet [3]. To compute $\theta_1^*(w)$, we get $w' = (21)1(21)(21)$, and $w_1' = 1 = 12^0$. Thus,

$$\theta_1^*(w) = 2\,3\,1\,3\,\underline{2}\,2\,1\,2\,1, \tag{61}$$

where the underlined letter is the subword w_1' replaced with $2 = 1^0 2$. To compute $\theta_2^*(w)$, we get $w' = 23(32)2$, $w_1' = 2(32)$, and $w_2' = 2 = 2^1 3^0$. Thus,

$$\theta_2^*(w) = \underline{3}\;3\;1\;3\;1\;2\;1\;2\;1, \tag{62}$$

where the underlined letter indicates the subword w_2' replaced with $3 = 2^0 3^1$. Therefore, we have

$$\begin{aligned} &\theta_1^*\theta_2^*(w) = 3\;3\;2\;3\;2\;2\;1\;2\;1,\\ &\theta_2^*\theta_1^*(w) = 3\;3\;1\;3\;2\;2\;1\;3\;1,\\ &\theta_1^*\theta_2^*\theta_1^*(w) = 3\;3\;2\;3\;2\;2\;1\;3\;1 = \theta_2^*\theta_1^*\theta_2^*(w). \end{aligned} \tag{63}$$

Let w be a Yamanouchi word over the alphabet [3] of evaluation (m_1, m_2, m_3). The set $\mathbb{W}^* = \{\theta^*(w) : \theta^* \in \langle\theta_1^*, \theta_2^*\rangle\}$ is called the set $*$-generated by w. In our example above, the elements $*$-generated by $w = 231312121$ are displayed in (61)–(63). Clearly, $\mathscr{S}_3$ acts on $\mathbb{W}^*$.

Given a group $G = \langle x_1, \ldots, x_{t-1}\rangle$ satisfying the Moore–Coxeter relations for $\mathscr{S}_t$, we say that $x \in G$ and $\sigma \in \mathscr{S}_t$ have the same word if there exist $i_1, \ldots, i_k \in \{1, \ldots, t-1\}$ such that $x = x_{i_1} \ldots x_{i_k}$ and $\sigma = s_{i_1} \ldots s_{i_k}$.

Let $\mathbb{H} = \{\mathscr{H}_\sigma : \sigma \in \mathscr{S}_3\}$ be the set of σ-Yamanouchi tableau words of evaluation σm. That is, $\mathscr{H}_\sigma = \theta^*(\mathscr{H}_{s_0})$ whenever θ^* and σ have the same word. Recall $w \equiv \mathscr{H}_{s_0}$ if and only if $\theta^*(w) \equiv \mathscr{H}_\sigma$. Indeed, given a word w over the alphabet $[t]$, for each $i = 1, \ldots, t-1$, we might have several parentheses matching operations θ_i on w. Some of them are giving rise to the same output as θ_i^* and others are not. From [6], we know that for every word w and for all $i = 1, \ldots, t-1$, $\theta_i(w_{|\{i,i+1\}}) \equiv \theta_i^*(w_{|\{i,i+1\}})$. Equivalently, $\theta_i(w_{|\{i,i+1\}}) = \theta_i^*(u')$, for some word $u' \equiv w_{|\{i,i+1\}}$ with u' over the subalphabet $\{i, i+1\}$. This means, that $\theta_i(w) = \theta_i^*(u)$, where u is the word obtained from w replacing $w_{|\{i,i+1\}}$ with u'. For $t > 2$, we may have $w \not\equiv u$, and, henceforth, $\theta_i(w) = \theta_i^*(u) \not\equiv \theta_i^*(w)$. It is easy to exhibit parentheses matching operations ξ_i, $i = 1, 2$, satisfying the Moore–Coxeter relations for $\mathscr{S}_3$ on a Yamanouchi word over the alphabet [3] which do not preserve the Knuth equivalence class $\mathscr{H}_\sigma$. For example, given the Yamanouchi word 3211,

$$\begin{aligned} &3211 \xleftrightarrow{\xi_2} 3211 \xleftrightarrow{\xi_1} \underline{3}\,\underline{2}\,1\,\underline{2} \xleftrightarrow{\xi_2} 3312\\ &3211 \xleftrightarrow{\xi_1} \underline{3}\,\underline{2}\,1\,\underline{2} \xleftrightarrow{\xi_2} 3312 \xleftrightarrow{\xi_1} 3312, \end{aligned}$$

and $3312 \equiv 31\,32 \not\equiv \theta_1^*\theta_2^*\theta_1^*(3211) = 321\,3 = \mathscr{H}_{s_1 s_2 s_1}$. Although, $\xi_2(322) = \theta_2^*(232) = 332$, with $322 \equiv 232$, we have $3212 \not\equiv 2312$ and, henceforth, $\theta_2^*(3212) = 3213 \not\equiv \theta_2^*(2312) = \xi_2(3212) = 3312$.

Definition 6.1. Given a Yamanouchi word w over the alphabet [3], the parentheses matching operations θ_i, $i = 1, 2$, satisfying the Moore–Coxeter relations for $\mathscr{S}_3$ on w, are said *plactic* if $\theta(w) \equiv \mathscr{H}_\sigma$, whenever $\theta \in \langle\theta_1, \theta_2\rangle$ and σ have the same word.

That is, putting $\mathbb{W} = \{\theta(w) : \theta \in \langle\theta_1, \theta_2\rangle\}$, called the *set generated* by w and $\langle\theta_1, \theta_2\rangle$, we have $\theta(w) \equiv \mathscr{H}_\sigma$, with θ and σ with the same word.

Using Theorem 4.1, we characterize a family of plactic parentheses matching operations θ_i, $i = 1, 2$, on a Yamanouchi word w over the alphabet [3]. The translation into words of the action generated by the decomposition given in Theorem 4.1 says:

- write the Yamanouchi word w of evaluation (m_1, m_2, m_3) as a shuffle of $0 \leqslant k \leqslant m_3$ words $v = 3\,1\,2\,1$, $m_3 - k$ words $w_3 = 3\,2\,1$, $m_2 - m_3$ words $w_2 = 2\,1$, and $m_1 - m_2 - k$ words $w_1 = 1$, that is,
$$w = \mathrm{sh}\big(w_3^{m_3-k}, w_2^{m_2-m_3}, w_1^{m_1-m_2-k}, v^k\big); \tag{64}$$
- compute $\theta^*(w_1)$, $\theta^*(w_2)$, and $\theta^*(v)$, with θ^* running over $\langle\theta_1^*, \theta_2^*\rangle$, as displayed below:
$$\begin{aligned}
&w_1 = 1 \xleftrightarrow{\theta_1^*} 2 \xleftrightarrow{\theta_2^*} 3 \xleftrightarrow{\theta_1^*} 3\\
&w_1 = 1 \xleftrightarrow{\theta_2^*} 1 \xleftrightarrow{\theta_1^*} 2 \xleftrightarrow{\theta_2^*} 3\\
&w_2 = 21 \xleftrightarrow{\theta_1^*} 21 \xleftrightarrow{\theta_2^*} 31 \xleftrightarrow{\theta_1^*} 32\\
&w_2 = 21 \xleftrightarrow{\theta_2^*} 31 \xleftrightarrow{\theta_1^*} 32 \xleftrightarrow{\theta_2^*} 32\\
&v = 3121 \xleftrightarrow{\theta_1^*} 3221 \xleftrightarrow{\theta_2^*} 3231 \xleftrightarrow{\theta_1^*} 3231\\
&v = 3121 \xleftrightarrow{\theta_2^*} 3121 \xleftrightarrow{\theta_1^*} 3221 \xleftrightarrow{\theta_2^*} 3231,
\end{aligned} \tag{65}$$
and note that the row word $w_3 = 321$ is invariant under θ_i^*;
- for each $\theta^* \in \langle\theta_1^*, \theta_2^*\rangle$, let
$$\mathrm{sh}\big(w_3^{m_3-k}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m_1-m_2-k}, (\theta^* v)^k\big) \tag{66}$$
be the word obtained by replacing in $\mathrm{sh}(w_3^{m_3-k}, w_2^{m_2-m_3}, w_1^{m_1-m_2-k}, v^k)$ (64), w_i with $\theta^* w_i$, $i = 1, 2$, and v with $\theta^* v$.

Considering (66), let $ij \ldots k$ be a word over the alphabet [2] and put
$$\theta_i\theta_j \ldots \theta_k(w) := \mathrm{sh}\big(w_3^{m_3-k}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m_1-m_2-k}, (\theta^* v)^k\big), \tag{67}$$
where $\theta^* = \theta_i^*\theta_j^* \cdots \theta_k^*$. Clearly, that θ_i, $i = 1, 2$, are matching operations satisfying the Moore–Coxeter relations for $\mathscr{S}_3$ on w. From Proposition 4.4, we have $\theta(w) = \mathrm{sh}(w_3^{m_3-k}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m_1-m_2-k}, (\theta^* v)^k) \equiv \mathscr{H}_\sigma$, σ and θ with the same word, and thus θ_i are plactic operations.

Reciprocally, let
$$\mathbb{W} = \big\{\mathrm{sh}\big(w_3^{m_3-k}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m_1-m_2-k}, (\theta^* v)^k\big) : \theta^* \in \langle\theta_1^*, \theta_2^*\rangle\big\}, \tag{68}$$

be the set generated by $\mathrm{sh}(w_3^{m_3-k}, w_2^{m_2-m_3}, w_1^{m_1-m_2-k}, v^k)$ with $m_1 - m_2, m_3 \geqslant k \geqslant 0$, and $\langle\theta_1, \theta_2\rangle$ defined in (67).

Fix arbitrarily indexing sets (F_1, F_2, F_3) of $w = \mathrm{sh}(w_3^{m_3-k}, w_2^{m_2-m_3}, w_1^{m_1-m_2-k}, v^k)$, and let $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma, F_3^\sigma) : \sigma \in \mathscr{S}_3\}$ such that

$$w(F_1^\sigma, F_2^\sigma, F_3^\sigma) = \mathrm{sh}\big(w_3^{m_3-k}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m_1-m_2-k}, (\theta^* v)^k\big), \tag{69}$$

where θ^* and σ have the same word. Translating to $\mathbb{W}$ the involution Θ_i, $i = 1, 2$, defined on $\mathbb{F}$, Corollary 4.2, we find that $\Theta(F_1, F_2, F_3)$ are indexing sets of $sh(w_3^{m_3-k}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m_1-m_2-k}, (\theta^* v)^k)$, where the word of $\Theta \in \langle\Theta_1, \Theta_2\rangle$ and $\theta^* \in \langle\theta_1^*, \theta_2^*\rangle$ is the same. That is, for each $i = 1,\ 2$, $\Theta_i(F_1^\sigma, F_2^\sigma, F_3^\sigma)$ are the indexing sets of $\theta_i(w(F_1^\sigma, F_2^\sigma, F_3^\sigma)) = \mathrm{sh}(w_3^{m_3-k}, (\theta_i^*\theta^* w_2)^{m_2-m_3}, (\theta_i^*\theta^* w_1)^{m_1-m_2-k}, (\theta_i^*\theta^* v)^k)$. Thus we have

$$\begin{aligned}\theta(w) &= \mathrm{sh}\big(w_3^{m_3-k}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m_1-m_2-k}, (\theta^* v)^k\big)\\ &= w(F_1^\sigma, F_2^\sigma, F_3^\sigma)\end{aligned}$$

where $\theta \in \langle\theta_1, \theta_2\rangle$, θ^* and σ have the same word.

In [6], it is shown that when $*$-generation is considered in Theorem 4.1, the action of symmetric group described in that theorem coincides with the action of the symmetric group generated by the involutions θ_i^*, $i = 1, 2$ on a Yamanouchi word w. Let us denote by

$$\mathrm{sh}^*\big(w_3^{m_3-k}, w_2^{m_2-m_3}, w_1^{m_1-m_2-k}, v^k\big)$$

any shuffle of w afforded by a decomposition of the indexing sets (F_1, F_2, F_3) given by $*$-generation. Then, $*$-generation by w corresponds to the $*$-generation by the class of indexing sets of w,

$$\begin{aligned}\mathbb{W}^* = \big\{\, &\theta^*(w) = sh^*\big(w_3^{m_3-k}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m_1-m_2-k}, (\theta^* v)^k\big) :\\ &\theta^* \in \langle\theta_1^*, \theta_2^*\rangle\big\}\end{aligned} \tag{70}$$

and, henceforth, the action of the symmetric group generated by the parentheses matching operation θ_i^* on a Yamanouchi word w is achieved. As we shall see below, in examples 1 and 2, $*$-generation on indexing sets may give rise to several decompositions of the indexing sets and, henceforth, to several shuffles of w. Nevertheless, all of them are giving rise to the same group action, that is $\theta^*(w) = sh^*(w_3^{m_3-k}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m_1-m_2-k}, (\theta^* v)^k)$ and among them there exists one that coincides with the parenthesization of θ^*.

We observe that the construction given by Theorem 4.1 does not give all plactic parentheses matching operations on a Yamanouchi word. For instance, consider the Knuth class $[3211] = \{3211, 3121, 1321\}$. The following diagram exhibits a family of plactic parentheses matching operations for each Yamanouchi word in

$[3211] = \{3211, 3121, 1321\}$. In particular, all the sets $\mathbb{W}$ (68) generated, according to Theorem 4.1, by the elements of the Knuth class [3211]

$$\begin{tikzcd}[column sep=small]
3211 \arrow[r, leftrightarrow, "\theta_1^*"] \arrow[dr, leftrightarrow, "\mu_1"] & 3212 \arrow[r, leftrightarrow, "\theta_2^*"] & 3213 \arrow[r, leftrightarrow, "\theta_1^*"] & 3213; & 3211 \arrow[r, leftrightarrow, "\theta_2^*"] & 3211 \arrow[r, leftrightarrow, "\theta_1^*"] \arrow[dr, leftrightarrow, "\mu_1"] & 3212 \arrow[r, leftrightarrow, "\theta_2^*"] & 3213 \\
3121 \arrow[r, leftrightarrow, "\theta_1^*"] & 3221 \arrow[r, leftrightarrow, "\theta_2^*"] \arrow[dr, leftrightarrow, "\mu_2"] & 3231 \arrow[r, leftrightarrow, "\theta_1^*"] & 3231; & 3121 \arrow[r, leftrightarrow, "\theta_2^*"] & 3121 \arrow[r, leftrightarrow, "\theta_1^*"] & 3221 \arrow[r, leftrightarrow, "\theta_2^*"] \arrow[dr, leftrightarrow, "\mu_2"] & 3231 \\
1321 \arrow[r, leftrightarrow, "\theta_1^*"] & 2321 \arrow[r, leftrightarrow, "\theta_2^*"] & 3321 \arrow[r, leftrightarrow, "\theta_1^*"] & 3321; & 1321 \arrow[r, leftrightarrow, "\theta_2^*"] & 1321 \arrow[r, leftrightarrow, "\theta_1^*"] & 2321 \arrow[r, leftrightarrow, "\theta_2^*"] & 3321
\end{tikzcd} \tag{71}$$

but

$$\begin{aligned}
&3211[\equiv 3121] \xleftrightarrow{\mu_1} 3221[\equiv 2321] \xleftrightarrow{\mu_2} 3\,3\,2\,1 \xleftrightarrow{\mu_1} 3321\\
&3211 \xleftrightarrow{\mu_2} 3\,2\,1\,1 \xleftrightarrow{\mu_1} 3221[\equiv 3212] \xleftrightarrow{\mu_2} 3321[\equiv 3231],
\end{aligned}$$

where μ_i, $i = 1, 2$, satisfy the Moore–Coxeter relations on 3211, shows that $\{\mu(3211) : \mu \in \langle\mu_1, \mu_2\rangle\}$ is not generated by Theorem 4.1.

We also observe that θ^* preserves the Q-tableau of a word, that is, $Q(w) = Q(\theta^* w)$ (see [13]). But in general $\theta \in \langle\theta_1, \theta_2\rangle$ (67) does not. For instance, considering $\langle\theta_1, \theta_2\rangle$ given by

$$\begin{aligned}
&3211 \xleftrightarrow{\mu_1} 3221 \xleftrightarrow{\theta_2^*} 3\,3\,3\,1 \xleftrightarrow{\theta_1^*} 3231\\
&3121 \xleftrightarrow{\theta_2^*} 3\,2\,1\,1 \xleftrightarrow{\mu_1^*} 3221 \xleftrightarrow{\theta_2^*} 3231,
\end{aligned}$$

we have $Q(3211) = 431\ 2 \neq Q(3221) = 421\ 3$.

Let $\mathbb{H} = \{\mathscr{H}_\sigma : \sigma \in \mathscr{S}_3\}$ be the set of σ-Yamanouchi tableau words of evaluation σm. That is $\mathscr{H}_\sigma = \theta^*(\mathscr{H}_{s0}) = w_3^{m_3}(\theta^* w_2)^{m_2-m_3}(\theta^* w_1)^{m_1-m_2}$, where the word of θ^* and σ is the same. Clearly, $\mathbb{H}$ is $*$-generated by $\mathscr{H}_{s0}$. Let $\mathbb{W} = \{\mathrm{sh}(w_3^{m_3-k}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m-1-m_2-k}, (\theta^* v)^k) : \theta^* \in \langle\theta_1^*, \theta_2^*\rangle\}$ as in (70), generated by (64), a shuffle of $\mathscr{H}_{s0}$. We address the question: How are the sets $\mathbb{H}$ and $\mathbb{W}$ related?

Note that from (65), $\theta^* v = \theta^*(3121) \equiv w_3(\theta^* w_1) \equiv (\theta^* w_1)w_3$, where $\theta^* \in \langle\theta_1^*, \theta_2^*\rangle$. For each $\theta^* \in \langle\theta_1^*, \theta_2^*\rangle$, replace in the word $\mathrm{sh}(w_3^{m_3-k}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m_1-m_2-k}, (\theta^* v)^k)$, $\theta^* v$ with $w_3\theta^*(w_1)$. We obtain a word $\widehat{\mathrm{sh}}(w_3^{m_3}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m_1-m_2})$. This defines a set

$$\widehat{\mathbb{W}} = \left\{\widehat{\mathrm{sh}}(w_3^{m_3}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m_1}) : \theta^* \in \langle\theta_1^*, \theta_2^*\rangle\right\}$$

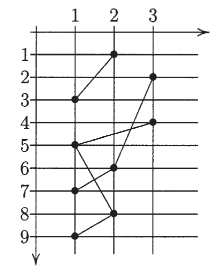
generated by the word $\hat{w}$ obtained replacing in w, $v = 3121$ with $w_3 w_1 = 3211$. Now, for each θ^*, we may again shuffle $\mathrm{sh}(w_3^{m_3}, (\theta^* w_2)^{m_2-m_3}, (\theta^* w_1)^{m_1-m_2})$ to get $\mathscr{H}_\sigma = w_3^{m_3}(\theta^* w_2)^{m_2-m_3}(\theta^* w_1)^{m_1-m_2}$, and, therefore, $\mathbb{H}$.

6.1. Examples

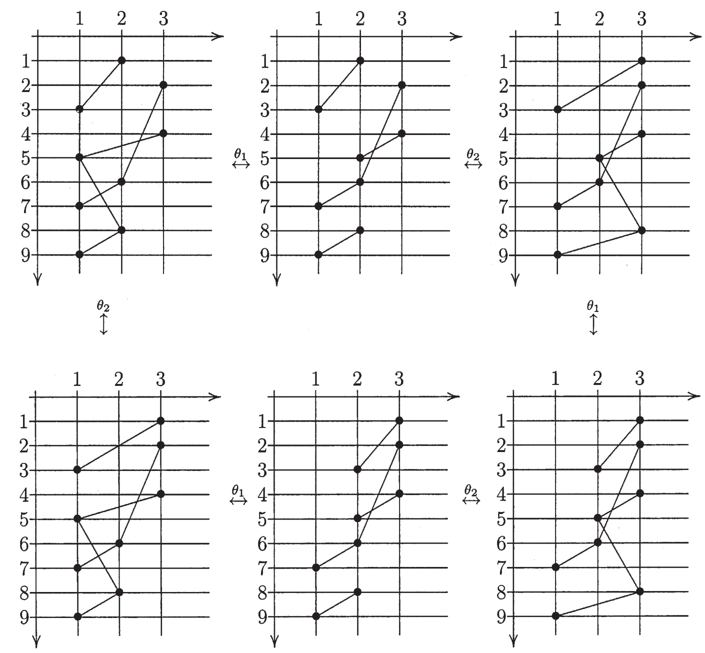
Consider again the word $w = 231312121$ and fix indexing sets $J_1 = \{3, 5, 7, 9\}$, $J_2 = \{1, 6, 8\}$ and $J_3 = \{2, 4\}$. The examples below exhibit several decompositions of the sequence (J_1, J_2, J_3) satisfying (b) of Theorem 4.1. In particular, using the procedure given in [6], Proposition 4.6, Examples 1 and 2, exhibit decompositions of (J_1, J_2, J_3) that give rise to the action of the symmetric group generated by the parentheses matching operations θ_i^* on w.

Example 1

The grid below exhibits a decomposition of the sequence (J_1, J_2, J_3) satisfying (b) of Theorem 4.1,



This decomposition of the indexing sets is equivalent to write the word $w = 231312121$ as a shuffle of the words $w_2 = w(\{3\}, \{1\}) = 21$, $w_3 = w(\{7\}, \{6\}, \{2\}) = 321$, and $v = w(\{5, 9\}, \{8\}, \{4\}) = 3121$. According to this decomposition we have the following action of $\mathscr{S}_3$:



The translation of this action into words yields

$$
\begin{array}{ccccc}
w = \bar{2}\,3\,\bar{1}\,\underline{3}\,\underline{1}\;2\,1\,\underline{2}\,\underline{1} & \overset{\theta_1}{\longleftrightarrow} & \bar{2}\,3\,\bar{1}\,\underline{3}\,\underline{2}\,2\,1\,\underline{2}\,\underline{1} & \overset{\theta_2}{\longleftrightarrow} & \bar{3}\,3\,\bar{1}\,\underline{3}\,\underline{2}\,2\,1\,\underline{3}\,\underline{1} \\
\theta_2 \updownarrow & & & & \updownarrow \theta_1 \\
\bar{3}\,3\,\bar{1}\,\underline{3}\,\underline{1}\,2\,1\,\underline{2}\,\underline{1} & \overset{\theta_1}{\longleftrightarrow} & \bar{3}\,3\,\bar{2}\,\underline{3}\,\underline{2}\,2\,1\,\underline{2}\,\underline{1} & \overset{\theta_2}{\longleftrightarrow} & \bar{3}\,3\,\bar{2}\,\underline{3}\,\underline{2}\,2\,1\,\underline{3}\,\underline{1}
\end{array}, \tag{72}
$$

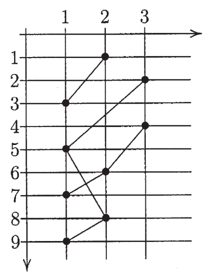
where the overlined letters define the word w_2 and its image under the operations θ_i, $i = 1, 2$, the underlined letters define v and its image under θ_i, $i = 1, 2$, and the remaining letters define w_3.

Below, we illustrate this action on a set of skew Young tableaux generated by an LR tableau $\mathscr{T}$ whose word is $w = 231312121$:

$$
\begin{array}{ccccc}
\mathscr{T} = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 2 \\ \bullet & \bullet & 1 & 3 & \\ \bullet & \bullet & 1 & 3 & \\ \bullet & 1 & 2 & & \\ 1 & 2 & & & \end{array} & \overset{\theta_1}{\longleftrightarrow} & \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 2 \\ \bullet & \bullet & 1 & 3 & \\ \bullet & \bullet & 2 & 3 & \\ \bullet & 1 & 2 & & \\ 1 & 2 & & & \end{array} & \overset{\theta_2}{\longleftrightarrow} & \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 3 \\ \bullet & \bullet & 1 & 3 & \\ \bullet & \bullet & 2 & 3 & \\ \bullet & 1 & 2 & & \\ 1 & 3 & & & \end{array} \\
\updownarrow^{\theta_2} & & & & \updownarrow^{\theta_1} \\
\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 3 \\ \bullet & \bullet & 1 & 3 & \\ \bullet & \bullet & 1 & 3 & \\ \bullet & 1 & 2 & & \\ 1 & 2 & & & \end{array} & \overset{\theta_1}{\longleftrightarrow} & \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 3 \\ \bullet & \bullet & 2 & 3 & \\ \bullet & \bullet & 2 & 3 & \\ \bullet & 1 & 2 & & \\ 1 & 2 & & & \end{array} & \overset{\theta_2}{\longleftrightarrow} & \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 3 \\ \bullet & \bullet & 2 & 3 & \\ \bullet & \bullet & 2 & 3 & \\ \bullet & 1 & 2 & & \\ 1 & 3 & & & \end{array}
\end{array}.
$$

Example 2

The decomposition of (J_1, J_2, J_3), in the previous example, gives rise to a matching operation θ_i which coincides with θ_i^*. Compare (72) with (61)–(63). The grid below exhibits another decomposition of (J_1, J_2, J_3), satisfying (b) of Theorem 4.1, giving rise to the symmetric group action described by Lascoux and Schutzenberger as well, but which corresponds to a different parentheses matching.



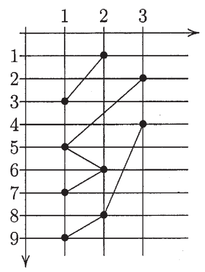
The translation into words of the action of $\mathscr{S}_3$ on the set generated by this decomposition of (J_1, J_2, J_3) gives

$$\begin{array}{ccccc}
w = \overline{2}\,\underline{3}\,\overline{1}\,3\,\underline{1}\,2\,1\,\underline{2}\,\underline{1} & \overset{\theta_1}{\longleftrightarrow} & \overline{2}\,\underline{3}\,\overline{1}\,3\,\underline{2}\,2\,1\,\underline{2}\,\underline{1} & \overset{\theta_2}{\longleftrightarrow} & \overline{3}\,\underline{3}\,\overline{1}\,3\,\underline{2}\,2\,1\,\underline{3}\,\underline{1} \\
\theta_2 \updownarrow & & & & \theta_1 \updownarrow \\
\overline{3}\,\underline{3}\,\overline{1}\,3\,\underline{1}\,2\,1\,\underline{2}\,\underline{1} & \overset{\theta_1}{\longleftrightarrow} & \overline{3}\,\underline{3}\,\overline{2}\,3\,\underline{2}\,2\,1\,\underline{2}\,\underline{1} & \overset{\theta_2}{\longleftrightarrow} & \overline{3}\,\underline{3}\,\overline{2}\,3\,\underline{2}\,2\,1\,\underline{3}\,\underline{1}
\end{array}, \tag{73}$$

where the overlined letters define the word $w_2 = w(\{3\}, \{2\}) = 21$ and its image under the operations θ_i, the underlined letters define $v = (\{5, 9\}, \{8\}, \{2\}) = 3121$ and its image under θ_i, $i = 1, 2$, and the remaining letters define $w_3 = w(\{7\}, \{6\}, \{4\}) = 321$. Although, the action of the symmetric group obtained by this decomposition of (J_1, J_2, J_3) coincides with the one in (61)–(63), the matching between letters 3 and letters 2 to their right, respectively, in $\theta_1 w$ and $\theta_1^* w$, and in $\theta_1\theta_2 w$ and $\theta_2^*\theta_1^* w$ is not the same.

Example 3

The next grid exhibits a decomposition of the indexing sets (J_1, J_2, J_3), satisfying (b) of Theorem 4.1, whose matching operation θ_i gives rise to an action of the symmetric group different from the one described by θ_1^* and θ_2^*,



According to this decomposition, we have $w = \overline{2}\,\underline{3}\,\overline{1}\,3\,\underline{1}\,\underline{2}\,\underline{1}\,2\,1$ as a shuffle of w_i, $i = 2, 3$, and v, which, by (65), leads to the following action of $\mathscr{S}_3$:

$$\begin{array}{ccccc}
w = \overline{2}\,\underline{3}\,\overline{1}\,3\,\underline{1}\,\underline{2}\,\underline{1}\,2\,1 & \overset{\theta_1}{\longleftrightarrow} & \overline{2}\,\underline{3}\,\overline{1}\,3\,\underline{2}\,\underline{2}\,\underline{1}\,2\,1 & \overset{\theta_2}{\longleftrightarrow} & \overline{3}\,\underline{3}\,\overline{1}\,3\,\underline{2}\,\underline{3}\,\underline{1}\,2\,1 \\
\theta_2 \updownarrow & & & & \theta_1 \updownarrow \\
\overline{3}\,\underline{3}\,\overline{1}\,3\,\underline{1}\,\underline{2}\,\underline{1}\,2\,1 & \overset{\theta_1}{\longleftrightarrow} & \overline{3}\,\underline{3}\,\overline{2}\,3\,\underline{2}\,\underline{2}\,\underline{1}\,2\,1 & \overset{\theta_2}{\longleftrightarrow} & \overline{3}\,\underline{3}\,\overline{2}\,3\,\underline{2}\,\underline{3}\,\underline{1}\,2\,1
\end{array}. \tag{74}$$

Below, we illustrate this action on a set of skew Young tableaux generated by the LR tableau $\mathscr{T}$ considered previously:

$$
\begin{array}{ccccc}
\mathscr{T} = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 2 \\ \bullet & \bullet & 1 & 3 & \\ \bullet & \bullet & 1 & 3 & \\ \bullet & 1 & 2 & & \\ 1 & 2 & & & \end{array} & \overset{\theta_1}{\longleftrightarrow} & \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 2 \\ \bullet & \bullet & 1 & 3 & \\ \bullet & \bullet & 2 & 3 & \\ \bullet & 1 & 2 & & \\ 1 & 2 & & & \end{array} & \overset{\theta_2}{\longleftrightarrow} & \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 3 \\ \bullet & \bullet & 1 & 3 & \\ \bullet & \bullet & 2 & 3 & \\ \bullet & 1 & 3 & & \\ 1 & 2 & & & \end{array} \\
\updownarrow^{\theta_2} & & & & \updownarrow^{\theta_1} \\
\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 3 \\ \bullet & \bullet & 1 & 3 & \\ \bullet & \bullet & 1 & 3 & \\ \bullet & 1 & 2 & & \\ 1 & 2 & & & \end{array} & \overset{\theta_1}{\longleftrightarrow} & \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 3 \\ \bullet & \bullet & 2 & 3 & \\ \bullet & \bullet & 2 & 3 & \\ \bullet & 1 & 2 & & \\ 1 & 2 & & & \end{array} & \overset{\theta_2}{\longleftrightarrow} & \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 3 \\ \bullet & \bullet & 2 & 3 & \\ \bullet & \bullet & 2 & 3 & \\ \bullet & 1 & 3 & & \\ 1 & 2 & & & \end{array}
\end{array}.
$$

Example 4

Finally, we consider a decomposition of the indexing sets (J_1, J_2, J_3) such that w is a shuffle of the row words $w_1 = w(\{5\}) = 1$, $w_2 = w(\{1, 3\}) = 21$, and $w_3^1 = w(\{7, 6, 2\}) = 321 = w_3^2 = (\{9, 8, 4\})$. According to this decomposition, we have $w = \overline{2}\,\underline{3}\,\overline{1}\,3\,\hat{1}\,\underline{2}\,\underline{1}\,21$ as a shuffle of w_i, $i = 1, 2$, where $w_1 = \hat{1}$, and w_3^1, w_3^2. Thus, by (65), the symmetric group acts on w in the following way:

$$
\begin{array}{ccccc}
w = \overline{2}\,\underline{3}\,\overline{1}\,3\,\hat{1}\,\underline{2}\,\underline{1}\,2\,1 & \overset{\theta_1}{\longleftrightarrow} & \overline{2}\,\underline{3}\,\overline{1}\,3\,\hat{2}\,\underline{2}\,\underline{1}\,2\,1 & \overset{\theta_2}{\longleftrightarrow} & \overline{3}\,\underline{3}\,\overline{1}\,3\,\hat{3}\,\underline{2}\,\underline{1}\,2\,1 \\
\overset{\theta_2}{\updownarrow} & & & & \overset{\theta_1}{\updownarrow} \\
\overline{3}\,\underline{3}\,\overline{1}\,3\,\hat{1}\,\underline{2}\,\underline{1}\,2\,1 & \overset{\theta_1}{\longleftrightarrow} & \overline{3}\,\underline{3}\,\overline{2}\,3\,\hat{2}\,\underline{2}\,\underline{1}\,2\,1 & \overset{\theta_2}{\longleftrightarrow} & \overline{3}\,\underline{3}\,\overline{2}\,3\,\hat{3}\,\underline{2}\,\underline{1}\,2\,1
\end{array}. \tag{75}
$$

This action clearly differs from the one considered in (61)–(63) but the output is still in the same Knuth class.

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