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On the computation of solutions of systems of interval polynomial equations

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Abstract

Systems of algebraic equations with interval coefficients are very common in several areas of engineering sciences. The computation of the solution of such systems is a central problem when the characterization of the variables related by such systems is desired.

In this paper we characterize the solution of systems of algebraic equations with real interval coefficients. The characterization is obtained considering the approach introduced in J. Comput. Appl. Math. 136 (2001) 271.

MSC: 65M06; 65M20; 65M15

Keywords: Interval polynomials; Systems of interval equations

1. Introduction

In certain areas of engineering sciences several behaviour laws of variables with a relevant role are formulated by algebraic expressions with perturbed coefficients which can be seen as real intervals [3,6–10]. The characterization of such variables is established solving systems of algebraic equations with interval coefficients. The case of one equation arises, for instance, in automatic control theory in [3], in dynamic systems in [6] and in astrophysics [4]. In this case in order to characterize the variable of interest, an interval polynomial equation must be solved. We mention that the computation of the roots of interval polynomials using the particular form of the polynomials was considered in [3]. In [1] the computation was made using the interval arithmetic. Attending that for certain interval

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equations, the last approach does not enable us to obtain the desired roots at least approximately [2], in [5] a new approach was introduced which can be considered for very general interval polynomial equations.

In this paper, we use the approach introduced in [5] in order to compute the solution of systems of interval polynomials equations. Using this approach we are able to characterize the set containing the solution of the mentioned system.

Attending that the characterization of the solution of a system of interval polynomial equations with interval coefficients is obtained characterizing the set of solutions of each equation we start in Section 2 by generalizing the approach considered by the authors before in [5] to interval polynomials in several variables. The arguments presented in Section 2 are used in Section 3 on the characterization of the solution of systems of equations involving interval polynomials in several variables.

2. Interval polynomials in several variables

Let $Q_{m_1,\dots,m_n}(z_1,\dots,z_n)$ be the interval polynomial in the variables z_i, $i=1,\dots,n$, of degree $m_i \in \mathbb{N}$, $i=1,\dots,n$, respectively, defined by

$$Q_{m_1,\dots,m_n}(z_1,\dots,z_n) = \sum_{i=1}^{n} \mathbb{P}_{m_i}(z_i) + A_0, \tag{1}$$

where $A_0 = [a_0, b_0]$ is a closed real interval, $\mathbb{P}_{m_i}(z_i)$ is an interval polynomial in the variable z_i, without independent term and of degree m_i, that is, $\mathbb{P}_{m_i}(z_i)$ is defined by

$$\mathbb{P}_{m_i}(z_i) = \sum_{j=1}^{m_i} A_{i,j} z_i^j, \quad A_{i,j} = [a_{i,j}, b_{i,j}], \quad j = 1,\dots,m_i$$

for $i=1,\dots,n$. The interval polynomials $Q_{m_1,\dots,m_n}(z_1,\dots,z_n)$ can be seen as a family of real polynomials in the variables $z_1,\dots,z_n$.

Definition 1. Let $Q_{m_1,\dots,m_n}(z_1,\dots,z_n)$ be a interval polynomial in the variables z_i, $i=1,\dots,n$, defined by (1). By $G(Q_{m_1,\dots,m_n})$ we denote the graph of $Q_{m_1,\dots,m_n}(z_1,\dots,z_n)$ which is given by

$$G(Q_{m_1,\dots,m_n}) = \{(\tilde{z}_1,\dots,\tilde{z}_n,\tilde{z}) \in \mathbb{R}^{n+1} : \exists p_{m_1,\dots,m_n}(z_1,\dots,z_n) \in Q_{m_1\dots m_n}(z_1,\dots,z_n),$$
$$\tilde{z} = p_{m_1,\dots,m_n}(\tilde{z}_1,\dots,\tilde{z}_n)\}.$$

Example 1. Let us consider the interval polynomial of degree two in the variables z_1, z_2 defined by

$$Q_{2,2}(z_1,z_2) = z_1^2 + [-4,-2]z_1 + z_2^2 + [-4,-2]z_2 + [1,4].$$

The graph of $Q_{2,2}(z_1,z_2)$, $G(Q_{2,2})$, is the set of all points of $\mathbb{R}^3$ between the two surfaces plotted in Fig. 1.

Using the approach introduced in [5], we can define certain functions such that their graphs are the boundary surfaces of $G(Q_{m_1,\dots,m_n})$.

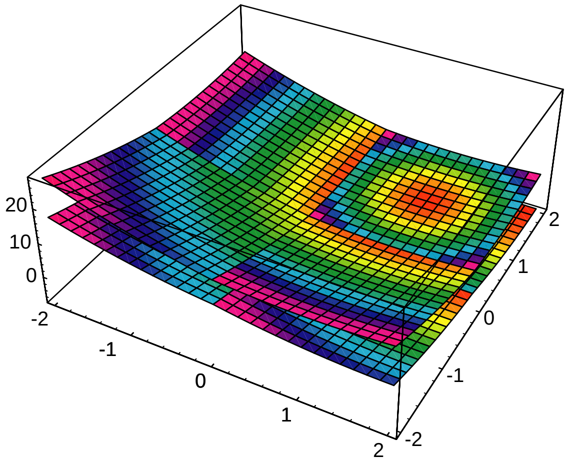

Fig. 1. $G(Q_{2,2})$ for $Q_{2,2}(z_1,z_2)=z_1^2+[-4,-2]z_1+z_2^2+[-4,-2]z_2+[1,4]$.

Following the mentioned approach we associate with $Q_{m_1,\dots,m_n}(z_1,\dots,z_n)$ the real polynomials $q_{m_1,\dots,m_n}(z_1,\dots,z_n)$ and $r_{m_1,\dots,m_n}(z_1,\dots,z_n)$ defined in what follows. We start by noting that $\mathbb{R}^n$ is the union of the sets of the type $\prod_{j=1}^n \mathbb{R}^{s_j}$ where $s_j \in \{+,-\}$ for $j=1,\dots,n$, and we assume that when all $s_j=+$ or all $s_j=-$, by $\mathbb{R}^{s_j}$ we represent $\mathbb{R}_0^{s_j}$. The expressions of $q_{m_1,\dots,m_n}(z_1,\dots,z_n)$ and $r_{m_1,\dots,m_n}(z_1,\dots,z_n)$ are depending of $\prod_{j=1}^n \mathbb{R}^{s_j}$.

We consider

$$q_{m_1,\dots,m_n}(z_1,\dots,z_n)=q_{m_1,\dots,m_n}^{s_1\cdots s_n}(z_1,\dots,z_n) \quad \text{if } (z_1,\dots,z_n)\in\prod_{j=1}^n \mathbb{R}^{s_j}, \tag{2}$$

where

$$q_{m_1,\dots,m_n}^{s_1\dots s_n}(z_1,\dots,z_n)=\sum_{i=1}^n\sum_{j=1}^{m_i} q_{i,j}z_i^j+q_0,$$

with $q_0=b_0$, and

$$q_{i,j}=\begin{cases} b_{i,j} & \text{if } s_i=+ \text{ or } s_i=- \text{ and } j \text{ even},\\ a_{i,j} & \text{if } s_i=- \text{ and } j \text{ odd}.\end{cases}$$

We represent by $r_{m_1,\dots,m_n}$ $(z_1,\dots,z_n)$ the following function:

$$r_{m_1,\dots,m_n}(z_1,\dots,z_n)=r_{m_1,\dots,m_n}^{s_1\dots s_n}(z_1,\dots,z_n) \quad \text{if } (z_1,\dots,z_n)\in\prod_{j=1}^n \mathbb{R}^{s_j}, \tag{3}$$

with

$$r_{m_1,\dots,m_n}^{s_1\dots s_n}(z_1,\dots,z_n)=\sum_{i=1}^n\sum_{j=1}^{m_i} r_{i,j}z_i^j+r_0,$$

$r_0 = a_0$, and

$$r_{i,j} = \begin{cases} a_{i,j} & \text{if } s_i = + \text{ or } s_i = - \text{ and } j \text{ even}, \\ b_{i,j} & \text{if } s_i = - \text{ and } j \text{ odd}. \end{cases}$$

We call $q_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ and $r_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$, respectively the upper and lower polynomials associated with $Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$, and together, these two polynomials will be called in the following, boundary polynomials associated with $Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$.

The following lemma is consequence of the above definitions.

Lemma 1. *Let $Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ be a interval polynomial in the variables z_i, $i=1,\ldots,n$, defined by (1) and $G(Q_{m_1,\ldots,m_n})$ its graph. We have*

$$G(Q_{m_1,\ldots,m_n}) = \{(\tilde{z}_1,\ldots,\tilde{z}_n,\tilde{z}) \in \mathbb{R}^{n+1} : r_{m_1,\ldots,m_n}(\tilde{z}_1,\ldots,\tilde{z}_n) \leqslant \tilde{z} \leqslant q_{m_1,\ldots,m_n}(\tilde{z}_1,\ldots,\tilde{z}_n)\}. \quad (4)$$

Let us suppose that we intent to characterize the set

$$\mathcal{N}(Q_{m_1,\ldots,m_n}) = \{(\tilde{z}_1,\ldots,\tilde{z}_n)\mathbb{C}^n : \exists p_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \in Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n), \\ p_{m_1,\ldots,m_n}(\tilde{z}_1,\ldots,\tilde{z}_n) = 0\},$$

where $\mathbb{C}^n = \{(z_1,\ldots,z_n) : z_i \in \mathbb{C},\ i=1,\ldots,n\}$ and $\mathbb{C}$ denotes the complex plan. This characterization can be easily made using the boundary polynomials $q_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ and $r_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ induced by $Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ and is established in the following theorem.

Theorem 1. *Let $Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ be a interval polynomial in the variables z_i, $i=1,\ldots,n$, defined by (1) and let $q_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ and $r_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ be the boundary polynomials associated with $Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$. We have*

1. *if $r_{m_1,\ldots,m_n}(z_1,\ldots,z_n) > 0$ for all $(z_1,\ldots,z_n) \in \mathbb{R}^n$, then $\mathcal{N}(Q_{m_1,\ldots,m_n}) \cap \mathbb{R}^n = \emptyset$;*
2. *if $q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) = 0$ for some $(z_1,\ldots,z_n) \in \mathbb{R}^n$, then $\mathcal{N}(Q_{m_1,\ldots,m_n}) \subset \mathbb{R}^n$;*
3. *if $q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) > 0$ for all $(z_1,\ldots,z_n) \in \mathbb{R}^n$, and $r_{m_1,\ldots,m_n}(z_1,\ldots,z_n) = 0$ for some $(z_1,\ldots,z_n) \in \mathbb{R}^n$, then $\mathcal{N}(Q_{m_1,\ldots,m_n}) = I_1 \cup I_2$ with $I_1 \subset \mathbb{R}^n$ and $I_2 \subset \mathbb{C}^n$.*

Example 2. Let us consider Example 1. For the interval polynomial $Q_{2,2}(z_1,z_2)$ consider there we have the following boundary polynomials:

1.

$$q_{2,2}(z_1,z_2) = \begin{cases} z_1^2 - 2z_1 + z_2^2 - 2z_2 + 4, & z_1 \geqslant 0,\ z_2 \geqslant 0, \\ z_1^2 - 2z_1 + z_2^2 - 4z_2 + 4, & z_1 \geqslant 0,\ z_2 < 0, \\ z_1^2 - 4z_1 + z_2^2 - 2z_2 + 4, & z_1 < 0,\ z_2 > 0, \\ z_1^2 - 4z_1 + z_2^2 - 4z_2 + 4, & z_1 < 0,\ z_2 < 0, \end{cases}$$

2.

$$r_{2,2}(z_1,z_2)=\begin{cases} z_1^2-4z_1+z_2^2-4z_2+1, & z_1\geqslant 0,\ z_2\geqslant 0,\\ z_1^2-4z_1+z_2^2-2z_2+1, & z_1\geqslant 0,\ z_2<0,\\ z_1^2-2z_1+z_2^2-4z_2+1, & z_1<0,\ z_2>0,\\ z_1^2-2z_1+z_2^2-2z_2+1, & z_1<0,\ z_2<0.\end{cases}$$

The graphics of the boundary polynomials are plotted in Fig. 1. We observe that $q_{2,2}(z_1,z_2)>0$ for all $(z_1,z_2)\in\mathbb{R}^2$ and $\{(z_1,z_2)\in\mathbb{R}^2 : r_{2,2}(z_1,z_2)=0\}$ is the set

$$\begin{aligned}&\{(z_1,z_2)\in\mathbb{R}^2 : z_1>0, z_2>0, (z_1-2)^2+(z_2-2)^2=7\}\\ &\cup\{(z_1,z_2)\in\mathbb{R}^2 : z_1>0, z_2<0, (z_1-2)^2+(z_2-1)^2=3\}\\ &\cup\{(z_1,z_2)\in\mathbb{R}^2 : z_1<0, z_2>0, (z_1-1)^2+(z_2-2)^2=4\}.\end{aligned}$$

and, by Theorem 1, $\mathcal{N}(Q_{2,2})=I_1\cup I_2$ with $I_1\subset\mathbb{R}^2$ and $I_2\subset\mathbb{C}^2$.

3. Systems of interval polynomial equations

In this section we consider systems of interval polynomial equations of the type (1), that is,

$$Q^{(\ell)}_{m_{1,\ell},\dots,m_{1,\ell}}(z_1,\dots,z_n)=0,\quad \ell=1,\dots,n, \tag{5}$$

where, for each ℓ,

$$Q^{(\ell)}_{m_{1,\ell},\dots,m_{1,\ell}}(z_1,\dots,z_n)=\sum_{i=1}^{n}\sum_{j=1}^{m_{i,\ell}}A^{(\ell)}_{i,j}z_i^j+A^{(\ell)}_0$$

and

$$A^{(\ell)}_{i,j}:=[a^{(\ell)}_{i,j},b^{(\ell)}_{i,j}],\quad A^{(\ell)}_0=[a^{(\ell)}_0,b^{(\ell)}_0].$$

Example 3. For example the system

$$\begin{aligned}&[1,2]z_1+[3,4]z_2=0,\\ &[2,3]z_1+[-2,-1]z_2+[-2,-1]=0\end{aligned} \tag{6}$$

is of type (5). Our aim is to compute $(\tilde z_1,\tilde z_2)\in\mathbb{R}^2$ verifying the two interval equations, that is, such that exist $p^{(1)}_{1,1}(z_1,z_2)\in[1,2]z_1+[3,4]z_2$ and $p^{(2)}_{1,1}(z_1,z_2)\in[2,3]z_1+[-2,-1]z_2+[-2,-1]$ verifying

$$p^{(1)}_{1,1}(\tilde z_1,\tilde z_2)=p^{(2)}_{1,1}(\tilde z_1,\tilde z_2)=0.$$

We intent to compute $(\tilde z_1,\dots,\tilde z_n)\in\mathbb{R}^n$ such that, for each $\ell\in\{1,\dots,n\}$, exists $p^{(\ell)}_{m_{1,\ell},\dots,m_{n,\ell}}(z_1,\dots,z_n)\in Q^{(\ell)}_{m_{1,\ell},\dots,m_{n,\ell}}(z_1,\dots,z_n)$ such that

$$p^{(\ell)}_{m_{1,\ell},\dots,m_{n,\ell}}(\tilde z_1,\dots,\tilde z_n)=0.$$

The set of all these points of $\mathbb{R}^n$ is denoted by $\mathcal{S}(Q^{(\ell)}_{m_{1,\ell},\dots,m_{n,\ell}},\ \ell=1,\dots,n)$.

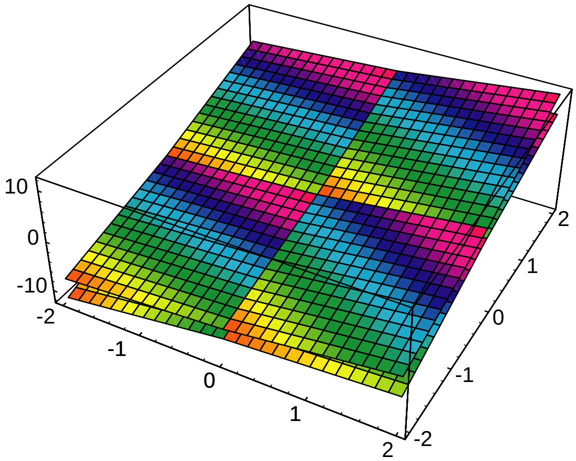

Fig. 2. $G(Q^{(1)}_{1,1})$ for $Q^{(1)}_{1,1}(z_1, z_2) = [1, 2]z_1 + [3, 4]z_2$.

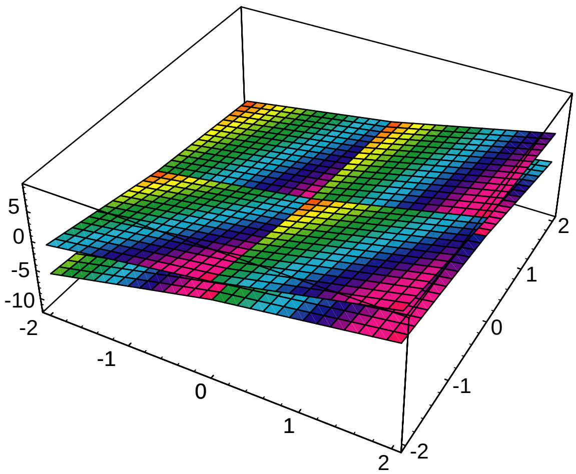

Fig. 3. $G(Q^{(2)}_{1,1})$ for $Q^{(2)}_{1,1}(z_1, z_2) = [2, 3]z_1 + [-2, -1]z_2 + [-2, -1]$.

Theorem 2. *Let us consider the system* (5) *and the sets* $\mathcal{N}(Q^{(\ell)}_{m_{1,\ell},\dots,m_{1,\ell}})$ *for* $\ell = 1, \dots, n$. *Then*

$$\mathcal{S}(Q^{(\ell)}_{m_{1,\ell},\dots,m_{n,\ell}}, \ell = 1, \dots, n) = \bigcap_{\ell=1}^{n} \mathcal{N}(Q^{(\ell)}_{m_{1,\ell},\dots,m_{1,\ell}}).$$

Applying Theorem 1 we are able to characterize each $\mathcal{N}(Q^{(\ell)}_{m_{1,\ell},\dots,m_{1,\ell}})$ and using Theorem 2, we obtain $\{(\tilde{z}_1, \dots, \tilde{z}_n) \in \mathbb{R}^n : Q^{(\ell)}_{m_{1,\ell},\dots,m_{1,\ell}}(\tilde{z}_1, \dots, \tilde{z}_n) = 0,\ \ell = 1, \dots, n\}$.

Example 4. Let us consider Example 3. The graphics of the two interval polynomials in the variables z_1 and z_2, $Q^{(1)}_{1,1}(z_1, z_2)$, $Q^{(2)}_{1,1}(z_1, z_2)$, defined by the two equations are the points of $\mathbb{R}^3$ between the two surfaces plotted in Figs. 2 and 3, respectively.

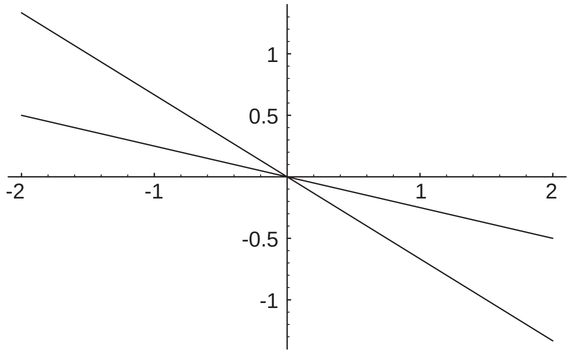

Fig. 4. $\mathcal{N}(Q_{1,1}^{(1)})$ for $Q_{1,1}^{(1)}(z_1, z_2) = [1,2]z_1 + [3,4]z_2$.

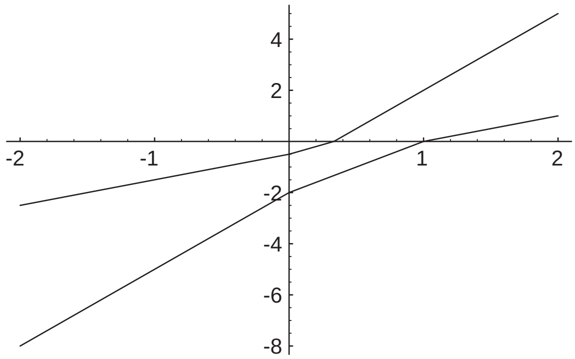

Fig. 5. $\mathcal{N}(Q_{1,1}^{(2)})$ for $Q_{1,1}^{(2)}(z_1, z_2) = [2,3]z_1 + [-2,-1]z_2 + [-2,-1]$.

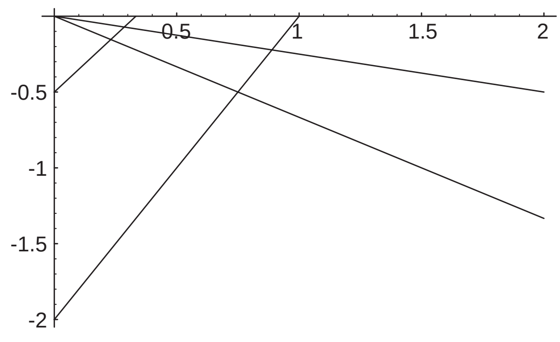

Fig. 6. $\mathcal{S}(Q_{1,1}^{(1)}, Q_{1,1}^{(2)}) = \mathcal{N}(Q_{1,1}^{(1)}) \cap \mathcal{N}(Q_{1,1}^{(2)})$.

In Figs. 4 and 5 we plot $\mathcal{N}(Q_{1,1}^{(1)})$ and $\mathcal{N}(Q_{1,1}^{(2)})$, respectively.
In Fig. 6 we plot $\mathcal{S}(Q_{1,1}^{(1)}, Q_{1,1}^{(2)}) = \mathcal{N}(Q_{1,1}^{(1)}) \cap \mathcal{N}(Q_{1,1}^{(2)})$.

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