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On first and second countable spaces and the axiom of choice

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Abstract

In this paper it is studied the role of the axiom of choice in some theorems in which the concepts of first and second countability are used. Results such as the following are established:

(1) In **ZF** (Zermelo–Fraenkel set theory without the axiom of choice), equivalent are:
  (i) every base of a second countable space has a countable subfamily which is a base;
  (ii) the axiom of countable choice for sets of real numbers.
(2) In **ZF**, equivalent are:
  (i) every local base at a point x, in a first countable space, contains a countable base at x;
  (ii) the axiom of countable choice (**CC**).
(3) In **ZF**, equivalent are:
  (i) for every local base system $(\mathcal{B}(x))_{x\in X}$ of a first countable space X, there is a local base system $(\mathcal{V}(x))_{x\in X}$ such that, for each $x \in X$, $\mathcal{V}(x)$ is countable and $\mathcal{V}(x) \subseteq \mathcal{B}(x)$;
  (ii) for every family $(X_i)_{i\in I}$ of non-empty sets there is a family $(A_i)_{i\in I}$ of non-empty, at most countable sets, such that $A_i \subseteq X_i$ for every $i \in I$ (ω-**MC**) and **CC**.

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1. Introduction

The idea that triggered the investigations on this paper was to find out the set theoretic status of the following Theorem of **ZFC**, i.e., *Zermelo–Fraenkel set theory including the axiom of choice*.

Theorem 1.1. (**ZFC**) *Every base of a second countable space has a countable subfamily which is a base.*

We will see that this theorem is not provable in **ZF**, *Zermelo–Fraenkel set theory without the axiom of choice*, by proving its equivalence to the axiom of countable choice for sets of reals.

It is clear that Theorem 1.1 provides an alternative definition of second countability that, in the absence of the axiom of choice, turns out to be non-equivalent to the familiar definition. Starting from these two definitions of second countability, we will discuss the consequences of replacing one by another in some well-known theorems. Namely, we will study the relations between this "new" class of second countable spaces, and the classes of separable, Lindelöf spaces.

In the literature it may be found a discussion of the equivalence, in **ZF**, of different ways of defining some well-known topological notions. As interesting examples of this kind of study, we have that the relations between different notions of compactness (e.g., [9,3]) or of Lindelöfness [18,10] were studied.

We also present two different attempts to generalize Theorem 1.1 to the class of first countable spaces, as well as their relations with the axiom of choice.

The following forms of choice will be used throughout this paper. Their definitions, as everything else in this work, take place in the setting of **ZF**.

Definition 1.2. The *axiom of countable choice* (**CC**) states that every countable family of non-empty sets has a choice function.

Definition 1.3. **CC**($\mathbb{R}$) is the axiom of countable choice restricted to families of sets of real numbers.

Proposition 1.4 ([6, p. 76], [11]). *Equivalent are*:

(i) **CC** (*respectively* **CC**($\mathbb{R}$));
(ii) *every countable family of non-empty sets* (*respectively subsets of* $\mathbb{R}$) *has an infinite subfamily with a choice function*;
(iii) *for every countable family* $(X_n)_n$ *of non-empty sets* (*respectively subsets of* $\mathbb{R}$), *there is a sequence that meets infinitely many of the* X_n*'s.*

Lemma 1.5.

(a) *If* $(X, \mathcal{T})$ *is a second countable space, then* $|\mathcal{T}| \leqslant |\mathbb{R}| = 2^{\aleph_0}$.
(b) *If* $(X, \mathcal{T})$ *is a second countable* T_0*-space, then* $|X| \leqslant |\mathbb{R}| = 2^{\aleph_0}$.

2. Second countable spaces

We start this section recalling some definitions.

Definitions 2.1.

(a) A topological space is *separable* if it contains an at most countable dense subset.
(b) A topological space X is *Lindelöf* if every open cover of X has an at most countable subcover.

The next lemma will play an important role in the proof the main result of the section, Theorem 2.3.

Lemma 2.2. *Equivalent are*:

(i) $\mathbf{CC}(\mathbb{R})$;
(ii) *the axiom of countable choice holds for families of dense subspaces of* $\mathbb{R}$;
(iii) *every subspace of* $\mathbb{R}$ *is separable*;
(iv) *every dense subspace of* $\mathbb{R}$ *is separable.*

Proof. The equivalence between (i) and (iii) was proved by Diener—cited in [5, p. 128] (see also [12]). That (i) implies (ii) and that (iii) implies (iv) is clear.

We consider the base of $\mathbb{R}$ consisting of open intervals $((q_n, r_n))_{n\in\mathbb{N}}$ with rational endpoints. For each $n \in \mathbb{N}$, one can define a bijection $f_n : \mathbb{R} \to (q_n, r_n)$ between $\mathbb{R}$ and (q_n, r_n).

(iv) ⇒ (i) Let $(A_n)_n$ be a countable family of non-empty subsets of $\mathbb{R}$ and define the sets $B_n := f_n(A_n)$ and $B := \bigcup_n B_n$. The space B is dense in $\mathbb{R}$. By (iv), there is $C := \{x_n\colon n \in \mathbb{N}\}$ countable and dense in B, which implies that it is also dense in $\mathbb{R}$.

Infinitely many of the sets $B_n \cap C$ are not empty, otherwise C would be bounded and then not dense in $\mathbb{R}$. For each element of $\mathbb{M} := \{n \in \mathbb{N}\colon\ B_n \cap C \neq \emptyset\}$, we define $\phi(m) := \min\{k \in \mathbb{N}\colon\ x_k \in B_m\}$. The set $\{f_m^{-1}(x_{\phi(m)})\colon\ m \in \mathbb{M}\}$ induces a choice function in the infinite subfamily $(A_m)_{m\in\mathbb{M}}$ of $(A_n)_{n\in\mathbb{N}}$. In view of Proposition 1.4, the proof is complete.

(ii) ⇒ (iv) Let A be a dense subspace of $\mathbb{R}$. For every $n \in \mathbb{N}$, $f_n^{-1}(A \cap (q_n, r_n))$ is dense in $\mathbb{R}$. A choice function in this family gives us a countable dense subspace of A. □

Theorem 2.3. *Equivalent are*:

(i) $\mathbf{CC}(\mathbb{R})$;
(ii) *every base of a second countable space has a countable subfamily which is a base*;
(iii) *every base for the open sets of* $\mathbb{R}$ *has a countable subfamily which is a base.*

Proof. (i) ⇒ (ii) Following the usual proof of (ii) (e.g., [2, 2.4.17], [8, 1.1.20]), we easily see that the only use of the axiom of choice is a countable choice in a family of subsets of the topology $\mathcal{T}$ of the second countable space. Lemma 1.5 says that $|\mathcal{T}| \leqslant |\mathbb{R}|$, and then **CC**($\mathbb{R}$) is enough to prove (ii).

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (ii) Let A be a dense subset of $\mathbb{R}$. By Lemma 2.2, it suffices to prove that A is separable. The fact that A is dense in $\mathbb{R}$ implies that $\mathcal{C} := \{(a, b)\colon a < b \text{ and } a, b \in A\}$ is a base for the open sets of $\mathbb{R}$. By (iii), there is a countable base $\{(a_n, b_n)\colon n \in \mathbb{N}\}$ contained in $\mathcal{C}$. The set $\{a_n\colon n \in \mathbb{N}\}$ is countable and dense in A. $\square$

It is well known that, in **ZFC**, for (pseudo)metric spaces the notions of second countability, separability and Lindelöfness are equivalent. Good and Tree [7] asked under which conditions these equivalences or implications remain valid in **ZF**. These questions are almost all answered (see [12,1,17]).

Motivated by condition (ii) of Theorem 2.3, we will introduce a definition of second countable space that is stronger than the usual one in **ZF**, but equivalent in **ZFC**.

We will look into the relations between this "new" class of topological spaces and the classes of separable, Lindelöf spaces.

Definition 2.4. A topological space is called *super second countable* (SSC) if every base has a countable subfamily which is a base.

Corollary 2.5. *Equivalent are*:

(i) **CC**($\mathbb{R}$);
(ii) $\mathbb{R}$ *is SSC*;
(iii) *every separable* (*pseudo*)*metric space is SSC.*

Note that, in **ZF**, every separable pseudometric space is second countable (see, e.g., [19, 16.11]).

The statement "*Every SSC topological* (*or pseudometric*) *space is separable*" is equivalent to **CC**. The proof remains the same as the one for second countable spaces [1]. It may seem surprising that, for subsets of $\mathbb{R}$, this implication is provable in **ZF**.

Theorem 2.6. *Every SSC subspace of* $\mathbb{R}$ *is separable.*

Proof. Let $A \subseteq \mathbb{R}$ be a SSC space. Without loss of generality, we consider that every point of A is an accumulation point of A. If $a \in A$ is not an accumulation point of A, $\{a\}$ must be in each base for the open sets of A.

The set $\mathcal{B} := \{(a, b) \cap A\colon a, b \in A\} \cup \{[c, d) \cap A\colon c, d \in A \text{ and } (\exists \delta > 0)\ (c - \delta, c) \cap A = \emptyset\} \cup \{(e, f] \cap A\colon e, f \in A \text{ and } (\exists \delta > 0)\ (f, f + \delta) \cap A = \emptyset\}$ is a base for the open sets of A. Since A is SSC, there is a countable base $(B_n)_n$ contained in $\mathcal{B}$. For $s_n := \inf B_n$, the set $\{s_n\colon n \in \mathbb{N}\}$ is countable and dense in A. $\square$

Since $\mathbb{R}$ is second countable and second-countability is hereditary, every second countable subspace of $\mathbb{R}$ is separable if and only if every subspace of $\mathbb{R}$ is separable, which turns out to be equivalent to **CC**($\mathbb{R}$)—Lemma 2.2.

This last fact, together with Lemma 1.5, implies that $\mathbf{CC}(\mathbb{R})$ is equivalent to: "*Every second countable metric* (*or* T_0) *space is separable*" (see also [17]).

In view of Theorem 2.6, the proof of this latter result cannot be adapted for SSC spaces.

After these considerations, one can ask the following questions:

(1) Is SSC hereditary?
(2) Are there non-separable SSC metric spaces? Are there uncountable SSC T_0-spaces?

The set theoretic status of the condition "*Every Lindelöf metric space is second countable*" is, to my knowledge, still unknown. It is known, however, that this condition implies the axiom of countable choice for finite sets [7,1,17]).

For SSC spaces, we can go further.

Theorem 2.7. *Every Lindelöf subspace of* $\mathbb{R}$ *is SSC if and only if* $\mathbf{CC}(\mathbb{R})$ *holds.*

Proof. If $\mathbf{CC}(\mathbb{R})$ holds, trivially, every subspace of $\mathbb{R}$ is SSC (Theorem 2.3).

One can prove similarly to the proof of Theorem 2.3, that $\mathbf{CC}(\mathbb{R})$ is equivalent to the fact that the closed interval $[0, 1]$ is SSC. So, if $\mathbf{CC}(\mathbb{R})$ fails, $[0, 1]$ is Lindelöf, but not SSC. □

Note that, if $\mathbf{CC}(\mathbb{R})$ fails, the only Lindelöf subspaces of $\mathbb{R}$ are the compact spaces, i.e., the closed and bounded ones (see [10]).

Corollary 2.8. *If every Lindelöf metric space is SSC, then* $\mathbf{CC}(\mathbb{R})$ *holds.*

$\mathbf{CC}(\mathbb{R})$ is equivalent to the condition "$\mathbb{N}$ *is Lindelöf*", and thus also equivalent to the condition "*Every second countable space is Lindelöf*" [12]. Correspondingly, "*Every SSC space is Lindelöf*" if and only if $\mathbf{CC}(\mathbb{R})$ holds, since $\mathbb{N}$ is SSC.

3. First countable spaces

It is natural to ask whether the result of Theorem 2.3 can be generalized to the class of first countable spaces.

There are two obvious ways of attempting this: a local one, considering a local base at a point, and a global one, considering, at the same time, a local base for each point of a first countable space. The next results are an attempt to answer these questions.

Theorem 3.1. *Equivalent are*:

(i) $\mathbf{CC}$;
(ii) *if a topological space has a countable local base at a point* x, *then every local base at* x *contains a countable base at* x;
(iii) *every local base at a point* x, *in a first countable space, contains a countable base at* x.

Proof. A proof that (i) implies (ii) can be seen in [2, 2.4.12] and (ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i) Let $(X_n)_n$ be a countable family of non-empty sets. Without loss of generality, we consider the sets X_n disjoint. By Proposition 1.4, it is enough to prove that there is a sequence that meets infinitely many of the X_n's.

Define $Y := \bigcup_n X_n \cup \{0\}$, with $0 \notin \bigcup_n X_n$, and for each $n \in \mathbb{N}$, $Y_n := \bigcup_{k=n+1}^{\infty} X_k \cup \{0\}$. The topology on Y, defined by the local base system:

$$\mathcal{B}(x) := \begin{cases} \{\{x\}\} & \text{if } x \neq 0, \\ \{Y_n\colon n \in \mathbb{N}\} & \text{if } x = 0, \end{cases}$$

is first countable.

Since, for all $n \in \mathbb{N}$ and $x \in X_{n+1}$, $Y_{n+1} \subseteq Y_{n+1} \cup \{x\} \subseteq Y_n$, the family $\mathcal{C}(0) := \{Y_n \cup \{x\}\colon x \in X_n,\ n \in \mathbb{N}\}$ is a local base at 0.

By (iii), there is a countable local base at 0, $\mathcal{D} := \{D_n\colon n \in \mathbb{N}\} \subseteq \mathcal{C}(0)$. Define, for every $n \in \mathbb{N}$, $\mathcal{C}_n := \{Y_n \cup \{x\}\colon x \in X_n\}$.

For each $n \in \mathbb{N}$, there is exactly one $\phi(n) \in \mathbb{N}$ such that $D_n \in C_{\phi(n)}$, because $\mathcal{C}(0)$ is the disjoint union of all $\mathcal{C}_n$'s. For every $n \in \mathbb{N}$, let x_n be the element of the singleton set $D_n \setminus Y_{\phi(n)}$. The sequence $(x_n)_n$ meets infinitely many of the X_n's, otherwise $\mathcal{D}$ would not be a base. □

Definition 3.2 ([13], [14, Form 76]). ω-**MC** states that, for every family $(X_i)_{i\in I}$ of non-empty sets, there is a family $(A_i)_{i\in I}$ of non-empty at most countable sets such that $A_i \subseteq X_i$ for every $i \in I$.

Theorem 3.3. *If ω-**MC** holds, then every first countable space X has a local base system $(\mathcal{D}(x))_{x\in X}$ such that, for each $x \in X$, $\mathcal{D}(x)$ is countable.*

Proof. Let X be a first countable space and consider the set $\mathsf{A}(x)$ of all functions $f:\mathbb{N} \to \mathcal{P}(X)$ such that $f(\mathbb{N})$ is a local base at $x \in X$. Since X is first countable, $(\mathsf{A}(x))_{x\in X}$ is a family of non-empty sets. So, by ω-**MC**, there is $(\mathsf{C}(x))_{x\in X}$, with $\mathsf{C}(x)$ countable and $\emptyset \neq \mathsf{C}(x) \subseteq \mathsf{A}(x)$ for each x in X.

Since $\mathsf{C}(x)$ is countable, one easily shows that $\mathcal{D}(x) := \{f(n)\colon\ f \in \mathsf{C}(x),\ n \in \mathbb{N}\}$ is also countable and then $(\mathcal{D}(x))_{x\in X}$ is a local base system with the local base at each point countable. □

Definition 3.4. The *countable union theorem* (**CUT**) says that countable unions of countable sets are countable.

Theorem 3.5. *Equivalent are*:

(i) ω-**MC** *and* **CC**;
(ii) ω-**MC** *and* **CUT**;
(iii) ω-**MC** *and the axiom of countable choice holds for families of countable sets* (**CC**$(\aleph_0)$);
(iv) *for every local base system $(\mathcal{B}(x))_{x\in X}$ of a first countable space X, there is a local base system $(\mathcal{V}(x))_{x\in X}$ such that, for each $x \in X$, $\mathcal{V}(x)$ is countable and $\mathcal{V}(x) \subseteq \mathcal{B}(x)$;*

(v) *if a topological space X has a local base system $(\mathcal{D}(x))_{x\in X}$ with each $\mathcal{D}(x)$ countable, then for every local base system $(\mathcal{B}(x))_{x\in X}$ of X, there is a local base system $(\mathcal{V}(x))_{x\in X}$ such that, for each $x \in X$, $\mathcal{V}(x)$ is countable and $\mathcal{V}(x) \subseteq \mathcal{B}(x)$.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) It is obvious that **CC** $\Rightarrow$ **CUT** $\Rightarrow$ **CC**($\aleph_0$), and if ω-**MC** holds, then **CC** is equivalent **CC**($\aleph_0$).

(iv) $\Rightarrow$ (v) Apparent.

(ii) $\Rightarrow$ (iv) Let $(\mathcal{B}(x))_{x\in X}$ be a local base system of a first countable space X. Theorem 3.3 says that X has a local base system $(\mathcal{D}(x))_{x\in X}$ with the local base at each point countable.

For each $x \in X$ and $U \in \mathcal{D}(x)$, define the sets $\mathsf{M}(U,x) := \{B \in \mathcal{B}(x)\colon\ B \subseteq U\}$ and $I := \bigcup_{x\in X} \mathcal{D}(x) \times \{x\}$.

Since each $\mathcal{B}(x)$ is a local base, it is clear that $(\mathsf{M}(U,x))_{(U,x)\in I}$ is a family of non-empty sets. Then ω-**MC** implies that there is a family $(\mathsf{E}(U,x))_{(U,x)\in I}$ such that each $\mathsf{E}(U,x)$ is countable and contained in $\mathsf{M}(U,x)$. Thus, by **CUT**, the sets $\mathcal{V}(x) := \bigcup_{U\in\mathcal{D}(x)} \mathsf{E}(U,x)$ are countable.

Finally, $(\mathcal{V}(x))_{x\in X}$ is a family of countable sets with $\mathcal{V}(x) \subseteq \mathcal{B}(x)$ for each $x \in X$, since $\mathsf{E}(U,x) \subseteq \mathsf{M}(U,x) \subseteq \mathcal{B}(x)$ for every pair $(U,x) \in I$. From the way it was defined, $(\mathcal{V}(x))_{x\in X}$ is also a local base system, which concludes the proof.

(v) $\Rightarrow$ (i) From Theorem 3.1 we know that condition (v) implies **CC**.

Let $(X_i)_{i\in I}$ be a family of non-empty sets. Without loss of generality, consider the family disjoint with its union disjoint from I.

Define the sets $Y_i := (X_i \times \mathbb{N}) \cup \{i\}$, $Y := \bigcup_{i\in I} Y_i$ and $D(i,n) := \{(x,k)\colon\ x \in X_i$ and $k \geqslant n+1\} \cup \{i\}$. The local base system

$$\mathcal{D}(x) := \begin{cases} \{\{x\}\} & \text{if } x \notin I, \\ \{D(x,n)\colon\ n \in \mathbb{N}\} & \text{if } x \in I. \end{cases}$$

defines a (first countable) topology on Y. It is clear that, for each point, the given local base is countable.

Since for each $x \notin I$, the singleton set $\{x\}$ must belong to every local base at x, for simplicity we consider $(\mathcal{B}(i) := \{D(i,n) \cup \{(x,n)\}\colon\ x \in X_i,\ n \in \mathbb{N}\})_{i\in I}$ as a local base system of Y.

By (v), there exists a family $(\mathcal{V}(i))_{i\in I}$ such that for every $i \in I$, $\mathcal{V}(i) \subseteq \mathcal{B}(i)$ and $\mathcal{V}(i)$ is at most countable and also non-empty, because it is a local base at i.

Finally, for each $i \in I$ we define the set $Y_i := \{x \in X_i\colon\ (\exists C \in \mathcal{V}(i))\ C \setminus D(i,n) = \{(x,n)\}$ for some $n \in \mathbb{N}\}$. This process gives a family $(Y_i)_{i\in I}$ of non-empty at most countable sets, with $Y_i \subseteq X_i$. $\square$

The equivalent conditions of Theorem 3.5 are properly weaker than the axiom of choice itself (Cohen/Pincus model—$\mathcal{M}1(\langle\omega_1\rangle)$ in [14]). In Part III of [14] other models with these characteristics can be found.

4. Hausdorff spaces

This section is motivated by the question: "*Are there first countable non-Hausdorff spaces in which every sequence has at most one limit*?" As we will see, the answer to this question is affirmative.

Theorem 4.1. *Equivalent are*:

(i) **CC**;
(ii) *a first countable space is Hausdorff if and only if every sequence has at most one limit.*

Proof. (i) ⇒ (ii) Condition (ii) is Proposition 1.6.17 in [4]. It is not difficult to see that no condition stronger than **CC** is used in the proof.

(ii) ⇒ (i) Let $(X_n)_n$ be a countable family of non-empty disjoint sets. In a similar way to the proof of Theorem 3.1, we construct the sets $Y_n := \bigcup_{k=n}^{\infty} X_k$ and $Y := \bigcup_n X_n \cup \{a, b\}$, with $a \neq b$ and both not in $\bigcup_n X_n$. The local base system

$$\mathcal{B}(x) := \begin{cases} \{\{x\}\} & \text{if } x \notin \{a, b\}, \\ \{Y_n \cup \{x\}\colon\ n \in \mathbb{N}\} & \text{if } x \in \{a, b\} \end{cases}$$

defines a first countable topology on Y.

Clearly, the space Y is not Hausdorff. Thus, by (ii), there is a sequence in Y with at least two limit points. Such a sequence must converge to a and to b. A sequence converging, simultaneously, to these two points meets infinitely many of the X_n's.

This fact together with Proposition 1.4 concludes the proof. □

Theorem 4.2. *Equivalent are*:

(i) **CC**$(\mathbb{R})$;
(ii) *a second countable space is Hausdorff if and only if every sequence has at most one limit.*

Proof. (i) ⇒ (ii) That in a Hausdorff space every sequence (net) has at most one limit is a theorem of **ZF** (cf. [4, 1.6.7]).

If, in a topological space X, every sequence has at most one limit, then X is a T_1-space (see, e.g., [4, 1.6.16]). Lemma 1.5 implies that, if X is a T_1-space with a countable base, then $|X| \leqslant |\mathbb{R}|$. The usual proof (see [4, 1.6.17]) only uses a countable choice for subsets of X.

(ii) ⇒ (i) Let $(X_n)_n$ be a countable family of non-empty subsets of $\mathbb{R}$. We may consider each X_n as a subset of $(\frac{1}{n+1}, \frac{1}{n})$. Define the sets Y and $(Y_n)_n$ as in the proof of Theorem 4.1.

We define a topology in Y in which $Y \setminus \{a, b\}$ is open and has the topology of subspace of $\mathbb{R}$, and the points a and b have the same local bases as before. With this topology Y is a second countable non-Hausdorff space. From this point, the proof proceeds as the proof of Theorem 4.1. □

It is well known that the condition (ii) of Theorem 4.1 may be generalized to the class of topological spaces, replacing sequences by filters (or nets). This result is still valid in **ZF**.

Under the *Ultrafilter Theorem*, i.e., every filter over a set can be extended to an ultrafilter, the convergence of ultrafilters may also be used. We will see that we cannot avoid the Ultrafilter Theorem.

The Ultrafilter Theorem is equivalent to the Boolean Prime Ideal Theorem (see [15, p. 17]).

Theorem 4.3. *Equivalent are*:

(i) *Ultrafilter Theorem*;
(ii) *a topological space X is Hausdorff if and only if, in X, every ultrafilter has at most one limit.*

Proof. (i) $\Rightarrow$ (ii) In [4, 1.6.7], (ii) is proved for filters (nets). If (i) does hold, it is clear that the proof can be done with ultrafilters.

(ii) $\Rightarrow$ (i) Let $\mathcal{F}$ be a free filter over X, and a, b two distinct points of X. Once again, we define a local base system for a topology on X:

$$\mathcal{B}(x) := \begin{cases} \{\{x\}\} & \text{if } x \notin \{a, b\}, \\ \{F \cup \{x\}\colon\ F \in \mathcal{F}\} & \text{if } x \in \{a, b\}. \end{cases}$$

With this topology, X is not Hausdorff. So, by (ii) there is an ultrafilter converging for two different points in X. These two points can only be a and b, which means that such an ultrafilter must contain $\mathcal{F}$. □

5. Countable products

The last part of this paper is devoted to the study of the countable productivity of the class of second countable spaces. Such a property is provable in **ZFC**. The question was studied by Keremedis [16] in the absence of the axiom of choice. He arrived at some interesting results, although not definitive ones. Indeed, an equivalence to a set-theoretic statement is missing. In Theorems 5.1 and 5.2 below, we will narrow the gap between the (known) necessary and sufficient conditions to prove of the countable productivity of the class of second countable spaces. We prove this property, using a choice principle properly weaker than **CC**.

Theorem 5.1. *If countable products of second countable spaces are second countable, then the countable union theorem does hold.*

Proof. Without loss of generality, let $(X_n)_n$ be a family of countable disjoint sets and consider the discrete spaces $Y_n := X_n \cup \{n\}$.

Clearly every Y_n is second countable and then, by hypothesis, $Y := \prod_n Y_n$ is also second countable. Let $\mathcal{B} := \{B_k\colon\ k \in \mathbb{N}\}$ be a base for Y. For each n in $\mathbb{N}$, $\{p_n(B_k)\colon\ k \in \mathbb{N}\}$ is a base for Y_n, since the projections p_n are open surjections. This induces the injective function

$f_n : X_n \to \mathbb{N}$ defined by $f_n(x) := \min\{k \in \mathbb{N}\colon\ p_n(B_k) = \{x\}\}$. Now, it is easy to see that $f : \bigcup_n X_n \to \mathbb{N} \times \mathbb{N}$ with $f(x) := (n, k)$ if $x \in X_n$ and $f_n(x) = k$ is an injection, which concludes the proof. □

Theorem 5.2. *If the axiom of countable choice holds for families of sets with cardinality at most* $2^{\aleph_0}$ ($\mathbf{CC}(\leqslant 2^{\aleph_0})$), *then countable products of second countable spaces are second countable.*

Proof. Let $((X_n, \mathcal{T}_n))_n$ be a family of second countable spaces. We will prove that $\prod_n (X_n, \mathcal{T}_n)$ has a countable base.

By Lemma 1.5, we know that $|\mathcal{T}_n| \leqslant 2^{\aleph_0}$, for every $n \in \mathbb{N}$. Consider the sets $C_n := \{(f : \mathbb{N} \to \mathcal{T}_n)\colon\ f(\mathbb{N})$ is a base of $(X_n, \mathcal{T}_n)\}$. We have that, for all $n \in \mathbb{N}$, $|C_n| \leqslant |(\mathcal{T}_n)^{\mathbb{N}}| \leqslant (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. By $\mathbf{CC}(\leqslant 2^{\aleph_0})$, there is $(f_n)_n$ with each f_n an element of C_n.

The subbase $\mathcal{C} := \{p_n^{-1}(f_n(k))\colon\ n, k \in \mathbb{N}\}$ of $\prod_n X_n$ is countable, and then the base generated by $\mathcal{C}$ is also countable. □

In a analogous way to the proofs of Theorems 5.1 and 5.2, one can prove the following corollary.

Corollary 5.3. *Equivalent are*:

(i) *the axiom of countable choice holds for families of finite sets* ($\mathbf{CC}$(fin));
(ii) *countable products of spaces with finite topologies are second countable.*

We recall that the countable union theorem for finite sets—Form 10 A in [14]—is equivalent to $\mathbf{CC}$(fin)—Form 10 in [14].

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