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Quadratic Malcev superalgebras

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Abstract

A quadratic Malcev superalgebra is a Malcev superalgebra $M=M_{\bar{0}}\oplus M_{\bar{1}}$ with a non-degenerate supersymmetric even invariant bilinear form B; B is called an invariant scalar product on M. In this paper, we obtain the inductive classifications of quadratic Malcev algebras and of Malcev superalgebras $M=M_{\bar{0}}\oplus M_{\bar{1}}$ such that $M_{\bar{0}}$ is a reductive Malcev algebra and the action of the $M_{\bar{0}}$ on $M_{\bar{1}}$ is completely reducible.

MSC: 15A63; 17A70

1. Introduction

In this work, we consider finite dimensional Malcev superalgebras over an algebraically closed commutative field K of characteristic 0. Let $M=M_{\bar{0}}\oplus M_{\bar{1}}$ be a Malcev superalgebra over K. We denote by $\mathrm{Der}(M)$ the Lie superalgebra of superderivations of M and by $Z(M)$ the center of M. The Malcev superalgebra M is called quadratic if there exists B, an even bilinear form on M, such that B is supersymmetric, non-degenerate and invariant. In this case B is called an invariant scalar product on M. Recall that abelian Malcev superalgebras, simple Malcev algebras and basic classical Lie superalgebras are quadratic [15,23]. In this paper, we give some

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examples of non-abelian and non-simple quadratic Malcev superalgebras and we study their supernucleus and superjacobian. An invariant scalar product on Lie superalgebras has been an important tool for developing the structure of quadratic Lie superalgebras [5–8,11–16,18,20,23]. The main results of this paper are the inductive classifications of quadratic Malcev algebras and of Malcev superalgebras $M = M_{\bar{0}} \oplus M_{\bar{1}}$ such that $M_{\bar{0}}$ is a reductive Malcev algebra and the action of $M_{\bar{0}}$ on $M_{\bar{1}}$ is completely reducible. These classifications are obtained in the last two sections of this work. In order to obtain these results we develop in the second section the notion of semi-direct product of Malcev superalgebras, we introduce the notion of central extension of Malcev superalgebras in the third section and in addition we generalize the notion of double extension to quadratic Malcev superalgebra. This notion was introduced by Medina and Revoy [18] to study quadratic Lie algebras and transferred to quadratic Lie superalgebras by Benamor and Benayadi [5].

Lie superalgebras with semisimple even part were studied by A. Elduque in [10] and with even part reductive such that the action of the even part on the odd part is completely reducible were studied by S. Benayadi in [7]. For the definitions and basic facts of the Theory of Malcev algebras and superalgebras we refer the reader to [1,2,9,20,21,24,25].

2. Some basic properties and examples

Definition. A superalgebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is called a Malcev superalgebra if

(1) $XY = -(-1)^{xy} YX, \quad \forall_{(X,Y)\in M_x \times M_y}$,

(2) $$(-1)^{yz}(XZ)(YT) = ((XY)Z)T + (-1)^{x(y+z+t)}((YZ)T)X + (-1)^{(x+y)(z+t)}((ZT)X)Y + (-1)^{t(x+y+z)}((TX)Y)Z, \quad \forall_{(X,Y,Z,T)\in M_x\times M_y\times M_z\times M_t}.$$

Definition. Let M be a Malcev superalgebra and B be a bilinear form on M. B is called supersymmetric if $B(X,Y) = (-1)^{xy}B(Y,X)$, $\forall_{(X,Y)\in M_x\times M_y}$; B is called invariant if $B(XY,Z) = B(X,YZ)$, $\forall_{X,Y,Z\in M}$; B is called even if $B(X,Y) = 0$, $\forall_{X\in M_{\bar{0}},Y\in M_{\bar{1}}}$.

Definition. A Malcev superalgebra M is called quadratic if there exists a bilinear form B on M such that B is supersymmetric, non-degenerate, even and invariant. In this case, B is called an invariant scalar product on M.

Remark. A quadratic Malcev algebra has been called a quasi-classical Malcev algebra [20].

Definition. Let M be a quadratic Malcev superalgebra and B be an invariant scalar product on M. A graded ideal J of M is called non-degenerate if the restriction of B to $J \times J$ is a non-degenerate bilinear form. M is called irreducible if M contains no non-trivial non-degenerate graded ideal.

Definition. Let M be a Malcev superalgebra.

(i) If $V = V_{\bar{0}} \oplus V_{\bar{1}}$ a Z_2-graded vector space and $\phi : M \to \mathrm{End}(V)$ an even linear map. ϕ is called a Malcev representation of M in V if

$$\begin{aligned}\phi((XY)Z) = {} & \phi(X)\phi(Y)\phi(Z) - (-1)^{x(y+z)}\phi(YZ)\phi(X) \\ & - (-1)^{z(x+y)}\phi(Z)\phi(X)\phi(Y) \\ & + (-1)^{x(y+z)}\phi(Y)\phi(ZX), \quad \forall_{(X,Y,Z)\in M_x\times M_y\times M_z}.\end{aligned}$$

(ii) If we have two Malcev representations ϕ and ψ of the Malcev superalgebra M, $\phi : M \to \mathrm{End}(V)$ and $\psi : M \to \mathrm{End}(W)$, where V and W are Z_2-graded vector spaces, we say that ϕ and ψ are equivalent if there is a bijective even linear map $\delta : V \to W$ such that $\delta \circ \phi(X) = \psi(X) \circ \delta$, $\forall_{X\in M}$.

Proposition 2.1. *Let M be a Malcev superalgebra and M^* be the dual of the vector space M. Let $\pi : M \to \mathrm{End}(M)$ and $\pi^* : M \to \mathrm{End}(M^*)$ be the linear maps defined by $\pi(X)(Y) = XY$, $\forall_{X,Y\in M}$ and $\pi^*(X)(F)(Y) = -(-1)^{xf}F(XY)$, $\forall_{(X,Y,F)\in M_x\times M_y\times M_f^*}$. Then π and π^* are Malcev representations of M. π (resp. π^*) is called the adjoint (resp. coadjoint) representation of M.*

Proof. It is known that π is a Malcev representation [1]. Now let be $X \in M_x$, $Y \in M_y$, $Z \in M_z$, $U \in M_u$, $F \in M_f^*$. If we recall the definition of a Malcev superalgebra we can write

$$\begin{aligned}F(((XY)Z)U) = {} & -(-1)^{x(y+z+u)}F(((YZ)U)X) - (-1)^{(x+y)(u+z)}F(((ZU)X)Y) \\ & - (-1)^{u(x+y+z)}F(((UX)Y)Z) + (-1)^{yz}F((XZ)(YU)).\end{aligned}$$

But,

$$F(((XY)Z)U) = -(-1)^{(x+y+z)f}\pi^*((XY)Z)(F)(U),$$

$$F(((YZ)U)X) = -(-1)^{x(u+f)+f(y+z)}\pi^*(YZ)\pi^*(X)(F)(U),$$

$$F((XZ)(YU)) = (-1)^{(x+z)f+(x+z+f)y}(\pi^*(Y)\pi^*(XZ))(F)(U),$$

$$F(((ZU)X)Y) = -(-1)^{(x+y+z)f+u(x+y)}\pi^*(Z)\pi^*(X)\pi^*(Y)(F)(U),$$

$$F(((UX)Y)Z) = (-1)^{(u+f)(x+y+z)}\pi^*(X)\pi^*(Y)\pi^*(Z)(F)(U).$$

Then we have for the map π^*,

$$\begin{aligned}\pi^*((XY)Z = {} & \pi^*(X)\pi^*(Y)\pi^*(Z) - (-1)^{x(y+z)}\pi^*(YZ)\pi^*(X) \\ & - (-1)^{z(x+y)}\pi^*(Z)\pi^*(X)\pi^*(Y) - (-1)^{xy}\pi^*(Y)\pi^*(XZ),\end{aligned}$$

what is equivalent to say that π^* is a Malcev representation of M. $\square$

In the next theorem we are going to obtain a characterization of quadratic Malcev superalgebras. In the case of Lie algebras, this characterization is obtained in [8,19].

Theorem 2.1. *Let M be a Malcev superalgebra. Then M is quadratic if and only if the adjoint and coadjoint representations of M are equivalent and* $\dim M_{\bar{1}}$ *is even.*

Proof. Let us start supposing that M is quadratic with an invariant scalar product B. Then $(M_{\bar{1}}, B/_{(M_{\bar{1}}\times M_{\bar{1}})})$ is a symplectic vector space and consequently $\dim M_{\bar{1}}$ is even. Consider $\phi: M \to M^*$ defined by $\phi(X)(Y) = B(X,Y)$, $\forall_{X,Y\in M}$. It is clear that ϕ is an even isomorphism of super-vector spaces. Let π be the adjoint representation of M and π^* the coadjoint representation of M. Let $X\in M_x$, $Y\in M_y$, $Z\in M_z$.

$$\begin{aligned}\phi(\pi(Y)X))(Z) &= B(\pi(Y)X,Z) = B(YX,Z)\\ &= -(-1)^{xy}B(X,YZ) = -(-1)^{xy}\phi(X)(YZ) = \pi^*(Y)(\phi(X))(Z).\end{aligned}$$

Then,

$$\phi(\pi(Y)(X)) = \pi^*(Y)(\phi(X)), \quad \forall_{X,Y\in M}$$

and $\phi\circ\pi(Y) = \pi^*(Y)\circ\phi$, $\forall_{Y\in M}$. So the representations π and π^* are equivalent. Now let us suppose that the representations π and π^* are equivalent. Then there is an isomorphism $\phi: M\to M^*$ of super-vector spaces of degree 0 such that $\phi\circ\pi(X) = \pi^*(X)\circ\phi$, $\forall_{X\in M}$. We may consider the bilinear form defined by

$$B: M\times M \to K,$$

$$(X,Y)\to B(X,Y) = \phi(X)(Y).$$

Because ϕ is an isomorphism of degree 0, we can conclude that B is even and non-degenerate. On the other hand,

$$\begin{aligned}B(XY,Z) &= \phi(XY)(Z) = \phi(\pi(X)(Y))(Z) = (\pi^*(X)\phi(Y))(Z)\\ &= -(-1)^{xy}\phi(Y)(\pi(X)(Z)) = -(-1)^{xy}B(Y,XZ), \forall_{X\in M_x,Y\in M_y,Z\in M}.\end{aligned}$$

Then $B(YX,Z)=B(Y,XZ), \forall_{X\in M_x,Y\in M_y,Z\in M}$ and B is invariant. It is clear that $B=B_a+B_s$ where B_s is the supersymmetric part of B and B_a is the superantisymmetric part of B, that is, $B_a(X,Y) = \frac{1}{2}(B(X,Y) - (-1)^{xy}B(Y,X))$ and $B_s(X,Y) = \frac{1}{2}(B(X,Y) + (-1)^{xy}B(Y,X))$, for all $X\in M_x, Y\in M_y$. As B is invariant, B_a and B_s are invariants and let us prove that $M^2 \subseteq \mathrm{Rad}(B_a) = \{X\in M\colon\ B_a(X,M) = \{0\}\}$. Consider $X\in M_x$, $Y\in M_y$, $Z\in M_z$. Then,

$$B_a(XY,Z) = -(-1)^{xy}B_a(Y,XZ) = (-1)^{(x+z)y+xy}B_a(XZ,Y) = (-1)^{zy}B_a(XZ,Y).$$

On the other hand,

$$B_a(XY,Z) = B_a(X,YZ) = -(-1)^{zy}B_a(X,ZY) = -(-1)^{zy}B_a(XZ,Y).$$

It follows that $B_a(XY,Z)=0$ so, $M^2\subseteq \mathrm{Rad}(B_a)$. As a consequence we have that $(M^2\cap \mathrm{Rad}(B_s))\subseteq \mathrm{Rad}(B)$. Then $(M^2\cap\mathrm{Rad}(B_s))=\{0\}$ because B is non-degenerate. Now, let V be a graded sub-vector space of M such that $M = V\oplus \mathrm{Rad}(B_s)$ and $M^2\subseteq V$. Then $\mathrm{Rad}\, B_s$ is a graded abelian ideal of M and V is a graded ideal of M such that $B_s/_{(V\times V)}$ is invariant and supersymmetric. Let $X\in V$ such that $B_s(X,V)=\{0\}$. Then $B_s(X,M)=\{0\}$ which proves that $X\in\mathrm{Rad}(B_s)$ and $X=0$. We have proved that $B_s/_{(V\times V)}$ is a scalar product invariant over V and $\dim V_{\bar{1}}$ is even. Consequently, $\dim(\mathrm{Rad}(B_s))_{\bar{1}}$ is even. As $\mathrm{Rad}(B_s)$ is abelian and $\dim(\mathrm{Rad}(B_s))_{\bar{1}}$ is even, then there is a bilinear form ϕ on $\mathrm{Rad}(B_s)$ that is even, supersymmetric, invariant and non-degenerate. Consider now the even bilinear form $T : M\times M \to K$, defined by $T/_{(V\times V)} = B_s/_{(V\times V)}$, $T/_{(\mathrm{Rad}(B_s)\times \mathrm{Rad}(B_s))} = \phi$, $T/_{(V\times \mathrm{Rad}(B_s))} = T/_{(\mathrm{Rad}(B_s)\times V)} = 0$. It is clear that T is an invariant scalar product on M. $\square$

Definition. Let M be a Malcev superalgebra. Let $\mathrm{SJ} : M\times M\times M \to M$ be the trilinear map defined by

$$\mathrm{SJ}(X,Y,Z) = (XY)Z + (-1)^{x(y+z)}(YZ)X + (-1)^{z(x+y)}(ZX)Y,$$

$$\forall_{(X,Y,Z)\in M_x\times M_y\times M_z}.$$

$\mathrm{SJ}(X,Y,Z)$ is called the superjacobian of X,Y,Z. The Superjacobian is the graded vector subspace of M spanned by $\{\mathrm{SJ}(X,Y,Z)/X,Y,Z\in M\}$. It is denoted by $\mathrm{SJ}(M,M,M)$. The supernucleus of M is the graded vector subspace of M: $SN(M) = \{X\in M/\mathrm{SJ}(X,M,M) = \{0\}\}$. Recall that $\mathrm{SJ}(M,M,M)$ and $\mathrm{SN}(M)$ are graded ideals of M [1].

Remark. The superjacobian and the supernucleus of a Malcev superalgebra are of great importance in the study of the theory of Malcev superalgebras (see for example [3,4]). In some sense we can say that these ideals measure how far away from a Lie superalgebra a Malcev superalgebra is situated. Recall that in a Lie superalgebra we have $\mathrm{SJ}(M,M,M) = \{0\}$ and $\mathrm{SN}(M) = M$.

Lemma 2.1. *If M is a Malcev superalgebra then for all $X\in M_x$, $Y\in M_y$, $Z\in M_z$,*

(1) $\mathrm{SJ}(X,Y,Z) = -(-1)^{xy}\mathrm{SJ}(Y,X,Z)$,
(2) $\mathrm{SJ}(X,Y,Z) = -(-1)^{yz}\mathrm{SJ}(X,Z,Y)$,
(3) $\mathrm{SJ}(Z,Y,X) = -(-1)^{z(x+y)+xy}\mathrm{SJ}(X,Y,Z)$.

Besides, if M admits an invariant bilinear form B, then $\forall_{X,Y,Z,T\in M}$,

$$B(\mathrm{SJ}(X,Y,Z),T) = -B(X,\mathrm{SJ}(Y,Z,T)).$$

Proof. Only calculations. $\square$

Proposition 2.2. *Let M be a quadratic Malcev superalgebra and B an invariant scalar product on M.*

(1) $\mathrm{SN}(M) = (\mathrm{SJ}(M,M,M))^{\perp}$,
(2) $\mathrm{SN}(M) = \{0\}$ *if and only if* $\mathrm{SJ}(M,M,M) = M$,
(3) *If* $M(\mathrm{SJ}(M,M,M)) = M$ *then* $\mathrm{SN}(M) = \{0\}$. *If* $M(\mathrm{SN}(M)) = M$ *then M is a Lie superalgebra*,
(4) $Z(M) = (M^2)^{\perp}$,
(5) *If M is a non-zero solvable superalgebra, then* $\mathrm{SN}(M) \neq \{0\}$.

Proof. (1) Let $X \in \mathrm{SN}(M)_x, (Y,Z,T) \in M_y \times M_z \times M_t$. By the preceding lemma $B(\mathrm{SJ}(Y,Z,T),X) = -B(Y,\mathrm{SJ}(Z,T,X)) = (-1)^{z(x+t)+xt}B(Y,\mathrm{SJ}(X,T,Z)) = 0$. Then $\mathrm{SN}(M) \subseteq (\mathrm{SJ}(M,M,M))^{\perp}$. Let $X \in (\mathrm{SJ}(M,M,M))^{\perp} \cap M_x, (Y,Z,T) \in M_y \times M_z \times M_t$, $B(\mathrm{SJ}(X,Y,Z),T) = -B(X,\mathrm{SJ}(Y,Z,T)) = 0$. Consequently $\mathrm{SJ}(X,Y,Z) = 0$. Then $X \in \mathrm{SN}(M)$. We conclude that $\mathrm{SN}(M) = (\mathrm{SJ}(M,M,M))^{\perp}$.

(2) is a direct consequence of 1).

(3) As B is invariant, $B(M(\mathrm{SN}(M)),\mathrm{SJ}(M,M,M)) = B(\mathrm{SN}(M),M(\mathrm{SJ}(M,M,M))) = B(M,(\mathrm{SN}(M))(\mathrm{SJ}(M,M,M))) = \{0\}$ because we know that $(\mathrm{SN}(M))(\mathrm{SJ}(M,M,M)) = \{0\}$ [1]. As B is non-degenerate we have the result.

(4) Let $X,Y \in M$ and $Z \in Z(M)$. Then, $B(XY,Z) = B(X,YZ) = 0$ and $XY \in Z(M)^{\perp}$ so $M^2 \subseteq Z(M)^{\perp}$. Now let $X \in (M^2)^{\perp}$ and let $Y,Z \in M$, $B(XY,Z) = B(X,YZ) = 0$. As B is non-degenerate, $XY = 0$. Then $(M^2)^{\perp} \subseteq Z(M)$ and so, $Z(M)^{\perp} \subseteq M^2$. So we can conclude that, $M^2 = Z(M)^{\perp}$.

(5) If M is solvable, then $M \neq M^2$. Consequently $(M^2)^{\perp} \neq \{0\}$ that is $Z(M) \neq \{0\}$. As $Z(M) \subseteq \mathrm{SN}(M)$ then $\mathrm{SN}(M) \neq \{0\}$. □

Remark. The last assertion proves that there are no non-zero solvable quadratic Malcev superalgebras with supernucleus zero. Recall that the same assertion is not true if we consider Malcev superalgebras that are not quadratic. Think in $M(1,2,v)$ that is an example of a solvable Malcev superalgebra with supernucleus zero [3].

In order to have some examples of quadratic Malcev non-Lie superalgebras of low dimension, we are going to determine the quadratic Malcev superalgebras in the list of Malcev non-Lie superalgebras of dim $\leqslant 4$ obtained in [3,17].

Example. Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Z_2-graded vector space such that dim $M_{\bar{0}} = 1$ and the dimension of $M_{\bar{1}}$ is even. Let $\{a\}$ be a basis of $M_{\bar{0}}$ and $\{v_1, v_2, \ldots, v_n, y_1, y_2, \ldots, y_n\}$ be a basis of $M_{\bar{1}}$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq K \setminus \{0\}$. If $i,j \in \{1,\ldots,n\}$, δ_{ij} denotes the Kronecker symbol, that is $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. We consider the following supersymmetric bilinear map. $: M \times M \to M$ defined by

$$a.v_i = y_i = -v_i.a; \quad i = 1,\ldots,n,$$

$$v_i.v_j = \delta_{ij}\alpha_i a; \quad i,j = 1,\ldots,n,$$

$$M.y_i = \{0\}; \quad i = 1,\ldots,n.$$

It is clear that $\forall_{X,Y,Z,T} \in M$, we have $(XZ)(YT) = ((XY)Z)T = 0$ and M with this multiplication is a Malcev superalgebra. Moreover $\forall_{i \in \{1,\ldots,n\}}$ we have $\mathrm{SJ}(v_i,v_i,v_i) = 3(v_i v_i)v_i = 3\alpha_i y_i \neq 0$ so M is not a Lie superalgebra. Now consider $B : M \times M \to K$ the bilinear symmetric bilinear form defined by $B(a,a) = 1$; $B(y_i,v_j) = \delta_{ij}\alpha_i = -B(v_j,y_i)$; $B(a,M_{\bar{1}}) = \{0\}$; $B(y_i,y_j) = B(v_i,v_j) = 0$, $\forall_{i,j \in \{1,\ldots,n\}}$. With some calculations we can see easily that B is a scalar product defined on M and consequently M is an example of a quadratic Malcev superalgebra. Note that if $n = 1$, then M is the Malcev superalgebra $M(1,2)$ of the list in [3].

Lemma 2.2. *Let M be a Malcev superalgebra such that* $\dim M_{\bar{0}} = 2$. *If there exists a basis $\{a, b\}$ of $M_{\bar{0}}$ such that $ab = b$, then M is not quadratic.*

Proof. Let B be an invariant, even bilinear form on M. Then

$$B(a,b) = B(a,ab) = B(a^2,b) = 0$$

and

$$B(b,b) = B(b,ab) = -B(b^2,a) = 0.$$

Consequently $(b, M) = \{0\}$. We conclude that B is degenerate. □

Corollary. *The Malcev superalgebras $M^2(2,2), M^4(2,2), M^0(2,2), M^3(2,2), M(2,2,v)$, $M^1(2,2,\gamma), M(2,2,2,1), M(2,2,2,0), M(2,2,1/2,\delta), M(2,2,1/2,0)$* [3] *are not quadratic.*

Lemma 2.3. *Let M be a Malcev superalgebra such that $M_{\bar{1}}^2 = M_{\bar{0}}$ and for some $a \in M_{\bar{0}}$, $aM_{\bar{1}} = \{0\}$. Then M is not quadratic.*

Proof. Let B be an even invariant bilinear form on M. In this case $B(a, M_{\bar{0}}) = B(a, M_{\bar{1}}^2) = B(aM_{\bar{1}}, M_{\bar{1}}) = \{0\}$ and $B(a, M) = \{0\}$. So B is degenerate. □

Corollary. *The Malcev superalgebras $M(v,2,2)$ and $M(2,2,1)$* [3] *are not quadratic.*

Theorem 2.2. *Let M be a non-Lie Malcev superalgebra such that $\dim M \leqslant 4$. Then M is quadratic if and only if M is isomorphic to $M(1,2)$ that is, $M_{\bar{0}} = \langle a \rangle$, $M_{\bar{1}} = \langle u, v \rangle$, such that $au = v, u^2 = a$.*

Proof. If M is a Malcev algebra of dim $\leqslant 4$ then M is a Lie algebra or M has a basis $\{e_1, e_2, e_3, e_4\}$ such that $e_1e_2 = e_3$; $e_1e_4 = e_1$; $e_2e_4 = e_2$; $e_3e_4 = -e_4$ [17]. With some calculations we see that if B is an even invariant bilinear form then $B(e_3, M) = \{0\}$ and B is degenerate. If M is a quadratic Malcev superalgebra with $M_{\bar{1}} \neq 0$, then by the above corollaries, M is isomorphic to $M(1,2)$ or $M^1(2,2)$ or $M(1,2,v)$. $M^1(2,2)$ is quadratic as we have seen in the example. Let us prove that $M^1(2,2)$ and $M(1,2,v)$ are not quadratic. Recall that the even (resp. odd) part of $M(1,2,v)$ has a basis $\{a\}$ (resp. $\{u, v\}$) such that $au = v$, $uv = va$ (with $v \neq 0$). Let B be an invariant even form of $M(1,2,v)$. $B(a,a) = v^{-1}B(a, va) = v^{-1}B(a, uv) = v^{-1}B(av, u) = 0$. Then $B(a, M(1,2,v)) = 0$ and B is degenerate. We conclude that $M(1,2,v)$ is not a quadratic Malcev superalgebra. Now, the even (resp. odd) part of $M^1(2,2)$ has a basis $\{a, b\}$(resp. $\{u, v\}$) such that $bv = u$ and $uv = a$. Let B be an invariant even bilinear form of $M^1(2,2)$. $B(a,a) = B(uv, a) = B(u, va) = B(u, 0) = 0$ and $B(a,b) = B(uv, b) = -B(v, ub) = B(v, 0) = 0$. Then $B(a, M^1(2,2)) = 0$ and so B is degenerate. We conclude that $M^1(2,2)$ is not a quadratic Malcev superalgebra. □

3. Semi-direct product of Malcev superalgebras

Definition. Let M be a Malcev superalgebra and let $\phi : M \to M$ be an endomorphism of M. We say that ϕ is a Malcev operator of M if

$$\phi((VW)S) = (\phi(V)W)S - (-1)^{vw}\phi(W)(VS) - (-1)^{s(v+w)}(\phi(S)V)W - (-1)^{v(w+s)}\phi(WS)V, \quad \forall_{V\in M_v, W\in M_w, S\in M_s}.$$

Proposition 3.1. *Let M be a Malcev superalgebra and let D be an even derivation of M. Then, D is a Malcev operator if and only if $D(\mathrm{SJ}(M,M,M)) = 0$.*

Proof. If D is a Malcev operator then,

$$D((XY)Z) = (D(X)Y)Z + XD(YZ) - (-1)^{xy}D(Y)(XZ) + (-1)^{zy}(XD(Z))Y,$$

$$D((YZ)X) = (D(Y)Z)X + YD(ZX) - (-1)^{yz}D(Z)(YX) + (-1)^{xz}(YD(X))Z,$$

$$D((ZX)Y) = (D(Z)X)Y + ZD(XY) - (-1)^{zx}D(X)(ZY) + (-1)^{xy}(ZD(Y))X,$$

$$\forall_{X\in M_x, Y\in M_y, Z\in M_z}.$$

Consequently,

$$D(\mathrm{SJ}(X,Y,Z)) = -\mathrm{SJ}(D(X),Y,Z) - \mathrm{SJ}(X,D(Y),Z) - \mathrm{SJ}(X,Y,D(Z)).$$

As D is an even derivation we have

$$D(\mathrm{SJ}(X,Y,Z)) = \mathrm{SJ}(D(X),Y,Z) + \mathrm{SJ}(X,D(Y),Z) + \mathrm{SJ}(X,Y,D(Z))$$

and $D(\mathrm{SJ}(X,Y,Z)) = 0$, $\forall_{X\in M_x, Y\in M_y, Z\in M_z}$.

Reciprocally, we know that

$$D((XY)Z) = D(\mathrm{SJ}(X,Y,Z)) - (-1)^{x(y+z)}D((YZ)X) - (-1)^{z(x+y)}D((ZX)Y)$$

and if

$$D(\mathrm{SJ}(X,Y,Z)) = 0$$

we have,

$$D((XY)Z) = -(-1)^{x(y+z)}D((YZ)X) - (-1)^{z(x+y)}D((ZX)Y).$$

But $\mathrm{SJ}(D(X),Y,Z) = 0$ and doing some calculations, we conclude that D is a Malcev operator. $\square$

Definition. Let M and V be two Malcev superalgebras and let π be a Malcev representation of M in V. We say that π is a Malcev admissible representation if $\bar{M} = M \oplus V$ with the product

$$(X+T)(Y+W) = XY + \pi(X)(W) - (-1)^{xy}\pi(Y)T + TW, \quad \forall_{X+T\in \bar{M}_x, Y+W\in \bar{M}_y}$$

is a Malcev superalgebra.

Theorem 3.1. *Let M and V be Malcev superalgebras and let π be a Malcev representation of M in V. π is Malcev admissible if and only if the following conditions are satisfied*:

(1) $\forall_{X\in M_x, Y\in M_y, T\in V_t, W\in V_w}$,

$$[\pi(XY)(T)]W = \pi(X)[\pi(Y)(T)W] + (-1)^{yt}[\pi(X)(T)][\pi(Y)(W)] - (-1)^{xy}\pi(Y)\pi(X)(TW) - (-1)^{xy+tw}[\pi(Y)\pi(X)(W)]T.$$

(2) $\forall_{X\in M_x, Z\in M_z, Y\in V_y, T\in V_t}$,

$$-(\pi(Z)(\pi(X)(Y)))T + (-1)^{x(y+z+t)}\pi(X)((\pi(Z)(Y))T) - (-1)^{(x+y)(z+t)}(\pi(X)(\pi(Z)(T))Y + (-1)^{t(x+y+z)}\pi(Z)(\pi(X)(T)Y) = (-1)^{yz}\pi(XZ)(YT).$$

(3) $\forall_{X\in M_x}$, *$\pi(X)$ is a Malcev operator of V.*

Proof. If these conditions are satisfied let us prove that $\bar{M} = M \oplus V$ with the product

$$(X + T)(Y + W) = XY + \pi(X)(W) - (-1)^{xy}\pi(Y)T + TW, \quad \forall_{X+T\in\bar{M}_x, Y+W\in\bar{M}_y},$$

is a Malcev superalgebra. It suffices to prove the axiom of definition of a Malcev superalgebra with $((XY)Z)T$ in the following cases: If all these elements are in M or in V there is no problem because M and V are Malcev superalgebras. The same for the case where one of these elements is in V and the other three in M because we know that π is a Malcev representation of M in V. For the case where $X\in M_x, Y\in M_y, Z\in V_z, T\in V_t$ we obtain the first condition that is verified. The second condition implies that the axiom is true in the case $X\in M_x,\ Y\in V_y,\ Z\in V_z,\ T\in V_t$, and the third condition implies the equality in the case $X\in M_x,\ Y\in V_y,\ Z\in M_z,\ T\in V_t$. Conversely, if $\bar{M}$ is a Malcev superalgebra these conditions are satisfied. $\square$

Definition. Let M and V be Malcev superalgebras and let π be a Malcev admissible representation of M in V. The Malcev superalgebra $\bar{M}=M\oplus V$ defined before is called the semi-direct product of V by M by means of π.

Example. If M and V are two Malcev superalgebras such that V is abelian then all Malcev representations of M in V are Malcev admissible.

We are going next to study two interesting examples of semi-direct products of Malcev superalgebras.

Proposition 3.2. *Let V be a Malcev superalgebra and let ϕ be an even endomorphism of V*: (1) *The map $\pi: Ke \to \mathrm{End}(V)$ defined by*: $\pi(\alpha e)(v) = \alpha\phi(v)$, $\forall_{\alpha\in K}$ (*where Ke is the one-dimensional Lie algebra*) *is a Lie* (*hence Malcev*) *representation.*

(2) *The map π is a Malcev admissible representation if and only if ϕ is a Malcev operator of V*, (a) $\phi^2(XY)=\phi(X)\phi(Y)+\phi(\phi(X)Y)+X\phi^2(Y)$, $\forall_{X,Y\in V}$ *and* (b) $\phi^2(X)Y-\phi(\phi(X)Y) = -(-1)^{xy}(\phi^2(Y)X - \phi(\phi(Y)X))$, $\forall_{X\in V_x, Y\in V_y}$.

Proof. If ϕ is an even endomorphism the map $\pi : Ke \to \mathrm{End}(V)$ defined by $\pi(\alpha e)(v) = \alpha\phi(v)$ is a Lie (then Malcev) representation. Then $Ke \oplus V$ is a Malcev superalgebra if and only if $\pi(e)(=\phi)$ is a Malcev perator and the conditions (1) and (2) of Theorem 3.1 for π and $M = Ke$ are satisfied that is,

$$\phi((XY)Z) = (\phi(X)Y)Z + X\phi(YZ) - (-1)^{xy}\phi(Y)(XZ) + (-1)^{xz}(X\phi(Z))Y,$$

$$\forall_{X \in V_x, Y \in V_y, Z \in V_z},$$

$$\phi^2(XY) = \phi(X)\phi(Y) + \phi(\phi(X)Y) + X\phi^2(Y), \quad \forall_{X,Y \in V},$$

$$\phi^2(X)Y - \phi(\phi(X)Y) = -(-1)^{xy}(\phi^2(Y)X - \phi(\phi(Y)X), \quad \forall_{X \in V_x, Y \in V_y}. \qquad \square$$

Definition. Let V be a Malcev superalgebra. An even endomorphism ϕ of V is called Malcev admissible operator if it is a Malcev operator and satisfies the conditions (a) and (b) of the last theorem.

Corollary. *Let V be a Malcev superalgebra and let ϕ be a Malcev admissible operator of V. Then the Z_2-graded vector space N defined by $N_0 = Ke \oplus V_{\bar{0}}$ and $N_1 = V_{\bar{1}}$ with the product*: $(\alpha e + t)(\beta e + w) = \alpha\phi(w) - \beta\phi(t) + tw$, $\forall_{\alpha,\beta \in K}, \forall_{w,t \in V}$, *is a Malcev superalgebra. This Malcev superalgebra is the semi-direct product of V by the one-dimensional Lie algebra by means of ϕ.*

Proposition 3.3. *Let V be a Malcev superalgebra and let D be an even derivation of V. D is a Malcev admissible operator of V if and only if D is a Malcev operator.*

Proof. It suffices to note that if D is an even derivation of V, then

$$D^2(XY) = D(X)D(Y) + D(D(X)Y) + XD^2(Y), \quad \forall_{X,Y \in V},$$

$$D^2(X)Y - D(D(X)Y) = -(-1)^{xy}D^2(Y)X - D(D(Y)X), \quad \forall_{X \in V_x, Y \in V_y}. \qquad \square$$

Next we are going to obtain a way to construct out of any Malcev superalgebra a quadratic Malcev superalgebra whose dimension is twice of the dimension of M. It will be another interesting example of a semi-direct product and very useful to obtain examples of quadratic Malcev superalgebras. In the ungraded case it has been called inessential T^* extension in [8].

Theorem 3.2. *Let M be a Malcev superalgebra and let M^* be the dual of M as a vector space considered as an abelian superalgebra. Denote by $\pi^* : M \to \mathrm{End}(M^*)$ the coadjoint representation of M. Then the semi-direct product of M^* by M of means by π^* with the bilinear form B defined by*

$$B(X + f, Y + g) = f(Y) + (-1)^{xy}g(X), \quad \forall_{X+f \in \bar{M}_x, Y+g \in \bar{M}_y},$$

is a quadratic Malcev superalgebra.

Proof. As we already said as M^* is an abelian superalgebra, it is enough to note that the map π^* is a Malcev representation of M in M^*. So $\bar{M} = M \oplus M^*$ with the multiplication defined by

$$(X+f)(Y+g) = XY + \pi^*(X)(g) - (-1)^{xy}\pi^*(Y)(f), \quad \forall_{X+f\in\bar{M}_x, Y+g\in\bar{M}_y},$$

is a Malcev superalgebra. On the other hand, the bilinear form B on $\bar{M}$ defined by

$$B(X+f, Y+g) = f(Y) + (-1)^{xy}g(X), \quad \forall_{X+f\in\bar{M}_x, Y+g\in\bar{M}_y}$$

is supersymmetric. Let us see that B is non-degenerate. Consider $X+f$ such that $B(X+f, Y+g) = 0$, $\forall_{Y+g\in\bar{M}_y}$. Then $B(X+f, Y) = f(Y) = 0$, $\forall_{Y\in M}$ and $f = 0$. On the other hand $B(X,g) = g(X)$, $\forall_{g\in\bar{M}_y}$ implies that $g(X) = 0, \forall_{g\in\bar{M}_y}$ and $X = 0$. Let $X+f\in\bar{M}_{\bar{0}}$ and $Y+g\in\bar{M}_{\bar{1}}$, $B(X+f, Y+g) = f(Y) + g(X)$ and if $(X+f)\in\bar{M}_{\bar{0}}$ and $Y+g\in\bar{M}_{\bar{1}}$ we have that $f(Y)\in K_{\bar{1}} = 0$. If $g\in M^*_{\bar{1}}$ and $X\in M_{\bar{0}}$ we have that $g(X)\in K_{\bar{1}} = 0$. Then B is even.

Now, let $X+f\in\bar{M}_x$, $Y+g\in\bar{M}_y$, $Z+h\in\bar{M}_z$,

$$\begin{aligned}
B((X+f)(Y+g), Z+h) &= B((XY + \pi^*(X)(g) - (-1)^{xy}\pi^*(Y)(f), Z+h)\\
&= \pi^*(X)(g)(Z) - (-1)^{xy}\pi^*(Y)(f)(Z)\\
&\quad + (-1)^{z(x+y)}h(XY)\\
&= -(-1)^{xy}g(XZ) + f(YZ) + (-1)^{z(x+y)}h(XY)\\
&= -(-1)^{x(y+z)+zy}\pi^*(Z)g(X) + f(YZ)\\
&\quad + (-1)^{z(x+y)+xy+yz}\pi^*(Y)(h)(X)\\
&= f(YZ) + (-1)^{x(y+z)}(\pi^*(Y)(h) - (-1)^{zy}\pi^*(Z)g)(X)\\
&= B(X+f, YZ + \pi^*(Y)(h) - (-1)^{zy}\pi^*(Z)(g))\\
&= B(X+f, (Y+g)(Z+h))
\end{aligned}$$

and we have concluded that B is invariant. □

Proposition 3.4. *Let M be a Malcev superalgebra and $\bar{M} = M \oplus M^*$ the semi-direct product of M^* by M by means of the coadjoint representation. Then*

(1) *If M is solvable then $\bar{M}$ is solvable.*
(2) $\mathrm{SN}(\bar{M}) = \mathrm{SN}(M) \oplus \{f\in M^*:\ f(\mathrm{SJ}(M,M,M)) = \{0\}\}$.

Proof. (1) It is a consequence of the following general result: For a non-associative algebra A, if I is a solvable ideal of A and A/I is solvable then A is solvable. (2) It is easy to see that $\mathrm{SJ}(\bar{M},\bar{M},\bar{M})$ is equal to $\mathrm{SJ}(M,M,M)\oplus K$, where K is the span of all elements of the form

$$\mathrm{SJ}(f,Y,Z) = (fY)Z + (-1)^{x(z+y)}(YZ)f + (-1)^{(x+y)z}(Zf)Y.$$

But by Proposition 2.2 (1) the orthogonal space of $\mathrm{SJ}(\bar{M},\bar{M},\bar{M})$ is $\mathrm{SN}(\bar{M})$, which is the direct sum of the annihilator space of $\mathrm{SJ}(M,M,M)$, that is $\mathrm{SN}(M)$, with the annihilator space of K that is $\{f \in M^* : f(\mathrm{SJ}(M,M,M)) = \{0\}\}$. □

Corollary. *Let M be a Malcev superalgebra and $\bar{M} = M \oplus M^*$ be the semi-direct product of M^* by M by means of the coadjoint representation. If* $\mathrm{SJ}(M,M,M) = M$ *then* $\mathrm{SN}(M \oplus M^*) = \mathrm{SN}(M)$.

Example. For the non-Lie simple Malcev superalgebra C, we know that $\mathrm{SJ}(C,C,C) = C$. Consider the semi-direct product of C^* by C by means of the coadjoint representation. Then $\bar{C}$ is a quadratic non-semi-simple Malcev superalgebra because C^* is an abelian ideal of $\bar{C}$ with $\mathrm{SN}(\bar{C}) = \mathrm{SN}(C) = \{0\}$. This example shows that there are quadratic non-semi-simple Malcev superalgebras with nucleus zero.

4. Central extension of Malcev superalgebras

Proposition 4.1. *Let M be a Malcev superalgebra and V be a super-vector space. Let $w: M \times M \to V$ be an even bilinear map. In the space $\tilde{M} = M \oplus V$ we consider the multiplication*: $(X+v)(Y+t) = XY + w(X,Y)$, $\forall_{(X,Y)\in M\times M,(v,t)\in V\times V}$.

$\tilde{M}$ with this multiplication is a Malcev superalgebra if and only if:

(1) $w(X,Y) = -(-1)^{xy} w(Y,X)$

(2) $$(-1)^{yz} w(XZ,YT) = w((XY)Z,T) + (-1)^{x(y+z+t)} w((YZ)T,X) + (-1)^{(x+y)(z+t)} w((ZT)X,Y) + (-1)^{t(x+y+z)} w((TX)Y,Z),$$

$\forall_{X\in M_x, Y\in M_y, Z\in M_z, T\in M_t}$. (*In this case $\tilde{M}$ is called the central extension of V by M by means of w.*)

Proof. We must recall the definition of a Malcev superalgebra and consider the identity:

$$(((X+v)(Y+w))(Z+u))(T+q) = ((XY)Z)T + w((XY)Z,T). \qquad \square$$

Definition. Let M be a Malcev superalgebra and V be a super-vector space. Let $w: M \times M \to V$ be a homogeneous bilinear map. If assertions (1) and (2) of the last theorem are satisfied we say that w is a Malcev 2-cocycle of M with values in V.

Proposition 4.2. *Let M be a Malcev superalgebra, V be a super-vector space and $w: M \times M \to V$ be an even Malcev 2-cocycle. Let $\tilde{M}$ be the central extension of V by M by means of w. Then,*

$$\mathrm{SJ}(X+P, Y+W, Z+T) = \mathrm{SJ}(X,Y,Z) + [w(XY,Z) + (-1)^{x(y+z)} w(YZ,X) + (-1)^{z(x+y)} w(ZX,Y)], \quad \forall_{X+P\in\tilde{M}_x, Y+W\in\tilde{M}_y, Z+T\in\tilde{M}_z}.$$

Proof. This result follows easily if we note that,

$$((X+P)(Y+W))(Z+T)=(XY)Z+w(XY,Z). \qquad \square$$

Corollary. *Let M be a Malcev superalgebra, V be a super-vector space and $w:M\times M\to V$ be an even Malcev 2-cocycle. Consider the graded super-vector space $A=A_{\bar 0}\oplus A_{\bar 1}$ with $A_x=\{X\in M_x:\ w(XY,Z)+(-1)^{x(y+z)}w(YZ,X)+(-1)^{z(x+y)}w(ZX,Y)=0,\ \forall_{Y\in M_y,Z\in M_z}\}$. Then $\mathrm{SN}(\tilde M)=(\mathrm{SN}(M)\cap A)\oplus V$, where $\tilde M$ is the central extension of V by M by means of w.*

Proposition 4.3. *Let M be a Malcev superalgebra, V be a super-vector space and $w:M\times M\to V$ be an even 2-cocycle. Let $\tilde M$ be the central extension of V by M by means of w. If M is nilpotent (resp. solvable), then $\tilde M$ is nilpotent (resp. solvable).*

Proof. Only calculations. $\square$

Definition. Let (M,B) be a quadratic Malcev superalgebra and $\phi\in\mathrm{End}(M)$ homogeneous of degree δ. We say that ϕ is B superantisymmetric if $\forall_{X\in M_x,Y\in M}$, $(-1)^{\delta x}B(X,\phi(Y))=-B(\phi(X),Y)$.

Proposition 4.4. *Let (M,B) be a quadratic Malcev superalgebra and let $w:M\times M\to K$ be a bilinear form of degree δ.*

(1) *There exists $\phi\in\mathrm{End}(M)$ homogeneous of degree δ such that $\forall_{X,Y\in M}, w(X,Y)=B(\phi(X),Y)$.*
(2) *w is a 2-cocycle if and only if ϕ is a Malcev operator and ϕ is B-superantisymmetric.*

Proof. (1) As B is non-degenerate there is a unique $\phi\in\mathrm{End}(M)$ such that $w(X,Y)=B(\phi(X),Y)$. (2) Let us begin by proving that w is superantisymmetric if and only if B is ϕ-superantisymmetric: $w(X,Y)=-(-1)^{xy}w(Y,X)$ if and only if $(-1)^{xy}B(\phi(Y),X)=-B(\phi(X),Y)$. But $B(\phi(Y),X)=(-1)^{(y+\delta)x}B(X,\phi(Y))$ then we can conclude that $B(\phi(X),Y)=-(-1)^{\delta x}B(X,\phi(Y))$. Now, suppose that the second condition of the definition of a 2-cocycle is satisfied by w, that is, $\forall_{X\in M_x,Y\in M_y,Z\in M_z,T\in M_t}$,

$$\begin{aligned}(-1)^{yz}B(\phi(XZ),YT)=&\,B(\phi((XY)Z),T)+(-1)^{x(y+z+t)}B(\phi((YZ)T),X)\\&+(-1)^{(x+y)(z+t)}B(\phi((ZT)X),Y)\\&+(-1)^{t(x+y+z)}B(\phi((TX)Y),Z).\end{aligned}$$

But

$$\begin{aligned}(-1)^{x(y+z+t)}B(\phi((YZ)T),X)&=-(-1)^{(x+\delta)(y+z+t)}B((YZ)T,\phi(X))\\&=(-1)^{(x+\delta)(y+z+t)+t(y+z)}B(T,(YZ)\phi(X))\\&=-B(\phi(X)(YZ),T).\end{aligned}$$

Analogously,

$$(-1)^{(x+y)(z+t)}B(\phi((ZT)X),Y) = -(-1)^{\delta x}B(x\phi(Y))Z,T),$$

$$(-1)^{t(x+y+z)}B(\phi((TX)Y),Z) = -(-1)^{(x+y)\delta)}B(X(Y\phi((Z)),T).$$

So, as B is non-degenerate we have

$$(-1)^{yz}\phi(XZ)Y$$
$$= \phi((XY)Z) - \phi(X)(YZ) - (-1)^{\delta x}(X\phi(Y))Z - (-1)^{\delta(x+y)}X(Y\phi(Z))$$

and ϕ is a Malcev operator. The reciprocal is immediate. □

Now, let M be a Malcev superalgebra. We denote by $\mathrm{Op}(M)_\delta$ the vector subspace of $\mathrm{End}(M)$ formed by the Malcev operators of degree δ. Then $\mathrm{Op}(M) = [\mathrm{Op}(M)]_{\bar{0}} \oplus [\mathrm{Op}(M)]_{\bar{1}}$. Let us denote by $(\mathrm{Op}_a(M))_\delta$ the sub-space of B-superantisymmetric elements of $\mathrm{Op}(M)_\delta$. Let us put $\mathrm{Op}_a(M) = [\mathrm{Op}_a(M)]_{\bar{0}} \oplus [\mathrm{Op}_a(M)]_{\bar{1}}$, which is a sub super-vector space of $\mathrm{Op}(M)$.

Theorem 4.1. *Let M_1 and M_2 be Malcev superalgebras with M_1 quadratic with an invariant scalar product B_1. Suppose that there is an admissible Malcev representation ψ of M_2 to M_1 such that $\forall_{X\in M_2}$, $\psi(X)\in \mathrm{Op}_a(M_1)$. Let us consider the bilinear map $\phi : M_1 \times M_1 \to M_2^*$ defined by*

$$\phi(X,Y)(Z) = (-1)^{z(x+y)}B_1(\psi(Z)(X),Y), \quad \forall_{Z\in(M_2)_z, X\in(M_1)_x, Y\in(M_1)_y}.$$

Then ϕ is a 2-cocycle of M_1 with values in M_2^* and consequently $M_1 \oplus M_2^*$ with the multiplication defined by

$$(X_1+f)(Y_1+g) = X_1Y_1 + \phi(X_1,Y_1), \quad \forall_{X_1+f, Y_1+g\in(M_1\oplus M_2^*)}$$

is the central extension of M_2^* by M_1 by means of ϕ.

Proof. Let us prove that ϕ is an even 2-cocycle of M_1 with values in M_2^*. Let $X\in(M_1)_x, Y\in(M_1)_y, Z\in(M_2)_z$,

$$\phi(X,Y)(Z) = (-1)^{z(x+y)}B_1(\psi(Z)(X),Y) = -(-1)^{z(x+y)+zx}B_1(X,\psi(Z)(Y))$$
$$= -(-1)^{zy+x(z+y)}B_1(\psi(Z)(Y),X) = -(-1)^{xy}\phi(Y,X)(Z).$$

On the other hand,

$$\phi(X,Y)(Z) = (-1)^{z(x+y)}B_1(\psi(Z)X,Y) \in K_{x+y+z}$$

and $\phi(X,Y)\in(M_2)^*_{x+y}$. Then we have concluded that ϕ is even and superantisymmetric. But as ψ is a Malcev operator then the bilinear form ϕ defined by $\phi(X,Y)(Z) = (-1)^{z(x+y)}B_1(\psi(Z)(X),Y)$, $\forall_{Z\in(M_2)_z, X\in(M_1)_x, Y\in(M_1)_y}$ is a 2 cocycle and consequently $M_1\oplus(M_2)^*$ with the defined product is a Malcev superalgebra, called the central extension of $(M_2)^*$ by M_1 by means of ϕ. □

Proposition 4.5. *Let (M,B) be a quadratic Malcev superalgebra, and let D be a homogeneous derivation of M of degree 0. Then the bilinear form $w : M \times M \to K$*

defined by $w(X,Y)=B(D(X),Y)$ *is a 2-cocycle if and only if* $D=-D^{\mathrm{T}}$ *and* $D(\mathrm{SJ}(X,Y,Z))=0,\ \forall_{X,Y,Z\in M}$.

Proof. We have proved already that $w(X,Y)$ is a 2-cocycle if and only if D is B-superantisymmetric and D is a Malcev operator. By Proposition 3.1 D is a Malcev operator if and only if $D(\mathrm{SJ}(X,Y,Z))=0$. $\square$

5. Double extension of Malcev superalgebras

Theorem 5.1. *Let* (M_1,B_1) *be a quadratic Malcev superalgebra and* M_2 *be a Malcev superalgebra. Let* $\psi: M_2\to \mathrm{End}(M_1)$ *be an admissible Malcev representation of* M_2 *in* M_1 *such that* $\forall_{X\in M_2}, \psi(X)\in \mathrm{Op}_a(M_1)$ *and the following condition is satisfied,*

$$\psi(SX)(YZ)=\psi(S)((\psi(X)Y)Z)-(-1)^{yz}(\psi(S)(\psi(X)Z))Y$$
$$+(-1)^{sx+yz}\psi(X)((\psi(S)Z)Y)+(-1)^{sx}(\psi(X)(\psi(S)Y))Z,$$
$$\forall_{X\in M_{2x},Y\in M_{1y},Z\in M_{1z},S\in M_{2s}}.$$

Let us consider the bilinear map $\phi: M_1\times M_1\to M_2^*$ *defined by*

$$\phi(X,Y)(Z)=(-1)^{z(x+y)}B_1(\psi(Z)(X),Y),\qquad \forall_{Z\in(M_2)_z,X\in(M_1)_x,Y\in(M_1)_y}.$$

(1) *Then* ϕ *is a 2-cocycle of* M_1 *with values in* M_2^* *and consequently* $M_1\oplus M_2^*$ *with the multiplication defined by*

$$(X_1+f)(Y_1+g)=X_1Y_1+\phi(X_1,Y_1),\qquad \forall_{X_1+f,Y_1+g\in(M_1\oplus M_2^*)}$$

is the central extension of M_2^* *by* M_1 *by means of* ϕ.

(2) *Let us consider the linear map* $\rho: M_2\to \mathrm{End}(M_1\oplus M_2^*)$ *defined by* $\rho(X_2)=\psi(X_2)+\pi^*(X_2)$ *that is,* $\rho(X_2)(X_1+F)=\psi(X_2)(X_1)+\pi^*(X_2)(F),\forall_{X_1\in M_1,X_2\in M_2,F\in M_2^*}$ *where* π^* *is the coadjoint representation of* M_2. *Then* ρ *is an admissible Malcev representation of* M_2 *in* $M_1\oplus M_2^*$ *and consequently* $M=M_2\oplus M_1\oplus M_2^*$ *with the multiplication*

$$(X_2+X_1+F)(Y_2+Y_1+G)=(X_2Y_2)+\psi(X_2)Y_1+\pi^*(X_2)(G)+(X_1Y_1)$$
$$+\phi(X_1,Y_1)-(-1)^{xy}\psi(Y_2)(X_1)$$
$$-(-1)^{xy}\pi^*(Y_2)F,\qquad \forall_{(X_2+X_1+F)\in M_x,(Y_2+Y_1+G)\in M_y}$$

is the semi-direct product of $M_1\oplus M_2^*$ *by* M_2 *by means of* ρ.

(3) *Let* γ *be an even supersymmetric invariant bilinear form on* M_2 *not necessarily non-degenerate. Then the bilinear form* B *on* $M=M_2\oplus M_1\oplus M_2^*$ *defined by*

$$B(X_2+X_1+F,Y_2+Y_1+G)=\gamma(X_2,Y_2)+B_1(X_1,Y_1)+F(Y_2)+(-1)^{xy}G(X_2),$$
$$\forall_{(X_2+X_1+F)\in M_x,(Y_2+Y_1+G)\in M_y}$$

is an invariant scalar product. The quadratic Malcev superalgebra $M_2\oplus M_1\oplus M_2^*$ *is called the double extension of* (M_1,B_1) *by* M_2 *by means of* ψ.

Proof. (1) is proved by the result of Theorem 4.1.

(2) It is easy to see that ρ is even. Let us prove now that ρ is a Malcev admissible representation of M_2 in $M_1 \oplus M_2^*$. As ρ is the direct sum of two Malcev representations it is a Malcev representation of M_2 in $M_1 \oplus (M_2)^*$. Now let $X \in (M_2)_x$, $Y \in (M_2)_y$, $V + F \in (M_1 \oplus M_2^*)_v$, $W + H \in (M_1 \oplus M_2^*)_w$. Then,

$$\begin{aligned}(\rho(XY)(V+F))(W+H) &= (\psi(XY)(V) + \pi^*(XY)(F))(W+H)\\ &= (\psi(XY)(V))(W) + \phi(\psi(XY)(V), W).\end{aligned}$$

Analogously, we can prove that,

$$\rho(X)(\rho(Y)(V+F)(W+H)) = \psi(X)(\psi(Y)(V).W) + \pi^*(X)\phi(\psi(Y)(V), W),$$

$$(\rho(X)(V+F))(\rho(Y)(W+H)) = \psi(X)(V)\psi(Y)(W) + \phi(\psi(X)(V), \psi(Y)(W)),$$

$$\rho(Y)\rho(X)((V+F)(W+H)) = \psi(Y)\psi(X)(VW) + \pi^*(Y)\pi^*(X)\phi(V, W),$$

$$(\rho(Y)\rho(X)(W+H))(V+F) = (\psi(Y)\psi(X)(W))(V) + \phi(\psi(Y)\psi(X)(W), V).$$

As ψ is Malcev admissible, ρ satisfies (1) of Theorem 3.1 if and only if,

$$\begin{aligned}\phi(\psi(XY)(V), W) = \pi^*(X)\phi(\psi(Y)(V), W) &+ (-1)^{yv}\phi(\psi(X)(V), \psi(Y)(W))\\ &- (-1)^{xy}\pi^*(Y)\pi^*(X)\phi(V, W)\\ &- (-1)^{xy+vw}\phi(\psi(Y)\psi(X)(W), V).\end{aligned}$$

But $\phi(\psi(XY)(V), W)(T) = (-1)^{t(x+y+v+w)}B_1(\psi(T)\psi(XY)(V), W)$ and analogously,

$$\pi^*(X)\phi(\psi(Y)(V), W)(T) = -(-1)^{t(y+v+w)}B_1(\psi(XT)\psi(Y)(V), W),$$

$$\begin{aligned}(-1)^{yv}\phi(\psi(X)(V), \psi(Y)(W))(T) = -(-1)^{x(t+y)+t(v+w)}B_1(\psi(Y)\\ \times\psi(T)\psi(X)(V), W),\end{aligned}$$

$$-(-1)^{xy}\pi^*(Y)\pi^*(X)\phi(V, W)(T) = -(-1)^{t(v+w)}B_1(\psi(X(YT))(V), W),$$

$$-(-1)^{xy+vw}\phi(\psi(Y)\psi(X)(W), V)(T) = (-1)^{t(v+w)}B_1(\psi(X)\psi(Y)\psi(T)(V), W).$$

As we know B_1 is non-degenerate then,

$$\begin{aligned}&(-1)^{t(x+y)}\psi(T)\psi(XY)\\ &\quad = -(-1)^{ty}(\psi(XT)\psi(Y)) - (-1)^{x(t+y)}\psi(Y)\psi(T)\psi(X)\\ &\qquad - \psi(X(YT)) + \psi(X)\psi(Y)\psi(T),\end{aligned}$$

what is equivalent to,

$$\begin{aligned}\psi((YT)X) = \psi(Y)\psi(T)\psi(X) &- (-1)^{x(y+t)}\psi(X)\psi(Y)\psi(T)\\ &+ (-1)^{ty+x(y+t)}\psi(XT)\psi(Y) + (-1)^{t(x+y)+x(y+t)}\psi(T)\psi(XY)\end{aligned}$$

and this last equality is verified because ψ is a Malcev representation of M_2 in M_1. Let us consider now, $X \in (M_2)_x$, $Z \in (M_2)_z$ and $(Y+F) \in (M_1 \oplus M_2^*)_y, (T+G) \in (M_1 \oplus (M_2)^*)_t$. Doing some calculations we conclude that

$$(\rho(Z)(\rho(X)(Y+F))(T+G)) = (\psi(Z)(\psi(X)(Y))T + \pi^*(Z)\phi(\psi(X)Y,T),$$

$$\rho(X)(\rho(Z)(Y+F)(T+G)) = \psi(X)((\psi(Z)Y)T) + \pi^*(X)\phi(\psi(Z)T,Y),$$

$$\rho(X)(\rho(Z)(T+G))(Y+F)) = \psi(X)(\psi(Z)T)Y + \phi(\psi(X)\psi(Z)T,Y),$$

$$\rho(Z)(\rho(X)(T+G)(Y+F)) = \psi(Z)(\psi(X)TY) + \pi^*(Z)(\phi(\psi(X)T,Y)),$$

$$\rho(XZ)((Y+F)(T+G)) = \psi(XZ)(YT) + \pi^*(XZ)\phi(Y,T).$$

As ψ is Malcev admissible, ρ satisfies (2) of Theorem 3.1 if and only if,

$$\begin{aligned}&-\pi^*(Z)\phi(\psi(X)Y,T) + (-1)^{x(y+z+t)}\pi^*(X)\phi(\psi(Z)Y,T)\\ &\quad -(-1)^{(x+y)(z+t)}\phi(\psi(X)(\psi(Z)T),Y)) + (-1)^{t(x+y+z)}\pi^*(Z)(\phi(\psi(X)T,Y)))\\ &\qquad = (-1)^{yz}(\pi^*(XZ)\phi(Y,T)).\end{aligned}$$

But this equality is satisfied because B_1 is non-degenerate and ψ is a Malcev representation of M_2 in M_1.

Finally, let us prove now that $\rho(X)$ is a Malcev operator, $\forall_{X \in (M_2)_x}$, that is, for $X \in (M_2)_x, Y+F \in (M_{\bar{1}} \oplus M_2^*)_y, Z+G \in (M_1 \oplus M_2^*)_z, T+H \in (M_1 \oplus M_2^*)_t$, we must have

$$\begin{aligned}\rho(X)(((Y+F)(Z+G))(T+H)) &= (\rho(X)(Y+F))(Z+G))(T+H))\\ &\quad -(-1)^{yz}\rho(X)(Z+G))((Y+F)(T+H))\\ &\quad -(-1)^{t(y+z)}(\rho(X)(T+H)(Y+F))(Z+G)\\ &\quad -(-1)^{y(t+z)}\rho(X)((Z+G)(T+H))(Y+F).\end{aligned}$$

But,

$$\begin{aligned}\rho(X)(((Y+F)(Z+G))(T+H)) &= \rho(X)(YZ + \phi(Y,Z))(T+H)\\ &= \rho(X)((YZ)T + \phi(YZ,T))\\ &= \psi(X)((YZ)T) + \pi^*(X)\phi(YZ,T).\end{aligned}$$

Analogously,

$$(\rho(X)(Y+F))(Z+G))(T+H)) = ((\psi(X)(Y))Z)T) + \phi(\psi(X)(Y)Z,T),$$

$$\rho(X)(Z+G))((Y+F)(T+H)) = (\psi(X)(Z))(YT) + \phi(\psi(X)Z,YT),$$

$$(\rho(X)(T+H)(Y+F))(Z+G) = (\psi(X)(T)Y)Z + \phi(\psi(X)(T)Y,Z),$$

$$\rho(X)((Z+G)(T+H))(Y+F) = ((\psi(X)(ZT))Y + \phi(\psi(X)(ZT),Y).$$

As $\psi(X)\in \mathrm{Op}(M_{\bar{1}})$ is a Malcev operator then $\rho(X)$ is a Malcev operator if

$$\pi^*(X)\phi(YZ,T)=\phi(\psi(X)(Y)Z,T)-(-1)^{yz}\phi(\psi(X)Z,YT)$$
$$-(-1)^{t(y+z)}\phi(\psi(X)(T)Y,Z)-(-1)^{y(z+t)}\phi(\psi(X)(ZT),Y).$$

Now let us verify if this last condition is satisfied. We know that for an element $S\in(M_2)_s$,

$$\pi^*(X)\phi(YZ,T)(S)=-(-1)^{s(y+z+t)}B_1(\psi(XS)(YZ),T),$$
$$\phi((\psi(X)Y)Z,T)(S)=(-1)^{s(x+y+z+t)}B_1((\psi(S)\psi(X)(Y)Z,T),$$
$$(-1)^{yz}\phi(\psi(X)Z,YT)(S)=(-1)^{s(x+y+z+t)+yz}B_1(\psi(S)(\psi(X)(Z)),YT),$$
$$(-1)^{t(y+z)}\phi(\psi(X)(T)Y,Z)(S)=(-1)^{t(y+z)+s(x+y+z+t)}B_1(\psi(S)((\psi(X)T)Y),Z),$$
$$(-1)^{y(z+t)}\phi(\psi(X)(ZT),Y)(S)=(-1)^{y(z+t)+s(x+y+z+t)}B_1(\psi(S)(\psi(X)(ZT)),Y).$$

Thus, we must prove that,

$$-(-1)^{sx}B_1(\psi(XS)(YZ),T)$$
$$=B_1(\psi(S)\psi(X)Y)Z,T)-(-1)^{yz}B_1(\psi(S)(\psi(X)Z)Y,T)$$
$$-(-1)^{t(y+z)}B_1(\psi(S)(\psi(X)T)Y,Z)-(-1)^{y(z+t)}B_1(\psi(S)\psi(X)(ZT),Y).$$

Isolating T, we see that the last condition is equivalent, as B_1 is non-degenerate, to the following identity:

$$\psi(SX)(YZ)=\psi(S)(\psi(X)Y)Z)-(-1)^{yz}\psi(S)(\psi(X)Z)Y)$$
$$+(-1)^{sx+yz}\psi(X)((\psi(S)Z)Y)+(-1)^{sx}(\psi(X)(\psi(S)Y))Z.$$

Let us define in M the following bilinear form:

$$B(X_2+X_1+f,Y_2+Y_1+G)=\gamma(X_2,Y_2)+B_1(X_1,Y_1)+F(Y_2)+(-1)^{xy}G(X_2),$$

where γ is an even supersymmetric invariant form over M_2 not necessarily non-degenerate. By construction we see that B is even and supersymmetric. Let us prove that it is an invariant, non-degenerate form. B is non-degenerate because if we consider $X_2+X_1+F\in \mathrm{rad}\,B$ and $G\in(M_2)^*_y$ we have, $B(X_2+X_1+F,G)=0$ which implies $G(X_2)=0$ and $X_2=0$. On the other hand, let $Y_1\in(M_2)_y$. $B(X_2+X_1+F,Y_1)=0$, $B_1(X_1,Y_1)=0$ and $X_1=0$. If $B(F,Y_2)=0$, $\forall_{Y_2\in(M_2^*)_y}$ then $F(Y_2)=0$, and $F=0$. Let us prove now that B is invariant. Consider $(X_2+X_1+F)\in M_x$, $(Y_2+Y_1+G)\in M_y$ and $(Z_2+Z_1+H)\in M_z$.

$$B((X_2+X_1+F)(Y_2+Y_1+G),(Z_2+Z_1+H))$$
$$=B(X_2Y_2+\psi(X_2)(Y_1)+\pi^*(X_2)(G)+X_1Y_1+\phi(X_1,Y_1)-(-1)^{xy}\psi(Y_2)X_1$$
$$-(-1)^{xy}\pi^*(Y_2)(F),Z_2+Z_1+H)$$

$$= B_1(\psi(X_2)(Y_1) + X_1Y_1 - (-1)^{xy}\psi(Y_2)(X_1), Z_1)$$
$$+ (\pi^*(X_2)(G) + \phi(X_1, Y_1) - (-1)^{xy}\pi^*(Y_2(F))(Z_2)$$
$$+ (-1)^{z(x+y)}H(X_2Y_2) + \gamma(X_2Y_2, Z_2).$$

On the other hand,

$$B(X_2 + X_1 + F, (Y_2 + Y_1 + G)(Z_2 + Z_1 + H))$$
$$= B(X_2 + X_1 + F, Y_2Z_2 + \psi(Y_2)(Z_1) + \pi^*(Y_2)H - (-1)^{yz}\psi(Z_2)(Y_1)$$
$$+ Y_1Z_1 + \phi(Y_1, Z_1) - (-1)^{yz}\pi^*(Z_2)(G)) = \gamma(X_2, Y_2Z_2) + B_1(X_1, \psi(Y_2)(Z_1))$$
$$- (-1)^{yz}B_1(X_1, \psi(Z_2)(Y_1)) + B_1(X_1, Y_1Z_1) + (-1)^{x(y+z)}\pi^*(Z_2)(H)(X_2)$$
$$+ B(\psi(X_2)(Y_1)(X_2) + F(Y_2Z_2) + (-1)^{(x(y+z)}(\pi^*(Y_2)(H)(X_2)$$
$$+ \phi(Y_1, Z_1)X_2 - (-1)^{yz}\pi^*(Z_2)G(X_2)) + F(Y_2Z_2).$$

As $\gamma(X_2Y_2, Z_2) = \gamma(X_2, Y_2Z_2)$ and by the known properties of the bilinear form B_1 we have,

$$B((X_2 + X_1 + F)(Y_2 + Y_1 + G), (Z_2 + Z_1 + H))$$
$$= B((X_2 + X_1 + F), (Y_2 + Y_1 + G)(Z_2 + Z_1 + H))$$

and B is invariant. $\square$

Corollary 1. *Let M be a quadratic Malcev superalgebra and B an invariant scalar product defined on M. Let $\phi \in (\mathrm{Op}_a(M))_{\bar{0}}$ such that,*

(1) $\phi^2(XY) = \phi(X)\phi(Y) + \phi(\phi(X)Y) + X\phi^2(Y)$,

(2) $\phi^2(X)Y - \phi(\phi(X)Y) = -(-1)^{xy}(\phi^2(Y)X - \phi(\phi(Y)X)) \quad \forall_{X\in M_x, Y\in M_y, Z\in M_z}$.

Then $N = Ke \oplus M \oplus (Ke)^*$ (where Ke is the one-dimensional Lie algebra and $(Ke)^* = Ke^*$ its dual) with the multiplication,

$$(ae + X + \alpha e^*)(be + Y + \beta e^*)$$
$$= a\phi(Y) - b\phi(X) + XY + B(\phi(X), Y)e^*, \quad \forall_{\alpha,a,\beta,b\in K,(X,Y)\in M\times M}$$

is a Malcev superalgebra. Moreover, the bilinear form $\tilde{B}: N \times N \to K$ defined by,

$$\tilde{B}(ae + X + \alpha e^*, be + Y + \beta e^*) = a\beta + \alpha b + B(X, Y),$$

is an invariant scalar product on N.

Proof. Since $\phi \in (\mathrm{Op}_a(M))_{\bar{0}}$ satisfies conditions (1) and (2) it follows that $\psi : Ke \to \mathrm{End}(M)$ defined by $\psi(\alpha e) = \alpha\phi$ is an admissible Malcev representation of the one-dimensional Lie algebra on M. Consequently by the last theorem we have the result. $\square$

Corollary 2. *Let M be a quadratic Malcev superalgebra and B an invariant scalar product defined on M. Let D be an even derivation of M such that $D \in (\mathrm{Op}_a(M))_{\bar{0}}$. Then $N = Ke \oplus M \oplus (Ke)^*$ (where Ke is the one-dimensional Lie algebra and $(Ke)^* = Ke^*$ its dual) with the multiplication,*

$$(ae + X + \alpha e^*)(be + Y + \beta e^*)$$
$$= aD(Y) - bD(X) + XY + B(D(X), Y)e^*$$

and the bilinear form $\tilde{B} : N \times N \to K$ defined by,

$$\tilde{B}(af + X + \alpha e^*, bf + Y + \beta e^*) = (a\beta + \alpha b) + B(X, Y),$$

is a quadratic Malcev superalgebra.

Proof. By the last corollary and the result of Proposition 3.3. □

Definition. Let (M, B) be a quadratic Malcev superalgebra.

(i) A graded ideal I of M is called non-degenerate (resp. degenerate), if the restriction of B to $I \times I$ is a non-degenerate (resp. degenerate) bilinear form.
(ii) (M, B) is called irreducible if M contains no non-trivial non-degenerate graded ideal.

Proposition 5.1. *Let (M, B) be a quadratic Malcev superalgebra and let I be a graded ideal of M. Then,*

(1) *$I^\perp$ (the orthogonal of I relatively to B) is a graded ideal of M;*
(2) *If I is non-degenerate, then $M = I \oplus I^\perp$ and $I^\perp$ is a non-degenerate ideal.*

Proof. (1) Let $X \in I^\perp$, $Y \in M$, $Z \in I$.

$$B(YX, Z) = B(Y, XZ) = \{0\},$$

which implies that $YX \in I^\perp$ and $I^\perp$ is an ideal of M. Is graded because I is graded and B is even.

(2) Let us suppose that I is non-degenerate. Then $I \cap I^\perp = \{0\}$ so $M = I \oplus I^\perp$ and $I^\perp$ is a non-degenerate ideal: if there exists $A \in I^\perp$ such that $B(A, I^\perp) = \{0\}$ then $B(A, M) = \{0\}$ and $A = 0$. □

Proposition 5.2. *Let (M, B) be a quadratic Malcev superalgebra. Then $M = \bigoplus_{i=1}^n M_i$ such that $\forall_{i \in \{1,\dots,n\}}$,*

(1) *M_i is a non-degenerate graded ideal;*
(2) *M_i contains no non-trivial non-degenerate graded ideal of M;*
(3) *for all $i \neq j$, M_i and M_j are orthogonal (that is, $B(M_i, M_j) = 0$) $\forall_{i,j \in \{1,\dots,n\}(i \neq j)}$.*

Proof. By induction on the dimension of M. The proof is analogously to the one in Lie case [7]. □

Theorem 5.2. *Let (M,B) be an irreducible quadratic Malcev superalgebra. Let I be a maximal graded ideal of M and suppose that there exists a sub superalgebra A of M such that $M = A \oplus I$. Then M is a double extension of the quadratic Malcev superalgebra $(I/I^{\perp}, \tilde{B})$ by A by means of ϕ, where $\tilde{B}$ is the bilinear form defined by (denote by $\bar{X} = X + I^{\perp}, X \in I$), $\tilde{B}(\bar{X}, \bar{Y}) = B(X,Y), \forall_{(X,Y)\in I\times I}$, and $\phi : A \to \mathrm{End}(I/I^{\perp})$ defined by $\phi(a)(\bar{X}) = \overline{aX}, \forall_{(a,X)\in A\times I}$.*

Proof. Since I is a maximal graded ideal of M it follows that $I^{\perp}$ is a minimal graded ideal of M. Consequently as M is irreducible then $I^{\perp} \subseteq I$. Therefore $M = I^{\perp} \oplus L^{\perp} \oplus A$ where $L = I^{\perp} \oplus A$ and $I = I^{\perp} \oplus L^{\perp}$. It is easy to see that $AL^{\perp} \subseteq L^{\perp}$. Now if $X, Y \in L^{\perp}$, then $XY = \alpha(X,Y) + \beta(X,Y)$ where $\alpha(X,Y) \in I^{\perp}$ and $\beta(X,Y) \in L^{\perp}$. It is clear that β induces a structure of Malcev superalgebra on $L^{\perp}$ which we denote by $[,]_{L^{\perp}}$. Moreover $B' = B_{L^{\perp}\times L^{\perp}}$ is invariant . Consequently $(L^{\perp}, [,]_{L^{\perp}}, B')$ is a quadratic Malcev superalgebra. It is clear that $I/_{I^{\perp}}$ is a Malcev superalgebra and the bilinear form $\bar{B}$ on $I/_{I^{\perp}}$ defined by $\bar{B}(\bar{X}, \bar{Y}) = B(X,Y)$, $\forall_{X,Y\in I}$ is a well defined and it is an invariant scalar product on $I/I^{\perp}$. We consider the map $v : I^{\perp} \to A^*$ defined by $v(X) = B(X,.)$, $\forall_{X\in I^{\perp}}$. This map is an isomorphism of Z_2-graded vector spaces. On $I^{\perp}$ and A^* there are structures of Malcev A-modules defined by

(1) $a.X = aX, \quad \forall_{a\in A, X\in I^{\perp}}$,

(2) $(X.F)(Y) = -(-1)^{xf}F(XY), \quad \forall_{(X,Y)\in A_x\times A_y, F\in (A^*)_f}$.

Let $\phi : A \to \mathrm{End}(I/_{I^{\perp}})$ defined by $\phi(a)(\bar{X}) = \overline{aX}$, $\forall_{(a,X)\in A\times I}$. It is easy to see that ϕ is an admissible Malcev representation of A in $(I/_{I^{\perp}})$ and $\phi(a) \in (\mathrm{Op}_a)(I/_{I^{\perp}})$, $\forall_{a\in A}$. Consider $\theta : L^{\perp} \to I/I^{\perp}$ be defined by $\theta(X) = \bar{X}$, $\forall_{X\in L^{\perp}}$. θ is an isomorphism of Malcev superalgebras. Denote by $N = A \oplus I/I^{\perp} \oplus A^*$ the double extension of $(I/I^{\perp}, \bar{B})$ by A by means of ϕ. Consider $\beta : M = A \oplus L^{\perp} \oplus I^{\perp} \to N = A \oplus I/I^{\perp} \oplus A^*$ defined by $\beta(a + X + Y) = a + \theta(X) + v(Y)$, $\forall_{(a,X,Y)\in A\oplus L^{\perp}\oplus I^{\perp}}$. This map is an isomorphism of Malcev superalgebras. Moreover if T is an invariant scalar product on N defined by

$$
\begin{aligned}
&T(a + \bar{X} + F, a' + \bar{X}' + F') \\
&\quad = \bar{B}(\bar{X}, \bar{X}') + F(a') + (-1)^{xx'}F'(a) \\
&\quad = B(X, X') + F(a') + (-1)^{xx'}F'(a), \quad \forall_{a+\bar{X}+F\in N_x, a'+\bar{X}'+F'\in N_{x'}},
\end{aligned}
$$

then

$$
\begin{aligned}
&T(\theta(a + X + Y), \theta(a', X', Y')) = B(a + X + Y, a' + X' + Y') \\
&\quad \forall_{(a+X+Y, a'+X'+Y')\in M\times M}. \qquad \square
\end{aligned}
$$

Corollary. *Let M be an irreducible quadratic Malcev superalgebra such that $\dim M \geqslant 2$ and B be an invariant scalar product defined on M. If $Z(M) \cap M_{\bar{0}} \neq \{0\}$, then M is a double extension of a quadratic Malcev superalgebra A such that $\dim A = \dim M - 2$ by the one-dimensional Lie algebra.*

Proof. Since $Z(M)\cap M_{\bar{0}} \neq \{0\}$ it follows that there exists $X\in Z(M)\cap M_{\bar{0}} \neq \{0\}$. Consequently $J = KX$ is a graded ideal of M and $J \neq M$ because dim $M \geqslant 2$. So $I=J^{\perp}$ is a maximal graded ideal of M. The fact that $X\in M_{\bar{0}}$ implies that $M_{\bar{1}} \subseteq J^{\perp}$ so there is a $Y\in M_{\bar{0}}$ such that $M=J\oplus KY$. The sub-vector space KY is a sub-superalgebra of M because $Y\in M_{\bar{0}}$ and it follows by the last theorem that M is a double extension of $(I/I^{\perp},\bar{B})$ by KY by means of $\psi : KY \to \mathrm{End}(I/I^{\perp})$ defined by $\psi(Y)(\bar{X})=\overline{YX}$, $\forall_{X\in I}$, where $\bar{B}(\bar{X},\bar{X}')=B(X,X')$, $\forall_{X,X'\in I}$. □

Theorem 5.3. *Let M be an irreducible quadratic Malcev algebra such that* $\dim M \geqslant 2$ *and B be an invariant scalar product defined on M. Then M is a double extension of a quadratic Malcev algebra A such that* $\dim A \leqslant \dim M - 2$ *by the one-dimensional Lie algebra or by the simple non-Lie Malcev algebra.*

Proof. If $Z(M)\neq \{0\}$ then by the last corollary, M is a double extension of a quadratic Malcev algebra A such that dim $A = \dim M - 2$ by the one-dimensional Lie algebra. Now suppose that $Z(M)=\{0\}$. As $M^2=Z(M)^{\perp}$ then $M^2=M$. Consequently $M=S\oplus R$ where S is a semi-simple subalgebra of M such that $S\neq\{0\}$ and R is the solvable radical of M. Since S is semi-simple then $S=S_1\oplus S_2\oplus\cdots\oplus S_n$ where S_i is a simple ideal of S, $\forall_{i\in\{1,\ldots,n\}}$. It is clear that $I=(S_2\oplus S_3\oplus\cdots\oplus S_n)\oplus R$ is an ideal of M. As M/I is a simple Malcev algebra then I is a maximal ideal of M. So, by Theorem 5.2, M is a double extension of the quadratic Malcev algebra $I/I^{\perp}$ by S_1 because $M=S_1\oplus I$ and S_1 is a subalgebra of M. □

Definition. Let (M,B) and (N,C) be two quadratic Malcev superalgebras. The quadratic Malcev superalgebra $M\oplus N$ with the invariant scalar product T defined by: $T/_{M\times M}=B$; $T/_{M\times N}=T/_{N\times M}=0$; $T/_{N\times N}=C$, is called the orthogonal direct sum of (M,B) and (N,C).

Let U be the set formed by $\{0\}$, the one-dimensional Lie algebra and by the simple Malcev algebra. Recall that if M is a simple Malcev algebra then either M is a simple Lie algebra or M is the non-Lie Malcev simple algebra C [20,22].

Corollary. *Let (M,B) be a quadratic Malcev algebra. Then either M is an element of U or M is obtained by the following way: we take $M_1,M_2,\ldots,M_n$ elements of U and we complete by double extensions by the one-dimensional Lie algebra or by a simple Malcev algebra and/or by orthogonal direct sums of quadratic Malcev algebras.*

Proof. This corollary is a result of the last theorem and Proposition 5.2. □

6. The action of the even part on the odd part is completely reducible

Lemma 6.1. *Let $M=M_{\bar{0}}+M_{\bar{1}}$ be a quadratic Malcev superalgebra and B the scalar invariant product on M, such that $M_{\bar{1}}$ is a completely reducible $M_{\bar{0}}$ module. Then*

$M_{\bar{1}} = \bigoplus_{i=1}^{n} U_i$ *such that,*

(1) U_i *is an* $M_{\bar{0}}$ *submodule of* $M_{\bar{1}}$ *such that* $B/_{(U_i \times U_i)}$ *is non-degenerate,* $\forall_{i=1,\dots,n}$;

(2) U_i *is irreducible or* $U_i = U_{i1} \oplus U_{i2}$ *where* U_{i1} *and* U_{i2} *are irreducible* $M_{\bar{0}}$ *submodules of* $M_{\bar{1}}$ *such that* $B(U_{i1}, U_{i1}) = B(U_{i2}, U_{i2}) = \{0\}$, $\forall_{i=\{1,\dots,n\}}$;

(3) $B(U_i, U_j) = \{0\}$, $\forall_{i,j\in\{1,\dots,n\}(i\neq j)}$.

Proof. We must remind that every representation of a semi-simple Malcev algebra is completely reducible, and that the Lemma of Schur stay true in the Malcev case. That is, if $\rho : M \to gl(V)$ is a simple representation of a Malcev algebra and $f \in \mathrm{End}(V)$ such that $\rho(m) \circ f = f \circ \rho(m)$, $\forall_{m\in M}$, then there is a $\gamma \in K$ such that, $f = \gamma Id$. Then the proof of this lemma is the same that the proof of the equivalent result in the Lie case [7]. □

Lemma 6.2. *Let* (M, B) *be an irreducible quadratic Malcev superalgebra. Let us suppose that the action of* $M_{\bar{0}}$ *over* $M_{\bar{1}}$ *is completely reducible and that* $M_{\bar{0}} \neq \{0\}$. *Then* $M_{\bar{0}}M_{\bar{1}} = M_{\bar{1}}$ *and* $Z(M) = Z(M) \cap M_{\bar{0}}$.

Proof. Let $M_{\bar{1}}^{M_{\bar{0}}} = \{m \in M_{\bar{1}}\colon M_{\bar{0}}m = \{0\}\}$. Then $M_{\bar{1}}^{M_{\bar{0}}}$ is a submodule of $M_{\bar{1}}$, so there is a submodule W of $M_{\bar{1}}$, such that $M_{\bar{1}} = M_{\bar{1}}^{M_{\bar{0}}} \oplus W$. Let $x \in M_{\bar{0}}$ and $m \in M_{\bar{1}}$. Then $m = v + w$ where $v \in M_{\bar{1}}^{M_{\bar{0}}}$, $w \in W$. Consequently, $xm = xv + xw \in W$ and $M_{\bar{0}}M_{\bar{1}} \subseteq W$. If we suppose that $M_{\bar{0}}M_{\bar{1}} \neq W$ then there is W' submodule of W such that $W = M_{\bar{0}}M_{\bar{1}} \oplus W'$ because $M_{\bar{0}}M_{\bar{1}}$ is a submodule of W. As $M_{\bar{0}}W' \subseteq M_{\bar{0}}M_{\bar{1}} \cap W'$ we have that $M_{\bar{0}}W' \subseteq \{0\}$ and consequently $W' \subseteq M_{\bar{1}}^{M_{\bar{0}}} \cap W = \{0\}$. Then $W' = \{0\}$ and $W = M_{\bar{0}}M_{\bar{1}}$ and $M_{\bar{1}} = M_{\bar{1}}^{M_{\bar{0}}} \oplus (M_{\bar{0}}M_{\bar{1}})$. As B is invariant then $B(M_{\bar{0}}M_{\bar{1}}, M_{\bar{1}}^{M_{\bar{0}}}) = \{0\}$. Let us consider $J = M_{\bar{0}} \oplus M_{\bar{0}}M_{\bar{1}}$. It is easy to see that J is a graded ideal of M and is non-degenerate because $B(M_{\bar{0}}M_{\bar{1}}, M_{\bar{1}}^{M_{\bar{0}}}) = \{0\}$ and B is even. As M is irreducible then $M = J$ because $J \neq \{0\}$. Consequently $M_{\bar{1}} = M_{\bar{0}}M_{\bar{1}}$ and $M_{\bar{1}}^{M_{\bar{0}}} = \{0\}$. On the other hand, as $Z(M) = (Z(M) \cap M_{\bar{0}}) \oplus (Z(M) \cap M_{\bar{1}})$ we have that $Z(M) = Z(M) \cap M_{\bar{0}}$ because $Z(M) \cap M_{\bar{1}} \subseteq M_{\bar{1}}^{M_{\bar{0}}} = \{0\}$. □

Theorem 6.1. *Let* (M, B) *be a non-zero irreducible quadratic Malcev superalgebra with reductive even part* $M_{\bar{0}} \neq 0$ *and the action of* $M_{\bar{0}}$ *on* $M_{\bar{1}}$ *is completely reducible. Then* M *is simple if and only if* $Z(M) = \{0\}$.

Proof. If M is simple then $Z(M) = \{0\}$. Reciprocally, if $M_{\bar{1}} = \{0\}$ then M was a semi-simple Malcev algebra and it follows that M is a simple Malcev algebra because M is irreducible. If $M_{\bar{1}} \neq \{0\}$ and M is not simple, there is a non-zero minimal graded ideal I of M. As I is minimal then $I \cap I^{\perp} = I$ or $I \cap I^{\perp} = \{0\}$. Since M is irreducible it follows that $I \cap I^{\perp} = I$ and $I^2 = \{0\}$ (because B is non-degenerate) with $M_{\bar{0}}I_{\bar{1}} \subseteq I_{\bar{1}}$. Consequently, there is P a $M_{\bar{0}}$ submodule of $M_{\bar{1}}$ such that $M_{\bar{1}} = P \oplus I_{\bar{1}}$. The fact that $I_{\bar{0}}P \subseteq P \cap I_{\bar{1}} = \{0\}$, implies that $I_{\bar{0}}M_{\bar{1}} = \{0\}$. As $I_{\bar{0}}M_{\bar{1}} = \{0\}$, then $I_{\bar{0}}$ is a graded ideal of M. As I is minimal then $I_{\bar{0}} = \{0\}$ or $I_{\bar{0}} = I$ and consequently, $I = I_{\bar{1}}$ or $I = I_{\bar{0}}$. If we suppose that $I = I_{\bar{1}}$, then $B(I, M_{\bar{1}}) = B(I_{\bar{1}}, M_{\bar{1}}) = B(I_{\bar{1}}, M_{\bar{0}}M_{\bar{1}})$ (by

Lemma 6.2, $M_{\bar{0}}M_{\bar{1}} = M_{\bar{1}}$). Thus, $B(I, M_{\bar{1}}) = B(I_{\bar{1}}M_{\bar{1}}, M_{\bar{0}}) = \{0\}$, as $I_1M_{\bar{1}} \subseteq I_{\bar{0}} = \{0\}$. Consequently, $I_{\bar{1}} = \{0\}$ because B is even and non-degenerate.Therefore $I = \{0\}$ which is a contradiction. Then $I = I_{\bar{0}}$. Let us prove now that I is a minimal ideal of $M_{\bar{0}}$. Let J be an ideal of $M_{\bar{0}}$ such that $J \subseteq I$. As $IM_{\bar{1}} = \{0\}$, then $JM_{\bar{1}} = 0$ and J is a graded ideal of M which implies that $J = \{0\}$ or $J = I$ because I is a graded minimal ideal of M. Then I is a minimal ideal of $M_{\bar{0}}$ with $I^2 = \{0\}$. By hypothesis $M_{\bar{0}}$ is reductive, then $M_{\bar{0}} = S \oplus Z(M_{\bar{0}})$ where S is the greatest semi-simple ideal of $M_{\bar{0}}$. $SI \subseteq S \cap I = \{0\}$ because $S \cap I$ is an abelian ideal of S. Consequently, $M_{\bar{0}}I = \{0\}$ and $MI = \{0\}$. Then $I \subseteq Z(M) = \{0\}$ that is a contradiction with $I \neq \{0\}$ and we can conclude that M is simple. □

Corollary. *Let (M, B) be a non-zero irreducible quadratic Malcev superalgebra with semi-simple even part $M_{\bar{0}} \neq \{0\}$. Then M is a simple Malcev algebra or a simple Lie superalgebra.*

Proof. If $M_{\bar{0}}$ is a semi-simple Malcev algebra then the $M_{\bar{0}}$-module $M_{\bar{1}}$ is completely reducible. Consequently by Lemma 6.1, we have that $Z(M) = Z(M) \cap M_{\bar{0}}$. It follows that $Z(M) \subseteq Z(M_{\bar{0}}) = \{0\}$ because $M_{\bar{0}}$ is semi-simple. Therefore by the last theorem, M is a simple Malcev superalgebra. As we know, simple Malcev superalgebras are simple Malcev algebras or simple Lie superalgebras [24]. □

Corollary. *Let (M, B) be a non-zero quadratic Malcev superalgebra with semi-simple even part M_0. Then $M = \bigoplus_{i=1}^{n} M_i$ such that $\forall_{i \in \{1,\dots,n\}}$,*

(1) *M_i is a non-degenerate graded ideal of M that is either a simple Malcev algebra or a simple Lie superalgebra or the two-dimensional Lie superalgebra $N = N_1$;*

(2) *$B(M_i, M_j) = 0$, $\forall_{i,j \in \{1,\dots,n\}(i \neq j)}$.*

Let us recall that a Malcev superalgebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is called trivial if $M_{\bar{1}}^2 = 0$.

Definition. Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Malcev superalgebra with $M_{\bar{0}} = A_{\bar{0}} \oplus (\text{ideal}(M_{\bar{1}}))_{\bar{0}}$, where $A_{\bar{0}}$ is a subalgebra of $M_{\bar{0}}$ and $(\text{ideal}(M_{\bar{1}}))_{\bar{0}}$ is the even part of the ideal of M generated by $M_{\bar{1}}$. If V is a Malcev module of $A_{\bar{0}}$, V becomes a Malcev module of M by imposing that the action of the (ideal $(M_{\bar{1}})) = \text{ideal}(M_{\bar{1}})_0 \oplus M_{\bar{1}}$ on V is trivial and that $V = V_{\bar{1}}$. Then the split extension $\tilde{M} = M \oplus V = M_{\bar{0}} \oplus (M_{\bar{1}} \oplus V)$ is a Malcev superalgebra called the elementary odd extension of M by the module V.

One of the main results of [4] is

Theorem 6.2. *Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Malcev superalgebra with reductive $M_{\bar{0}}$ and completely reductible action of $M_{\bar{0}}$* in *$M_{\bar{1}}$. Then $M = A \oplus B$ such that*

(1) *A and B are graded ideals of M*;

(2) *A is a trivial Malcev superalgebra*;

(3) *B is an elementary odd extension of a Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$, with reductive $L_{\bar{0}}$ and completely reducible action of $L_{\bar{0}}$ in $L_{\bar{1}}$, by a Malcev module.*

Theorem 6.3. *Let (M,B) be an irreducible quadratic Malcev superalgebra such that $M_{\bar 0} \neq \{0\}$ is a reductive Malcev algebra and the action of $M_{\bar 0}$ on $M_{\bar 1}$ is completely reducible. Then M is either a simple Malcev algebra or a Lie superalgebra.*

Proof. By the last theorem, $M = A \oplus T$ where A and T are graded ideals of M. Moreover A is a trivial Malcev superalgebra and T is an elementary odd extension of a Lie superalgebra $L = L_{\bar 0} \oplus L_{\bar 1}$ with reductive $L_{\bar 0}$ and completely reducible action of $L_{\bar 0}$ in $L_{\bar 1}$ by a Malcev module $V = V_{\bar 1}$. Consequently, $B(A_{\bar 1}A_{\bar 0}, M_{\bar 1}) = B(A_{\bar 0}, A_{\bar 1}M_{\bar 1}) = \{0\}$ and $B(VT_{\bar 0}, M_{\bar 1}) = B(T_0, VM_{\bar 1}) = \{0\}$ because B is invariant, $A_{\bar 1}M_{\bar 1} = \{0\}$ and $VM_{\bar 1} = \{0\}$. Then $A_{\bar 1}A_{\bar 0} = VT_0 = 0$. It follows that $(A_{\bar 1} \oplus V)M = \{0\}$ because $A_{\bar 1}A_{\bar 1} = A_{\bar 1}T = VT_{\bar 1} = VA = \{0\}$. Since $Z(M) \subseteq M_{\bar 0}$ it follows that $A_{\bar 1} \oplus V = \{0\}$. Therefore $A = A_{\bar 0}$ is a reductive Malcev algebra and $T = L$ is a Lie superalgebra. Consider S the greatest semi-simple ideal of A. Then S is non-degenerate and $S = \{0\}$ or $A = S = M$ because M is irreducible. If $S \neq \{0\}$ then $M = S$ is a simple Malcev algebra. Now if $S = \{0\}$ then A is an abelian Malcev algebra and $M = A \oplus L$ is a Lie superalgebra. $\square$

Now, consider (L,B) be a quadratic Lie superalgebra such that the $L_{\bar 0}$-module $L_{\bar 1}$ is completely reducible. In the following we are going to recall the notion of an elementary double extension of (L,B) by the one-dimensional Lie algebra Ke [7]. If $L_{\bar 1} = \{0\}$ we take $\psi = 0$. If $L_{\bar 1} \neq \{0\}$ by Lemma 6.1, $L_{\bar 1} = \bigoplus_{i=1}^{n} U_i$ such that,

(1) U_i is an $L_{\bar 0}$ submodule of $L_{\bar 1}$ such that $B/_{(U_i \times U_i)}$ is non-degenerate, for all $i \in \{1,\ldots,n\}$;
(2) There exists $m \leqslant n$ such that U_i is an irreducible $L_{\bar 0}$-submodule of $L_{\bar 1}$, $\forall_{i=1,\ldots,n}$ and $U_i = U_{i1} \oplus U_{i2}$, where U_{i1} and U_{i2} are irreducible $L_{\bar 0}$-submodules of $L_{\bar 1}$ such that $B(U_{i1}, U_{i1}) = B(U_{i2}, U_{i2}) = 0$, $\forall_{i \in \{\{m+1,\ldots,n\}}$;
(3) $B(U_i, U_j) = 0$, $\forall_{i,j \in \{1,\ldots,n\}(i \neq j)}$.

Let $D : L \to L$ be the linear map defined by

(1) $D(L_{\bar 0}) = D(\bigoplus_{i=1}^{m} U_i) = \{0\}$;
(2) $D/_{U_{i1}} = k_i Id_{U_{i1}}$ and $D/_{U_{i2}} = -k_i Id_{U_{i2}}$, where $k_i \in K$, for all $i \in \{m+1,\ldots,n\}$.

It is easy to see that D is a homogeneous superantisymmetric superderivation of degree 0 of L. Consider the linear map $\psi : Ke \to Der_a(L) \subseteq Der(L)$ defined by $\psi(\alpha e) = \alpha D$, $\forall_{\alpha \in K}$ (Ke is the one-dimensional Lie algebra).

Let G be the set formed by $\{0\}$, the basic classical Lie superalgebras [15,23], the one-dimensional Lie algebra, and the two-dimensional Lie superalgebra $N = N_1$.

Theorem 6.4 (Benayadi [7]). *Let (L,B) be a quadratic Lie superalgebra such that $L_{\bar 0}$ is a reductive Lie algebra and the $L_{\bar 0}$ module $L_{\bar 1}$ is completely reducible. Then either L is an element of G or L is obtained by a sequence of elementary double extensions by the one-dimensional Lie algebra and/or by orthogonal direct sums of quadratic Lie superalgebras from a finite number of elements of G.*

We know that if M is a simple Malcev algebra then either M is a simple Lie algebra or M is the Malcev non-Lie simple algebra $C(\dim C = 7)$. Then by the last theorem and the result of Theorem 6.3 we have an inductive classification of the quadratic Malcev superalgebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$ such that $M_{\bar{0}}$ is a reductive Malcev algebra and the action of $M_{\bar{0}}$ on $M_{\bar{1}}$ is completely reducible. More precisely we obtain the following result:

Theorem 6.5. *Let (M, B) be a quadratic Malcev superalgebra such that $M_{\bar{0}}$ is a reductive Malcev algebra and the $M_{\bar{0}}$ module $M_{\bar{1}}$ is completely reducible. Then, $M = A \oplus L$ where A and L are non-degenerate graded ideals of M such that*:

(1) *A is a direct sum of copies of C, where C is the non-Lie Malcev simple algebra*;
(2) *L is a Lie superalgebra with $L_{\bar{0}}$ reductive and the $L_{\bar{0}}$ module $L_{\bar{1}}$ is completely reducible such that $L \in G$ or L is obtained by a sequence of elementary double extensions by the one-dimensional Lie algebra and/or by orthogonal direct sums of quadratic Lie superalgebras from a finite number of elements of G.*

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