An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 42 (2001) 1177–1188
www.elsevier.nl/locate/camwa

Deflation for Block Eigenvalues of Block Partitioned Matrices with an Application to Matrix Polynomials of Commuting Matrices

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Abstract—A method for computing a complete set of block eigenvalues for a block partitioned matrix using a generalized form of Wielandt's deflation is presented. An application of this process is given to compute a complete set of solvents of matrix polynomials where the coefficients and the variable are commuting matrices.

Keywords—Block eigenvalues deflation, Matrix polynomials.

1. INTRODUCTION

The definition of block eigenvalue (see Definition 1.1) was presented in a technical report by Dennis, Traub and Weber [1] in a method for computing one solvent of matrix polynomials. We extend such a method for finding a complete solution; see the definition of a complete set of solvents (Definition 4.5), for this sake: first we develop a block version of the classical Wielandt deflation, and then we apply this block deflation to matrix polynomials with some restrictions. Our approach casts in a new special form of matrix which we baptize as an L-shaped matrix.

DEFINITION 1.1. *A matrix X of order n is a block eigenvalue of order n of a matrix A of order mn if there exists a block vector V of full rank, such that $AV = VX$. V is a block eigenvector of A.*

The matrix A is partitioned into $m \times m$ blocks of order n, and the block vector V is of type $mn \times n$ (a column of m blocks of order n). This is a particular case of the classical problem

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The authors wish to express their deep thanks to R. Serôdio for the help in putting this paper in the final format.

0898-1221/01/$ - see front matter Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
PII: S0898-1221(01)00231-0

$AX = XB$ (see [2]) with the order of A being a multiple of the order of B (X in our notation) and X (the block vector V in our notation) being an invariant subspace of A (see [3]).

We will work with matrices having entries in the field of complex numbers, and sometimes we will name scalar eigenvalues the eigenvalues in order to distinguish from block eigenvalues. We will denote the j^{th} block ($n \times n$) of a block vector V by $(V)_j$ and the j^{th} block row ($n \times mn$) of a block partitioned matrix A by $(A)_j$ as follows:

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_j \\ \vdots \\ V_m \end{bmatrix}, \quad (V)_j = V_j \quad \text{and} \quad A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{j1} & \cdots & A_{jm} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix}, \quad (A)_j = [A_{j1} \quad \cdots \quad A_{jm}],$$

and unless otherwise explicitly referred to, we will assume the order of a block matrix to be mn, the order of a block eigenvalue to be n, and the type of a block eigenvector to be $mn \times n$.

Furthermore, the following known results and definitions related to the Definition 1.1 will be used in this paper.

THEOREM 1.1. *If $AV = VX$, with V of full rank, then all the eigenvalues of X are eigenvalues of A [1].*

DEFINITION 1.2. *A set of block eigenvalues of a block matrix is a complete set if the set of all the eigenvalues of these block eigenvalues is the set of eigenvalues of the matrix [1].*

THEOREM 1.2. *Every matrix, of order mn, has a complete set of block eigenvalues of order n [1].*

DEFINITION 1.3. *In a complete set of block eigenvalues, one of them is weakly dominant if all its eigenvalues are greater than or equal (in moduli) to the eigenvalues of any other block eigenvalue in the complete set [1].*

THEOREM 1.3. *Every block matrix has a complete set of block eigenvalues with one of them weakly dominant [1].*

THEOREM 1.4. *Any matrix similar to a block eigenvalue of a matrix A is also a block eigenvalue of A [1].*

THEOREM 1.5. $\det(Ip - RS) = \det(Iq - SR)$, *where R and S are arbitrary $p \times q$ and $q \times p$ matrices [3].*

We summarize the context of this paper. In Section 2, we define ordered sets of block eigenvalues. In Section 3, we develop a block deflation process for an ordered set of block eigenvalues. We present the basic theory of matrix polynomials in Section 4. In Section 5, we apply the deflation to the block companion matrix of a matrix polynomial of commuting matrices, and in Section 6 we give a numerical example.

2. ORDERED SETS OF BLOCK EIGENVALUES

In the scalar eigenvalue problem, every matrix of order n has n eigenvalues, including multiplicities, and for any matrix, a typical deflation process to compute the n eigenvalues consists of

(1) find iteratively the dominant eigenvalue of the matrix of order n, and then get the matrix of order $n - 1$ which has only the remaining eigenvalues, and
(2) repeat Step 1 with the deflated matrix until the order 1.

We will generalize this process to compute a complete set of block eigenvalues for a block partitioned matrix under certain conditions, being aware that, unlike in the scalar case, a partitioned matrix into $m \times m$ blocks generally has more than m block eigenvalues and hence more than one complete set.

Let us suppose now that we have computed the mn scalar eigenvalues of a partitioned matrix A. We can construct a complete set of block eigenvalues by taking m matrices of order n in Jordan form where the diagonal elements are those scalar eigenvalues. Furthermore, if the scalar eigenvalues of A are distinct, these m matrices are diagonal matrices as is shown in the following construction:

$$\begin{aligned}
X_1 &= \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \\
X_2 &= \begin{bmatrix} \lambda_{n+1} & & \\ & \ddots & \\ & & \lambda_{2n} \end{bmatrix}, \\
&\vdots \\
X_m &= \begin{bmatrix} \lambda_{(m-1)n+1} & & \\ & \ddots & \\ & & \lambda_{mn} \end{bmatrix},
\end{aligned} \tag{1}$$

where the λ_i, $i = 1, \ldots, mn$, are the eigenvalues of A. The proof that the matrices X_j, $j = 1, \ldots, m$, are a complete set of block eigenvalues of A is in [1, p. 74].

We stress that our goal is to compute the block eigenvalues without knowing its scalar eigenvalues.

DEFINITION 2.1. *In a complete set of block eigenvalues, one of them is dominant if all its eigenvalues are greater (in moduli) than the eigenvalues of any other block eigenvalue in the complete set.*

DEFINITION 2.2. *If a matrix A of order mn has kn distinct scalar eigenvalues ordered as follows:*

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{kn}|$$

and $X_1, X_2, \ldots, X_k$ are block eigenvalues of A, where the eigenvalues of

$$\begin{aligned}
X_1 &\text{ are } \lambda_1, \ldots, \lambda_n, \\
X_2 &\text{ are } \lambda_{n+1}, \ldots, \lambda_{2n}, \\
&\vdots \\
\text{and } X_k &\text{ are } \lambda_{(k-1)n+1}, \ldots, \lambda_{kn},
\end{aligned}$$

then $X_1, X_2, \ldots, X_k$ are an ordered set of block eigenvalues of A with k elements. And if $k = m$, then $X_1, X_2, \ldots, X_m$ are a complete ordered set of block eigenvalues of A.

We are now in position for presenting our first proposition.

PROPOSITION 2.1. *Every block matrix with distinct scalar eigenvalues has a complete ordered set of block eigenvalues, the first of them being weakly dominant.*

PROOF. The result follows from construction (1) where the scalar eigenvalues λ_i are ordered according to Definition 2.2 and X_1 is, by Definition 1.3, the weakly dominant block eigenvalue. ∎

3. THE BLOCK DEFLATION PROCESS

Let us consider a matrix A of order mn partitioned into blocks of order n. Let us assume that A is nonsingular and that all of its mn eigenvalues are distinct. Denote by X_1 a weakly dominant block eigenvalue and by V_1 the corresponding block eigenvector.

Consider a matrix B defined as follows:

$$B = A - V_1 X_1 U, \tag{2}$$

where U is any block vector of type $n \times mn$ (i.e., a row of m blocks of order n) verifying the relation

$$UV_1 = I_n. \tag{3}$$

Multiplying both sides of (2) on the right by V_1 and taking (3) into account, we have

$$BV_1 = AV_1 - V_1 X_1 U V_1 = 0. \tag{4}$$

In relation (4), it is shown that V_1 is also a block eigenvector of B corresponding to the null block eigenvalue. We will show that the matrix B has the remaining block eigenvalues of the matrix A. This procedure is a generalized form of the Wielandt's deflation [3,4]. Householder [5] and Wilkinson [3, p. 599] have considered a similar deflation process, nevertheless not related to the block eigenvalue problem, as their work dealt with the deflation of several scalar eigenvalues at one step.

The deflation process is stated in the following proposition.

PROPOSITION 3.1. *If*

(1) *A is a matrix of order mn having nonzero distinct scalar eigenvalues,*
(2) *$X_1, \dots, X_m$ are a complete ordered set of block eigenvalues of A,*
(3) *V_1 is a block eigenvector corresponding to the weakly dominant block eigenvalue X_1,*
(4) *B is a matrix of order mn defined by*

$$B = A - V_1 X_1 U,$$

where U is any block vector of type $n \times mn$ such that

$$UV_1 = I_n,$$

then

(a) *0_n is a block eigenvalue of B corresponding to V_1,*
(b) *$X_2, \dots, X_m$ are block eigenvalues of B,*
(c) *B has an $m-1$ ordered set of block eigenvalues that form an ordered set of block eigenvalues of A, as well.*

PROOF OF PART (a).

$$\begin{aligned} BV_1 &= AV_1 - V_1 X_1 U V_1 \\ &= V_1 X_1 - V_1 X_1 = V_1 (X_1 - X_1) = V_1 0_n. \end{aligned}$$

∎

PROOF OF PART (b). Let us define the block vector Z_i of type $mn \times n$ by

$$Z_i = V_i - V_1 X_1 U V_i X_i^{-1}, \qquad i = 2, 3, \dots, m, \tag{5}$$

where V_i is a block eigenvector of A corresponding to the block eigenvalue X_i.

Now from (2) and (5), we have

$$BZ_i = AV_i - AV_1 X_1 U V_i X_i^{-1} - V_1 X_1 U V_i + V_1 X_1 U V_1 X_1 U V_i X_i^{-1},$$

and from (3) and $AV_1 = V_1 X_1$, we obtain

$$\begin{aligned} BZ_i &= AV_i - V_1 X_1 U V_i \\ &= V_i X_i - V_1 X_1 U V_i \\ &= \left(V_i - V_1 X_1 U V_i X_i^{-1}\right) X_i \\ &= Z_i X_i; \end{aligned}$$

hence, X_i is also a block eigenvalue of B corresponding to the block eigenvector Z_i. ∎

To prove Part (c), we need the following lemma.

LEMMA 3.1. *B has the scalar eigenvalues of A except those n ones which are eigenvalues of X_1 that are replaced by zeros.*

PROOF. Let λ_i, $i = n+1, \dots, nm$, be an eigenvalue of A that it is not an eigenvalue of X_1, and let $\lambda_\epsilon = \lambda_i + \epsilon$, for any $\epsilon > 0$, such that λ_ϵ is not an eigenvalue of A; then from

$$AV_1 = V_1 X_1,$$

it follows that

$$\lambda_\epsilon V_1 - AV_1 = \lambda_\epsilon V_1 - V_1 X_1,$$
$$(\lambda_\epsilon I_{mn} - A) V_1 = V_1 (\lambda_\epsilon I_n - X_1),$$

or

$$V_1 (\lambda_\epsilon I_n - X_1)^{-1} = (\lambda_\epsilon I_{mn} - A)^{-1} V_1. \tag{6}$$

Now we have

$$\begin{aligned}
\det(\lambda_\epsilon I_{mn} - B) &= \det(\lambda_\epsilon I_{mn} - A + V_1 X_1 U) \\
&= \det(\lambda_\epsilon I_{mn} - A) \det\left(I_{mn} + (\lambda_\epsilon I_{mn} - A)^{-1} V_1 X_1 U\right) \\
&= \det(\lambda_\epsilon I_{mn} - A) \det\left(I_{mn} + V_1 (\lambda\epsilon I_n - X_1)^{-1} X_1 U\right) && \text{from (6)} \\
&= \det(\lambda_\epsilon I_{mn} - A) \det\left(I_n + X_1 U V_1 (\lambda_\epsilon I_n - X_1)^{-1}\right) && \text{from Theorem 1.5} \\
&= \det(\lambda_\epsilon I_{mn} - A) \det\left(I_n + X_1 (\lambda_\epsilon I_n - X_1)^{-1}\right) && \text{from (3)} \\
&= \det(\lambda_\epsilon I_{mn} - A)\, \lambda_\epsilon^n \det(\lambda_\epsilon I_n - X_1)^{-1},
\end{aligned}$$

and thus by continuity of the characteristic polynomial of B and taking into account that, by hypothesis, $\det(\lambda_i I_{mn} - A) = 0$ and $\det(\lambda_i I_n - X_1) \neq 0$, we have

$$\det(\lambda_i I_{mn} - B) = 0, \qquad \text{as } \epsilon \to 0;$$

hence, λ_i is an eigenvalue of B.

Furthermore, if the eigenvalues of the block eigenvalue 0_n are eigenvalues of B, then B has n vanishing eigenvalues and the proof is complete. ∎

PROOF OF PART (c). From Lemma 3.1, it follows that B has $mn - n$ nonzero distinct scalar eigenvalues from A, and therefore, we can construct an $m-1$ ordered set of block eigenvalues of B

$$Y_1, Y_2, \dots, Y_{m-1},$$

such that the eigenvalues of

$$\begin{aligned}
&Y_1 \text{ are those of } X_2, \\
&Y_2 \text{ are those of } X_3, \\
&\quad\vdots \\
&Y_{m-1} \text{ are those of } X_m;
\end{aligned}$$

hence, the n scalar eigenvalues of Y_i are the same of X_{i+1}, $i = 1, \dots, m-1$, and as these eigenvalues are distinct, Y_i and X_{i+1} have the same Jordan form and then they are similar and, by Theorem 1.4, Y_i is also a block eigenvalue of A. ∎

COROLLARY 3.1. *If*

(1) V_1 *is normalized by putting the* j^{th} *block* $(V_1)_j = I_n$,
(2) $X_1 U = (A)_j$,
(3) *we construct the matrix* A^1 *of order* $mn - n$ *from* B *by taking off both the* j^{th} *block row and the* j^{th} *block column,*

then $Y_1, Y_2, \dots, Y_{m-1}$ *are a complete ordered set of block eigenvalues of* A^1.

PROOF. For a fixed j, the relation

$$AV_1 = V_1 X_1$$

implies

$$(A)_j V_1 = (V_1)_j X_1.$$

From condition (2), we have

$$UV_1 = X_1^{-1}(A)_j V_1 = X_1^{-1}(V_1)_j X_1 = I_n,$$

and hence, we can write

$$B = A - V_1 (A)_j,$$

where

$$(B)_j = (A)_j - (V_1)_j (A)_j = 0, \tag{7}$$

so that the j^{th} block row of B is null.

Consequently, by relation (5), we have for Z_i, the block eigenvectors of B corresponding to the block eigenvalues X_i

$$\begin{aligned} Z_i &= V_i - V_1 (A)_j V_i X_i^{-1}, \\ Z_i &= V_i - V_1 (V_i)_j X_i X_i^{-1}, \\ Z_i &= V_i - V_1 (V_i)_j, \end{aligned}$$

and using condition (1) we get

$$(Z_i)_j = 0, \qquad \text{for } i = 2, \dots, m. \tag{8}$$

On the other hand, when proving Proposition 3.1, it has been shown that Y_i is similar to X_{i+1}, $i = 1, \dots, m-1$, and hence, we can write

$$P_i Y_i = X_{i+1} P_i,$$

where P_i is a nonsingular matrix. From

$$BZ_i = Z_i X_i,$$

it follows that

$$BZ_{i+1} = Z_{i+1} P_i Y_i P_i^{-1}, \tag{9}$$

$$BZ_{i+1} P_i = Z_{i+1} P_i Y_i, \tag{10}$$

and by putting

$$W_i = Z_{i+1} P, \qquad i = 1, \dots, m-1, \tag{11}$$

we obtain from (10)

$$BW_i = W_i Y_i, \qquad i = 1, \dots, m-1,$$

which leads us to conclude that W_i is a block eigenvector of B corresponding to the block eigenvalue Y_i; from (8) and (11), we get

$$(W_i)_j = 0, \qquad i = 1, \dots, m-1, \tag{12}$$

and hence, the j^{th} block of W_i is null.

Now for the sake of simplicity, let us take $j = 1$. From (7) and (12), it follows that from the relation

$$BW_i = W_i Y_i, \qquad i = 1, \dots, m-1,$$

we can write

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \vdots & \vdots & & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix} \begin{bmatrix} 0 \\ W_2 \\ \vdots \\ W_m \end{bmatrix}_i = \begin{bmatrix} 0 \\ W_2 \\ \vdots \\ W_m \end{bmatrix}_i Y_i, \qquad i = 1, \dots, m-1; \tag{13}$$

then we construct the matrix A^1 according to condition (3), as follows:

$$A^1 = \begin{bmatrix} B_{22} & \cdots & B_{2m} \\ \vdots & & \vdots \\ B_{m2} & \cdots & B_{mm} \end{bmatrix},$$

and from (13), it is easy to verify that

$$\begin{bmatrix} B_{22} & \cdots & B_{2m} \\ \vdots & & \vdots \\ B_{m2} & \cdots & B_{mm} \end{bmatrix} \begin{bmatrix} W_2 \\ \vdots \\ W_m \end{bmatrix}_i = \begin{bmatrix} W_2 \\ \vdots \\ W_m \end{bmatrix}_i Y_i, \qquad i = 1, \dots, m-1.$$ ∎

COROLLARY 3.2. *Under the hypothesis of Corollary 3.1, the matrix A^1 has nonzero distinct eigenvalues.*

PROOF. The eigenvalues of A^1 are the eigenvalues of Y_i, $i = 1, \dots, m-1$ (by Theorem 1.1), and they are distinct and nonzero by Proposition 3.1. ∎

Now we can continue the process. So, Corollaries 3.1 and 3.2 show that the matrix A^1 of order $mn - n$ obeys the same conditions as the original matrix A for the application of the subsequent deflation.

4. MATRIX POLYNOMIALS

In this section, we give a brief summary of the basic theory of matrix polynomials as developed in [1,6].

DEFINITION 4.1. *Given n by n matrices $A_1, \dots, A_m$, a matrix polynomial $M(X)$ is the matrix expression*

$$M(X) = X^m + A_1 X^{m-1} + \cdots + A_m,$$

where the variable X is also of order $n \times n$.

We consider the powers of X at the right of the coefficients. We also remark that if we consider $X = \lambda I_n$, $\lambda \in \mathbb{C}$, then the matrix polynomial becomes a lambda-matrix $M(\lambda)$, and we indicate that most of the available theory is concerned with matrix polynomials considered as lambda-matrices (see [7]).

DEFINITION 4.2. *A matrix S is a (right) solvent of the matrix polynomial $M(X)$ if $M(S) = 0$.*

DEFINITION 4.3. *A solvent is dominant if all its eigenvalues are greater than (in moduli) to the eigenvalues of any other solvent.*

DEFINITION 4.4. *Given a matrix polynomial*

$$M(X) = X^m + A_1 X^{m-1} + \cdots + A_m,$$

its (bottom) block companion matrix is

$$C = \begin{bmatrix} 0 & I & & \cdots & 0 \\ 0 & 0 & I & & \\ \vdots & & & \ddots & \\ 0 & & & & I \\ -A_m & -A_{m-1} & \cdots & -A_2 & -A_1 \end{bmatrix}.$$

The relation between solvents and block eigenvalues of C are stated in the next two theorems.

THEOREM 4.1. *If S is a solvent of $M(X)$, then S is a block eigenvalue of C associated to the normalized block eigenvector*

$$V = \begin{bmatrix} I \\ S \\ S^2 \\ \vdots \\ S^{m-1} \end{bmatrix}.$$

THEOREM 4.2. *If $CV = VX$ with V of full rank and $(V)_1$ is nonsingular, then $S = (V)_1 X (V)_1^{-1}$ is a solvent of $M(X)$.*

DEFINITION 4.5. *A set of m solvents of matrix polynomial $M(X)$ is a complete set if the set of mn eigenvalues of the m solvents is the same, counting multiplicities, as the set of mn eigenvalues of the block companion matrix C.*

This is a simplification of the original definition [1] in terms of $M(\lambda)$, and an equivalent definition (by Definition 1.2) is: a set of solvents is a complete set if they are a complete set of block eigenvalues of C. We note that $(V)_1$ is the first block of the block eigenvector V, and if a block matrix has a dominant block eigenvalue, this one will be also a dominant solvent (see [1]).

Thus, it means that the existence of a dominant block eigenvalue of C is equivalent to the existence of a dominant solvent of $M(X)$. Now we see the importance of our construction of an ordered set of block eigenvalues, for the application of the deflation process in order to obtain a complete set of solvents of $M(X)$.

5. APPLICATION TO MATRIX POLYNOMIALS OF COMMUTING MATRICES

Let us suppose that $M(X)$ is a matrix polynomial where the matrix coefficients $A_1, \ldots, A_m$ pairwisely commute and the solvents $S_1, \ldots, S_m$, are ordered according Definition 2.2, and let the block companion matrix of $M(X)$ be

$$C = \begin{bmatrix} 0 & I & & \cdots & 0 \\ 0 & 0 & I & & \\ \vdots & & & \ddots & \\ 0 & & & & I \\ -A_m & -A_{m-1} & \cdots & -A_2 & -A_1 \end{bmatrix},$$

where

$$
\begin{aligned}
A_1 &= -(S_1 + \cdots + S_m),\\
A_2 &= S_1S_2 + S_1S_3 + \cdots + S_1S_m + S_2S_3 + \cdots + S_2S_m \cdots + S_{m-1}S_m,\\
&\;\;\vdots\\
A_{m-1} &= (-1)^{m-1}\left(S_2S_3 \ldots S_m + S_1S_3 \ldots S_m + S_1S_2 \ldots S_{m-1}\right),\\
A_m &= (-1)^m\left(S_1S_2 \ldots S_m\right).
\end{aligned}
$$

Notice that if we do not work with commuting matrices, we cannot write the A_i in the classical Girard-Newton-Viète formulae.

We have from Theorem 4.1 that S_1 (the dominant solvent) is a block eigenvalue and V_1 is a normalized block eigenvector in the form

$$
V_1 = \begin{bmatrix} (V_1)_1 \\ (V_1)_2 \\ (V_1)_3 \\ \vdots \\ (V_1)_m \end{bmatrix} = \begin{bmatrix} I \\ S_1 \\ S_1^2 \\ \vdots \\ S_1^{m-1} \end{bmatrix}.
$$

Now we apply the deflation process to the block companion matrix with $S_1U = (C)_1$ ($j = 1$ in condition (2) of Corollary 3.1). The suitability of this choice lies in the fact that the deflated matrix C^1 (see condition (3) of Corollary 3.1) has a special L form, which we call L-shaped matrix

$$
L^1 = C^1 = \begin{bmatrix} -(V_1)_2 & I & & \cdots & 0 \\ -(V_1)_3 & 0 & I & & \\ \vdots & & & \ddots & \\ -(V_1)_{m-1} & & & & I \\ -A_{m-1} - (V_1)_m & -A_{m-2} & \cdots & -A_2 & -A_1 \end{bmatrix}.
$$

For S_2, the second solvent (and dominant of L^1), the associated block eigenvector is

$$
V_2 = \begin{bmatrix} (V_2)_1 \\ (V_2)_2 \\ (V_2)_3 \\ \vdots \\ (V_2)_{m-1} \end{bmatrix} = \begin{bmatrix} (V_1)_1 \\ (V_1)_2 \\ (V_1)_3 \\ \vdots \\ (V_1)_{m-1} \end{bmatrix} + S_2 \begin{bmatrix} 0 \\ (V_2)_1 \\ (V_2)_2 \\ \vdots \\ (V_2)_{m-2} \end{bmatrix}.
$$

Now we choose $S_2U = (L^1)_1$, and we get

$$
L^2 = C^2 = \begin{bmatrix} -(V_2)_2 & I & & \cdots & 0 \\ -(V_2)_3 & 0 & I & & \\ \vdots & & & \ddots & \\ -(V_2)_{m-2} & & & & I \\ -A_{m-2} - (V_2)_{m-1} & -A_{m-3} & \cdots & -A_2 & -A_1 \end{bmatrix}.
$$

We see that the L form is preserved, and hence, in the end

$$
V_{m-1} = \begin{bmatrix} (V_{m-1})_1 \\ (V_{m-1})_2 \end{bmatrix} = \begin{bmatrix} (V_{m-2})_1 \\ (V_{m-2})_2 \end{bmatrix} + S_{m-1} \begin{bmatrix} 0 \\ (V_{m-1})_1 \end{bmatrix}
$$

is associated with the $(m-1)^{\text{th}}$ solvent S_{m-1}.

The last deflated matrix is

$$L^{m-1} = C^{m-1} = [-A_1 - (V_{m-1})_2].$$

We have

$$\begin{aligned}(V_1)_2 &= S_1,\\ (V_2)_2 &= S_1 + S_2,\\ &\vdots\\ (V_{m-1})_2 &= S_1 + S_2 + \cdots + S_{m-1},\end{aligned}$$

and

$$A_1 = -(S_1 + \cdots + S_m);$$

hence,

$$L^{m-1} = S_m.$$

The generic form of the block eigenvectors is

$$V_1 = \begin{bmatrix} (V_1)_1 \\ (V_1)_2 \\ (V_1)_3 \\ \vdots \\ (V_1)_m \end{bmatrix} = \begin{bmatrix} I \\ S_1 \\ S_1^2 \\ \vdots \\ S_1^{m-1} \end{bmatrix},$$

where S_1 is the dominant block eigenvalue, hence the dominant solvent, and

$$V_j = \begin{bmatrix} (V_j)_1 \\ (V_j)_2 \\ (V_j)_3 \\ \vdots \\ (V_j)_{(m-j)+1} \end{bmatrix} = \begin{bmatrix} (V_{j-1})_1 \\ (V_{j-1})_2 \\ (V_{j-1})_3 \\ \vdots \\ (V_{j-1})_{(m-j)+1} \end{bmatrix} + S_j \begin{bmatrix} 0 \\ (V_j)_1 \\ (V_j)_2 \\ \vdots \\ (V_j)_{m-j} \end{bmatrix}, \qquad j = 2, \dots, m-1,$$

where S_j are the respective block eigenvalues and the remaining solvents.

The generic form for the L-shaped matrices is

$$L^j = \begin{bmatrix} -(V_j)_2 & I & & \cdots & 0 \\ -(V_j)_3 & 0 & I & & \\ \vdots & & & \ddots & \\ -(V_j)_{m-j} & & & & I \\ -A_{m-j} - (V_j)_{(m-j)+1} & -A_{(m-j)-1} & \cdots & -A_2 & -A_1 \end{bmatrix}, \qquad j = 1, \dots, m-1.$$

6. NUMERICAL EXAMPLE

In the next example, we use a generalization of the power method [1] to compute a normalized block eigenvector associated to a dominant block eigenvalue.

Let

$$M(X) = X^5 + \begin{bmatrix} -20 & 10 \\ -5 & -35 \end{bmatrix} X^4 + \begin{bmatrix} 120 & -220 \\ 110 & 450 \end{bmatrix} X^3 + \begin{bmatrix} -100 & 1700 \\ -850 & -2650 \end{bmatrix} X^2$$
$$+ \begin{bmatrix} -1006 & -5390 \\ 2695 & 7079 \end{bmatrix} X + \begin{bmatrix} 1950 & 5790 \\ -2895 & -6735 \end{bmatrix}$$

be a fifth degree matrix polynomial of commuting matrices of order 2 ($m = 5$ and $n = 2$). We have $CV_1 = V_1X_1$ where C is the block companion matrix (of order 10).

A dominant block eigenvalue (and a dominant solvent) is

$$X_1 = S_1 = \begin{bmatrix} 8 & -2 \\ 1 & 11 \end{bmatrix},$$

and the corresponding normalized block eigenvector is

$$V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 8 & -2 \\ 1 & 11 \\ 62 & -38 \\ 19 & 119 \\ 458 & -542 \\ 271 & 1271 \\ 3122 & -6878 \\ 3439 & 13439 \end{bmatrix}.$$

Taking $X_1U = (C)_1$, we have

$$B = C - V_1(C)_1$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -11 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -62 & 38 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -19 & -119 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -458 & 542 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -271 & -1271 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1950 & -5790 & -2116 & 12268 & 100 & -1700 & -120 & 220 & 20 & -10 \\ 2895 & 6735 & -6134 & -20518 & 850 & 2650 & -110 & -450 & 5 & 35 \end{bmatrix}.$$

Now we omit the first block row and the first block column of B and we obtain

$$L^1 = \begin{bmatrix} -8 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -11 & 0 & 1 & 0 & 0 & 0 & 0 \\ -62 & 38 & 0 & 0 & 1 & 0 & 0 & 0 \\ -19 & -119 & 0 & 0 & 0 & 1 & 0 & 0 \\ -458 & 542 & 0 & 0 & 0 & 0 & 1 & 0 \\ -271 & -1271 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2116 & 12268 & 100 & -1700 & -120 & 220 & 20 & -10 \\ -6134 & -20518 & 850 & 2650 & -110 & -450 & 5 & 35 \end{bmatrix}.$$

Continuing, we get

$$X_2 = S_2 = \begin{bmatrix} 6 & -2 \\ 1 & 9 \end{bmatrix},$$

the corresponding normalized block eigenvector being

$$V_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 14 & -4 \\ 2 & 20 \\ 142 & -102 \\ 51 & 295 \\ 1208 & -1744 \\ 872 & 3824 \end{bmatrix},$$

and

$$L^2 = \begin{bmatrix} -14 & 4 & 1 & 0 & 0 & 0 \\ -2 & -20 & 0 & 1 & 0 & 0 \\ -142 & 102 & 0 & 0 & 1 & 0 \\ -51 & -295 & 0 & 0 & 0 & 1 \\ -1108 & 44 & -120 & 220 & 20 & -10 \\ -22 & -1174 & -110 & -450 & 5 & 35 \end{bmatrix}.$$

Continuing the process, we get from the subsequent deflations

$$S_3 = \begin{bmatrix} 4 & -2 \\ 1 & 7 \end{bmatrix},$$
$$S_4 = \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix},$$
$$S_5 = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix};$$

thus, S_1, S_2, S_3, S_4, and S_5 are a complete set of solvents of $M(X)$.

7. REMARKS AND CONCLUSIONS

The main purpose of this paper is to present a block deflation procedure for computing a complete set of block eigenvalues of a matrix partitioned into blocks, in order to effectively compute the solvents of a matrix polynomial. We have generalized a scalar version procedure as exposed in [3,4]. We have adapted the proof of Lemma 3.1 from [3] and [5], by presenting a formal way to invert lambda-matrices (see [8]).

The Wielandt deflation may be considered unused by some authors, but we see in the recent work of Saad [9] an important application to sparse matrices, and using the first block row of the matrices for the deflation led us to the special L-shaped matrices, which preserves the sparse characteristic of the companion matrix. The commuting matrices of the numerical example were constructed based in the work of Jaffar [10].

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