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Computing the Zeros of Quaternion Polynomials

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Abstract—A method is developed to compute the zeros of a quaternion polynomial with all terms of the form $q_k X^k$. This method is based essentially in Niven's algorithm [1], which consists of dividing the polynomial by a characteristic polynomial associated to a zero. The information about the trace and the norm of the zero is obtained by an original idea which requires the companion matrix associated to the polynomial. The companion matrix is represented by a matrix with complex entries. Three numerical examples using Mathematica 2.2 version are given.

Keywords—Right (and left) quaternionic eigenvalues, Quaternionic polynomials, Companion quaternionic matrix.

1. INTRODUCTION

Concerning the unilateral quaternion polynomials, the problem of counting the number of zeros is well studied by several authors [1–5]. The only reference which gives an explicit way to compute the zeros is Niven's [1]. He arrives at a very simple and neat formula which only depends on two parameters (besides the coefficients of the quaternion polynomial). But Niven's method to obtain these parameters turned, as he said, the algorithm into an unpractical one. Our aim is to show an alternative way to obtain these parameters to apply Niven's formula and make this algorithm practical.

We begin in this section with the definitions of quaternions and quaternion polynomials. In Section 2, we present Niven's algorithm. In Section 3, the definition of similarity is given and some known results concerning quaternion matrices are presented. In Section 4, we define the quaternion companion matrix and present new results which lead, in Section 5, to the proposed

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method. Some results about zeros of quaternion polynomials are also presented. Three numerical examples are given in Section 6.

Throughout this paper, $\mathbb{R}$ and $\mathbb{C}$ will denote the fields of the real and complex numbers, respectively.

DEFINITION 1. *(See [6].) A quaternion is an element of the division ring over the reals denoted by* $\mathbb{H}$*, and generated by the elements* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *satisfying*

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

and

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \qquad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

A real quaternion, simply called quaternion, can be written in a unique form

$$q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

where $a_i \in \mathbb{R}$ $(i = 0, \dots, 3)$. Every quaternion q has a conjugate

$$\overline{q} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}.$$

We now define the quaternion polynomials.

DEFINITION 2. *Given the quaternions* $q_0, \dots, q_{m-1}$*, a monic*[1] *quaternion polynomial, or simply a quaternion polynomial, of degree* m *is the expression*

$$p(x) = x^m + q_{m-1}x^{m-1} + \cdots + q_1x + q_0$$

in the quaternion indeterminate x.

DEFINITION 3. *Given a quaternion polynomial* $p(x)$*, we say that* λ *is a zero if* $p(\lambda) = 0$.

The fundamental theorem of algebra over the algebra of real quaternions was first considered by Eilenberg and Niven [1,7]. They proved that every quaternion polynomial has at least one zero in $\mathbb{H}$. Later on, Beck [2] proved that if the number of zeros of a polynomial of degree m is finite, then there are exactly m zeros counting possible repetition.

2. NIVEN'S ALGORITHM

This section summarizes the more important steps in Niven's work. We start with two fundamental definitions [1].

DEFINITION 4. *The trace,* t*, of a quaternion is the sum of the quaternion and its conjugate.*

DEFINITION 5. *The norm,* n*, of a quaternion is the product of the quaternion by its conjugate.*

Every quaternion q satisfies a second degree polynomial equation with real coefficients

$$x^2 - tx + n = 0, \tag{1}$$

where t and n are the trace and the norm, respectively, of q. This is the characteristic equation associated to a quaternion [2].

If we divide a quaternion polynomial

$$p(x) = x^m + q_{m-1}x^{m-1} + \cdots + q_1 + q_0$$

[1] If the polynomial is not monic, we can always factorize. For example, if $q(x) = a_mx^m + \cdots + a_0$ with $a_m \neq 0$, then $q(x) = a_mp(x)$ and $p(x)$ is monic.

from the right by the characteristic polynomial $x^2 - tx + n$, we obtain[2]

$$p(x) = Q(x)\left(x^2 - tx + n\right) + f(q_i, t, n)x + g(q_i, t, n), \tag{2}$$

where f and g are polynomials in t, n, and q_i. The nature of $Q(x)$ does not interest us, because if q is a zero of $p(x)$, then substituting in (2)

$$0 = f(q_i, t, n)\, q + g(q_i, t, n),$$

where q, t, and n satisfy (1).

If $f \neq 0$, then, paying attention to the noncommutativity, we get, in the final step of Niven's algorithm, the formula

$$q = -\frac{1}{f} \cdot g. \tag{3}$$

One can write the last equation as

$$q = -\frac{1}{f\overline{f}}\, \overline{f} g. \tag{4}$$

Conjugating q, we have

$$\overline{q} = -\frac{1}{f\overline{f}}\, \overline{g} f, \tag{5}$$

since the conjugate of the product is the product of the conjugates in the reversed order. By multiplication and addition of (4) and (5), Niven has got an expression for the trace and the norm of q,

$$n = \frac{1}{\overline{f}f} \cdot \overline{g}g \qquad \text{and} \qquad t = -\frac{1}{\overline{f}f}\left(\overline{f}g + \overline{g}f\right).$$

These equations may be written in the form

$$N(t, n) = n\overline{f}f - \overline{g}g = 0 \qquad \text{and} \qquad T(t, n) = t\overline{f}f + \overline{f}g + \overline{g}f = 0. \tag{6}$$

Niven proved that every real solution of these two equations gives the trace and the norm of the solutions of the polynomial equation $p(x) = 0$, and vice-versa. A problem arising in Niven's method is that computing the trace and the norm using (6) is not very practical, since it involves the simultaneous solving of two real equations of degree $2m - 1$. We use a new approach to overcome this problem. For that, we need some known results about quaternion matrices.

3. EQUIVALENCE CLASSES OF QUATERNIONS

DEFINITION 6. *(See [6].) Two quaternion x and y are said to be similar if there exists a nonzero quaternion σ such that $\sigma^{-1}x\sigma = y$. This is written as $x \sim y$.*

The relation $\sim$ is an equivalence relation on the quaternions. We denote by $[x]$ the equivalence class containing x or simply the class containing x.

The next result is very important as it relates any quaternion to an equivalence class generated by a special quaternion isomorphic to a complex number with nonnegative imaginary part.

THEOREM 1. *(See [6].) If $q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, with $a_i \in \mathbb{R}$ $(i = 0, \dots, 3)$, then q and $a_0 + \mathbf{i}\sqrt{a_1^2 + a_2^2 + a_3^2}$ are similar, namely, $q \in [a_0 + \mathbf{i}\sqrt{a_1^2 + a_2^2 + a_3^2}]$.*

The notion of eigenvalue is slightly different in $\mathbb{H}$ since the scalar multiplication is not commutative.

[2] Niven called this relation a right-division algorithm. It can easily be proved that it is valid for every quaternion polynomial (Definition 2).

DEFINITION 7. *(See [6].) A quaternion λ is said to be a left (right) eigenvalue of a matrix A provided that $Av = \lambda v$ with $v \neq 0$ ($Av = v\lambda$). To each λ, the associated v is called the left (right) eigenvector.*

Since $Av = v\lambda$ implies $A(vq) = v\lambda q = (vq)q^{-1}\lambda q$ for any nonzero q, it follows that if λ is a right eigenvalue, then so is $q^{-1}\lambda q$. Thus, if λ is a right eigenvalue, then all elements in $[\lambda]$ are also right eigenvalues. This property is not verified, in general, for the left eigenvalues.

The following result states the number of right eigenvalues of a square matrix with quaternion entries.

THEOREM 2. *(See [8].) Any $m \times m$ matrix with quaternion entries has exactly m right eigenvalues which are complex numbers with nonnegative imaginary part.*

Note that to each complex right eigenvalue corresponds an equivalence class of quaternions which are also right eigenvalues. So, if we know the m complex right eigenvalues, then we know all the right eigenvalues. A way to compute these complex right eigenvalues is given in [9].

THEOREM 3. *(See [9].) If we write $A = A_1 + A_2\mathbf{j}$, where A_1, A_2 are $m \times m$ matrices with complex entries, the right eigenvalues of A are the eigenvalues (in the classical sense) of the $2m \times 2m$ matrix with complex entries*

$$\Phi(A) = \begin{bmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{bmatrix}.$$

We are now ready to introduce our results.

4. THE QUATERNION COMPANION MATRIX

The quaternion companion matrix gives us the needed information about the trace and norm of the zeros, in order to apply Niven's algorithm (3) in a very practical way, as presented in Section 5.

DEFINITION 8. *Given the quaternion polynomial $p(x) = x^m + q_{m-1}x^{m-1} + \cdots + q_0$, the matrix*

$$C_p = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0 & -q_1 & \cdots & -q_{m-1} \end{bmatrix}$$

is called the (bottom) companion matrix associated with the quaternion polynomial $p(x)$.

In the next result, we show that a basic property of the companion matrix [10] is preserved within the quaternions.

PROPOSITION 1. *If λ is a left eigenvalue of the companion matrix associated with the quaternion polynomial $p(x)$, then*

(i) *λ is a zero of $p(x)$;*
(ii) *$v = [1 \ \ \lambda \ \ \lambda^2 \ \ \cdots \ \ \lambda^{m-1}]^\top$ is an associated left eigenvector.*

PROOF.

(i) Let us consider the companion matrix C_p of the polynomial $p(x) = x^m + q_{m-1}x^{m-1} + \cdots + q_0$ and suppose λ is a left eigenvalue of C_p. Then we can write $C_pv = \lambda v$, which yields

$$\begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0 & -q_1 & \cdots & -q_{m-1} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_{m-1} \\ v_m \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ \vdots \\ v_{m-1} \\ v_m \end{bmatrix}. \tag{7}$$

Multiplying both sides, we obtain the system of linear equations

$$\begin{aligned} v_2 &= \lambda v_1, \\ v_3 &= \lambda v_2, \\ &\vdots \\ v_m &= \lambda v_{m-1}, \\ -q_0 v_1 - q_1 v_2 - \cdots - q_{m-1} v_m &= \lambda v_m. \end{aligned}$$

Consecutive substitutions of the $(m-1)^{\text{st}}$ equations yield

$$v_i = \lambda^{i-1} v_1, \qquad i = 2, \ldots, m. \tag{8}$$

Substituting this result in the last system's equation, we get

$$-q_0 v_1 - q_1 \lambda v_1 - q_2 \lambda^2 v_1 - \cdots - q_{m-1} \lambda^{m-1} v_1 = \lambda^m v_1, \tag{9}$$

and, after factorization,

$$\left(\lambda^m + q_{m-1}\lambda^{m-1} + \cdots + q_2\lambda^2 + q_1\lambda + q_0\right) v_1 = 0.$$

Since $v_1 \neq 0$, as v cannot be the null vector, we conclude that

$$\lambda^m + q_{m-1}\lambda^{m-1} + \cdots + q_2\lambda^2 + q_1\lambda + q_0 = 0,$$

and the left eigenvalue λ is a zero of the polynomial $p(x)$.

(ii) Let λ be a left eigenvalue of the companion matrix associated with the polynomial $p(x) = x^m + x^{m-1} + \cdots + q_0$. Then we choose $v_1 = 1$, and by (8) we obtain

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{m-1} \end{bmatrix},$$

and we conclude that $\begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{m-1} \end{bmatrix}^\top$ is a left eigenvector associated to the eigenvalue λ. ∎

COROLLARY 1. *If λ is a left eigenvalue of a companion matrix, then it is also a right eigenvalue.*

PROOF. Let λ be a left eigenvalue of the companion matrix associated with the polynomial $p(x) = x^m + q_{m-1}x^{m-1} + \cdots + q_0$. Then,

$$C_p v = \lambda v.$$

By part (ii) of Proposition 1, we have

$$\begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0 & -q_1 & \cdots & -q_{m-1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix},$$

and since λ commutes with itself

$$\begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0 & -q_1 & \cdots & -q_{m-1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix} \lambda.$$

So, we conclude that $C_p v = v\lambda$ and λ is also a right eigenvalue. ∎

With this theory, we can obtain the following result, which is a corollary of the fundamental theorem of algebra in $\mathbb{H}$ for polynomials as considered in Definition 3.

PROPOSITION 2. *If $p(x)$ is a quaternion polynomial of degree m, then the set of zeros belongs to at most m equivalence classes of quaternions.*

PROOF. Let us consider a polynomial of degree m, $p(x) = x^m + q_{m-1}x^{m-1} + \cdots + q_0$, $q_i \in \mathbb{H}$. The companion matrix associated to this polynomial is of order $m \times m$, and by Theorem 2, has exactly m right complex eigenvalues. Suppose λ is such an eigenvalue. Then, we have the relation $C_p v = v\lambda$, and following the proof of Proposition 1, we obtain a similar relation to (9) with the λ at the right side. Multiplying this relation on the right side by v_1^{-1}, we obtain

$$-q_0 - q_1 v_1 \lambda v_1^{-1} - \cdots - q_{m-1} v_1 \lambda^{m-1} v_1^{-1} = v_1 \lambda^m v_1^{-1}.$$

Putting $v_1 \lambda v_1^{-1} = \sigma$, we obtain

$$\sigma^m + q_{m-1}\sigma^{m-1} + \cdots + q_1\sigma + q_0 = 0.$$

Thus, for each right complex eigenvalue, there exists at least one similar quaternion which is a left eigenvalue. By Proposition 1, it is also a zero of $p(x)$. Since there exist only m right complex eigenvalues, we conclude that the zeros must belong to no more than m different equivalent classes of quaternions. ∎

5. THE METHOD

The method consists in applying Niven's formula

$$x = -\frac{1}{f(q_i, t, n)} \cdot g(q_i, t, n), \tag{10}$$

with the trace t and the norm n obtained through the eigenvalues of the companion matrix.

In Proposition 2, we have shown that the zeros of a polynomial of degree m belong at most to m different classes of equivalence and that these classes are generated by m nonnegative complex eigenvalues of the corresponding companion matrix. Hence, by Theorem 1, we can obtain from these m eigenvalues, the traces and the norms of the zeros.

In Niven's algorithm, after the process of division, it might happen that f vanishes for a particular pair (t, n). In this case, g vanishes too (cf. [1]). We cannot apply formula (10). For this, we have the following result.

PROPOSITION 3. *If $f(q_i, t, n)$ and $g(q_i, t, n)$ vanish for a particular pair (t, n), then all quaternions belonging to the class $[t/2 + \mathbf{i}\sqrt{n - (t/2)^2}]$ are zeros of the polynomial $p(x)$.*

PROOF. Suppose $f(q_i, t, n)$ and $g(q_i, t, n)$ vanish for a particular pair (t, n). Then

$$p(x) = q(x)\left(x^2 - tx + n\right).$$

Let x_1 be the zero corresponding to such pair. Then

$$p(x_1) = q(x_1)\left(x_1^2 - tx_1 + n\right) = 0.$$

Consider a quaternion $x_2 = \sigma x_1 \sigma^{-1}$ similar to x_1. Then

$$\begin{aligned} p(x_2) &= q(x_2)\left(x_2^2 - tx_2 + n\right) \\ &= q(x_2)\left[\left(\sigma x_1 \sigma^{-1}\right)^2 - t\left(\sigma x_1 \sigma^{-1}\right) + n\right] \end{aligned}$$

since t and n are real, and $\sigma\sigma^{-1} = 1$,

$$\begin{aligned} p(x_2) &= q(x_2)\sigma\left(x_1^2 - tx_1 + n\right)\sigma^{-1} \\ &= q(x_2)\sigma.0.\sigma^{-1} \\ &= 0; \end{aligned}$$

hence, x_2 is also a zero. As we considered an arbitrary σ, we conclude that all quaternions similar to x_1 are roots of the polynomial equation $p(x) = 0$.

If $x_1 = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then $t = 2a_0$ and $n = a_0^2 + a_1^2 + a_2^2 + a_3^2$. Or, writing in another way, $a_0 = t/2$ and $a_1^2 + a_2^2 + a_3^2 = n - (t/2)^2$. By Theorem 1, we conclude that $x_1 \in [t/2 + \mathbf{i}\sqrt{n - (t/2)^2}]$ and all quaternions belonging to this class are zeros of the polynomial. ∎

In the following, a known result [3,4] is proved directly.

COROLLARY 2. *If a polynomial has two nonequal zeros belonging to the same class of equivalence, then all the quaternions of this class are also zeros of the polynomial.*

PROOF. Suppose that a polynomial has two nonequal zeros belonging to the same class of equivalence. Then, by Theorem 1, they have the same trace and norm. Now, substituting these values in Niven's formula, consider that f and g do not vanish. Then we would obtain a unique solution, which contradicts the hypothesis that there exist two solutions.

Hence, f and g must vanish and, by Proposition 3, all the quaternions of this class are also zeros of the polynomial. ∎

REMARK. We can develop an analogous theory for quaternion polynomials with coefficient on the right of the indeterminate

$$\widetilde{p}(x) = x^m + x^{m-1}q_{m-1} + \cdots + xq_1 + q_0. \tag{11}$$

Dividing the polynomial (11) on the left by the characteristic polynomial $x^2 - tx + n$, we obtain the formula

$$x = -g(q_i, t, n) \cdot \frac{1}{f(q_i, t, n)}.$$

In this case, the trace and the norm are obtained in a similar way with the (right) companion matrix

$$C_{\widetilde{p}} = \begin{bmatrix} 0 & \cdots & 0 & -q_0 \\ 1 & & 0 & -q_1 \\ & \ddots & & \vdots \\ 0 & & 1 & -q_{m-1} \end{bmatrix}$$

of the polynomial $\widetilde{p}$, since the left eigenvalues of $C_{\widetilde{p}}$ and the zeros of $\widetilde{p}$ are the same.

6. NUMERICAL EXAMPLES

We consider three polynomials: the first having finite many zeros and the other two with infinite many zeros, one with quaternion coefficients and the other with real coefficients.

EXAMPLE 1. Let us consider the polynomial

$$p(x) = x^{10} + (1 + 2\mathbf{i} - 4\mathbf{j})x^9 - (3.1\mathbf{i} + \mathbf{k})x^8 + (2.5\mathbf{j} + 2.1\mathbf{k})x^7 + (3 - \mathbf{i})x^6 - 1.7x^5 - (\mathbf{i} + \mathbf{j})x^4 - 7.2x^3 - \mathbf{j}x + 2.9(\mathbf{j} - \mathbf{k}) - 4.$$

The eigenvalues of the (bottom) companion matrix are presented in Table 1.

Table 1.

i	λ_i	t	n
1	$-1.26112 \pm 4.56864\mathbf{i}$	-2.52223	22.46292
2	$0.93019 \pm 0.73780\mathbf{i}$	1.86037	1.40960
3	$-1.07301 \pm 0.49536\mathbf{i}$	-2.14602	1.39674
4	$1.14205 \pm 0.12193\mathbf{i}$	2.28409	1.31914
5	$-0.65287 \pm 0.92714\mathbf{i}$	-1.30574	1.28583
6	$0.21713 \pm 1.06495\mathbf{i}$	0.43427	1.18127
7	$-0.38874 \pm 0.96394\mathbf{i}$	-0.77748	1.08031
8	$0.28474 \pm 0.81268\mathbf{i}$	0.56948	0.74152
9	$0.60157 \pm 0.57023\mathbf{i}$	1.20314	0.68705
10	$-0.79994 \pm 0.18175\mathbf{i}$	-1.59988	0.67294

Table 2.

$x_1 =$	$-1.26112 - 1.92564\mathbf{i} + 4.10523\mathbf{j} - 0.55813\mathbf{k}$
$x_2 =$	$0.93019 + 0.27871\mathbf{i} - 0.51178\mathbf{j} - 0.45249\mathbf{k}$
$x_3 =$	$-1.07301 + 0.46409\mathbf{i} - 0.09237\mathbf{j} - 0.14655\mathbf{k}$
$x_4 =$	$1.14205 + 0.08057\mathbf{i} + 0.07784\mathbf{j} - 0.04811\mathbf{k}$
$x_5 =$	$-0.65287 - 0.01858\mathbf{i} + 0.88395\mathbf{j} - 0.27907\mathbf{k}$
$x_6 =$	$0.21713 - 0.24588\mathbf{i} + 1.02275\mathbf{j} + 0.16627\mathbf{k}$
$x_7 =$	$-0.38874 + 0.08867\mathbf{i} - 0.46591\mathbf{j} - 0.83920\mathbf{k}$
$x_8 =$	$0.28474 - 0.45581\mathbf{i} - 0.33143\mathbf{j} + 0.58552\mathbf{k}$
$x_9 =$	$0.60157 + 0.21245\mathbf{i} + 0.44266\mathbf{j} - 0.28997\mathbf{k}$
$x_{10} =$	$-0.79994 - 0.03749\mathbf{i} + 0.16559\mathbf{j} - 0.06488\mathbf{k}$

Substituting the values of the trace and the norm of each eigenvalue in Niven's algorithm, we obtain the zeros shown in Table 2.

EXAMPLE 2. Let us now consider the polynomial

$$p(x) = x^6 + (\mathbf{i} + 3\mathbf{k})x^5 + (3 + \mathbf{j})x^4 + (5\mathbf{i} + 15\mathbf{k})x^3 + (-4 + 5\mathbf{j})x^2 + (6\mathbf{i} + 18\mathbf{k})x - 12 + 6\mathbf{j}.$$

The eigenvalues of the (bottom) companion matrix are shown in Table 3.

Table 3.

i	λ_i	t	n
1	$\pm\sqrt{5}\mathbf{i}$	0	5
2	$\pm\sqrt{3}\mathbf{i}$	0	3
3	$\pm\sqrt{3}\mathbf{i}$	0	3
4	$\pm\sqrt{2}\mathbf{i}$	0	2
5	$\pm\sqrt{2}\mathbf{i}$	0	2
6	$\pm\mathbf{i}$	0	1

Substituting the values of the trace and the norm of each eigenvalue in Niven's algorithm, we get

$$\begin{aligned} x_1 &= -\mathbf{i} - 2\mathbf{k}, \\ x_2 &= \left[\mathbf{i}\sqrt{3}\right], \\ x_3 &= \left[\mathbf{i}\sqrt{2}\right], \\ x_4 &= -0.6\mathbf{i} - 0.8\mathbf{k}. \end{aligned}$$

EXAMPLE 3. Let us consider a polynomial with real coefficients

$$p(x) = x^4 + 2x^3 + 5x^2 + 8x + 12.$$

The eigenvalues of the (bottom) companion matrix are shown in Table 4.

Table 4.

i	λ_i	t	n
1	$0.359481 \pm 1.91107i$	0.718963	3.78143
2	$-1.359481 \pm 1.15118i$	-2.718963	3.17340

Substituting these values, we get

$$x_1 = [0.359481 + 1.91107\,\mathbf{i}],$$
$$x_2 = [-1.35948 + 1.15118\,\mathbf{i}].$$

Note that the number of zeros is infinite, divided in two classes, and conform Spira's result [5] for quaternion polynomials with real coefficients and imaginary zeros in $\mathbb{C}$ (the λs in this example).

The above examples were developed in a Pentium 133 MHz with Mathematica 2.2 version. We have worked with polynomials of degree up to 20 and, in general, the time of computing did not exceed four seconds.

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