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A Pollard Type Result for Restricted Sums

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Let $\mathbb{F}$ be an arbitrary field. Let p be the characteristic of $\mathbb{F}$ in case of finite characteristic and ∞ if $\mathbb{F}$ has characteristic 0. Let A be a finite subset of $\mathbb{F}$. By $\bigwedge^2 A$ we denote the set $\{a+b \mid a, b \in A \text{ and } a \neq b\}$. For $c \in \bigwedge^2 A$, let $\nu_c^{(R)}$ be one-half of the cardinality of the set of pairs (a, b) satisfying $a \neq b$ and $a+b=c$. Denote by $\mu_i^{(R)}$ the cardinality of the set $\{c \in \bigwedge^2 A \mid \nu_c^{(R)} \geqslant i\}$. We prove that, for $t=1, ..., \lfloor |A|/2 \rfloor$, $\sum_{i=1}^{t} \mu_i^{(R)} \geqslant t \min\{p, 2(|A|-t)-1\}$. For $\mathbb{F}=\mathbb{Z}_p$ and $t=1$ we get the Erdős–Heilbronn conjecture, first proved by J. A. Dias da Silva and Y. O. Hamidoune (*Bull. London Math. Soc.* **26**, 1994, 140–146).

1. INTRODUCTION

Let $\mathbb{F}$ be an arbitrary field. Let p be the characteristic of $\mathbb{F}$ in case of finite characteristic and ∞ if $\mathbb{F}$ has characteristic 0. Given $b \in \mathbb{R}$ we write $\lceil b \rceil$ ($\lfloor b \rfloor$) for the smallest integer greater than or equal to b (the greatest integer less than or equal to b). For $a \in \mathbb{N}$ let $[1, a]$ denote the set $\{x \in \mathbb{N}: 1 \leqslant x \leqslant a\}$. Let A and B be nonempty finite subsets of $\mathbb{F}$. By $A+B$ we denote the set of elements $a+b$ with $a \in A$ and $b \in B$. For each element $c \in \mathbb{F}$, let $\nu_c(A, B)$ be the cardinality of the set of pairs (a, b) such that $a+b=c$. Let i be a positive integer. We denote by $\mu_i(A, B)$, or briefly

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by μ_i, the cardinality of the set of the elements $c \in A+B$ for which $v_c(A, B)$ is greater than or equal to i.

Let X be a set. We denote by $|X|$ the cardinality of X. If $|X|=k$ we say that X is a k-set. In [1] we prove the following theorem:

1.1. THEOREM. *Let A and B be finite nonempty subsets of $\mathbb{F}$. Then for $t=1, 2, ..., \min\{|A|, |B|\}$ we have*

$$\sum_{i=1}^{t} \mu_i \geqslant t \min\{p, |A|+|B|-t\}.$$

This result is an extension to an arbitrary field of a theorem proved by Pollard [4, 5], for $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, where p is a prime number. Notice that the case where $t=1$ is the well known Cauchy–Davenport Theorem.

In this paper we prove, for restricted sums, an analogue of Theorem 1.1. Let A be a finite subset of $\mathbb{F}$. We denote by $\bigwedge^2 A$ the set

$$\{a+b \mid a, b \in A \text{ and } a \neq b\}.$$

For $c \in \bigwedge^2 A$, let $v_c^{(R)}$ be one-half of the cardinality of the set of pairs (a, b) satisfying $a \neq b$ and $a+b=c$. Denote by $\mu_i^{(R)}$ the cardinality of the set

$$\{c \in \textstyle\bigwedge^2 A \mid v_c^{(R)} \geqslant i\}.$$

We prove that, for $t=1, ..., \lfloor |A|/2 \rfloor$,

$$\sum_{i=1}^{t} \mu_i^{(R)} \geqslant t \min\{p, 2(|A|-t)-1\}. \tag{1}$$

This lower bound is tight and the equality in (1) is attained when A is an arithmetic progression.

For $\mathbb{F} = \mathbb{Z}_p$ and $t=1$ we get the Erdős–Heilbronn conjecture [3], first proved in [2].

2. COMBINATORIAL BACKGROUND

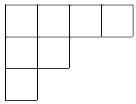
A sequence of integers $\lambda = (\lambda_1, ..., \lambda_t)$ will be called a *partition* if $0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_t$. We say that λ is a partition of k if

$$\sum_{i=1}^{t} \lambda_i = k.$$

The length of the partition λ is the number of its nonzero terms. Let s be a positive integer. The set of the partitions of k of length at most s is denoted by $\mathscr{P}_{k,s}$. Let μ be a partition of k and let λ be a partition of $k+1$. We write $\mu \to \lambda$ if there exists j such that $\lambda_i = \mu_i + \delta_{ij}$ for all i, where δ_{ij} is the Kronecker symbol.

To each partition $\lambda = (\lambda_1, ..., \lambda_t)$ of k there corresponds a Young Diagram, $[\lambda]$, which consists of k boxes in t rows starting in the same column, where the ith row consists of λ_{t-i+1} boxes, $1 \leqslant i \leqslant t$.

EXAMPLE. Let $k = 7$ and $\lambda = (1, 2, 4)$. Then

$[\lambda] =$ 

The box that lies in ith row and jth column of $[\lambda]$ is called the (i, j)-*box* of $[\lambda]$. The (i, j)-*hook* of $[\lambda]$, $H^{\lambda}_{i,j}$, is the collection of boxes consisting of the (i, j)-box along with the boxes of the same row to the right and the boxes of the same column under it. The number of boxes of $H^{\lambda}_{i,j}$ is denoted by $h^{\lambda}_{i,j}$. For a partition of k of length t, $\lambda = (\lambda_1, ..., \lambda_t)$, let

$$P(\lambda) = \prod_{i=1}^{t} \prod_{j=1}^{\lambda_{t-i+1}} h^{\lambda}_{i,j}.$$

In [2] the following result is established:

2.2. PROPOSITION. *Let* $\lambda \in \mathscr{P}_{k+1,s}$. *Then*

$$\sum_{\substack{\mu \in \mathscr{P}_{k,s} \\ \mu \to \lambda}} \frac{1}{P(\mu)} = \frac{k+1}{P(\lambda)}.$$

Using this proposition it is easy to see that, if λ is a partition of k, then $k!/P(\lambda)$ is an integer. The next result is easy to prove, so its proof will be left to the reader.

2.3. PROPOSITION. *For* $\mu = (\mu_1, \mu_2)$ *we have*

$$P(\mu) = \frac{(\mu_2 + 1)!\, \mu_1!}{\mu_2 - \mu_1 + 1}.$$

3. AUXILIARY RESULTS

Let $\mathbb{F}$ be an arbitrary field and denote by $\overline{\mathbb{F}}$ the algebraic closure of $\mathbb{F}$. Let $V \neq \{0\}$ be an m-dimensional vector space over the field $\mathbb{F}$ and let f be a linear operator on V. We use P_f to denote the minimal polynomial of f and $\alpha_{f,1} \mid \cdots \mid \alpha_{f,m} = P_f$ to denote the invariant factors of f (so that each $\alpha_{f,i}$ divides all subsequent polynomials $\alpha_{f,i+1}, ..., \alpha_{f,m}$). For every $v \in V$, $\mathscr{C}_f(v)$ is the f-cyclic space of v, i.e.,

$$\mathscr{C}_f(v) = \langle f^i(v) \colon i \in \mathbb{Z}^+ \rangle,$$

where $\langle X \rangle$ denotes the linear span of X and $\mathbb{Z}^+$ denotes the set of nonnegative integers. We use $\sigma(f)$ to denote the spectrum of f, i.e., $\sigma(f)$ is the family of the m roots of the characteristic polynomial of f in $\overline{\mathbb{F}}$. Let i be a positive integer. We denote by $m_i(f)$ the number of distinct roots of the characteristic polynomial of f with algebraic multiplicity greater than or equal to i. Notice that $m_1(f)$ is the number of distinct eigenvalues of f and for a diagonal linear operator f,

$$m_i(f) = \deg(\alpha_{f,m-i+1}), \qquad i = 1, ..., m.$$

3.4. Definition. Let $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ be two sequences of nonnegative integers. Denote by $(\bar{a}_1, ..., \bar{a}_n)$ and $(\bar{b}_1, ..., \bar{b}_n)$ the reordering, in a nonincreasing way, of a and b, respectively. We say that a *weakly dominates* b and we write

$$a \sqsupseteq b$$

if

$$\sum_{i=1}^{k} \bar{a}_i \geqslant \sum_{i=1}^{k} \bar{b}_i, \qquad k = 1, ..., n.$$

If $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ we say that a *dominates* b and we write $a \succcurlyeq b$.

In [1] the following result was proved:

3.5. Lemma. *Let V be a finite dimensional vector space over the field $\mathbb{F}$ of dimension m. Let f be a linear operator on V. Let $s_1, ..., s_t$ be positive integers. If there exist $v_1, ..., v_t \in V$ such that*

$$\bigcup_{i=1}^{t} \{v_i, f(v_i), f^2(v_i), ..., f^{s_i-1}(v_i)\}$$

is a linearly independent $(s_1 + \cdots + s_t)$*-set then*

$$(s_1, ..., s_t) \sqsubseteq (\deg(\alpha_{f,m}), ..., \deg(\alpha_{f,m-t+1})).$$

For the benefit of the reader we reproduce here the proof of this lemma. For this we need some definitions and results.

3.6. DEFINITION. Let $v_1, ..., v_t \in V$ and let f be a linear operator on V. The subspace

$$\mathscr{C}_f(v_1, ..., v_t) = \langle f^j(v_i) \colon j \in \mathbb{Z}^+, i = 1, ..., t \rangle$$

will be called the *generalized f-cyclic subspace associated to* $v_1, ..., v_t$. We say that the pair $((v_1, ..., v_t), f)$, or the generalized f-cyclic subspace $\mathscr{C}_f(v_1, ..., v_t)$, is *completely controllable* if

$$\mathscr{C}_f(v_1, ..., v_t) = V.$$

3.7. DEFINITION. Let f be a linear operator on V and let $v_1, ..., v_t \in V$. A basis, $\mathscr{B}$, of $\mathscr{C}_f(v_1, ..., v_t)$ selected from

$$\{f^j(v_i) \colon j \in \mathbb{Z}^+, i = 1, ..., t\}$$

is *nice* if, for $0 \leqslant i \leqslant k-1$, $f^i(v_j) \in \mathscr{B}$ provided that $f^k(v_j) \in \mathscr{B}$.

Let

$$\mathscr{B} = \bigcup_{i=1}^{t} \{v_i, f(v_i), f^2(v_i), ..., f^{r_i - 1}(v_i)\}$$

be a nice basis of $\mathscr{C}_f(v_1, ..., v_t)$. We say that the nonnegative integers $r_1, ..., r_t$ are *indices of* $\mathscr{B}$.

Let $v_1, ..., v_t \in V$. If

$$\mathscr{I} = \bigcup_{i=1}^{t} \{v_i, f(v_i), f^2(v_i), ..., f^{s_i - 1}(v_i)\}$$

is a linearly independent $(s_1 + \cdots + s_t)$-set, we say that $\mathscr{I}$ is a $((v_1, ..., v_t), f)$-*nice independent set* and we call the nonnegative integers $s_1, ..., s_t$ *indices of* $\mathscr{I}$.

Notice that it is possible to associate more than one list of indices to a $((v_1, ..., v_t), f)$-nice independent set. For instance if v_1, v_2, v_3 are linearly independent vectors of V and $f(v_1) = v_2$,

$$\mathscr{I} = \{v_1, v_2, v_3\} = \{v_1, f(v_1), v_3\}$$

is a $((v_1, v_2, v_3), f)$-nice independent set and both $(1, 1, 1)$ and $(2, 0, 1)$ are lists of indices of $\mathscr{I}$.

In [6], the following result is proved.

3.8. PROPOSITION. *Let A be an $\ell \times \ell$ matrix and let $\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_\ell$ be its invariants factors. Let m be a positive integer satisfying $m > \ell$. Let $\gamma_1, ..., \gamma_m$ be monic polynomials over $\mathbb{F}$ such that $\deg(\gamma_1 \cdots \gamma_m) = m$ and $\gamma_1 \mid \cdots \mid \gamma_m$. Then there exist $C \in \mathbb{F}^{(m-\ell)\times \ell}$ and $D \in \mathbb{F}^{(m-\ell)\times(m-\ell)}$ such that the $m \times m$ matrix*

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$$

has invariant factors $\gamma_1, ..., \gamma_m$, if and only if

$$\gamma_i \mid \alpha_i \mid \gamma_{i+m-\ell}, \qquad i = 1, ..., \ell.$$

The next theorem is proved in [8, Corollary 2.2].

3.9. THEOREM. *Let V be an m-dimensional vector space over the field $\mathbb{F}$. Let f be a linear operator on V and let $r_1, ..., r_t$ be positive integers. Then there exist linearly independent vectors $v_1, ..., v_t$, such that $\mathscr{C}_f(v_1, ..., v_t)$ is completely controllable, and a nice basis of $\mathscr{C}_f(v_1, ..., v_t)$ with indices $r_1, ..., r_t$ if and only if the following conditions hold:*

$$\alpha_{f,i} = 1, \qquad i = 1, ..., m - t,$$

and

$$(r_1, ..., r_t) \preccurlyeq (\deg(\alpha_{f,m}), ..., \deg(\alpha_{f,m-t+1})).$$

The next theorem is proved in [1] and states a necessary condition for the existence of nice bases with prescribed indices when the constraint of complete controllability is skipped.

3.10. THEOREM. *Let V be an m-dimensional vector space over the field $\mathbb{F}$ and let f be a linear operator on V. Let $r_1, ..., r_t$ be positive integers. If there exist linearly independent vectors $v_1, ..., v_t$ and a nice basis of $\mathscr{C}_f(v_1, ..., v_t)$ with indices $r_1, ..., r_t$, then*

$$(r_1, ..., r_t) \sqsubseteq (\deg(\alpha_{f,m}), ..., \deg(\alpha_{f,m-t+1})).$$

Proof. Let $U = \mathscr{C}_f(v_1, ..., v_t)$ and let $\ell = \dim(U)$. As usual, let $f_{|U}$ denote the restriction of f to U. Clearly, $\mathscr{C}_{f|U}(v_1, ..., v_t)$ is completely controllable and from Theorem 3.9 we have

$$(r_1, ..., r_t) \preccurlyeq (\deg(\alpha_{f_{|U}, \ell}), ..., \deg(\alpha_{f_{|U}, \ell - t + 1})). \tag{2}$$

By the transposed version of Proposition 3.8 we know that

$$\alpha_{f, i} \mid \alpha_{f_{|U}, i} \mid \alpha_{f, i+m-\ell}, \qquad i = 1, ..., \ell.$$

Therefore

$$\alpha_{f_{|U}, \ell} \alpha_{f_{|U}, \ell - 1} \cdots \alpha_{f_{|U}, \ell - j} \mid \alpha_{f, m} \alpha_{f, m-1} \cdots \alpha_{f, m-j}, \qquad j = 0, ..., \ell - 1. \tag{3}$$

Taking degrees in (3) we have

$$\sum_{i=0}^{j} \deg(\alpha_{f_{|U}, \ell - i}) \leqslant \sum_{i=0}^{j} \deg(\alpha_{f, m-i}), \qquad j = 0, ..., \ell - 1. \tag{4}$$

Since $v_1, ..., v_t \in U$ are linearly independent vectors we have $t \leqslant \dim(U) = \ell$. Therefore, from (4) and (2) we get

$$(r_1, ..., r_t) \sqsubseteq (\deg(\alpha_{f, m}), ..., \deg(\alpha_{f, m-t+1})). \quad \blacksquare$$

Proof of Lemma 3.5. Let $s_1, ..., s_t$ be positive integers and suppose that

$$\bigcup_{i=1}^{t} \{v_i, f(v_i), f^2(v_i), ..., f^{s_i - 1}(v_i)\}$$

is a linearly independent $(s_1 + \cdots + s_t)$-set. In order to use Theorem 3.10, we complete this set to a nice basis of $\mathscr{C}_f(v_1, ..., v_t)$. For each $q \in \{1, ..., t\}$, let r_q be the positive integer such that

$$\left(\bigcup_{j=1}^{q} \{v_j, f(v_j), ..., f^{r_j - 1}(v_j)\}\right) \cup \left(\bigcup_{i=q+1}^{t} \{v_i, f(v_i), f^2(v_i), ..., f^{s_i - 1}(v_i)\}\right)$$

is a linearly independent $(r_1 + \cdots + r_q + s_{q+1} + \cdots + s_t)$-set and

$$f^{r_q}(v_q) \in \left\langle \left(\bigcup_{j=1}^{q} \{v_j, f(v_j), ..., f^{r_j - 1}(v_j)\}\right) \cup \left(\bigcup_{i=q+1}^{t} \{v_i, f(v_i), f^2(v_i), ..., f^{s_i - 1}(v_i)\}\right)\right\rangle.$$

It is obvious, from the definitions, that

$$f\left(\left\langle \bigcup_{i=1}^{t} \{v_i, ..., f^{r_i-1}(v_i)\} \right\rangle\right) \subseteq \left\langle \bigcup_{i=1}^{t} \{v_i, ..., f^{r_i-1}(v_i)\} \right\rangle. \tag{5}$$

We now show that

$$\bigcup_{i=1}^{t} \{v_i, ..., f^{r_i-1}(v_i)\}$$

is a maximal linearly independent set contained in $\mathscr{C}_f(v_1, ..., v_t)$. Assume, for a contradiction, that for some $i \in \{1, ..., t\}$ and some $r \in \mathbb{N}$,

$$f^r(v_i) \notin \left\langle \bigcup_{i=1}^{t} \{v_i, ..., f^{r_i-1}(v_i)\} \right\rangle. \tag{6}$$

Without loss of generality we can suppose that r is the smallest integer with this property. Then

$$f^{r-1}(v_i) \in \left\langle \bigcup_{i=1}^{t} \{v_i, ..., f^{r_i-1}(v_i)\} \right\rangle$$

and

$$f^r(v_i) \in f\left(\left\langle \bigcup_{i=1}^{t} \{v_i, ..., f^{r_i-1}(v_i)\} \right\rangle\right).$$

Using (5) we get

$$f^r(v_i) \in \left\langle \bigcup_{i=1}^{t} \{v_i, ..., f^{r_i-1}(v_i)\} \right\rangle,$$

which contradicts (6).

By Theorem 3.10 we conclude that

$$(r_1, ..., r_t) \sqsubseteq (\deg(\alpha_{f,m}), ..., \deg(\alpha_{f,m-t+1})).$$

But, since by construction, we have $s_i \leqslant r_i$, $i = 1, ..., t$, we get from the former inequalities

$$(s_1, ..., s_t) \sqsubseteq (\deg(\alpha_{f,m}), ..., \deg(\alpha_{f,m-t+1})). \quad \blacksquare$$

Let $\bigwedge^2 V$ be the second exterior power of V. Let f be a linear operator on V. We denote by $D(f)$ the induced operator on $\bigwedge^2 V$, defined by

$$D(f)(v_1 \wedge v_2) = f(v_1) \wedge v_2 + v_1 \wedge f(v_2), \qquad v_1, v_2 \in V.$$

3.11. Proposition. *Given a finite subset $A \subseteq \mathbb{F}$, let V be a vector space over $\mathbb{F}$ of dimension $|A|$. Let f be a linear operator on V with spectrum $\sigma(f) = A$. Then*

$$m_i(D(f)) = \mu_i^{(R)}, \qquad i \in \mathbb{N}.$$

Proof. Suppose $A = \{a_1, ..., a_n\}$. It is easily derived from the definitions that the spectrum of $D(f)$ is the family

$$(a_j + a_k)_{1 \leqslant j < k \leqslant n}.$$

Then for $i \in \mathbb{N}$ we have

$$\begin{aligned} m_i(D(f)) &= |\{x \in \textstyle\bigwedge^2 A \colon |\{(j, k) \colon 1 \leqslant j < k \leqslant n \text{ and } a_j + a_k = x\}| \geqslant i\}| \\ &= \mu_i^{(R)}. \quad \blacksquare \end{aligned}$$

Let f be a linear operator on V and let $v \in V$ be such that $n = \dim \mathscr{C}_f(v) \geqslant 2$.

3.12. Definition. Let $x \in \bigwedge^2 \mathscr{C}_f(v)$. We define the *weight of* x as the maximal element of the set

$$\{i + j \colon x \text{ has a nonzero coefficient of } f^i(v) \wedge f^j(v)\}.$$

The following results will allow us to evaluate the weight of $D(f)^k\,(f^{j-1}(v) \wedge f^j(v))$.

3.13. Lemma. *For every $k \in \mathbb{Z}^+$ and $j \in \mathbb{N}$*

$$D(f)^k\,(f^{j-1}(v) \wedge f^j(v)) = \sum_{\lambda \in \mathscr{P}_{k,2}} \frac{k!}{P(\lambda)}\, f^{\lambda_1 + j - 1}(v) \wedge f^{\lambda_2 + j}(v).$$

Proof. We use induction on k. For $k = 0$ the result is trivial. Observe now that, for $\lambda \in \mathscr{P}_{k,2}$,

$$D(f)(f^{\lambda_1 + j - 1}(v) \wedge f^{\lambda_2 + j}(v)) = \sum_{\substack{\beta \in \mathscr{P}_{k+1,2} \\ \lambda \to \beta}} f^{\beta_1 + j - 1}(v) \wedge f^{\beta_2 + j}(v). \tag{7}$$

We have now, using the induction hypothesis,

$$
\begin{aligned}
D(f)^{k+1}&\,(f^{j-1}(v) \wedge f^{j}(v)) \\
&= \sum_{\lambda \in \mathscr{P}_{k,2}} \frac{k!}{P(\lambda)} \sum_{\substack{\beta \in \mathscr{P}_{k+1,2} \\ \lambda \to \beta}} f^{\beta_1+j-1}(v) \wedge f^{\beta_2+j}(v) \\
&= \sum_{\beta \in \mathscr{P}_{k+1,2}} \left(\sum_{\substack{\lambda \in \mathscr{P}_{k,2} \\ \lambda \to \beta}} \frac{k!}{P(\lambda)} \right) f^{\beta_1+j-1}(v) \wedge f^{\beta_2+j}(v) \\
&= \sum_{\beta \in \mathscr{P}_{k+1,2}} \frac{(k+1)!}{P(\beta)} f^{\beta_1+j-1}(v) \wedge f^{\beta_2+j}(v).
\end{aligned}
$$

The last equality follows from Proposition 2.2. ∎

3.14. Lemma. *For $k \in \mathbb{Z}^+$ and $j \in \mathbb{N}$ there exists a family of elements of $\mathbb{F}$, $(b_\nu)_{\substack{0 \leqslant \nu_1 \leqslant \nu_2 \leqslant n-2 \\ \nu_1+\nu_2 \leqslant k+2j-3}}$, such that*

$$
D(f)^{k}\,(f^{j-1}(v) \wedge f^{j}(v)) = \sum_{\substack{\lambda \in \mathscr{P}_{k,2} \\ \lambda_2 \leqslant n-j-1}} \frac{k!}{P(\lambda)} f^{\lambda_1+j-1}(v) \wedge f^{\lambda_2+j}(v)
$$

$$
+ \sum_{\substack{0 \leqslant \nu_1 \leqslant \nu_2 \leqslant n-2 \\ \nu_1+\nu_2 \leqslant k+2j-3}} b_\nu f^{\nu_1}(v) \wedge f^{\nu_2+1}(v).
$$

Proof. We use Lemma 3.13 and isolate the terms $f^{\lambda_1+j-1}(v) \wedge f^{\lambda_2+j}(v)$ with $\lambda_2 + j \geqslant n$. Clearly, each of these terms can be written as a linear combination of $f^{\nu_1}(v) \wedge f^{\nu_2}(v)$, where $0 \leqslant \nu_1 < \nu_2 \leqslant n-1$ and $\nu_1+\nu_2 \leqslant (\lambda_1+j-1)+(\lambda_2+j)-1 = k+2j-2$. The result follows. ∎

3.15. Theorem. *For $j \in \mathbb{N}$ and $0 \leqslant k \leqslant \min\{p-1, 2n-2j-2\}$, the weight of*

$$
D(f)^{k}\,(f^{j-1}(v) \wedge f^{j}(v))
$$

is $k+2j-1$.

Proof. Clearly the weight does not exceed $k+2j-1$. On the other hand, let $\lambda = (\lfloor k/2 \rfloor, \lceil k/2 \rceil) \in \mathscr{P}_{k,2}$. Since $k \leqslant 2n-2j-2$ we have $\lambda_2 \leqslant n-j-1$. We now use Lemma 3.14 and notice, that the coefficient $k!/P(\lambda)$ is not 0 in $\mathbb{F}$ as $k < p$. ∎

3.16. LEMMA. *Let $\mathbb{F}$ and p be as usual. Let $a, b, k \in \mathbb{Z}^+$ satisfy $b+2k \leqslant a < p$. Then the $(k+1)\times(k+1)$ matrix over $\mathbb{F}$, $C(a, b, k) = [c_{ij}]$ where*

$$c_{ij} = \begin{cases} \dfrac{(a-i+1)!\,(b+i-1)!}{(a-i-j+2)!\,(b+i-j)!} & \text{if} \quad b+i-j \geqslant 0 \\ 0 & \text{if} \quad b+i-j < 0, \end{cases}$$

is invertible.

Proof. We proceed by induction on k. If $k=0$ we have

$$C(a, b, 0) = [1].$$

Assume now that $k \geqslant 1$. Let J be the $(k+1)\times(k+1)$ matrix, with the $(i+1, i)$ entries, $i = 1, ..., k$, equal to 1, and the remaining entries equal to 0. We have

$$(I_{k+1} - J)\, C(a, b, k) = \left[\begin{array}{c|ccc} 1 & c_{12} & \cdots & c_{1,k+1} \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right],$$

where $B = (b_{ij})$ is the $k \times k$ matrix whose (i, j)-entry is $b_{ij} = -c_{i,j+1} + c_{i+1,j+1}$, $i, j = 1, ..., k$.

If $b+i-j<0$ both $c_{i,j+1}$ and $c_{i+1,j+1}$ are zero. Then $b_{ij}=0$.

If $b+i-j=0$ then $c_{i,j+1}=0$ and

$$\begin{aligned} b_{ij} &= c_{i+1,j+1} \\ &= \frac{(a-i)!\,(b+i)!}{(a-i-j)!\,(b+i-j)!} \\ &= \frac{(a-i)!\,(b+i-1)!\; j(a-i-j+1)}{(a-i-j+1)!\,(b+i-j)!} \\ &= \frac{(a-i)!\,(b+i-1)!\; j(a-b-2i+1)}{(a-i-j+1)!\,(b+i-j)!}. \end{aligned}$$

For the third equality we have used the fact that $(b+i)! = j(b+i-1)!$ and we have multiplied both the numerator and the denominator by $a-i-j+1 = a-b-2i+1 > 0$. If $b+i-j > 0$ we have

$$b_{ij} = \frac{(a-i)!\,(b+i-1)!}{(a-i-j+1)!\,(b+i-j)!}$$

$$\times\left[-(a-i+1)(b+i-j)+(b+i)(a-i-j+1)\right]$$

$$= \frac{(a-i)!\,(b+i-1)!\; j(a-b-2i+1)}{(a-i-j+1)!\,(b+i-j)!}.$$

Then there exist two invertible matrices P and Q such that $PBQ = C(a-1, b, k-1)$. Therefore using the induction hypothesis, we conclude that $C(a, b, k)$ is invertible. ∎

4. MAIN RESULTS

4.17. *Notation.* Let A be a finite subset of the field $\mathbb{F}$. Recall that if i is a positive integer, $\mu_i^{(R)}$ is the cardinality of the set

$$\{x \in \wedge^2 A\colon \nu_x^{(R)} \geqslant i\}.$$

It is easy to see that $\mu_i^{(R)} = 0$ for $i > |A|/2$.

4.18. THEOREM. *Let V be a nonzero m-dimensional vector space over the field $\mathbb{F}$. Let f be a linear operator on V with minimal polynomial P_f and assume that* $\deg(P_f) \geqslant 2$ *and* $1 \leqslant t \leqslant \lfloor \deg(P_f)/2 \rfloor$. *Then we have*

$$\sum_{i=1}^{t} \deg(\alpha_{D(f),\binom{m}{2}-i+1}) \geqslant t \min\{p, 2(\deg(P_f)-t)-1\}.$$

4.19. COROLLARY. *Let A be a finite subset of the field $\mathbb{F}$ and $1 \leqslant t \leqslant \lfloor |A|/2 \rfloor$. Assume that $|A| \geqslant 2$. Then we have*

$$\sum_{i=1}^{t} \mu_i^{(R)} \geqslant t \min\{p, 2(|A|-t)-1\}.$$

Proof of Corollary 4.19. Let $n=|A|$ and let f be a diagonal linear operator on $\mathbb{F}^n$ whose spectrum is A. Then $D(f)$ is diagonal with spectrum $\bigwedge^2 A$. Using Proposition 3.11 we obtain

$$\sum_{i=1}^{t} \mu_i^{(R)} = \sum_{i=1}^{t} \deg(\alpha_{D(f), \binom{n}{2}-i+1}), \qquad t=1, 2, ..., \binom{n}{2}.$$

Then using Theorem 4.18 we conclude that, for $1 \leqslant t \leqslant \lfloor n/2 \rfloor$,

$$\sum_{i=1}^{t} \mu_i^{(R)} \geqslant t \min\{p, 2(n-t)-1\}. \quad \blacksquare$$

Remark. If x is an integer, denote by $\bar{x}$ the canonical image of x in $\mathbb{F}$. Suppose that $A \subseteq \mathbb{F}$ is an arithmetic progression with $|A| \geqslant 3$. Then $p \geqslant |A| \geqslant 3$.

Let $A' = \{\bar{0}, \bar{1}, ..., \overline{|A|-1}\}$. For $\bar{x} \in \bigwedge^2 A'$, let $\hat{v}_{\bar{x}}^{(R)}$ be one-half of the cardinality of the set of pairs $(\bar{a}, \bar{b}) \in A' \times A'$ satisfying $\bar{a} \neq \bar{b}$ and : $\bar{a}+\bar{b}=\bar{x}$. Denote by $\hat{\mu}_i^{(R)}$ the cardinality of the set

$$\{\bar{x} \in \textstyle\bigwedge^2 A' \mid \hat{v}_{\bar{x}}^{(R)} \geqslant i\}.$$

It is easy to see that

$$\mu_i^{(R)} = \hat{\mu}_i^{(R)}, \qquad i \in \mathbb{N}.$$

For $\bar{x} \in \bigwedge^2 A' = \{\bar{1}, ..., \overline{\min\{p, 2\,|A|-3\}}\}$ we have:

If $p \leqslant 2\,|A|-4$ then

$$\hat{v}_{\bar{x}}^{(R)} = \begin{cases} |A| - \dfrac{p+1}{2} & \text{if } \bar{x} \in \{\bar{1}, ..., \overline{2\,|A|-p-3}\} \\ \left\lceil \dfrac{x}{2} \right\rceil & \text{if } \bar{x} \in \{\overline{2\,|A|-p-2}, ..., \overline{|A|-1}\} \\ |A| - \left\lfloor \dfrac{x}{2} \right\rfloor - 1 & \text{if } \bar{x} \in \{\overline{|A|}, ..., \bar{p}\}. \end{cases}$$

If $p \geqslant 2\,|A|-3$ then

$$\hat{v}_{\bar{x}}^{(R)} = \begin{cases} \left\lceil \dfrac{x}{2} \right\rceil & \text{if } \bar{x} \in \{\bar{1}, ..., \overline{|A|-1}\} \\ |A| - \left\lfloor \dfrac{x}{2} \right\rfloor - 1 & \text{if } \bar{x} \in \{\overline{|A|}, ..., \overline{2\,|A|-3}\} \end{cases}.$$

Then, for $i=1, ..., \lfloor |A|/2 \rfloor$, we have

$$\begin{aligned}
\mu_i^{(R)} &= \hat{\mu}_i^{(R)} \\
&= |\{\bar{x} \in \wedge^2 A' : \hat{v}_{\bar{x}}^{(R)} \geqslant i\}| \\
&= \begin{cases} p & \text{if} \quad 1 \leqslant i \leqslant |A| - \dfrac{p+1}{2} \\ 2\,|A| - 4i + 1 & \text{if} \quad \max\left\{1, |A| - \dfrac{p-1}{2}\right\} \leqslant i \leqslant \left\lfloor \dfrac{|A|}{2} \right\rfloor. \end{cases}
\end{aligned}$$

It follows that, for $t=1, 2, ..., \lfloor |A|/2 \rfloor$,

$$\sum_{i=1}^{t} \mu_i^{(R)} = \begin{cases} tp & \text{if} \quad t \leqslant |A| - \dfrac{p+1}{2} \\ t(2\,|A| - 2t - 1) & \text{if} \quad t \geqslant \max\left\{1, |A| - \dfrac{p-1}{2}\right\} \end{cases}$$

and thus equality holds in Corollary 4.19.

5. PROOF OF THEOREM 4.18

Let $v \in V$ be such that $\dim \mathscr{C}_f(v) = \deg(P_f) = n \geqslant 2$. Let $\mathscr{B}$ be the basis of $\wedge^2 \mathscr{C}_f(v)$ defined by

$$\begin{aligned}
\mathscr{B} &= \{ f^{\lambda_1}(v) \wedge f^{\lambda_2+1}(v) : 0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant n-2 \} \\
&= \left\{ f^{\lambda_1}(v) \wedge f^{\lambda_2+1}(v) : \lambda = (\lambda_1, \lambda_2) \in \bigcup_{s \in \mathbb{Z}^+} \mathscr{P}_{s,2} \text{ and } \lambda_2 \leqslant n-2 \right\}.
\end{aligned}$$

Let $1 \leqslant t \leqslant \lfloor n/2 \rfloor$. For $k \in \mathbb{Z}^+$ and $1 \leqslant j \leqslant t$ define

$$z_{k,j} = D^k(f)(f^{j-1}(v) \wedge f^j(v)).$$

Let $u = t \min\{p, 2n-2t-1\}$. We shall prove that

$$\mathscr{C} = \{ z_{k,j} : 1 \leqslant j \leqslant t, 0 \leqslant k \leqslant \min\{p-1, 2n-2t-2\} \}$$

is a linear independent u-set and use Lemma 3.5 to conclude that

$$(\deg(\alpha_{D(f), \binom{m}{2}}), \deg(\alpha_{D(f), \binom{m}{2}-1}), ..., \deg(\alpha_{D(f), \binom{m}{2}-t+1}))$$

weakly dominates the t-tuple

$$(\min\{p, 2n-2t-1\}, ..., \min\{p, 2n-2t-1\}),$$

thereby obtaining the result.

In order to prove that $\mathscr{C}$ is a linearly independent set we split it into several linearly independent and pairwise disjoint subsets and prove that the linear span of $\mathscr{C}$ is the direct sum of the linear spans of those subsets. These subsets will be obtained by grouping together the elements of $\mathscr{C}$ with the same weight.

From Theorem 3.15 it is easy to see that the maximum weight of the vectors of $\mathscr{C}$ is

$$M_t = \min\{p+2t-2, 2n-3\}.$$

For $r=1, ..., M_t$ let $\mathscr{S}_r$ be the index set of the subset of the elements of $\mathscr{C}$ of weight r. That is,

$$\begin{aligned}\mathscr{S}_r = \{&(k, j) \in \mathbb{Z}^+ \times \mathbb{N}\colon 1 \leqslant j \leqslant t, 0 \leqslant k \leqslant \min\{p-1, 2n-2t-2\}\\ &\text{and } k+2j-1=r\}\\ = \{&(r-2j+1, j) \in \mathbb{Z}^+ \times \mathbb{N}\colon a_r \leqslant j \leqslant b_r\},\end{aligned}$$

where

$$a_r = \max\left\{1, \left\lceil \frac{r-p}{2} \right\rceil + 1, \left\lceil \frac{r+1}{2} \right\rceil - n + t + 1\right\}$$

and

$$b_r = \min\left\{t, \left\lfloor \frac{r+1}{2} \right\rfloor\right\}$$

We have

$$\mathscr{C} = \dot{\bigcup_{r=1}^{M_t}} \{z_{k,j}\colon (k, j) \in \mathscr{S}_r\}. \tag{8}$$

Claim 1. *For any fixed $r \in [1, M_t]$, the set $\{z_{k,j}\colon (k, j) \in \mathscr{S}_r\}$ is linearly independent.*

Proof. Let $q_r = |\mathscr{S}_r| = b_r - a_r + 1$. We denote by $\mathscr{B}_r$ the set of those elements of $\mathscr{B}$ with weight r:

$$\mathscr{B}_r = \left\{ f^i(v) \wedge f^{r-i}(v) \colon \max\{0, r-n+1\} \leqslant i \leqslant \left\lfloor \frac{r-1}{2} \right\rfloor \right\}.$$

Let π_r be the projection of $\bigwedge^2 \mathscr{C}_f(v)$ onto $\langle \mathscr{B}_r \rangle$, along $\bigoplus_{s=1, s \neq r}^{2n-3} \langle \mathscr{B}_s \rangle$.

Let $(k, j) \in \mathscr{S}_r$. From Lemma 3.14 we have

$$\pi_r(z_{k,j}) = \sum_{\substack{\lambda \in \mathscr{P}_{k,2} \\ \lambda_2 \leqslant n-j-1}} \frac{k!}{P(\lambda)} f^{\lambda_1 + j - 1}(v) \wedge f^{\lambda_2 + j}(v). \tag{9}$$

We order the elements of $\{\pi_r(z_{k,j}) : (k, j) \in \mathscr{S}_r\}$ by writing

$$y_j = \pi_r(z_{r-2j-2a_r+3,\, j+a_r-1}), \qquad j = 1, 2, ..., q_r.$$

To prove Claim 1 it is sufficient to prove

Claim 1′. *$\{y_1, ..., y_{q_r}\}$ is linearly independent.*

Proof. Let

$$\left\{ \theta_j \colon \max\{0, r-n+1\} \leqslant j \leqslant \left\lfloor \frac{r-1}{2} \right\rfloor \right\}$$

be the dual basis of $\mathscr{B}_r$; i.e., θ_j are linear functions on $\bigwedge^2 \mathscr{C}_f(v)$, satisfying

$$\theta_j(f^i(v) \wedge f^{r-i}(v)) = \delta_{ij}, \qquad \max\{0, r-n+1\} \leqslant i, j \leqslant \left\lfloor \frac{r-1}{2} \right\rfloor,$$

where δ_{ij} is the Kronecker symbol.

We prove that the $|\mathscr{B}_r| \times q_r$ matrix of the coefficients of $y_1, ..., y_{q_r}$ with respect to the basis $\mathscr{B}_r$, that is,

$$[\theta_i(y_j)]_{\substack{i=1, ..., |\mathscr{B}_r| \\ j=1, ..., q_r}},$$

has an invertible $q_r \times q_r$ submatrix, to conclude that $\{y_1, ..., y_{q_r}\}$ is linearly independent.

We consider two cases.

Case 1. $a_r \geqslant r-n+2$. For all $i \in \{1, 2, ..., q_r\}$ we have

$$\max\{0, r-n+1\} \leqslant a_r - 1 \leqslant i + a_r - 2 \leqslant b_r - 1 \leqslant \left\lfloor \frac{r-1}{2} \right\rfloor,$$

so we can consider $X_i = \theta_{i+a_r-2}$.

From (9) it follows that

$$\begin{aligned} X_i(y_j) &= \theta_{i+a_r-2}(\pi_r(z_{r-2j-2a_r+3,\, j+a_r-1})) \\ &= \sum_{\substack{\lambda \in \mathscr{P}_{r-2j-2a_r+3,\,2} \\ \lambda_2 \leqslant n-j-a_r}} \frac{(r-2j-2a_r+3)!}{P(\lambda)} \\ &\quad \times \theta_{i+a_r-2}(f^{\lambda_1+j+a_r-2}(v) \wedge f^{\lambda_2+j+a_r-1}(v)) \\ &= \sum_{\substack{\lambda \in \mathscr{P}_{r-2j-2a_r+3,\,2} \\ \lambda_2 \leqslant n-j-a_r}} \frac{(r-2j-2a_r+3)!}{P(\lambda)} \delta_{\lambda_1,\, i-j}, \end{aligned} \tag{10}$$

for $1 \leqslant i, j \leqslant q_r$.

If $i < j$, then all $\delta_{\lambda_1,\, i-j}$ at the right vanish and $X_i(y_j) = 0$.

Suppose $i \geqslant j$. Then

$$2a_r + 2i \leqslant 2a_r + 2q_r.$$

Bearing in mind that $q_r = b_r - a_r + 1$ we have

$$2a_r + 2i \leqslant 2b_r + 2.$$

Since $b_r = \min\{t, \lfloor (r+1)/2 \rfloor\}$, we get from the former equality

$$2a_r + 2i \leqslant 2\left\lfloor \frac{r+1}{2} \right\rfloor + 2 \leqslant r+3.$$

Then $2i \leqslant r - 2a_r + 3$ and

$$i - j = 2i - (i+j) \leqslant r - i - j - 2a_r + 3 \leqslant n - a_r - j.$$

Thus, if $i \geqslant j$ then for $\lambda_1 = i - j$, $\lambda_2 = r - i - j - 2a_r + 3$ we have $(\lambda_1, \lambda_2) \in \mathscr{P}_{r-2j-2a_r+3,\,2}$. Next, from the assumption $a_r \geqslant r-n+2$ we get $\lambda_2 = r - i - j - 2a_r + 3 \leqslant n - a_r - j$, and hence by (10) we have

$$X_i(y_j) = \frac{(r-2j-2a_r+3)!}{P((\lambda_1, \lambda_2))}.$$

It follows that $[X_i(y_j)]_{i,\, j=1,2,\ldots,q_r}$ is a triangular matrix with the elements on the principal diagonal equal to 1, and so $\{y_1, \ldots, y_{q_r}\}$ is linearly independent.

Case 2. $a_r \leqslant r-n+1$. In this case $r-n+1 \geqslant a_r \geqslant 1$ and then

$$\mathscr{B}_r = \left\{ f^i(v) \wedge f^{r-i}(v) \colon r-n+1 \leqslant i \leqslant \left\lfloor \frac{r-1}{2} \right\rfloor \right\}.$$

Observe that since, by definition we have $q_r = b_r - a_r + 1$, we get

$$q_r \leqslant t - \left(\left\lceil \frac{r+1}{2} \right\rceil - n + t + 1 \right) + 1.$$

Therefore

$$q_r + r - n \leqslant \left\lfloor \frac{r-1}{2} \right\rfloor,$$

and so we can define $X_i = \theta_{i+r-n}$, $i = 1, 2, \ldots, q_r$.

For $i, j = 1, 2, \ldots, q_r$,

$$\begin{aligned} X_i(y_j) &= \theta_{i+r-n}(\pi_r(z_{r-2j-2a_r+3,\, j+a_r-1})) \\ &= \sum_{\substack{\lambda \in \mathscr{P}_{r-2j-2a_r+3,\, 2} \\ \lambda_2 \leqslant n-j-a_r}} \frac{(r-2j-2a_r+3)!}{P(\lambda)} \\ &\qquad \times \theta_{i+r-n}(f^{\lambda_1+j+a_r-2}(v) \wedge f^{\lambda_2+j+a_r-1}(v)) \\ &= \sum_{\substack{\lambda \in \mathscr{P}_{r-2j-2a_r+3,\, 2} \\ \lambda_2 \leqslant n-j-a_r}} \frac{(r-2j-2a_r+3)!}{P(\lambda)} \delta_{\lambda_1,\, i-j-a_r+r-n+2}. \end{aligned} \tag{11}$$

If $i-j-a_r+r-n+2 < 0$, then all $\delta_{\lambda_1,\, i-j-a_r+r-n+2}$ vanish and $X_i(y_j) = 0$.

Suppose $i-j-a_r+r-n+2 \geqslant 0$. Since $i \geqslant 1$, we have $n-i-j-a_r+1 \leqslant n-j-a_r$. Also, from $i \leqslant q_r$ we get

$$\begin{aligned} 2i &\leqslant 2b_r - 2a_r + 2 \\ &\leqslant 2t - 2\left(\left\lceil \frac{r+1}{2} \right\rceil - n + t + 1 \right) + 2 \\ &\leqslant -2\left\lceil \frac{r+1}{2} \right\rceil + 2n \\ &\leqslant -r - 1 + 2n, \end{aligned}$$

and thus

$$i-j-a_r+r-n+2 \leqslant n-i-j-a_r+1.$$

Then, for $\lambda_1=i-j-a_r+r-n+2$, $\lambda_2=n-i-j-a_r+1$ we have $(\lambda_1, \lambda_2) \in \mathscr{P}_{r-2j-2a_r+3,\,2}$. Clearly $\lambda_2 \leqslant n-a_r-j$ and by (11) we have

$$X_i(y_j)=\frac{(r-2j-2a_r+3)!}{P((\lambda_1, \lambda_2))}.$$

Using Proposition 2.3 we conclude that

$$X_i(y_j)=\begin{cases} 0 & \text{if } \; i-j-a_r+r-n+2<0 \\ \dfrac{(r-2j-2a_r+3)!\,(2n-2i-r)}{(n-i-j-a_r+2)!\,(i-j-a_r+r-n+2)!} & \\ \qquad \text{if } \; i-j-a_r+r-n+2 \geqslant 0. & \end{cases}$$

Then there exist two invertible matrices P and Q such that $P[X_i(y_j)]_{i,\,j=1,\,\ldots,\,q_r}$ $Q=C(n-a_r, r-n-a_r+2, q_r-1)$.

Next we verify that the conditions for application of Lemma 3.16 to the matrix $C(n-a_r, r-n-a_r+2, q_r-1)$ are fulfilled.

From $r \leqslant M_t \leqslant 2n-3$ and $t \leqslant n/2$ we have

$$\left\lceil \frac{r-p}{2} \right\rceil +1 \leqslant \frac{r-p}{2}+1 \leqslant n-\frac{p-1}{2}<n$$

and

$$\left\lceil \frac{r+1}{2} \right\rceil -n+t+1 \leqslant t<n.$$

Then, by the definition of a_r we get $a_r<n$, that is, $n-a_r \geqslant 1$.

From the assumption $a_r \leqslant r-n+1$ we get $r-n-a_r+2 \geqslant 1$.

From the definitions of a_r, b_r, and q_r we have

$$2q_r-2 \leqslant 2t-2\left(\left\lceil \frac{r+1}{2} \right\rceil -n+t+1\right) \leqslant 2n-r-2,$$

and thus $(r-n-a_r+2)+2(q_r-1) \leqslant n-a_r$. Also, from the definition of a_r we have $p \geqslant r-2a_r+2$. Since $r \geqslant a_r+n-1$ it follows that $p \geqslant n-a_r+1$.

Thus we can apply Lemma 3.16 and conclude that $C(n-a_r, r-n-a_r+2,$ $q_r-1)$ is an invertible matrix. Then also $[X_i(y_j)]_{i,\,j=1,\,\ldots,\,q_r}$ is invertible and $\{y_1, \ldots, y_{q_r}\}$ is linearly independent.

From (8) we have

$$\langle \mathscr{C} \rangle = \sum_{r=1}^{M_t} \langle z_{kj} \colon (k, j) \in \mathscr{S}_r \rangle. \tag{12}$$

This proves Claims 1′ and 1.

Next we prove that the sum in (12) is direct. Suppose

$$\sum_{r=1}^{M_t} \sum_{(k,j) \in \mathscr{S}_r} u_{k,j} z_{k,j} = 0.$$

Then

$$\sum_{r=1}^{M_t} \sum_{(k,j) \in \mathscr{S}_r} u_{k,j} \pi_{M_t}(z_{k,j}) = 0. \tag{13}$$

For $(k, j) \notin \mathscr{S}_{M_t}$ the vector $z_{k,j}$ has weight $k + 2j - 1 < M_t$ and thus $\pi_{M_t}(z_{k,j}) = 0$. Then, by (13) we have

$$\sum_{(k,j) \in \mathscr{S}_{M_t}} u_{k,j} \pi_{M_t}(z_{k,j}) = 0.$$

From Claim 1′ it follows that $u_{k,j} = 0$, for all $(k, j) \in \mathscr{S}_{M_t}$.

If we repeat this procedure with π_s, $s = M_t - 1, M_t - 2, ..., 1$, we conclude that

$$u_{k,j} = 0, \qquad (k, j) \in \mathscr{S}_r, \qquad r = 1, ..., M_t.$$

Then the sum in (12) is direct and $\mathscr{C}$ is linearly independent, which proves Theorem 4.18. ∎

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