Electronic version of an article published as Int. J. Geom. Methods Mod. Phys., 12, (9), 2015, 1560018, 8, doi: 10.1142/S021988781560018X © copyright World Scientific Publishing Company www.worldscientific.com/worldscinet/jgmmp

COVARIANT LIE DERIVATIVES AND FRÖLICHER-NIJENHUIS BRACKET ON LIE ALGEBROIDS

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Abstract. We define covariant Lie derivatives acting on vector-valued forms on Lie algebroids and study their properties. This allows us to obtain a concise formula for the Frölicher-Nijenhuis bracket on Lie algebroids.

1. Introduction

The Frölicher-Nijenhuis calculus was developed in the seminal article [2] and extended to Lie algebroids in [10]. It has proven to be an indispensable tool of Differential Geometry. Indeed, different kinds of curvatures and obstructions to integrability are computed by the Frölicher-Nijenhuis bracket. For example, if $J : TM \to TM$ is an almost-complex structure, then J is complex structure if and only if the Nijenhuis tensor $\mathcal{N}_J = \frac{1}{2}[J, J]_{FN}$ vanishes (this is the celebrated Newlander-Nirenberg theorem [9]). If $F : TM \to TM$ is a fibrewise diagonalizable endomorphism with real eigenvalues and of constant multiplicity, then the eigenspaces of F are integrable if and only if $[F, F]_{FN} = 0$ (see [4]). Further, if $P : TE \to TE$ is a projection operator on the tangent spaces of a fibre bundle $E \to B$, then $[P, P]_{FN}$ is a version of the Riemann curvature (see [5], page 78). Finally, given a Lie algebroid $\mathcal{A}$ and $N \in \Gamma(\mathcal{A}^* \otimes \mathcal{A})$ such that $[N, N]_{FN} = 0$, one can construct a new (deformed) Lie algebroid $\mathcal{A}_N$ (cf. [3, 6]). Moreover, Frölicher-Nijenhuis calculus is useful in geometric mechanics where it allows to give an intrinsic formulation of Euler-Lagrange equations. In this field, Lie algebroids have also been shown to be a useful tool to deal with systems with some kinds of symmetries.

In [8], P. Michor obtained a short expression for the Frölicher-Nijenhuis bracket on manifolds in terms of the covariant Lie derivatives. A formula for the Frölicher-Nijenhuis bracket on Lie algebroids in supergeometric language was obtained by P. Antunes in [1]. In this paper we define some operators relevant for Frölicher-Nijenhuis calculus in the setting of Lie algebroids, including the covariant Lie derivative, and study their properties. In this way we are able to extend Michor's formula for Frölicher-Nijenhuis bracket to Lie algebroids.

2. Covariant Lie derivative on Lie algebroids

Let $(\mathcal{A}, [\ ,\ ], \rho)$ be a Lie algebroid over a manifold M, and E a vector bundle over M. We write $\Omega^k(\mathcal{A}, E) = \Gamma(\wedge^k \mathcal{A}^* \otimes E)$ for the space of skew-symmetric E-valued k-forms on $\mathcal{A}$. If $E = M \times \mathbb{R}$ is the trivial line bundle over M, we denote $\Omega^k(\mathcal{A}, E)$ by $\Omega^k(\mathcal{A})$.

We write Σ_m for the permutation group on $\{1, \dots, m\}$. For k and s such that $k + s = m$, we denote by $\mathrm{Sh}_{k,s}$ the subset of (k, s)-shuffles in Σ_m. Thus $\sigma \in \mathrm{Sh}_{k,s}$

Research partially supported by CMUC, funded by the European program COMPETE/FEDER, by FCT (Portugal) grants PEst-C/MAT/UI0324/2011 (A.D.N. and I.Y.), by MICINN (Spain) grants MTM2011-15725-E, MTM2012-34478 (A.D.N.), and exploratory research project in the frame of Programa Investigador FCT IF/00016/2013.

if and only if

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k), \qquad \sigma(k+1) < \cdots < \sigma(k+s).$$

Similarly, for a triple (k, l, s), such that $k + l + s = m$, we denote by $\mathrm{Sh}_{k,l,s}$ the subset of (k, l, s)-shuffles in Σ_m, that is the set of permutations σ, such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(k+l),$$
$$\sigma(k+l+1) < \cdots < \sigma(k+l+s).$$

For a k-form $\omega \in \Omega^k(\mathcal{A})$ and $\phi \in \Omega^p(\mathcal{A}, E)$, we define the form $\omega \overline{\wedge} \phi \in \Omega^{k+p}(\mathcal{A}, E)$ by

$$(\omega \overline{\wedge} \phi)(Z_1, \ldots, Z_{p+k}) = \sum_{\sigma \in \mathrm{Sh}_{k,p}} (-1)^\sigma \omega\left(Z_{\sigma(1)}, \ldots, Z_{\sigma(k)}\right) \phi(Z_{\sigma(k+1)}, \ldots, Z_{\sigma(k+p)}).$$

Here and everywhere in this paper $Z_1, \ldots, Z_{p+k}$ denote arbitrary sections of the Lie algebroid $\mathcal{A}$. If $E = M \times \mathbb{R}$ is the trivial line bundle over M, we denote $\overline{\wedge}$ by $\wedge$, and $\Omega^*(\mathcal{A})$ becomes a commutative graded algebra with the multiplication given by $\wedge$. Further, note that $\Omega^*(\mathcal{A}, E)$ is an $\Omega^*(\mathcal{A})$-module with the action given by $\overline{\wedge}$. For any $\omega \in \Omega^k(\mathcal{A})$ we define the operator ϵ_ω on $\Omega^*(\mathcal{A}, E)$ by

$$\begin{aligned} \epsilon_\omega : \Omega^*(\mathcal{A}, E) &\to \Omega^{*+k}(\mathcal{A}, E) \\ \phi &\mapsto \omega \overline{\wedge} \phi \end{aligned}$$

Sometimes, given a operator A we will use $\omega \wedge A$ as an alternative notation for $\epsilon_\omega A$.

Let $\phi \in \Omega^p(\mathcal{A}, \mathcal{A})$. For any vector bundle E over M, we define the operator i_ϕ on $\Omega^*(\mathcal{A}, E)$ by

$$(1) \quad (i_\phi \psi)(Z_1, \ldots, Z_{p+k}) = \sum_{\sigma \in \mathrm{Sh}_{p,k}} (-1)^\sigma \psi\left(\phi(Z_{\sigma(1)}, \ldots, Z_{\sigma(p)}), Z_{\sigma(p+1)}, \ldots, Z_{\sigma(p+k)}\right)$$

where $\psi \in \Omega^{k+1}(\mathcal{A}, E)$.

We say that $\nabla \colon \Gamma(\mathcal{A}) \times \Gamma(E) \to \Gamma(E)$ is an $\mathcal{A}$-*connection* on E (see [7]) if

1) ∇_X is an $\mathbb{R}$-linear endomorphism of $\Gamma(E)$;
2) ∇s is a $\mathcal{C}^\infty(M)$-linear map from $\Gamma(\mathcal{A})$ to $\Gamma(E)$;
3) $\nabla_X(fs) = (\rho(X)f)s + f\nabla_X s$ for any $f \in \mathcal{C}^\infty(M)$, $X \in \Gamma(\mathcal{A})$, and $s \in \Gamma(E)$.

The curvature of an $\mathcal{A}$-connection ∇ is defined by

$$R(X, Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

It is easy to check that R is tensorial and skew-symmetric in the first two arguments, thus we can consider R as an element of $\Omega^2(\mathcal{A}, \mathrm{End}(E))$, where $\mathrm{End}(E)$ is the endomorphism bundle of E.

Given an $\mathcal{A}$-connection on a vector bundle E, we define the covariant exterior derivative on $\Omega^*(\mathcal{A}, E)$ by

$$\begin{aligned} (d^\nabla \phi)(Z_1, \ldots, Z_{p+1}) = & \sum_{\sigma \in \mathrm{Sh}_{1,p}} (-1)^\sigma \nabla^E_{Z_{\sigma(1)}} \left(\phi(Z_{\sigma(2)}, \ldots, Z_{\sigma(p+1)})\right) \\ & - \sum_{\sigma \in \mathrm{Sh}_{2,p-1}} (-1)^\sigma \phi\left(\left[Z_{\sigma(1)}, Z_{\sigma(2)}\right], Z_{\sigma(3)}, \ldots, Z_{\sigma(p+1)}\right). \end{aligned}$$

Note that d^∇ is related to the curvature R of ∇^E by the formula

$$((d^\nabla)^2 \phi)(Z_1, \ldots, Z_{p+2}) = \sum_{\sigma \in \mathrm{Sh}_{2,p}} (-1)^\sigma R(Z_{\sigma(1)}, Z_{\sigma(2)})\left(\phi(Z_{\sigma(3)}, \ldots, Z_{\sigma(p+2)})\right).$$

Definition 1. *A* derivation *of degree k on $\Omega^*(\mathcal{A}, E)$ is a linear map $D\colon \Omega^*(\mathcal{A}, E) \to \Omega^{*+k}(\mathcal{A}, E)$ such that*

$$D(\omega \overline{\wedge} \phi) = \overline{D}(\omega) \overline{\wedge} \phi + (-1)^{kp} \omega \overline{\wedge} D(\phi)$$

for all $\omega \in \Omega^p(\mathcal{A})$ and $\phi \in \Omega^*(\mathcal{A}, E)$, where $\overline{D}\colon \Omega^*(\mathcal{A}) \to \Omega^*(\mathcal{A})$ is some map.

For any derivation D on $\Omega^*(\mathcal{A}, E)$ and $\alpha \in \Omega^*(\mathcal{A})$, we have

$$[D, \epsilon_\alpha] = \epsilon_{\overline{D}\alpha}.$$

In particular, the map $\overline{D}$ is unique for a given derivation D on $\Omega^*(\mathcal{A}, E)$. Let $\omega_1 \in \Omega^{p_1}(\mathcal{A})$, $\omega_2 \in \Omega^{p_2}(\mathcal{A})$. From the following computation

$$\begin{aligned} D((\omega_1 \wedge \omega_2) \overline{\wedge} \phi) &= \overline{D}(\omega_1 \wedge \omega_2) \overline{\wedge} \phi + (-1)^{k(p_1+p_2)} \omega_1 \wedge \omega_2 \overline{\wedge} D(\phi) \\ D(\omega_1 \overline{\wedge} (\omega_2 \overline{\wedge} \phi)) &= \overline{D}(\omega_1) \wedge \omega_2 \overline{\wedge} \phi + (-1)^{kp_1} \omega_1 \overline{\wedge} D(\omega_2 \overline{\wedge} \phi) \\ &= \overline{D}(\omega_1) \wedge \omega_2 \overline{\wedge} \phi + (-1)^{kp_1} \omega_1 \wedge \overline{D}(\omega_2) \overline{\wedge} \phi + (-1)^{k(p_1+p_2)} \omega_1 \wedge \omega_2 \overline{\wedge} D(\phi) \end{aligned}$$

one can see that $\overline{D}$ is a derivation on $\Omega^*(\mathcal{A})$.

It is easy to check that for any given $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$, i_ϕ is a derivation of degree $k-1$, and d^∇ is a derivation of degree 1 on $\Omega^*(\mathcal{A}, E)$. The *covariant Lie derivative* $\mathcal{L}^\nabla_\phi$ is defined as the *graded commutator* $[i_\phi, d^\nabla] = i_\phi d^\nabla + (-1)^k d^\nabla i_\phi$. The graded commutator of two derivations of degree k and l is a derivation of degree $k+l$. In particular, $\mathcal{L}^\nabla_\phi$ is a derivation of degree k for any $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$.

Suppose we have an $\mathcal{A}$-connection ∇ on $\mathcal{A}$. We will say that ∇ is torsion-free if $\nabla_X Y - \nabla_Y X = [X, Y]$ for all X, $Y \in \Gamma(\mathcal{A})$. On every algebroid $(\mathcal{A}, [\ ,\ ], \rho)$, there exists a torsion-free $\mathcal{A}$-connection. Namely, one can take an arbitrary bundle metric on $\mathcal{A}$ and the associated Levi-Civita connection on $\mathcal{A}$. Given $\mathcal{A}$-connections $\nabla^{\mathcal{A}}$ on $\mathcal{A}$ and ∇^E on E, we define $\nabla_X s \in \Omega^p(\mathcal{A}, E)$ for every $s \in \Omega^p(\mathcal{A}, E)$ by

$$(\nabla_X s)(Z_1, \ldots, Z_p) := \nabla^E_X(s(Z_1, \ldots, Z_p)) - \sum_{t=1}^{p} s(Z_1, \ldots, \nabla^{\mathcal{A}}_X Z_t, \ldots, Z_p).$$

It is easy to check that for any $s \in \Omega^k(\mathcal{A}, E)$, $X \in \Gamma(\mathcal{A})$, and a torsion-free $\mathcal{A}$-connection on $\mathcal{A}$, we have $\mathcal{L}^\nabla_X s = \nabla_X s + i_{\nabla X} s$ and $\nabla X = d^\nabla X$. In other words $\nabla_X = \mathcal{L}^\nabla_X - i_{d^\nabla X}$. Motivated by this relation, we define for $\phi \in \Omega^p(\mathcal{A}, \mathcal{A})$ an operator ∇_ϕ on $\Omega^*(\mathcal{A}, E)$ by

$$\nabla_\phi := \mathcal{L}^\nabla_\phi - (-1)^p i_{d^\nabla \phi}. \tag{2}$$

Note that ∇_ϕ depends on two connections: an $\mathcal{A}$-connection on E and a torsion-free $\mathcal{A}$-connection on $\mathcal{A}$. Since ∇_ϕ is a linear combination of two derivations of degree p, we see that ∇_ϕ is a derivation of degree p. The following proposition shows that for $s \in \Omega^*(A, E)$ the map $\nabla s\colon \Omega^*(\mathcal{A}, \mathcal{A}) \to \Omega^*(\mathcal{A}, E)$ is a homomorphism of $\Omega^*(\mathcal{A})$-modules.

Proposition 2. *For any $\omega \in \Omega^p(\mathcal{A})$, $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$, and $s \in \Omega^*(\mathcal{A}, E)$, we have*

$$\nabla_{\omega \overline{\wedge} \phi} s = (\omega \wedge \nabla_\phi) s = \epsilon_\omega \nabla_\phi s = \omega \overline{\wedge} (\nabla_\phi s).$$

Proof. The equation

$$\mathcal{L}^\nabla_{\omega \overline{\wedge} \phi} = \left[i_{\omega \overline{\wedge} \phi}, d^\nabla\right] = \left[\omega \wedge i_\phi, d^\nabla\right] = (-1)^{k+p} (d\omega) \wedge i_\phi + \omega \wedge \mathcal{L}^\nabla_\phi$$

implies that $\omega \wedge \mathcal{L}^\nabla_\phi = \mathcal{L}^\nabla_{\omega \overline{\wedge} \phi} - (-1)^{p+k} i_{(d\omega) \overline{\wedge} \phi}$. Now we have

$$\begin{aligned} \omega \wedge \nabla_\phi &= \omega \wedge \mathcal{L}^\nabla_\phi - (-1)^p \omega \wedge i_{d^\nabla \phi} = \mathcal{L}^\nabla_{\omega \overline{\wedge} \phi} - (-1)^{p+k} i_{(d\omega) \overline{\wedge} \phi} - (-1)^p i_{\omega \overline{\wedge} d^\nabla \phi} \\ &= \mathcal{L}^\nabla_{\omega \overline{\wedge} \phi} - (-1)^{p+k} i_{d\omega \overline{\wedge} \phi + (-1)^k \omega \overline{\wedge} d^\nabla \phi} = \nabla_{\omega \overline{\wedge} \phi}. \end{aligned}$$

∎

It was proven in [10] that the commutator $[i_\phi, i_\psi]$ for $\phi \in \Omega^k(\mathcal{A},\mathcal{A})$ and $\psi \in \Omega^l(\mathcal{A},\mathcal{A})$ is given by the formula

$$[i_\phi, i_\psi] = i_{i_\phi \psi} - (-1)^{(k-1)(l-1)} i_{i_\psi \phi}. \tag{3}$$

Theorem 3. *Let ∇ be a torsion-free $\mathcal{A}$-connection on $\mathcal{A}$ and ∇^E be an $\mathcal{A}$-connection on a vector bundle E. For $\phi \in \Omega^k(\mathcal{A},\mathcal{A})$ and $\psi \in \Omega^l(\mathcal{A},\mathcal{A})$ we have on $\Omega^*(\mathcal{A},E)$*

$$[\nabla_\phi, i_\psi] = i_{\nabla_\phi \psi} - (-1)^{k(l-1)} \nabla_{i_\psi \phi}. \tag{4}$$

Proof. First we check the claim for $\phi = X \in \Gamma(\mathcal{A})$ and $\psi = Y \in \Gamma(\mathcal{A})$. Let $s \in \Omega^{p+1}(\mathcal{A},E)$. We get

$$\begin{aligned}(\nabla_X i_Y s)(Z_1,\dots,Z_p) &= \nabla^E_X(s(Y,Z_1,\dots,Z_p)) - \sum_{t=1}^{p} s(Y,Z_1,\dots,\nabla_X Z_t,\dots,Z_p)\\ &= (\nabla_X s)(Y,Z_1,\dots,Z_p) + s(\nabla_X Y, Z_1,\dots,Z_p)\\ &= (i_Y \nabla_X s)(Z_1,\dots,Z_p) + (i_{\nabla_X Y} s)(Z_1,\dots,Z_p).\end{aligned}$$

Thus $[\nabla_X, i_Y] = i_{\nabla_X Y}$. Since (4) is additive in ϕ and ψ, it is enough to prove it for $\phi = \alpha\overline{\wedge}X$, $\psi = \beta\overline{\wedge}Y$, where $\alpha \in \Omega^k(\mathcal{A})$, $\beta \in \Omega^l(\mathcal{A})$, and X, $Y \in \Gamma(\mathcal{A})$. Repeatedly using Proposition 2 and $[\nabla_X, i_Y] = i_{\nabla_X Y}$, we get

$$\begin{aligned}\left[\nabla_{\alpha\overline{\wedge}X}, i_{\beta\overline{\wedge}Y}\right] &= [\alpha\wedge\nabla_X, \beta\wedge i_Y] = [\epsilon_\alpha, \beta\wedge i_Y]\,\nabla_X + \epsilon_\alpha\,[\nabla_X, \beta\wedge i_Y]\\ &= (-1)^{kl}\epsilon_\beta\,[\epsilon_\alpha, i_Y]\,\nabla_X + \epsilon_\alpha[\nabla_X,\epsilon_\beta] i_Y + \epsilon_\alpha\epsilon_\beta[\nabla_X, i_Y]\\ &= -(-1)^{kl-l}\epsilon_\beta\epsilon_{i_Y\alpha}\nabla_X + \epsilon_\alpha\epsilon_{\nabla_X\beta} i_Y + \epsilon_\alpha\epsilon_\beta i_{\nabla_X Y}\\ &= i_{\alpha\wedge\nabla_X\beta\overline{\wedge}Y + \alpha\wedge\beta\overline{\wedge}\nabla_X Y} + (-1)^{(k-1)l}\nabla_{\beta\wedge i_Y\alpha\overline{\wedge}X}\\ &= i_{\alpha\overline{\wedge}\nabla_X(\beta\overline{\wedge}Y)} + (-1)^{(k-1)l}\nabla_{\beta\wedge i_Y(\alpha\overline{\wedge}X)}\\ &= i_{\nabla_{\alpha\overline{\wedge}X}(\beta\overline{\wedge}Y)} + (-1)^{(k-1)l}\nabla_{i_{\beta\overline{\wedge}Y}(\alpha\overline{\wedge}X)}.\end{aligned}$$

□

To formulate the next result, we extend the definition of R by defining for any $\phi \in \Omega^k(\mathcal{A},\mathcal{A})$ and $\psi \in \Omega^l(\mathcal{A},\mathcal{A})$ the form $R(\phi,\psi) \in \Omega^{k+l+1}(\mathcal{A},\mathcal{A})$ as follows

$$\begin{aligned}&R(\phi,\psi)(Y_1,\dots,Y_{k+l+1}) =\\ &\qquad= \sum_{\sigma\in\mathrm{Sh}_{k,l,1}} R(\phi(Y_{\sigma(1)},\dots,Y_{\sigma(p)}), \psi(Y_{\sigma(p+1)},\dots,Y_{\sigma(p+q)}))Y_{\sigma(p+q+1)}.\end{aligned}$$

Theorem 4. *Let ∇ be a torsion-free $\mathcal{A}$-connection on $\mathcal{A}$ and ∇^E a flat $\mathcal{A}$-connection on a vector bundle E over M (i.e. ∇^E is a representation of $\mathcal{A}$). Then for any $\phi \in \Omega^k(\mathcal{A},\mathcal{A})$, $\psi \in \Omega^l(\mathcal{A},\mathcal{A})$, we have the following equality on $\Omega^*(\mathcal{A},E)$*

$$[\nabla_\phi, \nabla_\psi] = \nabla_{\nabla_\phi\psi} - (-1)^{kl}\nabla_{\nabla_\psi\phi} - i_{R(\phi,\psi)}. \tag{5}$$

Proof. First we prove (5) for $\phi = X, \psi = Y \in \Gamma(\mathcal{A})$. For $s \in \Omega^p(\mathcal{A})$, we get

$$\begin{aligned}(\nabla_X\nabla_Y s)(Z_1,\dots,Z_p) &= \nabla^E_X(\nabla^E_Y s(Z_1,\dots,Z_p)) - \sum_{s=1}^{p}\nabla^E_Y s(Z_1,\dots,\nabla_X Z_s,\dots,Z_p)\\ &= \nabla^E_X\nabla^E_Y(s(Z_1,\dots,Z_p)) - \sum_{s=1}^{p}\nabla^E_X(s(Z_1,\dots,\nabla_Y Z_s,\dots,Z_p))\\ &\quad - \sum_{s=1}^{p}\nabla^E_Y(s(Z_1,\dots,\nabla_X Z_s,\dots,Z_p) + \sum_{s=1}^{p} s(Z_1,\dots,\nabla_Y\nabla_X Z_s,\dots,Z_p)\\ &\quad + \sum_{s\neq t} s(Z_1,\dots,\nabla_Y Z_t,\dots,\nabla_X Z_s,\dots,Z_p).\end{aligned}$$

By anti-symmetrization of the above formula in X and Y and using that ∇^E is flat, we get

$$[\nabla_X,\nabla_Y]\, s(Z_1,\dots,Z_p) = \nabla^E_{[X,Y]}(s(Z_1,\dots,Z_p)) - \sum_{s=1}^{p} s(Z_1,\dots,[\nabla_X,\nabla_Y]Z_s,\dots,Z_p).$$

Further

$$(\nabla_{\nabla_X Y} - \nabla_{\nabla_Y X})s(Z_1,\dots,Z_p) = \nabla^E_{\nabla_X Y - \nabla_Y X}(s(Z_1,\dots,Z_p)) - \sum_{s=1}^{p} s(Z_1,\dots,(\nabla_{\nabla_X Y} - \nabla_{\nabla_Y X})Z_s,\dots,Z_p).$$

Taking the difference of the last two formulas and using the definition of R and that ∇ torsion-free, we have

$$(([\nabla_X,\nabla_Y] - \nabla_{\nabla_X Y} + \nabla_{\nabla_Y X})s)(Z_1,\dots,Z_p) = (-i_{R(X,Y)}s)(Z_1,\dots,Z_p).$$

Since (5) is additive in ϕ and ψ, it is enough to prove it for $\phi = \alpha\overline{\wedge}X$ and $\psi = \beta\overline{\wedge}Y$, where $\alpha \in \Omega^k(\mathcal{A})$, $\beta \in \Omega^l(\mathcal{A})$, and X, $Y \in \Gamma(\mathcal{A})$. Using the already proved case and Proposition 2, we get

$$\begin{aligned}[\nabla_{\alpha\overline{\wedge}X},\nabla_{\beta\overline{\wedge}Y}] &= [\alpha\wedge\nabla_X,\beta\wedge\nabla_Y] = [\epsilon_\alpha,\beta\wedge\nabla_Y]\nabla_X + \epsilon_\alpha[\nabla_X,\beta\wedge\nabla_Y]\\ &= (-1)^{kl}\epsilon_\beta[\epsilon_\alpha,\nabla_Y]\nabla_X + \epsilon_\alpha[\nabla_X,\epsilon_\beta]\nabla_Y + \epsilon_\alpha\epsilon_\beta[\nabla_X,\nabla_Y]\\ &= -(-1)^{kl}\epsilon_\beta\epsilon_{\nabla_Y\alpha}\nabla_X + \epsilon_\alpha\epsilon_{\nabla_X\beta}\nabla_Y + \epsilon_\alpha\epsilon_\beta(\nabla_{\nabla_X Y} - \nabla_{\nabla_Y X} - i_{R(X,Y)}).\end{aligned}$$

Repeatedly using Proposition 2, we see that $[\nabla_{\alpha\overline{\wedge}X},\nabla_{\beta\overline{\wedge}Y}]$ can be written as $\nabla_\theta + i_\tau$, where

$$\begin{aligned}\theta &= -(-1)^{kl}\beta\wedge\nabla_Y\alpha\overline{\wedge}X + \alpha\wedge\nabla_X\beta\overline{\wedge}Y + \alpha\wedge\beta\overline{\wedge}\nabla_X Y - \alpha\wedge\beta\overline{\wedge}\nabla_Y X\\ &= \alpha\overline{\wedge}\,\nabla_X(\beta\,\overline{\wedge}Y) - (-1)^{kl}(\beta\,\overline{\wedge}\,\nabla_Y(\alpha\overline{\wedge}X)) = \nabla_\phi\psi - (-1)^{kl}\nabla_\psi\phi\end{aligned}$$

and

$$\tau = -\alpha\wedge\beta\overline{\wedge}R(X,Y) = -R(\alpha\overline{\wedge}X,\beta\overline{\wedge}Y) = -R(\phi,\psi).$$

This finishes the proof. □

Note that the connection $\nabla^\rho_X f := \rho(X)f$ defined on the trivial line bundle $M\times\mathbb{R}\to M$ is obviously flat. Thus (5) holds on $\Omega^*(\mathcal{A})$, if ∇ is defined via ∇^ρ and any torsion-free connection on $\mathcal{A}$.

3. The Frölicher-Nijenhuis bracket on Lie algebroids

In [10], Nijenhuis defined the Frölicher-Nijenhuis bracket on Lie algebroids of $\phi\in\Omega^k(\mathcal{A},\mathcal{A})$ and $\psi\in\Omega^l(\mathcal{A},\mathcal{A})$ by an equality of operators on $\Omega^*(\mathcal{A})$ equivalent to

$$[\mathcal{L}^\nabla_\phi, i_\psi] = i_{[\phi,\psi]_{FN}} - (-1)^{k(l-1)}\mathcal{L}^\nabla_{i_\psi\phi}. \tag{6}$$

He also obtained a formula for computing $[\phi,\psi]_{FN}$. In the next theorem we give an alternative formula using the covariant Lie derivatives, which extends the one obtained in [8] to the Lie algebroids setting.

Theorem 5. *Let $\phi\in\Omega^k(\mathcal{A},\mathcal{A})$ and $\psi\in\Omega^l(\mathcal{A},\mathcal{A})$. Suppose ∇ be a torsion-free $\mathcal{A}$-connection on $\mathcal{A}$. Then*

$$[\phi,\psi]_{FN} = \mathcal{L}^\nabla_\phi\psi - (-1)^{kl}\mathcal{L}^\nabla_\psi\phi.$$

Proof. By (2) we have

$$[\mathcal{L}^{\nabla}_{\phi}, i_{\psi}] = [\nabla_{\phi} + (-1)^k i_{d^{\nabla}\phi}, i_{\psi}] = [\nabla_{\phi}, i_{\psi}] + (-1)^k [i_{d^{\nabla}\phi}, i_{\psi}].$$

Hence, using (3) and (4) we get

$$[\mathcal{L}^{\nabla}_{\phi}, i_{\psi}] = i_{\nabla_{\phi}\psi} - (-1)^{k(l-1)} \nabla_{i_{\psi}\phi} + (-1)^k i_{i_{d^{\nabla}\phi}\psi} - (-1)^{kl} i_{i_{\psi} d^{\nabla}\phi}.$$

Next, using (2) in the second summand we have

$$\begin{aligned}[\mathcal{L}^{\nabla}_{\phi}, i_{\psi}] = &- (-1)^{k(l-1)} \left(\mathcal{L}^{\nabla}_{i_{\psi}\phi} - (-1)^{k+l-1} i_{d^{\nabla} i_{\psi}\phi}\right) \\ &+ i_{\nabla_{\phi}\psi} + (-1)^k i_{i_{d^{\nabla}\phi}\psi} - (-1)^{kl} i_{i_{\psi} d^{\nabla}\phi}.\end{aligned}$$

Notice that the subscripts of $\mathcal{L}^{\nabla}$ in (6) and in the above formula are the same. Hence, due to the injectivity of $\phi \mapsto i_{\phi}$, we get by comparing the subscripts of i that

$$\begin{aligned}[\phi, \psi]_{FN} =&(-1)^{k(l-1)}(-1)^{k+l-1} d^{\nabla} i_{\psi}\phi + \nabla_{\phi}\psi + (-1)^k i_{d^{\nabla}\phi}\psi - (-1)^{kl} i_{\psi} d^{\nabla}\phi \\ =&\nabla_{\phi}\psi + (-1)^k i_{d^{\nabla}\phi}\psi - (-1)^{kl}(i_{\psi} d^{\nabla}\phi - (-1)^{l-1} d^{\nabla} i_{\psi}\phi)\end{aligned}$$

Finally, using the definitions of ∇_{ϕ} and of $\mathcal{L}^{\nabla}_{\psi}$ we get the claimed result. □

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