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Capacitated single allocation hub location problem— A bi-criteria approach

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Abstract

A different approach to the capacitated single allocation hub location problem is presented. Instead of using capacity constraints to limit the amount of flow that can be received by the hubs, we introduce a second objective function to the model (besides the traditional cost minimizing function), that tries to minimize the time to process the flow entering the hubs. Two bi-criteria single allocation hub location problems are presented: in a first model, total time is considered as the second criteria and, in a second model, the maximum service time for the hubs is minimized. To generate non-dominated solutions an interactive decision-aid approach developed for bi-criteria integer linear programming problems is used. Both bi-criteria models are tested on a set of instances, analyzing the corresponding non-dominated solutions set and studying the reasonableness of the hubs flow charge for these non-dominated solutions. The increased information provided by the non-dominated solutions of the bi-criteria model when compared to the unique solution given by the capacitated hub location model is highlighted.

Keywords: Hub location; Bi-criteria problems; Interactive methods

1. Introduction

The problem of locating hub facilities arises when some traffic must be transported from a set of origins to a set of destinations, but when it is impossible to establish a direct link between each pair of nodes or when it is too expensive to create such links. In this situation, generally a group of nodes is chosen to locate hubs, which will serve as consolidation, switching and distribution centers for traffic. The origins and destinations will be allocated to this set of hubs. So, if i is connected to hub k and j is connected to hub m, flows from i to j (W_{ij}) will be routed from i to k, from k to m and from m to j. It is assumed that hubs are fully interconnected and that the non-hub nodes can be linked directly to, at least, one hub [1–3]. One of the main characteristics of hub systems is that, by gathering the flows, they enable economies of scale resulting in lower transportation costs [4].

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This is a general definition of the basic hub location problem, but there are many variants about this issue: non-hub nodes may be allocated only to one hub—single allocation [5–7] or to several hubs—multiple allocation [8]; the use of direct links between non-hub nodes may be allowed; the location of some (or all) hubs may be fixed; hub nodes may be allowed to be located anywhere in a continuous region—continuous hub location problems [9] or may be chosen from a discrete set of places—discrete hub location problems; there may exist a constraint fixing the number of nodes that will be selected as hubs—p-hub problem [10,11] or a fixed cost for establishing a hub may be considered; instead of choosing nodes to locate hubs, we can select the arcs connecting the non-hub nodes to hubs—hub arc location problem [12,13]. Capacities in hub location problems may assume different aspects: there can be capacities on the hub nodes (limiting the volume of flow into the hub [14–16] or for the total flow through the hub) as well as on the flows between hubs or between hubs and non-hubs; on other hand, a minimum flow value needed to allow service on the link between a non-hub node and a hub may exist [17,18]. Many different cost functions have been studied, for example, flow-dependent cost functions [19]; time functions—latest arrival hub location problem [20].

The scheme chosen for the hub location problem should depend and reflect the reality of the distribution system considered. For more details on the classification of hub location problems, see Campbell [21], Campbell et al. [3] and Bryan et al. [22].

Hub location problems have important applications in transportation and telecommunication systems. The phenomenal growth of the air express package delivery business has been linked to the use of hub-and-spoke networks. Hub location in telecommunication systems has also received a lot of interest, for example, in designing backbone networks and locating concentrators [23]. This problem also arises in many different applications, such as, postal delivery services, airline services (air passenger travel, air freight travel), communication networks (telephone networks, video teleconferences and computer communications), emergency services and logistic systems [21,24,3].

In this paper we will study the capacitated single allocation hub location problem (CSAHLP) (the basic problem, with single allocation of non-hub nodes to hubs, and with upper bounds upon the amount of flow hubs can receive). The paper is organized as follows: Section 2 describes two bi-criteria models for the hub location problem; Section 3 explains the interactive approach used to calculate non-dominated solutions (n.d.s.); Section 4 presents and analyzes computational results and in Section 5 some conclusions are drawn.

2. The bi-criteria model

2.1. The purpose of the bi-criteria model

Bi-criteria models are especially adequate to deal with hub location problems, namely due to the usual conflict between the quality and the cost of the solutions. In this paper we take advantage of a new bi-criteria model to deal with the limitations of a classical capacitated hub location problem. In fact, if a mono-criterion model with capacity constraints is chosen, the options of the decision maker (DM) are limited to the acceptance or rejection of the optimal solution, many times without awareness of the simplifications/omissions of the model. The consideration of several criteria enables the stable part of the DM's preference structure to be fixed [25]. The use of the proposed bi-criteria model will allow the DM to consider the model as the core of a learning oriented decision support tool, enabling a reflection on the different n.d.s. and allowing negotiation with all the actors of the decision process while tolerating hesitations and ambiguities (dealing with the uncertainties associated with the aggregation of the preferences expressed by each criterion). The interactive process looks for a progressive and selective learning of the non-dominated solutions set, clarifying the criteria values aggregation meaning and consequences. Although in some situations it is possible to opt for one alternative in many others the interactive process just enables the elimination of great part of the feasible solutions reducing the final choice to a small part of the non-dominated ones.

In this case, if necessary, these alternatives can be scrutinized using another multi-criteria analysis tool dedicated to discrete problems, where the alternatives are known explicitly and in small number. Of course, this stage looks for a more detailed analysis of this sub-set of the non-dominated alternatives. However, it does not enable the combinatorial nature of feasible solutions to be explored. So, it just should be used for a deeper study of alternatives filtered by the phase one of the process. Furthermore, in some circumstances one must not restrict this second phase of the process to the analysis of the n.d.s. selected by our bi-criteria interactive decision support tool. Dominated solutions, very close, in terms of the objective function values, to one or more of the selected n.d.s. sub-set, should also be considered in the second phase, taking into account that a deeper scrutiny will be done and that they are acceptable in terms of the first phase of the process.

In this paper we propose a bi-criteria decision support tool dedicated to the above referred to first phase of the process. Besides the cost it considers alternatively the minimization of the time the hubs take to process the flow or the minimization of the maximum service time of the hubs. It seems to be suitable and simple enough to be accepted as relevant by the DM and other actors, possibly associated with the decision process. Furthermore, note that usually the classical cost minimizing criterion generates solutions that are characterized by an excessive concentration of the flow on a small number of nodes. To avoid this situation, the classical model restricts the amount of flow that can be redirected or sorted by a hub, generating a problem with capacity constraints. However, this is not free of inconveniences. Let us take the case of a mail distribution center with a limited flow capacity. If during one day they receive slightly more flow than the estimated capacity, they will need extra time to be able of processing the extra flow. It would not be reasonable to refuse the extra amount of flow. This will be even more obvious if we consider an emergency service. Obviously, an emergency service cannot refuse the emergency flow even if it exceeds its capacity.

To deal with these situations, we consider that a more realistic approach would be to try to minimize the total time that hubs take to process the flow or to try to minimize the maximum service time of the hubs, depending on the cases. So, we believe that a bi-criteria approach to the problem would be more helpful to the DM, instead of limiting sharply *a priori* the amount of flow the hubs can process. The analysis of the different n.d.s. will allow a clarification of the uncertainty associated with the capacity. It can be said that capacity rather than a “hard constraint” is considered as a “soft constraint”.

2.2. Mathematical formulation

Using the notation in Ernst and Krishnamoorthy [15], let us define the following variables: Y^i_{km} is the total amount of flow from location i (origin) that is routed via hubs k and m; $Z_{ik}=1$ if node i is allocated to a hub located at node k and 0 otherwise (in particular, $Z_{kk}=1$ implies that node k is selected as a hub). The input data are given as: n—number of locations; W_{ij}—flow from location i to location j ($O_i=\Sigma_j W_{ij}$ and $D_i=\Sigma_j W_{ji}$); d_{ij}—distance between node i and node j; χ—coefficient of the collection cost (per unit flow) from any non-hub node to any hub node; δ—coefficient of the distribution cost from any hub node to any non-hub node; α ($0\leqslant\alpha\leqslant 1$; $\alpha<\chi$ and $\alpha<\delta$)—coefficient of the transfer cost between any two hubs; F_k—fixed cost of establishing a hub at node k; Γ_k—capacity of collecting flow at hub k (flow into hub k).

Different authors have proposed many formulations for the CSAHLP. Our model is based on the formulation presented by Ernst and Krishnamoorthy [15], which seems to be the most effective approach for this problem.

CSAHLP:

$$\min \quad \sum_{i=1}^{n}\sum_{k=1}^{n} d_{ik}Z_{ik}(\chi O_i+\delta D_i)+\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{m=1}^{n}\alpha d_{km}Y^i_{km}+\sum_{k=1}^{n}F_kZ_{kk}$$

s.t.

$$\sum_{k=1}^{n} Z_{ik}=1, \quad i=1,\ldots,n, \tag{1}$$

$$Z_{ik}\leqslant Z_{kk}, \quad i=1,\ldots,n;\ \ k=1,\ldots,n, \tag{2}$$

$$\sum_{i=1}^{n} O_iZ_{ik}\leqslant \Gamma_k Z_{kk}, \quad k=1,\ldots,n, \tag{3}$$

$$\sum_{m=1}^{n} Y^i_{km}-\sum_{m=1}^{n} Y^i_{mk}=O_iZ_{ik}-\sum_{j=1}^{n} W_{ij}Z_{jk}, \quad i=1,\ldots,n;\ \ k=1,\ldots,n, \tag{4}$$

$$Z_{ik}\in\{0,1\}, \quad i=1,\ldots,n;\ \ k=1,\ldots,n, \tag{5}$$

$$Y^i_{km}\geqslant 0, \quad i=1,\ldots,n;\ \ k=1,\ldots,n;\ \ m=1,\ldots,n. \tag{6}$$

The objective function sums the transportation cost over all (i, j) pairs and the fixed cost of establishing a hub. Eq. (1) together with condition (5) enforces single allocation for each node; constraints (2) assure that no node is assigned to a location unless a hub is opened at that site; constraints (3) are the capacity constraints that limit the amount of flow processed by hub k; Eqs. (4) are the divergence equations for commodity i at node k in a complete graph, when the demand and supply at the node is determined by the allocations Z_{ik}.

The main idea behind this formulation is to track flows on arcs for each specific origin. It appears to be an approach with fewer variables and constraints, leading to a slightly weaker formulation but with faster solution times, therefore allowing for larger problems, than other models reported in the literature, to be solved.

In this work the idea is to remove the capacity constraints (3) and introduce a second objective function that measures the time hubs take to process the flow. Two approaches were considered for the second objective function: summing the total time for processing the flow gathered by the hubs that, of course, has to be minimized (BSAHLP-1); and minimizing the maximum service time on the hubs (BSAHLP-2). In this study, the capacity constraints are discarded and the hubs flow charge is analyzed for the different n.d.s. obtained during the interactive search.

Besides the data already defined, we also use the following additional data: T_k—time hub k takes to process one unit of flow; P_k—fixed time to initiate the service at hub k.

The formulations for the two models proposed in this paper are:

BSAHLP-1:

$$\min \quad \sum_{i=1}^{n}\sum_{k=1}^{n} d_{ik} Z_{ik}(\chi O_i + \delta D_i) + \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{m=1}^{n} \alpha d_{km} Y_{km}^{i} + \sum_{k=1}^{n} F_k Z_{kk}$$

$$\min \quad \sum_{i=1}^{n}\sum_{k=1}^{n} O_i T_k Z_{ik} + \sum_{k=1}^{n} P_k Z_{kk}$$

s.t.

$$\sum_{k=1}^{n} Z_{ik} = 1, \quad i = 1, \ldots, n, \tag{1}$$

$$Z_{ik} \leqslant Z_{kk}, \quad i = 1, \ldots, n; \quad k = 1, \ldots, n, \tag{2}$$

$$\sum_{m=1}^{n} Y_{km}^{i} - \sum_{m=1}^{n} Y_{mk}^{i} = O_i Z_{ik} - \sum_{j=1}^{n} W_{ij} Z_{jk}, \quad i = 1, \ldots, n; \quad k = 1, \ldots, n, \tag{4}$$

$$Z_{ik} \in \{0, 1\}, \quad i = 1, \ldots, n; \quad k = 1, \ldots, n, \tag{5}$$

$$Y_{km}^{i} \geqslant 0, \quad i = 1, \ldots, n; \quad k = 1, \ldots, n; \quad m = 1, \ldots, n. \tag{6}$$

BSAHLP-2:

$$\min \quad \sum_{i=1}^{n}\sum_{k=1}^{n} d_{ik} Z_{ik}(\chi O_i + \delta D_i) + \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{m=1}^{n} \alpha d_{km} Y_{km}^{i} + \sum_{k=1}^{n} F_k Z_{kk}$$

$$\min \max_{k} \quad \left(\sum_{i=1}^{n} O_i T_k Z_{ik} + P_k Z_{kk} \right)$$

s.t.

$$\sum_{k=1}^{n} Z_{ik} = 1, \quad i = 1, \ldots, n, \tag{1}$$

$$Z_{ik} \leqslant Z_{kk}, \quad i = 1, \ldots, n; \quad k = 1, \ldots, n, \tag{2}$$

$$\sum_{m=1}^{n} Y_{km}^{i} - \sum_{m=1}^{n} Y_{mk}^{i} = O_i Z_{ik} - \sum_{j=1}^{n} W_{ij} Z_{jk}, \quad i = 1, \ldots, n; \quad k = 1, \ldots, n, \tag{4}$$

$$Z_{ik} \in \{0, 1\}, \quad i = 1, \ldots, n; \quad k = 1, \ldots, n, \tag{5}$$

$$Y_{km}^{i} \geqslant 0, \quad i = 1, \ldots, n; \quad k = 1, \ldots, n; \quad m = 1, \ldots, n. \tag{6}$$

For the second model, after linearizing the second objective function, we get

BSAHLP-2*:

$$\min \quad \sum_{i=1}^{n}\sum_{k=1}^{n} d_{ik} Z_{ik}(\chi O_i + \delta D_i) + \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{m=1}^{n} \alpha d_{km} Y_{km}^{i} + \sum_{k=1}^{n} F_k Z_{kk}$$

$$\min \quad \beta$$

s.t.

$$\beta \geqslant \sum_{i=1}^{n} O_i T_k Z_{ik} + P_k Z_{kk}, \quad k = 1, \ldots, n, \tag{7}$$

$$\sum_{k=1}^{n} Z_{ik} = 1, \quad i = 1, \ldots, n, \tag{1}$$

$$Z_{ik} \leqslant Z_{kk}, \quad i = 1, \ldots, n; \quad k = 1, \ldots, n, \tag{2}$$

$$\sum_{m=1}^{n} Y_{km}^{i} - \sum_{m=1}^{n} Y_{mk}^{i} = O_i Z_{ik} - \sum_{j=1}^{n} W_{ij} Z_{jk}, \quad i = 1, \ldots, n; \quad k = 1, \ldots, n, \tag{4}$$

$$Z_{ik} \in \{0, 1\}, \quad i = 1, \ldots, n; \quad k = 1, \ldots, n, \tag{5}$$

$$Y_{km}^{i} \geqslant 0, \quad i = 1, \ldots, n; \quad k = 1, \ldots, n; \quad m = 1, \ldots, n. \tag{6}$$

3. Solution approaches for the bi-criteria model

We start this section by introducing some definitions that will be used throughout this paper.

3.1. Terminology for multi-criteria theory

A multiple objective linear problem (MOLP) can be written as

$$\begin{aligned}
\min \quad & f_1(x) \\
\min \quad & f_2(x) \\
& \vdots \\
\min \quad & f_k(x) \\
\text{s.t.} \quad & x \in S,
\end{aligned}$$

where k is the number of objectives, $f_i(x)$ is the ith objective function ($i = 1, 2, \ldots, k$) and S is the feasible region.

Definition 1 (*Steuer [26]*). Let $z^1 = (f_1(x^1), f_2(x^1), \ldots, f_k(x^1))$ and $z^2 = (f_1(x^2), f_2(x^2), \ldots, f_k(x^2)) \in \Re^k$ be two criteria vectors. Then, z^1 dominates z^2 if and only if $z^1 \leqslant z^2$ and $z^1 \neq z^2$ (i.e., $f_i(x^1) \leqslant f_i(x^2)$ for all i and $f_i(x^1) \neq f_i(x^2)$ for at least one i).

In other words, a criterion vector is non-dominated if it is not dominated by any other feasible criterion vector.

Definition 2 (*Steuer [26]*). A feasible solution x^{e} to problem MOLP is said to be efficient if and only $f(x^{\mathrm{e}})$ is non-dominated.

Definition 3 (*Steuer [26]*). The ideal point or ideal vector for problem MOLP is a point in the objective space, $f^* = [f_1^*, f_2^*, \ldots, f_k^*]$, such that f_i^* ($i = 1, 2, \ldots, k$) is the optimal objective function value for the problem

$$\begin{aligned}
\min \quad & f_i(x) \\
\text{s.t.} \quad & x \in S.
\end{aligned}$$

When we consider the optimal objective function value for each problem individually, we are referring to the lexicographic minimums, f_i^* $(i = 1, 2, \ldots, k)$. These values altogether, $f^* = [f_1^*, f_2^*, \ldots, f_k^*]$ form the ideal vector.

MOLPs rarely have points that simultaneously optimize all the objectives (e.g. the ideal point is not feasible). Rather, they have a set of efficient solutions, from which the DM must choose the one he/she prefers the most.

Definition 4. A reference point, which is an aspiration level, for problem MOLP is a point in the objective space, $f' = [f_1', f_2', \ldots, f_k']$, such that f_i' $(i = 1, 2, \ldots, k)$ is the value the DM wishes to obtain for the objective function *i*.

Usually the reference point is the ideal point, but it may correspond to other values of interest to the DM.

Definition 5. A reservation level, for problem MOLP is a point in the objective space, $\overline{f} = [\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_k]$, such that $\overline{f}_i$ $(i = 1, 2, \ldots, k)$ is the worst acceptable value for the objective function *i*.

3.2. The interactive approach

In order to analyze the information provided by the bi-criteria models we use an interactive method based upon the progressive and selective learning of the n.d.s. set (see [27,28]).

According to Larichev and Nikiforov [29], good interactive methods are those that use information relative to the objective function values in the dialogue with the DM, in order to create a system of preferences. This may lead to two different ways of acting. According to the terminology of Nakayama [30]:

- to fix aspiration levels (the objective function values the DM wishes to achieve), in general, close to the ideal value (optimistic point of view);
- to fix reservation levels (the worst acceptable values for the objective functions—pessimistic point of view).

The interactive framework (see Fig. 1) developed to this type of bi-criteria approach has several characteristics, namely:

(i) there are no irrevocable decisions along the process;
(ii) the method is not too demanding with respect to the information required from the user in each interaction, i.e., in the interaction phase the user is able to indicate a sub-region to carry on the search for n.d.s. in two (simple) different ways:
  - by indicating aspiration levels (optimistic reference points) concerning the objective function values;
  - by indicating reservation levels (pessimistic point of view) concerning the objective function values;
(iii) it enables the user to find any non-dominated solution of the problem, namely:
  - supported n.d.s. (those located on the frontier of the convex hull);
  - non-supported n.d.s. (in the duality 'gaps' between the adjacent points that define the convex hull);
(iv) on the operational side, a single criterion mixed integer programming problem (whose structure remains almost unchanged, with computational advantages) has to be solved at each step.

This method (inspired by Ferreira et al. [31]) starts by calculating the two lexicographic minima and involves two main phases: a dialogue phase with the DM and a calculation phase.

During the dialogue phase, the DM is asked to give indications about the sub-region to carry on the search for new n.d.s. If using reservation levels concerning the objective function values, then the information can be transmitted in two different ways: by indicating upper bounds on the values of both objective functions or by indicating two non-dominated solutions candidate to be adjacent. If using aspiration levels concerning the objective function values, then the DM can transmit the information by indicating the reference point he/she would like to achieve.

In the calculation phase, two procedures were used to determine n.d.s. on the region of interest, according to the procedure used to indicate the sub-region to carry on the search:

- the minimization of a Chebyshev distance to a reference point;
- the minimization of a weighted sum of the objective functions.

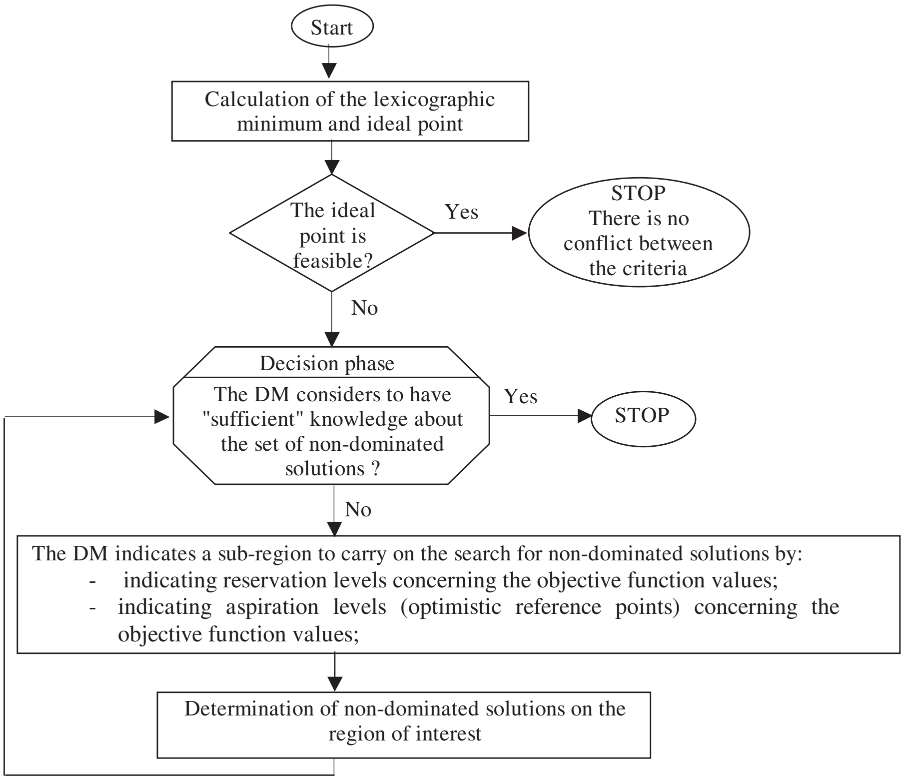

Fig. 1. General diagram of the interactive procedure inspired by Ferreira et al. [31]

3.2.1. *Minimization of a Chebyshev distance to a reference point*

To use the interactive procedure minimizing the Chebyshev distance to a reference point (see [26,32]), the first reference point will be the ideal vector (this is adequate in our case, but other reference points could be used). So, we have to calculate the ideal vector z^{**} as follows:

$$
\begin{aligned}
z_i^{**} &= z_i^* - \varepsilon_i \\
&= \min\{f_i(x) : x \in S\} - \varepsilon_i, \quad i = 1, 2.
\end{aligned}
$$

We have used $\varepsilon_i = 0.2$.

With a z^{**} ideal criterion vector and the weighting vectors

$$\theta \in \Theta = \left\{ \theta \in \Re^k : \theta_i > 0, \sum_{i=1}^{k} \theta_i = 1 \right\},$$

we can define the weighted Chebyshev metrics as (recall that we are considering a minimization criterion):

$$\|z - z^{**}\|_\infty^\theta = \max_{(i=1,\ldots,k)} \{\theta_i |z_i - z_i^{**}|\},$$

measuring the distance between $z \in \Re^k$ and z^{**}.

To measure the distance between $z \in \Re^k$ and z^{**}, we may also define the augmented weighted Chebyshev metrics as

$$|||z - z^{**}|||_\infty^\theta = \|z - z^{**}\|_\infty^\theta + \rho \sum_{i=1}^{k} |z_i - z_i^{**}|,$$

$$|||z - z^{**}|||_\infty^\theta = \max_{(i=1,\ldots,k)} \{\theta_i |z_i - z_i^{**}|\} + \rho \sum_{i=1}^{k} |z_i - z_i^{**}|$$

with $\rho > 0$ sufficiently small.

As $(z_i - z_i^{**})$ will never be negative, we can drop the absolute value signs, writing:

$$|||z - z^{**}|||_\infty^\theta = \max_{(i=1,\ldots,k)} \{\theta_i (z_i - z_i^{**})\} + \rho \sum_{i=1}^{k} (z_i - z_i^{**}).$$

The program for finding the points in Z closest to z^{**} according to the augmented weighted Chebyshev metric is

$$\begin{aligned}
\min \quad & \left\{\mu + \rho \sum_{i=1}^{k} (z_i - z_i^{**})\right\} \\
\text{s.t.} \quad & \mu \geqslant \theta_i (z_i - z_i^{**}) \quad (1 \leqslant i \leqslant k), \\
& f_i(x) = z_i \quad (1 \leqslant i \leqslant k), \\
& x \in S.
\end{aligned}$$

Specifying for our model BSAHLP-1 (considering $\rho = 0.000001$), and recalling that z_1 and z_2 have no bounds,

$$\begin{aligned}
\min \quad & \mu + 0.000001 w_1 + 0.000001 w_2 \\
\text{s.t.} \quad & \mu - \theta_1 w_1 \geqslant 0, \\
& \mu - \theta_2 w_2 \geqslant 0, \\
& w_1 - z_1^+ + z_1^- = z_1^{**}, \\
& w_2 - z_2^+ + z_2^- = z_2^{**}, \\
& z_1^+ - z_1^- - f_1(x) = 0, \\
& z_2^+ - z_2^- - f_2(x) = 0, \\
& x \in S, \\
& \mu, w_1, w_2, z_1^+, z_1^-, z_2^+, z_2^- \geqslant 0,
\end{aligned}$$

where S stands for the decision space defined by constraints (1), (2), (4)–(6).

For the weighting vectors $\theta \in \Theta$, we have considered (accordingly to Steuer [26]),

$$\theta_i = \frac{1}{(z_i - z_i^{**})} \cdot \left(\sum_{i=1}^{2} \frac{1}{(z_i - z_i^{**})}\right)^{-1} \quad (i = 1, 2).$$

When the reference point is changed, the process is repeated considering z^{**} as the new reference point. The interactive procedure would continue, changing the reference point (as it will be explained in the example of paragraph 3.2.3) and repeating the process.

3.2.2. Minimization of a weighted sum of the objective functions

Although the reference point procedure of the previous paragraph could be adapted to the interactive approach using reservation levels, we decided to introduce another procedure based on the optimization of weighted sums of

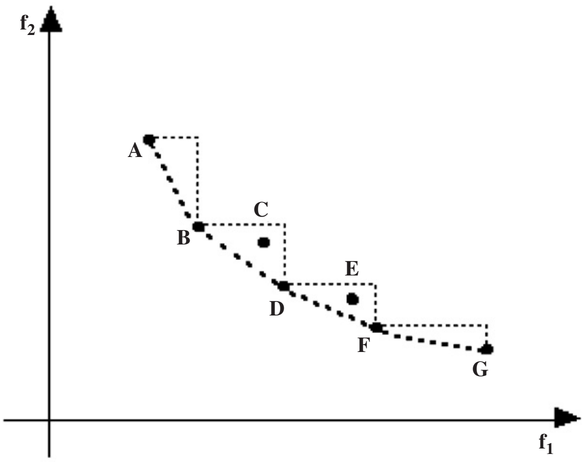

Fig. 2. Graphic representation of the objective space.

the objective functions considering two extra constraints (regarding the reservation levels). Note that the weights are just operational parameters that can be fixed by the analyst or even by default using equal weights. In any case they are not elicited during the dialogue phase with the DM. This procedure seems to be more easily understood by the DM in the case of asking him/her to fix reservation levels interactively. Furthermore, it enables the comparison of the computational efficiency of both procedures.

In bi-criteria integer linear programming (BILP) models, besides the n.d.s. on the efficient frontier of the convex hull in the objective space (supported non-dominated solutions—A, B, D, F and G) other n.d.s. may exist, called unsupported (C and E), inside the duality gaps, dominated by convex combinations of the supported ones, as can be seen in Fig. 2.

In discrete problems, the optimization of weighted sums of the different objectives will only guarantee the generation of part of the n.d.s. set. As presented by Ross and Soland [33], to obtain all n.d.s. of a BILP model like, for example,

$$\begin{aligned} \min \quad & f_1(x) \\ \min \quad & f_2(x) \\ \text{s.t.} \quad & x \in S \end{aligned}$$

a parametric constrained problem of the type

$$\begin{aligned} \min \quad & \lambda_1 f_1(x) + \lambda_2 f_2(x) \\ \text{s.t.} \quad & f_1(x) \leqslant \phi_1, \\ & f_2(x) \leqslant \phi_2, \\ & x \in S \end{aligned}$$

can be solved for different values of (ϕ_1, ϕ_2) and considering (λ_1, λ_2) such that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 > 0, \lambda_2 > 0$. Obviously, some unsupported n.d.s. can be the most appropriate for the DM. So, this sub-set of n.d.s. must be considered.

In the calculation phase we optimize a single criterion problem representing the weighted sum of both objective functions, imposing limits on their values accordingly to the preferences expressed by the DM during the dialog phase. This corresponds to the optimization of the problem below, where $\varepsilon > 0$ is sufficiently small to allow the calculation of a new non-dominated solution distinct from $Z^{(t)}$ and $Z^{(r)}$. We use f_1 and f_2 as the two objective functions of the

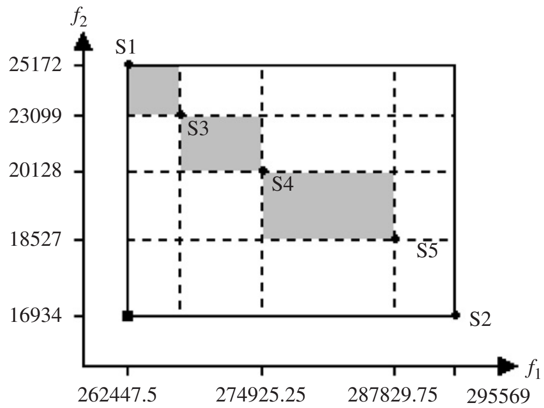

Fig. 3. Graphic representation of the objective space.

proposed bi-criteria model:

$$\begin{aligned} \min \quad & \lambda f_1(x) + (1-\lambda) f_2(x) \\ \text{s.t.} \quad & f_1(x) \leqslant f_1^{(t)} - \varepsilon, \\ & f_2(x) \leqslant f_2^{(r)} - \varepsilon, \\ & x \in S, \\ & 0 < \lambda < 1. \end{aligned}$$

We assume that $f_1^{(r)} < f_1^{(t)}$ and $f_2^{(t)} < f_2^{(r)}$; S stands for the decision space defined by constraints (1), (2), (4)–(6). If this problem has an optimal solution, then it will be a new non-dominated solution to the bi-criteria model. Otherwise, it is possible to conclude that the non-dominated solutions $Z^{(t)}$ and $Z^{(r)}$ are adjacent.

A general solver for mixed integer linear programming CPLEX was used to solve this single criterion problem.

A graphic representation of the objective space is presented to the DM with the non-dominated solutions already known, indicating those that are adjacent and showing the regions where new n.d.s. may still exist.

In Fig. 3, we show an example of a graphic representation of the objective space presented to the DM, and indicate the n.d.s. calculated for BSAHLP-1.

In this example, five n.d.s. have been calculated; S_1 and S_2 stand for the lexicographic minima of f_1 and f_2; the white region illustrates the inexistence of new non-dominated solutions (S_5 and S_2 are adjacent n.d.s.) and the gray region indicates the unexplored areas where new n.d.s. may still exist.

Let us take the case where the DM wishes to continue the search for new non-dominated solution in the region between S_4 and S_5. Then, in the calculation phase, we would solve the mono-criterion problem:

$$\begin{aligned} \min \quad & \lambda f_1(x) + (1-\lambda) f_2(x) \\ \text{s.t.} \quad & f_1(x) \leqslant 287\,829.50, \\ & f_2(x) \leqslant 20\,127, \\ & x \in S, \\ & 0 < \lambda < 1. \end{aligned}$$

The optimal solution to this problem would be S_6 with $f_1(S_6) = 280\,542.25$ and $f_2(S_6) = 19\,414$. After updating the objective space, the DM is presented with Fig. 4.

These two phases go on alternately, only ending when the DM considers having sufficient knowledge of the n.d.s. set. The DM has the option to calculate the entire n.d.s. set.

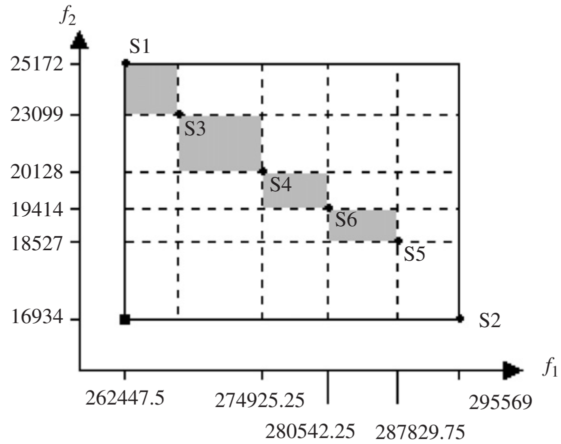

Fig. 4. Updated representation of the objective space.

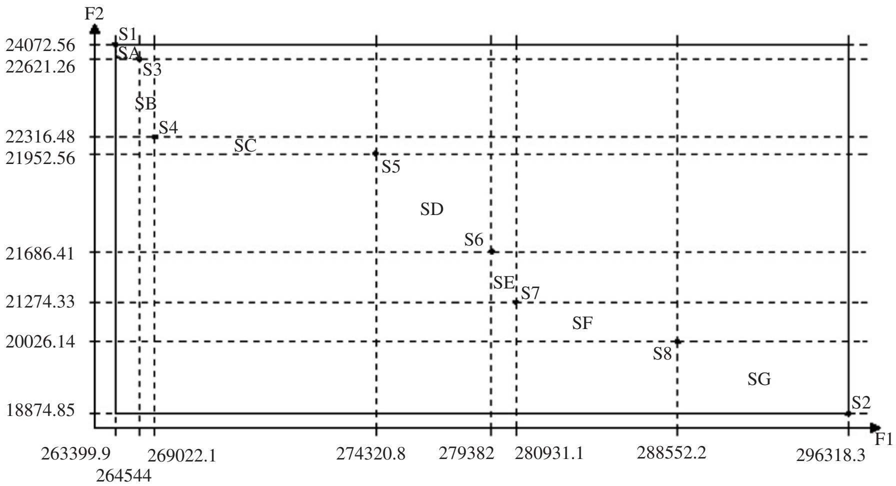

Fig. 5. Objective space for problem $10TL$ (using the optimization of a weighted sum of both objective functions).

3.2.3. Comparison between both interactive procedures

Obviously, whether using the optimization of a weighted sum of both objective functions or using the minimization of an augmented weighted Chebyshev distance during the calculation phase, the n.d.s. set is exactly the same.

Let us consider problem $10TL$ (problem with 10 nodes, fixed costs of type T and capacities of type L—for more details on problem classification, see Section 4.1.). When using the weighted sum of objective functions to calculate the n.d.s. for the first model, we get the objective space presented in Fig. 5. Notice that this problem was entirely explored with a total of eight n.d.s. We begin by calculating the two lexicographic minima and then, as explained in Section 3.2.2, we obtain the remaining non-dominated solutions. The white areas marked as SA, SB, SC, SD, SE, SF and SG, represent explored regions where there are no n.d.s. Notice that it was necessary to explore these areas in order to be sure that they did not include any other non-dominated solution.

If we use the minimization of a Chebyshev distance to a reference point we will, obviously, get the same n.d.s. set. The difference appears on the white areas and the searching process. It is possible to eliminate some of these areas of the objective space without having to explore the eventual existence of new n.d.s. Let us see how.

Once again, we begin by calculating the two lexicographic minima and the correspondent ideal point (z^{**}) (see Fig. 6).

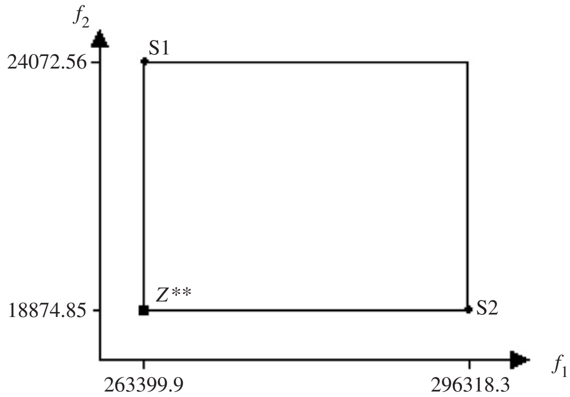

Fig. 6. Lexicographic minimums and the ideal point.

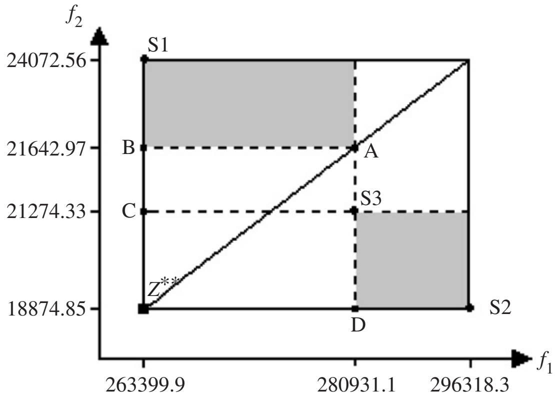

Fig. 7. Solution S_3.

Then, we minimize an augmented weighted Chebyshev distance to the ideal point, solving the following problem:

$$
\begin{aligned}
\min \quad & \mu + 0.000001W_1 + 0.000001W_2 \\
\text{s.t.} \quad & \mu - 0.136W_1 \geqslant 0, \\
& \mu - 0.864W_2 \geqslant 0, \\
& W_1 - Z_1^+ + Z_1^- = -263401, \\
& W_2 - Z_2^+ + Z_2^- = -18875.0, \\
& f_1(x) - Z_1^+ + Z_1^- = 0, \\
& f_2(x) - Z_2^+ + Z_2^- = 0, \\
& x \in S, \\
& \mu, W_1, W_2, Z_1^+, Z_1^-, Z_2^+, Z_2^- \geqslant 0
\end{aligned}
$$

obtaining the optimal solution S_3 (see Fig. 7).

Observing the diagonal equation and as $f_1(S_3) = 280\,931.1$, we can calculate $f_2(A) = 21\,642.97$.

The optimality of S_3 implies that there will not exist any solution on the rectangle $ABZ^{**}D$. So, it is possible to consider explored also the area $ABCS_3$. In the previous method, we would have to explore also this area. It is possible, for every non-dominated solution, to find this area through the calculation of the diagonal equation.

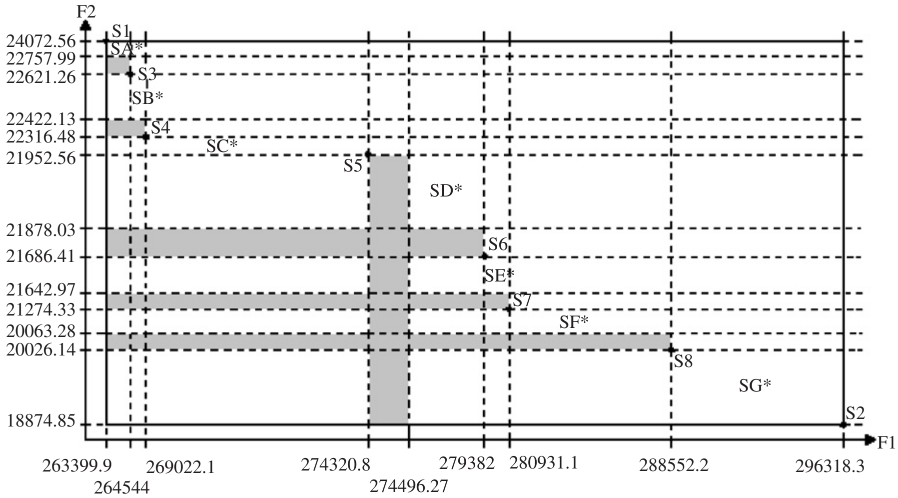

Fig. 8. Objective space for problem $10TL$ (using the minimization of an augmented weighted Chebyshev distance).

Table 1
CPU times (in seconds) for both interactive procedures

Solutions	First method (WCD)	Second method (WSOF)
S1	1.92	1.92
S3	0.36	0.99
S4	0.98	0.62
S5	1.28	0.91
S6	1.91	1.68
S7	1.65	1.49
S8	1.19	0.84
S2	0.17	0.17
SA*/SA	1.68	2.63
SB*/SB	1.12	1.15
SC*/SC	1.50	1.62
SD*/SD	2.08	2.38
SE*/SE	1.44	1.15
SF*/SF	2.03	1.91
SG*/SG	1.63	1.12

Next, the DM would be asked to indicate whether he/she wishes to continue the search for new n.d.s. He/she would have the opportunity to indicate one of two possible reference points: point B or point D (these are the best possible objective values in the gray rectangles where n.d.s. may still exist). This procedure would go on until the DM wishes to stop. For this problem in particular, and as said before, we have studied the entire n.d.s. set presented in Fig. 8.

The gray areas represent those regions that are eliminated from the search process by the diagonal equation. The white areas marked as SA*, SB*, SC*, SD*, SE*, SF* and SG*, as before, represent regions where there are no n.d.s. These white areas are smaller than before because they were reduced by the diagonal equation.

If we observe that the mono-criterion problem to solve in each interaction uses four more constraints than the weighted sum of the objective functions, we would expect higher processing times. In Table 1 we present the CPU times for both interactive procedures.

We can conclude that when comparing the CPU times, the second procedure (WSOF—optimization of a weighted sum of both objective functions) generally takes less time when calculating n.d.s., but takes a longer time when exploring the white areas (those areas where there are no new n.d.s.). This probably occurs because in the first procedure (WCD—minimization of a Chebyshev distance to a reference point) those areas are reduced by the diagonal equation.

Although we only present the comparison of the two procedures considering problem $10TL$ and model BSAHLP-1, the same type of results was obtained for all the problems tested.

4. Computational results

4.1. Test data

The computational results were carried on using the AP data set, electronically available from OR_Library (http://mscmga.ms.ic.ac.uk/info.html).

This data set consists of 200 nodes representing postcode districts and their coordinates and flow volumes. It also includes capacities and fixed costs on the nodes. There are two types of capacities and fixed costs: tight (T) and loose (L). Problems with values of type T are more difficult to solve. For every problem size there are four types of problems: *LL*, *LT*, *TL* and *TT*. The problems are denoted as *nFC*, where *n* stands for the number of nodes, *F* is the type of fixed cost and *C* is the type of capacity. The AP data set considers $\chi = 3$, $\alpha = 0.75$ and $\delta = 2$.

Our bi-criteria approach was applied to the data instances with 10, 20, 25 and 40 nodes. A total of 16 problems were analyzed (for each model BSAHLP-1 and BSAHLP-2).

To build the second objective function we needed to generate values for the times to process the flow (T_k and P_k) as the AP data set does not contemplate these values. This data was generated as a function of the capacity values available in the AP data set. For the time hub k takes to process one unit of flow (T_k), we assumed that hub k capacity was expressed in units of flow for one day comprised of eight hours of work. Choosing to express time in seconds, we calculated the time needed to process one unit of flow for each hub. For example, if we consider a node with capacity equal to 2199 units of flow per day (in 28 800 s), the results show that one unit of flow would take 13 s to be processed. For the fixed time to initiate service at hub k (P_k), we considered that it should be an increasing function (in a concave way) on the capacity of each node (values available in Appendix A).

4.2. N.d.s. set analysis for the bi-criteria models

After using the interactive procedures to calculate the n.d.s. set for the two bi-criteria models we established some comparisons between this set of solutions and the unique solution obtained with formulation CSAHLP. We observed that the 16 problems could be placed together in three different groups: a first one representing the problems for which the optimal solution of model CSAHLP is one of the n.d.s. of the bi-criteria model; a second one that groups those situations where the optimal solution of model CSAHLP corresponds to the lexicographic minimum of the first objective function of the bi-criteria model and a third group where the optimal solution of CSAHLP is a dominated solution of the bi-criteria model. We decided to calculate the entire set of non-dominated solutions in order to discuss the advantages of this type of approach when compared to the capacitated hub location model.

To compare all these situations we calculated the tradeoffs between the values of the two objective functions for the different n.d.s. set, relatively to the optimal solution of CSAHLP. These tradeoffs were calculated as

$$\frac{\Delta f_2^B / f_2^C}{\Delta f_1^B / f_1^C},$$

where f_1^C and f_2^C stand for the values of both objective functions for the optimal solution of CSAHLP; Δf_1^B and Δf_2^B represent, for both objective functions, the difference between the values of the current solution of BSAHLP and the optimal solution of CSAHLP. This measure allows us to analyze how much f_2 changes when f_1 varies in one unit. We can examine how much we have to penalize the value of one objective function, in order to improve the other. Obviously, all tradeoffs will be non-positive, indicating that both criteria values are inversely proportional: the decrease of one objective function will make the increase of the other.

When we analyze solutions with lower value for f_2 and higher value for f_1 (than the optimal solution of CSAHLP), we will be interested in higher tradeoffs, meaning that the increase of the value of f_1 by 1% is intended to produce the biggest reduction possible on the value of f_2. On the other hand, when we observe solutions with better value for f_1 and worse value for f_2, we will choose smaller tradeoffs, meaning that in order to reduce f_1 by 1%, we will penalize the value of f_2 the least possible.

Table 2
Main characteristics of problems in Group I (BSAHLP-1)

Problem	Number of n.d.s.	N.d.s. that improve f_1	N.d.s. that improve f_2	Number of n.d.s. with flow excess
$10LT$	22	11	10	17
$20LT$	17	10	6	16
$25LL$	31	5	25	8
$25TL$	29	1	27	4
$40LL$	42	3	38	9

We have also calculated the amount (if any) of flow that exceeds the capacity of the hubs.

Let us analyze in more detail the three groups of problems.

4.2.1. The BSAHLP-1 model

Group I: This group contains those problems where the optimal solution of the single criterion model belongs to the set of n.d.s. of the bi-criteria model. In this case the bi-criteria model identifies alternative solutions that allow the DM to improve the value of one of the two criteria: the DM will know a set of solutions with lower costs (and higher service times) than the unique original solution and an alternative set characterized by lower service times (and higher costs). The DM will be able to analyze the tradeoffs between the two criteria, study the excess of flow the hubs will have to process and, according to his/her preferences, he/she can choose a final solution that reflects his/her value system.

Five problems are in this situation: $10LT$, $20LT$, $25LL$, $25TL$ and $40LL$. Their main characteristics can be summarized in Table 2.

Let us observe problem $25LL$: in a total of 31 n.d.s., five allow improving cost values and 25 allow improving service time values. Analyzing the tradeoffs of four of these solutions, we can say that by increasing the value of service times by 0.92% (solution S5) the DM can reduce the costs by 1%. This is possible by exceeding the capacity of hub 18 by 6.1%.

Other possibility (solution S13) is to increase service times by 0.75% generating a reduction on the cost of 1%, with a slightly excessive flow (0.76%) on hub 18.

Alternatively, the DM may choose to improve service times: by increasing the value of total costs by 1%, he/she may reduce the value of service times by 7.27% (solution S3) or by 6.73% (solution S7). None of these solutions present an excess of flow in the hubs.

So, it is possible to see that the DM has some interesting alternatives to the unique solution of CSAHLP, which will improve the value of one criterion without a major increase on the value of the other one. This is possible, in some situations, even without exceeding the hubs capacity.

A similar analysis could be made for the remaining solutions of this problem and also for the other four problems of Group I.

As it is not possible to present the entire non-dominated solution set, we show in Appendix B the characteristics of some selected n.d.s. for the five problems in this group.

Group II: In this group we consider the problems for which the optimal solution of the single criterion model corresponds to the lexicographic minimum of the cost function for the bi-criteria model. In this situation, we can conclude that the capacity constraints from the original model are not restrictive and can be eliminated. The entire n.d.s. set of BSAHLP-1 presents solutions with lower service times and greater values on the costs. The DM will be able to know the alternatives characterized by an improvement in service quality.

Five problems are in this situation: $10LL$, $10TL$, $10TT$, $20LL$ and $20TL$. Only problem $10TT$ presents some excessive flow concentration in the hubs. Their special features are summarized in Table 3.

For problem $10LL$ (see Appendix B), we calculated a total of 13 n.d.s., none with excess of flow in the hubs. The DM is presented with a considerable set of solutions that allow reducing the service times significantly without major increase of the costs. There is a situation where it is possible to improve service times by 8.19%, by increasing the costs only by 0.77% (solution S3). Solution S6 gives an alternative with 10.31% less on service times and just 1.58% more on costs. Also, the DM can reduce service times by 20.48% (solution S7), by increasing cost values by 3.76%.

For a more detailed analysis of problems in this group, see Appendix B.

Table 3
Main characteristics of problems in Group II (BSAHLP-1)

Problem	Number of n.d.s.	Number of n.d.s. with flow excess
10LL	13	0
10TL	8	0
10TT	12	7
20LL	23	0
20TL	19	0

Table 4
Main characteristics of problems in Group III (BSAHLP-1)

Problem	Number of n.d.s.	N.d.s. that improve f_1 and f_2	N.d.s. that improve only f_1	N.d.s. that improve only f_2	Number of n.d.s. with flow excess
20TT	6	1	5	0	6
25LT	38	9	15	14	36
25TT	25	18	1	6	25
40LT	52	11	19	22	52
40TL	16	1	0	15	2
40TT	26	9	8	9	26

Group III: This set includes the problems where the optimal solution of CSAHLP is dominated by some of the n.d.s. of BSAHLP-1. Those solutions that dominate the original solution have excessive flow concentration on (one or several) hubs. Otherwise, they would be considered to be the optimal solution of CSAHLP. In this situation, the DM is presented with a set of alternatives that make it possible to improve simultaneously the value of costs and the value of service times. The DM should analyze those solutions that dominate the original one, comparing the improvements on the values of both criteria with the excessive flow incurred.

We found six problems in this situation: 20TT, 25LT, 25TT, 40LT, 40TL and 40TT. Their main characteristics are summarized in Table 4.

Problem 20TT was entirely explored and has a unique solution that dominates the optimal solution of CSAHLP. In this situation, the DM can conclude that, by using only hub 10, the cost values will decrease by 6.04% and service times by 19.15%. This improvement on both criteria is possible by an excess of flow on hub 10 of 82.21% (solution S2).

The analysis of problem 25TT is similar to the previous one but with a greater number of solutions that dominate the solution of model CSAHLP: a total of 18. So, the DM has a considerable set of solutions that grant the improvement on both objective function values. There is a situation where it is possible to reduce cost values by 1.88% and service times by 20.57%. Hub 25 capacity will be exceeded by 28.10% (solution S16). If the DM chooses to implement solution S18, he/she will improve cost values by 1.51% and service times by 20.92 %, exceeding hub 25 capacity by 34.40%.

The analysis of problems 25LT, 40LT, 40TL and 40TT is alike to the former problems with a total of, respectively, 9, 11, 1 and 9 solutions dominating the original one.

To examine the characteristics of the n.d.s. for problems of Group III, see Appendix B.

Once again, the final choice of the solution to be implemented belongs to the DM that should analyze the model, compare the different solutions, reflect and clear some ambiguities that he/she might have.

4.2.2. The BSAHLP-2 model

The interactive procedure using the minimization of a weighted sum of the objective functions was chosen to calculate the n.d.s. of the BSAHLP-2 model. This time, we noticed that all the 16 problems would be placed together in only two groups: Groups I and II described previously in Section 4.2 for the BSAHLP-1 model.

As before, in Group I we include those problems where the DM is faced with a set of solutions that allow the reduction of the costs, and another set that provides the reduction of the maximum service time. The majority of problems fall in this category. Their main characteristics are summarized in Table 5.

Table 5
Main characteristics of problems in Group I (BSAHLP-2)

Problem	Number of n.d.s.	N.d.s. that improve f_1	N.d.s. that improve f_2	Number of n.d.s. with flow excess
$10LT$	18	8	9	8
$20LT$	31	6	24	6
$20TT$	30	12	17	12
$25LL$	33	4	28	4
$25LT$	40	20	19	20
$25TL$	32	1	30	1
$25TT$	33	17	15	17
$40LL$	35	2	32	2
$40LT$	47	20	26	20
$40TL$	27	2	24	2
$40TT$	36	23	12	23

Table 6
Main characteristics of problems in Group II (BSAHLP-2)

Problem	Number of n.d.s.	Number of n.d.s. with flow excess
$10LL$	9	0
$10TL$	11	0
$10TT$	12	0
$20LL$	36	0
$20TL$	27	0

The DM should examine the different solutions, comparing the improvements on the objective function values with the excessive flow incurred. Notice that the solutions with reduced maximum service times (and worse costs) have no excessive flow on the hubs. For example, in problem $10LT$, the DM knows eight alternatives with lower costs and nine with reduced maximum service times. The DM can choose, for example, to reduce the value of costs by 1%, by increasing the maximum service time by 43.78% (solution S10). Excess on the capacity of hub 3 of 47.94% accompanies this situation. Alternatively, he/she can choose to reduce the maximum service time by 17.47%, by increasing the total costs by 1% (solution S7). This is possible with no excess of flow on the hubs.

A similar analysis is possible for the other 10 remaining problems. We have selected four illustrative solutions for each of these 11 problems and present their main characteristics in Appendix C.

In Group II, as in Section 4.2, we have considered those problems where the DM knows a set of alternatives with lower maximum service times (and higher costs). For these problems the DM knows that the capacity constraints are not limiting the optimal solution of the original model. It is important to notice that none of the n.d.s. calculated presents flow excess on the hubs. Five problems belong to this group (Table 6).

Once again, the analysis of these results is identical to the one made for the previous problems. For example, in problem $10LL$ the DM has information that allows him/her to improve maximum service times significantly, with little cost aggravation. He/she may choose to reduce maximum service times by 4.07% by increasing cost values by only 0.30% (solution S3). He/she can also decide for a major modification: improving maximum service time by 46.15%, by aggravating cost values by 6.71% (solution S13). The rest of the problems are summarized in Appendix C.

As previously, the DM chooses the final solution accordingly to his/her preferences.

5. Final remarks

In this paper we have presented a bi-criteria approach to the CSAHLP. We have studied two alternative bi-criteria models: a first one that minimizes the total service time and a second one that minimizes the maximum service time on the hubs. We chose to discard the capacity constraints and analyze the impact of these limits on the different n.d.s.

If in some problems the bi-criteria model was not able to give interesting alternatives to the unique solution of the CSAHLP, because those solutions had major excess of flow on the hubs or they assumed big increases in the value of the costs to reduce just slightly time values (and vice versa), it was not always like that. In the majority of the problems analyzed, the bi-criteria model revealed of great utility, because from its application new interesting and viable solutions were calculated. These new solutions give to the DM alternatives that enable him/her to improve significantly one criterion without a major increase in the other. In many situations, we have seen solutions that allow improving both simultaneously, without exceeding hub capacities significantly.

While it can be argued that model BSAHLP-1 presents some problems with many solutions with excessive flow on the hubs, this is not true for the BSAHLP-2 model. In the first model, we were concerned with the total processing time which, in order to be minimized, originated some solutions that concentrate flow on few hubs. As for the second model, since it tries to minimize the maximum service time, the solutions are characterized by a greater number of hubs, spreading the flow more evenly through the hubs and avoiding the flow excess.

Notice that when the capacity constraints represent real limitations for the hub system, the alternative solutions presented by the bi-criteria models will imply a readjustment on the value of these capacities. This is true for those situations where the n.d.s. are characterized by an excessive amount of flow in the hubs. We have seen many situations where the capacity constraints were not exceeded. Even if they are exceeded, we should note that, in some cases, hub capacities are only estimated values that can allow some changes.

In this paper, we have studied a model that interacts with the DM, giving him/her a major importance in the final decision, while allowing him/her to choose, from a set of significant alternatives, the solution that suits him/her better and the reality of his/her institution.

Appendix A. Values for the times to process the flow

Problems with 10 nodes:

k	Capacities type L		Capacities type T	
	T_K	P_K	T_K	P_K
1	13	610	188	508
2	5	1016	250	441
3	13	613	447	330
4	6	874	144	582
5	7	865	103	686
6	5	986	135	601
7	5	1016	929	229
8	4	1042	214	477
9	6	895	123	629
10	5	1027	100	697

Problems with 20 nodes:

k	Capacities type L		Capacities type T	
	T_K	P_K	T_K	P_K
1	15	562	25	443
2	56	295	33	385
3	9	725	59	287
4	9	756	19	506

k	Capacities type L		Capacities type T	
	T_K	P_K	T_K	P_K
5	5	947	14	597
6	8	789	18	523
7	6	902	123	199
8	9	748	28	415
9	7	835	16	548
10	6	876	13	608
11	14	591	1371	60
12	11	663	16	559
13	15	579	22	472
14	8	768	15	575
15	7	817	297	128
16	13	606	26	431
17	6	899	45	328
18	10	695	436	106
19	7	844	18	516
20	12	645	27	428

Problems with 25 nodes:

k	Capacities type L		Capacities type T	
	T_K	P_K	T_K	P_K
1	18	517	29	407
2	66	271	39	354
3	11	666	70	264
4	10	695	22	466
5	6	871	16	549
6	9	725	21	481
7	7	829	145	183
8	10	688	33	382
9	8	767	19	503
10	8	805	16	559
11	16	544	1600	55
12	13	609	18	514
13	17	532	26	434
14	10	706	17	528
15	9	751	351	118
16	16	558	31	396
17	7	827	53	302
18	12	639	514	97
19	8	776	22	474
20	14	593	31	394
21	8	794	50	311
22	20	493	220	149
23	8	761	32	391
24	69	265	37	364
25	21	486	15	568

Problems with 40 nodes:

k	Capacities type L		Capacities type T	
	T_K	P_K	T_K	P_K
1	23	464	36	366
2	82	244	48	318
3	14	599	86	237
4	12	624	28	418
5	8	782	20	493
6	11	651	26	432
7	9	745	180	164
8	13	618	41	343
9	10	689	24	452
10	9	723	19	502
11	20	488	2057	49
12	16	547	23	462
13	21	478	32	390
14	12	635	22	474
15	11	674	436	106
16	19	501	38	356
17	9	742	66	271
18	15	574	640	87
19	10	697	27	426
20	17	532	39	354
21	10	714	63	279
22	25	443	272	134
23	10	683	40	351
24	86	238	46	327
25	26	437	19	510
26	16	552	20	498
27	30	404	702	83
28	70	264	41	343
29	8	760	34	380
30	8	787	21	480
31	38	359	47	323
32	28	416	17	527
33	85	240	98	223
34	11	656	30	401
35	10	712	28	414
36	8	783	443	105
37	21	484	74	256
38	11	664	25	445
39	13	612	38	359
40	55	298	18	520

Appendix B. Selected non-dominated solutions for BSAHLP-1

Group I:

Problem	Solution	Δf_1 (%)	Δf_2 (%)	Tradeoffs	Maximum excess of flow (%)	Number of hubs with flow excess	Total number of hubs
10LT	S1	−10.65	520.72	−48.87	752.28	2	3
	S10	−1.79	64.83	−36.21	47.94	1	3
	S4	1.30	−6.24	−4.81	0.00	0	3
	S7	2.90	−7.89	−2.72	0.00	0	3
20LT	S4	−2.72	0.85	−0.31	15.04	1	2
	S9	−1.31	0.29	−0.22	4.44	1	2
	S10	0.47	−0.56	−1.21	2.95	1	2
	S2	8.53	−6.33	−0.74	82.21	1	1
25LL	S5	−0.97	0.94	−0.92	6.10	1	2
	S13	−0.26	0.19	−0.75	0.76	1	2
	S3	2.60	−18.87	−7.27	0.00	0	2
	S7	2.84	−19.12	−6.73	0.00	0	2
25TL	S1	−4.72	97.92	−20.74	137.34	1	1
	S31	12.30	−9.25	−0.75	0.00	0	2
	S12	15.66	−10.94	−0.70	0.00	0	2
	S7	18.76	−16.20	−0.86	0.00	0	1
40LL	S1	−0.40	296.22	−940.27	534.82	1	2
	S8	−0.29	287.75	−1273.21	516.86	1	2
	S3	0.07	−21.33	−414.59	0.00	0	2
	S7	0.23	−21.96	−123.61	0.00	0	2

Group II:

Problem	Solution	Δf_1 (%)	Δf_2 (%)	Tradeoffs	Maximum excess of flow (%)	Number of hubs with flow excess	Total number of hubs
10LL	S3	0.77	−8.19	−10.62	0.00	0	3
	S6	1.58	−10.31	−6.52	0.00	0	3
	S5	2.57	−18.36	−7.15	0.00	0	2
	S7	3.76	−20.48	−5.44	0.00	0	2
10TL	S3	0.43	−6.03	−13.88	0.00	0	2
	S5	2.13	−7.29	−3.42	0.00	0	2
	S7	4.15	−8.81	−2.12	0.00	0	2
	S11	9.55	−16.81	−1.76	0.00	0	2
10TT	S4	1.53	−1.76	−1.15	0.00	0	3
	S3	4.04	−6.32	−1.57	0.00	0	2
	S5	5.23	−6.50	−1.24	0.00	0	2
	S8	7.07	−6.51	−0.92	0.00	0	2

Problem	Solution	Δf_1 (%)	Δf_2 (%)	Tradeoffs	Maximum excess of flow (%)	Number of hubs with flow excess	Total number of hubs
20LL	S6	4.26	−7.02	−1.65	0.00	0	2
	S11	4.60	−7.46	−1.62	0.00	0	2
	S8	6.53	−11.13	−1.70	0.00	0	2
	S10	6.98	−11.50	−1.65	0.00	0	2
20TL	S3	0.39	−0.42	−1.08	0.00	0	2
	S7	1.58	−0.84	−0.53	0.00	0	2
	S4	2.60	−5.65	−2.18	0.00	0	1
	S18	15.57	−10.82	−0.70	0.00	0	2

Group III:

Problem	Solution	Δf_1 (%)	Δf_2 (%)	Tradeoffs	Maximum excess of flow (%)	Number of hubs with flow excess	Total number of hubs
20TT	S6	−8.16	226.79	−27.81	476.09	2	2
	S9	−7.36	206.99	−28.13	423.06	2	2
	S5	−6.93	196.13	−28.29	393.94	2	2
	S2	−6.04	−19.15	3.17	82.21	1	1
25LT	S39	−4.73	−0.41	0.09	30.56	1	3
	S36	−4.71	−0.51	0.11	30.56	1	3
	S33	−0.81	−4.25	5.22	21.53	2	3
	S34	−0.73	−4.38	5.98	21.53	2	3
25TT	S3	−9.98	−9.37	0.94	154.70	1	1
	S7	−8.78	−10.24	1.17	74.77	1	2
	S16	−1.88	−20.57	10.93	28.10	1	2
	S18	−1.51	−20.92	13.81	34.40	1	2
40LT	S57	−4.58	−1.43	0.31	108.36	1	2
	S46	−4.57	−1.51	0.33	105.66	1	2
	S47	−3.08	−2.49	0.81	70.94	2	2
	S58	−2.62	−2.61	0.99	66.93	2	2
40TL	S1	−1.93	−6.78	3.52	38.48	1	1
	S3	2.24	−10.99	−4.90	32.06	1	1
	S6	2.95	−11.15	−3.78	0.00	0	2
	S13	3.44	−11.52	−3.35	0.00	0	2
40TT	S28	−4.17	−10.60	2.54	55.54	2	2
	S19	−4.05	−11.00	2.72	61.72	2	2
	S24	−0.21	−12.94	61.56	90.99	1	2
	S34	−0.04	−12.95	313.28	91.20	1	2

Appendix C. Selected non-dominated solutions for BSAHLP-2

Group I:

Problem	Solution	Δf_1 (%)	Δf_2 (%)	Tradeoffs	Maximum excess of flow (%)	Number of hubs with flow excess	Total number of hubs
$10LT$	S14	−6.38	414.38	−64.99	326.86	2	4
	S10	−1.79	78.38	−43.78	47.97	1	3
	S7	0.50	−8.65	−17.47	0.00	0	4
	S23	4.62	−24.57	−5.31	0.00	0	4
$20LT$	S3	−3.70	15.73	−4.25	15.04	1	3
	S10	−1.31	5.25	−4.01	4.44	1	2
	S12	3.55	−20.87	−5.88	0.00	0	3
	S9	4.78	−26.13	−5.46	0.00	0	3
$20TT$	S10	−2.71	9.12	−3.37	6.23	2	2
	S18	−1.52	9.06	−5.96	6.16	2	2
	S13	1.88	−6.38	−3.40	0.00	0	3
	S22	7.21	−19.78	−2.74	0.00	0	3
$25LL$	S4	−0.71	7.56	−10.70	6.10	1	2
	S6	−0.26	2.25	−8.72	0.76	1	2
	S8	2.78	−26.69	−9.61	0.00	0	2
	S3	2.87	−28.28	−9.85	0.00	0	2
$25LT$	S22	−2.50	11.51	−4.60	9.56	2	4
	S24	−0.42	3.05	−7.29	1.13	1	4
	S25	1.31	−6.67	−5.09	0.00	0	3
	S26	2.59	−12.98	−5.02	0.00	0	3
$25TL$	S1	−4.72	198.53	−42.05	137.34	1	1
	S4	0.06	−3.14	−55.52	0.00	0	2
	S3	0.26	−7.53	−29.52	0.00	0	2
	S7	0.97	−12.24	−12.65	0.00	0	2
$25TT$	S13	−4.79	26.47	−4.79	24.35	2	2
	S21	−0.31	10.53	−34.28	8.45	1	3
	S15	2.35	−12.98	−5.53	0.00	0	3
	S14	2.61	−15.16	−5.80	0.00	0	3
$40LL$	S1	−0.40	577.83	−1442.08	534.82	1	2
	S5	−0.29	558.67	−1943.55	516.86	1	2
	S4	0.07	−14.49	−221.48	0.00	0	2
	S3	0.09	−15.70	−179.58	0.00	0	2
$40LT$	S8	−1.04	5.55	−5.34	5.30	2	3
	S13	−0.20	0.99	−5.06	0.68	2	3
	S23	0.53	−1.96	−3.70	0.00	0	3
	S7	0.76	−3.39	−4.46	0.00	0	3
$40TL$	S1	−1.93	38.45	−19.97	38.48	1	1
	S10	−0.02	0.97	−42.00	0.34	1	2
	S6	2.52	−29.48	−11.68	0.00	0	2
	S3	2.67	−31.14	−11.65	0.00	0	2

Problem	Solution	Δf_1 (%)	Δf_2 (%)	Tradeoffs	Maximum excess of flow (%)	Number of hubs with flow excess	Total number of hubs
$40TT$	S26	−4.92	11.75	−2.39	11.84	3	3
	S52	−4.17	11.26	−2.70	11.35	3	3
	S54	17.02	−32.22	−1.89	0.00	0	5
	S18	19.77	−36.52	−1.85	0.00	0	5

Group II:

Problem	Solution	Δf_1 (%)	Δf_2 (%)	Tradeoffs	Maximum excess of flow (%)	Number of hubs with flow excess	Total number of hubs
$10LL$	S3	0.30	−4.07	−13.76	0.00	0	3
	S8	3.44	−22.78	−6.63	0.00	0	3
	S5	5.20	−37.01	−7.12	0.00	0	4
	S13	6.71	−46.15	−6.88	0.00	0	5
$10TL$	S3	0.46	−7.94	−17.19	0.00	0	3
	S8	7.32	−25.01	−3.42	0.00	0	3
	S7	7.73	−30.06	−3.89	0.00	0	3
	S11	9.97	−32.94	−3.31	0.00	0	4
$10TT$	S5	0.46	−8.65	−18.74	0.00	0	3
	S7	5.32	−24.57	−4.62	0.00	0	3
	S15	7.32	−26.69	−3.65	0.00	0	3
	S13	9.74	−35.79	−3.34	0.00	0	3
$20LL$	S3	0.85	−5.18	−6.08	0.00	0	2
	S7	2.23	−10.41	−4.67	0.00	0	2
	S5	6.41	−30.07	−4.69	0.00	0	3
	S12	9.99	−45.74	−4.58	0.00	0	3
$20TL$	S3	0.39	−5.17	−13.27	0.00	0	2
	S5	1.58	−10.29	−6.52	0.00	0	2
	S7	4.19	−22.05	−5.26	0.00	0	2
	S8	10.49	−41.58	−3.96	0.00	0	3

References

[1] Aykin T. On the location of hub facilities. Transportation Science 1988;22(2):155–7.

[2] Campbell JF. Hub location and the p-hub median problem. Operations Research 1996;44(6):923–35.

[3] Campbell JF, Ernst AT, Krishnamoorthy M. Hub location problems. In: Drezner Z, Hamacher H, editors. Facility location: applications and theory. Berlin: Springer; 2002. p. 373–406.

[4] Klincewicz JG. Enumeration and search procedures for a hub location problem with economies of scale. Annals of Operations Research 2002;110:107–22.

[5] Ernst AT, Krishnamoorthy M. Efficient algorithms for the uncapacitated single allocation p-hub median problem. Location Science 1996;4(3): 139–54.

[6] Ernst AT, Krishnamoorthy M. An exact solution approach based on shortest paths for p-hub median problems. INFORMS Journal on Computing 1996;10(2):149–62.
[7] O'Kelly ME. A quadratic integer program for the location of interacting hub facilities. European Journal of Operational Research 1987;32: 393–404.
[8] Ernst AT, Krishnamoorthy M. Exact and heuristic algorithms for the uncapacitated multiple allocation p-hub median problem. European Journal of Operational Research 1998;104:100–12.
[9] O'Kelly ME, Miller HJ. Solution strategies for the single facility minimax hub location problem. Papers in Regional Science 1991;70(4): 367–80.
[10] Klincewicz JG. Heuristics for the p-hub location problem. European Journal of Operational Research 1991;53:25–37.
[11] Skorin-Kapov D, Skorin-Kapov J, O'Kelly ME. Tight linear programming relaxations of uncapacitated p-hub median problems. European Journal of Operational Research 1996;94:582–93.
[12] Campbell JF, Ernst AT, Krishnamoorthy M. Hub arc location problems: part I—introduction and results. Management Science 2005;51(10): 1540–55.
[13] Campbell JF, Ernst AT, Krishnamoorthy M. Hub arc location problems: part II—formulation and optimal algorithms. Management Science 2005;51(10):1556–71.
[14] Ebery J, Krishnamoorthy M, Ernst AT, Boland N. The capacitated multiple allocation hub location problem: formulations and algorithms. European Journal of Operational Research 2000;120:614–31.
[15] Ernst AT, Krishnamoorthy M. Solution algorithms for the capacitated single allocation hub location problem. Annals of Operations Research 1999;86:141–59.
[16] Aykin T. Lagrangean relaxation based approaches to capacitated hub-and-spoke network design problems. European Journal of Operational Research 1994;79:501–23.
[17] Lee H, Shi Y, Nazem S, Kang S, Park T, Sohn M. Multicriteria hub decision making for rural area telecommunication networks. European Journal of Operational Research 2001;133:483–95.
[18] Skorin-Kapov D, Skorin-Kapov J. On tabu search for the location of interacting hub facilities. European Journal of Operational Research 1994;73:502–9.
[19] O'Kelly ME, Bryan D. Interfacility interaction in models of hub and spoke networks. Journal of Regional Science 2002;42(1):145–64.
[20] Kara BY, Tansel B. The latest arrival hub location problem. Management Science 2001;47(10):1408–20.
[21] Campbell JF. Integer programming formulations of discrete hub location problems. European Journal of Operational Research 1994;72: 387–405.
[22] Bryan DL, O'Kelly ME. Hub-and-spoke networks in air transportation: an analytical review. Journal of Regional Science 1999;39:275–95.
[23] Klincewicz JG. Hub location in backbone/tributary network design: a review. Location Science 1998;6:307–35.
[24] Campbell JF. Location and allocation for distribution systems with transshipments and transportation economies of scale. Annals of Operations Research 1992;40:77–99.
[25] Bouyssou D. Modelling inaccurate determination, uncertainty, imprecision using multiple criteria. In: Lockett AG, Islei G, editors. Improving decision making in organisations. Heidelberg: Springer; 1989. p. 78–87.
[26] Steuer RE. Multiple criteria optimization: theory, computation and application. USA: Wiley; 1986.
[27] Clímaco JN, Antunes CH, Alves MJ. Programação linear multiobjectivo. Coimbra: Imprensa da Universidade de Coimbra; 2003.
[28] Ferreira C, Santos BS, Captivo ME, Clímaco J, Silva CC. Multiobjective location of unwelcome or central facilities involving environmental aspects—a prototype of a decision support system. JORBEL 1996;36(2–3):159–72.
[29] Larichev O, Nikiforov A. Analytical survey of procedures for solving multicriteria mathematical programming problems. Towards interactive and intelligent decision support systems 1. Berlin: Springer; 1987 p. 95–104.
[30] Nakayama H. On the components in interactive multiobjective programming methods. Plural rationality and interactive decision processes. Berlin: Springer; 1985.
[31] Ferreira C, Clímaco J, Paixão J. The location-covering problem: a bicriterion interactive approach. Investigación Operativa 1994;4(2):119–39.
[32] Wierzbicki AP, Makowski M, Wessels J, editors. Model-based decision support methodology with environmental applications. Netherlands: Kluwer Academic Publishers; 2000.
[33] Ross G, Soland R. A multicriteria approach to the location of public facilities. European Journal of Operational Research 1980;4:307–21.