ON THE INERTIA SETS OF SOME SYMMETRIC SIGN PATTERNS

C. M. da Fonseca, Coimbra

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Abstract. A matrix whose entries consist of elements from the set $\{+,-,0\}$ is a sign pattern matrix. Using a linear algebra theoretical approach we generalize of some recent results due to Hall, Li and others involving the inertia of symmetric tridiagonal sign matrices.

Keywords: inertia, sign pattern matrix, tridiagonal matrix

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1. Introduction

Several authors have studied properties of a matrix based on combinatorial and qualitative information such as the signs of entries in the matrix. A matrix whose entries are from the set $\{+,-,0\}$ is called a *sign pattern matrix* (or simply, *sign pattern*). For each $n\times n$ sign pattern A there is a natural class of real matrices whose entries have the signs indicated by A, i.e., the *sign pattern class* of a sign pattern A is defined by

$$Q(A)=\{B;\ \operatorname{sign} B=A\}.$$

We are interested in symmetric matrices and in the sign symmetric classes

$$Q_{\mathrm{SYM}}(A)=\{B;\ \operatorname{sign} B=A \text{ and } B=B^T\}.$$

Define the inertia of an $n\times n$ real symmetric matrix H as the triple $\operatorname{In}(H)=(\pi,\nu,\delta)$, where π is the number of positive eigenvalues, ν is the number of negative eigenvalues and $\delta=n-\pi-\nu$ is the number of zero eigenvalues. For a symmetric sign pattern A, we define the *inertia* (*set*) of A to be $\operatorname{In}(A)=\{\operatorname{In}(B);\ B\in Q_{\mathrm{SYM}}(A)\}$.

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We say the sign pattern A *requires unique inertia* and is *sign nonsingular* if every real matrix in $Q(A)$ has the same inertia and is nonsingular, respectively. If two sign patterns A_1 and A_2 are congruent, i.e., if for all $B_1 \in Q_{\mathrm{SYM}}(A_1)$ and $B_2 \in Q_{\mathrm{SYM}}(A_2)$ there exists a nonsingular real matrix S such that $B_1 = SB_2S^T$, then we say that A_1 and A_2 are *sign congruent* and write $A_1 \approx A_2$.

By Sylvester's law of inertia we may say that two sign congruent patterns have the same inertia. For example, the symmetric sign pattern

$$\begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$$

is sign congruent to

$$\begin{bmatrix} 0 & + & 0 \\ + & 0 & 0 \\ 0 & 0 & - \end{bmatrix}$$

and, therefore, requires the unique inertia $(1, 2, 0)$ and, consequently, is sign nonsingular. On the other hand, the tridiagonal sign pattern

$$\begin{bmatrix} + & + & 0 \\ + & + & + \\ 0 & + & + \end{bmatrix}$$

is sign congruent to

$$\begin{bmatrix} + & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & + \end{bmatrix},$$

where $*$ is 0, $+$ or $-$, and, therefore, requires the inertias $(2, 0, 1)$, $(3, 0, 0)$, and $(2, 1, 0)$.

A diagonal sign pattern each of whose entries is $+$ or $-$ is called a *signature pattern*. The square of a signature pattern is a signature pattern with all nonzero entries equal to $+$. A sign pattern such that there is exactly one entry in each row and each column equal to $+$ and all other entries are 0 is called a *permutation pattern*. Two sign congruent patterns by the way of a signature pattern and of a permutation pattern are called, respectively, *signature congruent* and *permutation congruent* patterns.

In this paper we generalize recent results on some symmetric sign patterns due to F. Hall, Z. Li and others (cf. [3], [4], [6], [7]). The results of these authors are based on a graph theoretical approach. Here we use mainly tools from congruences between matrices developed, e.g., by B. Cain and E. Marques de Sá (cf. [1], [2]).

2. Symmetric tridiagonal sign patterns

Given a symmetric tridiagonal sign pattern, the inertia does not depend on the sign of the off-diagonal elements, since two sign patterns under these conditions are signature congruent. Let us denote these entries by $\pm$.

The symmetric tridiagonal sign pattern

$$\begin{bmatrix} 0 & \pm & & & & \\ \pm & * & \pm & & & \\ & \pm & 0 & \pm & & \\ & & \pm & * & \pm & \\ & & & \pm & 0 & \pm \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}_{n\times n} \tag{1}$$

is congruent to

$$\begin{bmatrix} 0 & \pm & & & & \\ \pm & * & 0 & & & \\ & 0 & 0 & \pm & & \\ & & \pm & * & 0 & \\ & & & 0 & 0 & \pm \\ & & & & \ddots & \ddots & \ddots \end{bmatrix},$$

i.e., it is congruent to the direct sum

$$\begin{bmatrix} 0 & \pm \\ \pm & * \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0 & \pm \\ \pm & * \end{bmatrix} \oplus [\,0\,]$$

if n is odd, and to

$$\begin{bmatrix} 0 & \pm \\ \pm & * \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0 & \pm \\ \pm & * \end{bmatrix}$$

if n is even. Since the inertia of each block

$$\begin{bmatrix} 0 & \pm \\ \pm & * \end{bmatrix}$$

is $(1,1,0)$, we can generalize now Proposition 3.1 in [6].

Proposition 2.1. *For the symmetric tridiagonal sign pattern defined in* (1),

(a) *if n is even, then A is sign nonsingular and* $\mathrm{In}(A) = (\frac{n}{2}, \frac{n}{2}, 0)$,

(b) *if n is odd, then A is sign singular and* $\mathrm{In}(A) = (\frac{n-1}{2}, \frac{n-1}{2}, 1)$.

Note that the above proposition is still true for the $n \times n$ sign pattern

$$\begin{bmatrix} * & \pm & & & & \\ \pm & 0 & \pm & & & \\ & \pm & * & \pm & & \\ & & \pm & 0 & \pm & \\ & & & \pm & * & \pm \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

provided n is even.

Let us consider the $n \times n$ sign pattern

$$\begin{bmatrix} + & \pm & & \\ \pm & + & \pm & \\ & \pm & + & \ddots \\ & & \ddots & \ddots \end{bmatrix}.$$

With the $+$ in the $(1,1)$-entry we can, by congruence operations, "eliminate" the off-diagonals entries $(1,2)$ and $(2,1)$. If the new $(2,2)$-entry is 0 and $n > 2$, then we can decompose the sign pattern so that the first block is

$$\begin{bmatrix} + & 0 & \\ 0 & 0 & \pm \\ & \pm & 0 \end{bmatrix},$$

which has inertia $(2,1,0)$. In the case of $n = 2$, the block is simply

$$\begin{bmatrix} + & 0 \\ 0 & 0 \end{bmatrix},$$

which has inertia $(1,0,1)$. If the new $(2,2)$-entry is a $-$, then we can decompose the sign pattern so that the first block is

$$\begin{bmatrix} + & 0 \\ 0 & - \end{bmatrix},$$

which has inertia $(1,1,0)$. The new $(3,3)$-entry is always a $+$ and we restart the procedure from here.

Otherwise, the $(2,2)$-entry is a $+$, and the first block of the composition is simply $[+]$ and we restart the procedure from that entry.

By the above algorithm we can establish the maxima and minima for the number of eigenvalues.

Proposition 2.2. *If*

$$A_+ = \begin{bmatrix} + & \pm & & & \\ \pm & + & \pm & & \\ & \pm & \ddots & \ddots & \\ & & \ddots & \ddots & \pm \\ & & & \pm & + \end{bmatrix}$$

is an $n \times n$ symmetric tridiagonal sign pattern, then $\mathrm{In}(A_\pm)$ *has the form*

$$(n-k,k,0),\ 0 \leqslant k \leqslant \left\lfloor \frac{n}{2} \right\rfloor, \quad \text{or} \quad (n-k,k-1,1),\ 1 \leqslant k \leqslant \left\lfloor \frac{n}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greater integer less than or equal to the real number x.

Given a sign pattern we say that the diagonal (i,i)-entry is in an odd (even) position when i is odd (even). The diagonal (i,i) and (j,j)-entries are said to be in ascending positions provided $i < j$ (not necessarily consecutive).

We can rewrite some results from [6] and [7], generalize them and give a straightforward proof.

Theorem 2.3. *For the symmetric tridiagonal sign pattern*

$$A_* = \begin{bmatrix} * & \pm & & & \\ \pm & * & \pm & & \\ & \pm & \ddots & \ddots & \\ & & \ddots & \ddots & \pm \\ & & & \pm & * \end{bmatrix},$$

where each diagonal entry is 0, $+$ *or* $-$,

(a) *if n is even, then A_* is sign nonsingular if and only if neither two $+$ nor two $-$ diagonal entries in A_* are in odd-even ascending positions, respectively. In this case* $\mathrm{In}(A_*) = (\frac{n}{2}, \frac{n}{2}, 0)$;

(b) *if n is odd, then A_* is sign nonsingular if there is at least one $+$ or $-$ diagonal entry in an odd position, but not both in odd positions, and neither three $+$ nor three $-$ diagonal entries are in odd-even-odd ascending positions, respectively. In this case* $\mathrm{In}(A_*) = (\frac{n+1}{2}, \frac{n-1}{2}, 0)$ *if there are $+$ in odd positions, or* $\mathrm{In}(A_*) = (\frac{n-1}{2}, \frac{n+1}{2}, 0)$ *if there are $-$ in odd positions.*

P r o o f. Suppose that n is even. Without loss of generality we may assume that the first and the last diagonal entries are non-zero. In order for A_* to require unique inertia, when we use congruence relations in order to eliminate the off-diagonal elements, the signs of the diagonal should alternate between + and −.

By Proposition 2.1, if n is odd and neither + nor − diagonal entries are in odd positions, then A_* requires unique inertia $(\frac{n-1}{2}, \frac{n-1}{2}, 1)$. Without loss of generality we may assume that the first diagonal entry is non-zero. Assume that it is a +. Then using the congruence elimination procedure, we can not have − in odd diagonal positions and no three + diagonal entries in odd-even-odd ascending positions. □

The sign pattern

$$\begin{bmatrix} + & + & & & & \\ + & - & - & & & \\ & - & 0 & + & & \\ & & + & 0 & + & \\ & & & + & + & - \\ & & & & - & 0 \end{bmatrix}$$

is congruent to

$$[+] \oplus [-] \oplus \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & - \\ - & 0 \end{bmatrix}$$

and, therefore, requires unique inertia $(3, 3, 0)$.

However, the sign pattern

$$\begin{bmatrix} + & + & & & & \\ + & - & - & & & \\ & - & 0 & + & & \\ & & + & + & + & \\ & & & + & + & - \\ & & & & - & 0 \end{bmatrix}$$

is congruent to

$$[+] \oplus [-] \oplus [+] \oplus [*] \oplus \begin{bmatrix} 0 & - \\ - & 0 \end{bmatrix}$$

and the inertia set is $\{(3, 2, 1), (4, 2, 0), (3, 3, 0)\}$.

Let us give another example. The sign pattern

$$\begin{bmatrix} + & + & & & & & \\ + & - & - & & & & \\ & - & + & + & & & \\ & & + & 0 & + & & \\ & & & + & 0 & - & \\ & & & & - & - & + \\ & & & & & + & + \end{bmatrix}$$

is sign congruent to

$$\begin{bmatrix} + & & & & & & \\ & - & & & & & \\ & & + & & & & \\ & & & - & & & \\ & & & & + & & \\ & & & & & - & \\ & & & & & & + \end{bmatrix}$$

and hence requires unique inertia $(4,3,0)$, but the sign pattern

$$\begin{bmatrix} + & + & & & & & \\ + & - & - & & & & \\ & - & + & + & & & \\ & & + & 0 & + & & \\ & & & + & - & - & \\ & & & & - & - & + \\ & & & & & + & + \end{bmatrix}$$

is congruent to

$$[+] \oplus [-] \oplus [+] \oplus [-] \oplus [*] \oplus [-] \oplus [+]$$

and the inertia set is $\{(3,3,1),(4,3,0),(3,4,0)\}$.

3. Symmetric star sign patterns

We now consider a symmetric tree sign pattern matrix whose associated graph is a star.

Theorem 3.1 [7]**.** *Up to permutation congruence, signature congruence, and negation, a symmetric star sign pattern*

$$A = \begin{bmatrix} * & + & + & \dots & + \\ + & * & & & \\ + & & * & & \\ \vdots & & & \ddots & \\ + & & & & * \end{bmatrix}_{n\times n},$$

where each diagonal entry is 0, $+$ *or* $-$, *requires unique inertia if and only if the diagonal of* A *has the following forms:*

$$(*,\dots,*,0),\ (0,+,\dots,+),\ (-,+,\dots,+).$$

P r o o f. With the exception of the $(1,1)$-entry, if one of the diagonal entries is zero, then

$$A \approx \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} \oplus \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix}_{n-2\times n-2},$$

and A requires unique inertia.

Suppose now that all the diagonal entries are nonzero, possibly with the exception of the $(1,1)$-entry. Then

$$A \approx [*] \oplus \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix}_{n-1\times n-1}.$$

In this case, A requires unique inertia if and only if all the diagonal entries different from the $(1,1)$-entry have the same sign and the $(1,1)$-entry has a sign different from the other diagonal elements or is equal to 0. □

4. Sign patterns with all + off-diagonal entries

Finally, let J_n be the $n \times n$ symmetric sign pattern with all entries equal to $+$. Then

$$J_n \approx [+] \oplus B,$$

where B is a symmetric sign pattern of order $n-1$. Then the set of possible inertias of J_n is

$$\{(\pi, \nu, n-\pi-\nu);\ 1 \leqslant \pi \leqslant n,\ \pi+\nu \leqslant n\}.$$

If one considers $\hat{J}_n$, the $n \times n$ symmetric sign pattern with zero diagonal and $+$ off-diagonal entries, then

$$\hat{J}_n \approx \begin{bmatrix} 0 & + & 0 \\ + & 0 & 0 \\ 0 & 0 & - \end{bmatrix} \oplus B,$$

where B is a symmetric sign pattern of order $n-3$. Therefore the set of possible inertia of $\hat{J}_n$ is

$$\{(\pi, \nu, n-\pi-\nu);\ 1 \leqslant \pi \leqslant n,\ 2 \leqslant \nu \leqslant n,\ \pi+\nu \leqslant n\}.$$

This last result was obtained recently by Gao and Shao [5] via a different approach.

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References

[1] *B. E. Cain and E. Marques de Sá*: The inertia of Hermitian matrices with a prescribed 2×2 block decomposition. Linear and Multilinear Algebra *31* (1992), 119–130. Zbl 0756.15024

[2] *B. E. Cain and E. Marques de Sá*: The inertia of certain skew-triangular block matrices. Linear Algebra Appl. *160* (1992), 75–85. Zbl 0752.15006

[3] *C. Eschenbach and C. R. Johnson*: A combinatorial converse to the Perron-Frobenius theorem. Linear Algebra Appl. *136* (1990), 173–180. Zbl 0739.15004

[4] *C. Eschenbach and C. R. Johnson*: Sign patterns that require real, nonreal or pure imaginary eigenvalues. Linear and Multilinear Algebra *29* (1991), 299–311. Zbl 0764.05015

[5] *Y. Gao and Y. Shao*: The inertia set of nonnegative symmetric sign pattern with zero diagonal. Czechoslovak Math. J. *53* (2003), 925–934.

[6] *F. J. Hall and Z. Li*: Inertia sets of symmetric sign pattern matrices. Numer. Math. J. Chinese Univ. (English Ser.) *10* (2001), 226–240. Zbl 0997.15010

[7] *F. J. Hall, Z. Li and Di Wang*: Symmetric sign pattern matrices that require unique inertia. Linear Algebra Appl. *338* (2001), 153–169. Zbl 0994.15028

[8] *R. A. Horn and C. R. Johnson*: Matrix Analysis, Cambridge University Press, Cambridge. 1985. Zbl 0576.15001

[9] *C. Jeffries and C. R. Johnson*: Some sign patterns that preclude matrix stability. SIAM J. Matrix Anal. Appl. *9* (1988), 19–25. Zbl 0676.15004

Author's address: Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal, e-mail: `cmf@mat.uc.pt`.