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Functorial Quasi-Uniformities on Frames

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Abstract. We present a unified study of functorial quasi-uniformities on frames by means of Weil entourages and frame congruences. In particular, we use the pointfree version of the Fletcher construction, introduced by the authors in a previous paper, to describe all functorial transitive quasi-uniformities.

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1. Introduction

The method of constructing transitive compatible quasi-uniformities for an arbitrary frame [11], extending classical results of Fletcher for quasi-uniform spaces [12], naturally raises the question of its functoriality. The purpose of the present paper is to address this question.

To put this in perspective, we recall that a topological space $(X, \mathcal{T})$ is *uniformizable* if there exists a uniformity $\mathcal{E}$ on X such that the corresponding induced topology $T(\mathcal{E})$ coincides with the given topology $\mathcal{T}$. As is well-known, the topological spaces that are uniformizable are precisely the completely regular ones. This result has a perfect analog in the two-sided theory of quasi-uniform spaces (where they are considered over their induced bitopologies): a bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *quasi-uniformizable*, i.e. there exists a quasi-uniformity $\mathcal{E}$ on X such that $T(\mathcal{E}) = \mathcal{T}_1$ and $T(\mathcal{E}^{-1}) = \mathcal{T}_2$, if and only if it is pairwise completely regular. However, in the one-sided theory, where a quasi-uniformity is considered over a single underlying topology, the resemblance with the symmetric case is over and one gets a striking result: every topological space is quasi-uniformizable. Indeed, for every topological space $(X, \mathcal{T})$, there exists a (transitive) quasi-uniformity $\mathcal{E}_P(\mathcal{T})$ on X, *admissible* on $(X, \mathcal{T})$, that is, which induces as its first topology the given topology $\mathcal{T}$ ([9, 18]). The quasi-uniformity $\mathcal{E}_P(\mathcal{T})$ is nowadays called the *Császár–Pervin quasi-uniformity*. So, every topological space $(X, \mathcal{T})$ gives rise

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to a bitopological space $(X, T(\mathcal{E}_P(\mathcal{T})), T(\mathcal{E}_P(\mathcal{T})^{-1}))$ where $T(\mathcal{E}_P(\mathcal{T})) = \mathcal{T}$, and one can easily see that $T(\mathcal{E}_P(\mathcal{T})^{-1})$ is the topology having the closed sets of $(X, \mathcal{T})$ as a base. The join of these two topologies thus coincides with the well-known *Skula modification* of the topological space $(X, \mathcal{T})$ (that is, the *b-topology* of Skula [24]) and the above bitopological space is referred to as the *Skula bitopological space*.

Let T denote the forgetful functor from the category QUnif of quasi-uniform spaces and uniformly continuous maps to the category Top of topological spaces and continuous maps which assigns to each $(X, \mathcal{E}) \in \mathsf{QUnif}$ its first topology $\mathcal{T}(\mathcal{E})$. A *functorial admissible quasi-uniformity* [5] on the topological spaces is a *T-section*, that is, a functor $F : \mathsf{Top} \to \mathsf{QUnif}$ such that $TF = 1_{\mathsf{Top}}$. In other words, F assigns an admissible quasi-uniformity to each topological space in such a way that continuous maps become uniformly continuous.

In [5], Brümmer proved that the Pervin quasi-uniformity defines the coarsest T-section $\mathcal{C}_1^* : \mathsf{Top} \to \mathsf{QUnif}$, and in [23] (pp. 37–39), Salbany proved that, for any T-section F, the bitopological space underlying $F(X, \mathcal{T})$ is precisely the Skula bitopological space of $(X, \mathcal{T})$ mentioned above. The most important examples of functorial admissible transitive quasi-uniformities have received a great deal of study (in particular, the *well-monotone* quasi-uniformity, which plays a pivotal rôle with respect to the bicompletion and reflective subcategories of the category of T_0-spaces [17]). Brümmer [7] described all functorial admissible transitive quasi-uniformities via a construction due to Fletcher [12]. This is a striking aspect of transitive quasi-uniformities: they can all be obtained by Fletcher's construction, in terms of the interior-preserving open covers of their underlying topological spaces. This, together with its functorial nature, implies that transitive quasi-uniform spaces have a simple and elegant theory and makes the respective category an important subcategory of QUnif.

The present paper is devoted to placing these results in a pointfree context. It is part of a larger program started in [11], motivated by Problem 3 of Brümmer [8], asking for a pointfree formulation of the classical theory of functorial transitive quasi-uniformities. After recalling some basics on frames and quasi-uniform frames (Section 2), we study general functorial frame quasi-uniformities (Section 3). In the remaining sections we apply the general method of constructing compatible transitive quasi-uniformities on an arbitrary frame, introduced in [11], to describe functorial transitive quasi-uniformities.

2. Preliminaries

In this section, we recall the specific notions and facts which will be used later on. For general concepts concerning frames we refer to Johnstone [16] or Pultr [22]. Additional information concerning biframes may be found in [1] and [3]. For details concerning frame entourages and uniform structures see Picado ([20, 21]).

2.1. FRAMES AND BIFRAMES

Pointfree topology is part of the study of *frames* (or *locales*), that is, complete lattices L satisfying the infinite distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$$

for every $x \in L$ and every $S \subseteq L$. This notion generalizes both the lattice of open sets of a topological space and that of a complete Boolean algebra. A *frame homomorphism* $f : L \to M$ is a map between frames which preserves finite meets (including the top element 1) and arbitrary joins (including the bottom element 0). The corresponding category will be denoted by Frm. If L is a frame and $x \in L$ then $x^* := \bigvee\{a \in L \mid a \wedge x = 0\}$ is the *pseudocomplement* of x. Obviously, if $x \vee x^* = 1$, x is complemented and we denote the complement x^* by $\neg x$. Note that, in any frame, the first De Morgan law $(\bigvee_{i \in I} x_i)^* = \bigwedge_{i \in I} x_i^*$ holds but of the second only the trivial inequality survives.

Recall also that a *biframe* is a triple (L_0, L_1, L_2) where L_1 and L_2 are subframes of the frame L_0, which together generate L_0. A *biframe homomorphism*, $f : (L_0, L_1, L_2) \to (M_0, M_1, M_2)$, is a frame homomorphism $f : L_0 \to M_0$ which maps L_i into M_i $(i = 1, 2)$ and BiFrm denotes the resulting category. We shall denote by $\mathsf{CRBiFrm}$ the full subcategory of completely regular biframes [3] and by $K_i : \mathsf{CRBiFrm} \to \mathsf{Frm}$ $(i = 0, 1, 2)$ the functor given by $K_i((L_0, L_1, L_2)) = L_i$.

Further, a biframe (L_0, L_1, L_2) is *strictly zero-dimensional* [1] if it satisfies the following condition or its counterpart with L_1 and L_2 reversed: each $x \in L_1$ is complemented in L_0, with complement in L_2, and L_2 is generated by these complements. Along this paper, we always assume that strictly zero-dimensional biframes satisfy this condition, not its counterpart with L_1 and L_2 reversed.

2.2. THE SKULA BIFRAME OF A FRAME L

The lattice of frame congruences on L under set inclusion is a frame, denoted by $\mathfrak{C}L$. This is the analogue, in pointfree context, of the Skula modification of a topological space (indeed, the spectrum of $\mathfrak{C}L$ is homeomorphic to the Skula modification of the spectrum of L and, consequently, the Skula topology of the spectrum of L is the spatial reflection of $\mathfrak{C}L$ [4]). A good presentation of the congruence frame is given by Frith [14]. Here, we shall need the following properties:

(1) For any $x \in L$, ∇_x and Δ_x are, respectively, the congruences defined by $\{(a, b) \in L \times L \mid a \vee x = b \vee x\}$ and $\{(a, b) \in L \times L \mid a \wedge x = b \wedge x\}$.
(2) Each ∇_x is complemented in $\mathfrak{C}L$ with complement Δ_x.
(3) $\nabla L := \{\nabla_x \mid x \in L\}$ is a subframe of $\mathfrak{C}L$. Let ΔL denote the subframe of $\mathfrak{C}L$ generated by $\{\Delta_x \mid x \in L\}$. Since $\theta = \bigvee\{\nabla_y \wedge \Delta_x \mid (x, y) \in \theta, x \leq y\}$, for every $\theta \in \mathfrak{C}L$, the triple $S(L) := (\mathfrak{C}L, \nabla L, \Delta L)$ is a biframe (usually referred

to as the *Skula biframe* of L [14]). This is the analogue, for frames, of the Skula bitopological space and it is, clearly, a strictly zero-dimensional biframe.

(4) The correspondence $x \mapsto \nabla_x$ defines an epimorphism and a monomorphism $\nabla_L : L \to \mathfrak{C}L$ and gives an isomorphism $L \to \nabla L$, whereas the map $x \mapsto \Delta_x$ is a dual poset embedding $L \to \Delta L$ taking finitary meets to finitary joins and arbitrary joins to arbitrary meets.

The following result from [14] will be crucial in the sequel.

LEMMA (J. Frith [14]). *Let $h : L \to M$ be a frame homomorphism. If each element of $h[L]$ is complemented then there exists a unique frame homomorphism h^∇ making the following diagram commutative:*

$$\begin{array}{ccc} L & \xrightarrow{\nabla_L} & \mathfrak{C}L \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle h^\nabla} \\ & & M \end{array}$$

Proof. Clearly, if there exists such an h^∇, we must have

$$h^\nabla(\nabla_x) = h(x), \tag{2.2.1}$$
$$h^\nabla(\Delta_x) = h^\nabla(\neg\nabla_x) = \neg h(x). \tag{2.2.2}$$

Then, for any $\theta \in \mathfrak{C}L$,

$$\begin{aligned} h^\nabla(\theta) &= h^\nabla\Big(\bigvee\{\Delta_x \wedge \nabla_y \mid (x, y) \in \theta, x \le y\}\Big) \\ &= \bigvee\{\neg h(x) \wedge h(y) \mid (x, y) \in \theta, x \le y\}. \end{aligned}$$

This defines a frame homomorphism $h^\nabla : \mathfrak{C}L \to M$ (for a proof see [14], Theorem 5.17). The uniqueness follows from the fact that ∇_L is an epimorphism. □

For any frame homomorphism $h : L \to M$, consider the map $\overline{h} := (\nabla_M \circ h)^\nabla$

$$\begin{array}{ccc} L & \xrightarrow{\nabla_L} & \mathfrak{C}L \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \overline{h}} \\ M & \xrightarrow[\nabla_M]{} & \mathfrak{C}M \end{array} \tag{2.2.3}$$

given by the lemma. Clearly, by (2.2.1) and (2.2.2), $\overline{h}$ is a biframe map $S(L) \to S(M)$. We refer to the functor

$$\begin{array}{rcl} S : \mathsf{Frm} & \longrightarrow & \mathsf{BiFrm} \\ L & \longmapsto & S(L) \\ (h : L \to M) & \longmapsto & \big(\overline{h} : S(L) \to S(M)\big) \end{array}$$

as the *Skula functor*. Clearly, one has the natural isomorphism $K_1 S \cong 1_{\mathsf{Frm}}$.

2.3. WEIL ENTOURAGES

For a frame L consider the frame $\mathcal{D}(L \times L)$ of all non-void decreasing subsets of $L \times L$, ordered by inclusion. The coproduct $L \oplus L$ will be represented as usual (cf. [16]), as the subset of $\mathcal{D}(L \times L)$ consisting of all C-*ideals*, that is, of sets A for which

$$\{x\} \times S \subseteq A \Rightarrow \left(x, \bigvee S\right) \in A$$

and

$$S \times \{y\} \subseteq A \Rightarrow \left(\bigvee S, y\right) \in A.$$

Since the premise is trivially satisfied whenever $S = \emptyset$, each C-ideal A contains $\mathbf{O} := \{(0, a), (a, 0) \mid a \in L\}$, and $\mathbf{O}$ is the bottom element of $L \oplus L$. Obviously, each $x \oplus y := \downarrow(x, y) \cup \mathbf{O}$ is a C-ideal. The coproduct injections $u_i^L : L \to L \oplus L$ are defined by $u_1^L(x) = x \oplus 1$ and $u_2^L(x) = 1 \oplus x$, so that $x \oplus y = u_1^L(x) \wedge u_2^L(y)$.

For any frame homomorphism $h : L \to M$, the definition of coproduct ensures us the existence (and uniqueness) of a frame homomorphism $h \oplus h : L \oplus L \to M \oplus M$ such that $(h \oplus h) \circ u_i^L = u_i^M \circ h$ $(i = 1, 2)$.

A *Weil entourage* [19] on L is just an element E of $L \oplus L$ for which $\bigvee\{x \in L \mid (x, x) \in E\} = 1$. The collection $\mathit{WEnt}(L)$ of all Weil entourages of L with the inclusion is a partially ordered set with finitary meets (including a unit $1 := L \oplus L$).

If E and F are elements of $L \oplus L$ then

$$E \circ F := \bigvee\{x \oplus y \mid \exists z \in L \setminus \{0\} : (x, z) \in E,\ (z, y) \in F\}.$$

A Weil entourage E is called *transitive* if $E \circ E = E$.

2.4. QUASI-UNIFORM FRAMES

For a system $\mathcal{E}$ (always understood to be non-void) of Weil entourages of a frame L, we write $x \overset{\mathcal{E}}{\lhd}_1 y$ if $E \circ (x \oplus x) \subseteq y \oplus y$, for some $E \in \mathcal{E}$. Similarly, we write $x \overset{\mathcal{E}}{\lhd}_2 y$ whenever $(x \oplus x) \circ E \subseteq y \oplus y$, for some $E \in \mathcal{E}$, and $\mathcal{E}$ is called *admissible* if, for every $x \in L$,

$$x = \bigvee\{y \in L \mid y \overset{\overline{\mathcal{E}}}{\lhd}_1 x\}, \tag{2.4.1}$$

where $\overline{\mathcal{E}}$ stands for $\mathcal{E} \cup \{E^{-1} \mid E \in \mathcal{E}\}$. Note that, since $\overline{\mathcal{E}}$ is symmetric, the partial orders $\overset{\overline{\mathcal{E}}}{\lhd}_1$ and $\overset{\overline{\mathcal{E}}}{\lhd}_2$ do coincide.

An admissible filter $\mathcal{E}$ of $\mathit{WEnt}(L)$ is a *quasi-uniformity* on L if it satisfies the following condition:

(QU) For every $E \in \mathcal{E}$ there exists $F \in \mathcal{E}$ such that $F \circ F \subseteq E$.

A *quasi-uniform frame* is just a pair $(L, \mathcal{E})$ where L is a frame and $\mathcal{E}$ is a quasi-uniformity on L. If $(L, \mathcal{E}_1)$ and $(M, \mathcal{E}_2)$ are quasi-uniform frames, $f : (L, \mathcal{E}_1) \to (M, \mathcal{E}_2)$ is a *quasi-uniform homomorphism* if $f : L \to M$ is a frame homomorphism such that $(f \oplus f)(E) \in \mathcal{E}_2$, for all $E \in \mathcal{E}_1$. The resulting category is denoted by QUFrm.

A quasi-uniform frame $(L, \mathcal{E})$ is called *transitive* if $\mathcal{E}$ has a base consisting of transitive entourages. For more information on transitive quasi-uniformities we refer to [15].

We note further that the partial orders $\overset{\mathcal{E}}{\lhd}_1$ and $\overset{\mathcal{E}}{\lhd}_2$ induce the following important subframes of L:

$$\mathcal{L}_1(\mathcal{E}) := \Big\{x \in L \mid x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd}_1 x\}\Big\},$$
$$\mathcal{L}_2(\mathcal{E}) := \Big\{x \in L \mid x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd}_2 x\}\Big\}.$$

It might be added that the *admissibility condition* (2.4.1) is equivalent to saying that the triple $(L, \mathcal{L}_1(\mathcal{E}), \mathcal{L}_2(\mathcal{E}))$ is a biframe [20]. Then, condition (QU) implies that this is a completely regular biframe [19]. This is the pointfree expression of the classical fact that each quasi-uniform space $(X, \mathcal{E})$ induces a completely regular bitopological structure $(\mathcal{T}_1(\mathcal{E}), \mathcal{T}_2(\mathcal{E})) = (\mathcal{T}(\mathcal{E}), \mathcal{T}(\mathcal{E}^{-1}))$ on X.

We also point out that $\overset{\mathcal{E}}{\lhd}_1$ and $\overset{\mathcal{E}}{\lhd}_2$ may be characterized in the following way [21]:

- $x \overset{\mathcal{E}}{\lhd}_1 y$ if and only if there exists $E \in \mathcal{E}$ such that
$$st_1(x, E) := \bigvee\{\alpha \in L \mid (\alpha, \beta) \in E, \beta \wedge x \neq 0\} \leq y; \tag{2.4.2}$$
- $x \overset{\mathcal{E}}{\lhd}_2 y$ if and only if there exists $E \in \mathcal{E}$ such that
$$st_2(x, E) := \bigvee\{\beta \in L \mid (\alpha, \beta) \in E, \alpha \wedge x \neq 0\} \leq y. \tag{2.4.3}$$

The elements $st_i(x, E)$, $i = 1, 2$, satisfy the following properties, for every $x, y \in L$ [19]:

(S1) $x \leq y \Rightarrow st_i(x, E) \leq st_i(y, E)$, for every $E \in L \oplus L$;
(S2) For every Weil entourage E, $x \leq st_1(x, E) \wedge st_2(x, E)$;
(S3) For every $E, F \in L \oplus L$, $st_i(x, E \cap F) \leq st_i(x, E) \wedge st_i(x, F)$;
(S4) For every $E, F \in L \oplus L$, $st_1(st_1(x, E), F) \leq st_1(x, F \circ E)$ and $st_2(st_2(x, E), F) \leq st_2(x, E \circ F)$;
(S5) For every quasi-uniformity $\mathcal{E}$, $st_i(x, E) \leq y$ for some $E \in \mathcal{E}$ implies the existence of $z \in \mathcal{L}_j(\mathcal{E})$, $j \neq i$, such that $z \wedge x = 0$ and $z \vee y = 1$;
(S6) For every $E \in L \oplus L$, $st_i(\bigvee_J x_j, E) = \bigvee_J st_i(x_j, E)$;
(S7) For every $E \in L \oplus L$ and every frame homomorphism $h : L \to M$,
$$st_i(h(x), (h \oplus h)(E)) \leq h(st_i(x, E)).$$

3. Functorial Compatible Quasi-Uniformities

3.1. THE FORGETFUL FUNCTOR T : QUFrm $\to$ Frm

As we recalled above, for every quasi-uniform frame $(L, \mathcal{E})$, the corresponding biframe $(L, \mathcal{L}_1(\mathcal{E}), \mathcal{L}_2(\mathcal{E}))$ is a completely regular biframe. This correspondence defines a faithful functor T_b : QUFrm $\to$ CRBiFrm. Indeed, for any quasi-uniform homomorphism $h : (L, \mathcal{E}_1) \to (M, \mathcal{E}_2)$, h maps $\mathcal{L}_1(\mathcal{E}_1)$ into $\mathcal{L}_1(\mathcal{E}_2)$: for any $x \in \mathcal{L}_1(\mathcal{E}_1)$, $x = \bigvee\{y \in L \mid y \overset{\mathcal{E}_1}{\lhd_1} x\}$, so $h(x) = \bigvee\{h(y) \mid y \overset{\mathcal{E}_1}{\lhd_1} x\}$; but, by property (S7), $h(y) \overset{\mathcal{E}_2}{\lhd_i} h(x)$ whenever $y \overset{\mathcal{E}_1}{\lhd_i} x$ $(i = 1, 2)$, thus

$$\begin{aligned} h(x) &= \bigvee\{h(y) \mid y \overset{\mathcal{E}_1}{\lhd_1} x\} \\ &\leq \bigvee\{h(y) \mid h(y) \overset{\mathcal{E}_2}{\lhd_1} h(x)\} \\ &\leq \bigvee\{z \in M \mid z \overset{\mathcal{E}_2}{\lhd_1} h(x)\} \leq h(x) \end{aligned}$$

and, consequently, $h(x) \in \mathcal{L}_1(\mathcal{E}_2)$. Similarly, h maps $\mathcal{L}_2(\mathcal{E}_1)$ into $\mathcal{L}_2(\mathcal{E}_2)$ and therefore h is a biframe map from $(L, \mathcal{L}_1(\mathcal{E}_1), \mathcal{L}_2(\mathcal{E}_1))$ into $(M, \mathcal{L}_1(\mathcal{E}_2), \mathcal{L}_2(\mathcal{E}_2))$.

Then, we can consider the functor $T := K_1T_b$: QUFrm $\to$ Frm. This is a faithful functor if we restrict it to the $(L, \mathcal{E})$ such that $T_b(L, \mathcal{E})$ is strictly zero-dimensional:

LEMMA. *Let $h, k : (L, \mathcal{E}_1) \to (M, \mathcal{E}_2)$ be quasi-uniform frame morphisms with $Th = Tk$. If the underlying biframe $T_b(L, \mathcal{E}_1)$ is strictly zero-dimensional, then $h = k$.*

Proof. By what we have seen above, T_bh and T_bk are just biframe maps

$$h, k : (L, \mathcal{L}_1(\mathcal{E}_1), \mathcal{L}_2(\mathcal{E}_1)) \to (M, \mathcal{L}_1(\mathcal{E}_2), \mathcal{L}_2(\mathcal{E}_2)).$$

Their restrictions to subframes $\mathcal{L}_1(\mathcal{E}_1)$ and $\mathcal{L}_2(\mathcal{E}_1)$ are frame maps, respectively $Th, Tk : \mathcal{L}_1(\mathcal{E}_1) \to \mathcal{L}_1(\mathcal{E}_2)$ and, say, $r, s : \mathcal{L}_2(\mathcal{E}_1) \to \mathcal{L}_2(\mathcal{E}_2)$. The strict zero-dimensionality of $T_b(L, \mathcal{E}_1)$ yields the set $C := \{\neg a \mid a \in \mathcal{L}_1(\mathcal{E}_1)\} \subseteq \mathcal{L}_2(\mathcal{E}_1)$, the complements $\neg a$ being taken in L, and such that C generates $\mathcal{L}_2(\mathcal{E}_1)$. Now consider any $b \in \mathcal{L}_2(\mathcal{E}_1)$. Since C is closed under finite meets, we have $b = \bigvee\{\neg a \mid a \in \mathcal{L}_1(\mathcal{E}_1) \text{ and } \neg a \leq b\}$ and thus

$$h(b) = \bigvee\{\neg h(a) \mid a \in \mathcal{L}_1(\mathcal{E}_1) \text{ and } \neg a \leq b\}. \tag{3.1.1}$$

But, for each $a \in \mathcal{L}_1(\mathcal{E}_1)$, we have $h(a) = (Th)(a) = (Tk)(a) = k(a)$. Thus, it follows from (3.1.1) that $h(b) = k(b)$. Consider any $x \in L$. Since $x = \bigvee\{a \wedge b \mid a \in \mathcal{L}_1(\mathcal{E}_1), b \in \mathcal{L}_2(\mathcal{E}_1), a \wedge b \leq x\}$, we have $h(x) = \bigvee\{h(a) \wedge h(b) \mid a \in \mathcal{L}_1(\mathcal{E}_1), b \in \mathcal{L}_2(\mathcal{E}_1), a \wedge b \leq x\}$. So, for the elements considered, we have $h(a) \wedge h(b) = k(a) \wedge k(b)$. It follows then that $h(x) = k(x)$. □

3.2. THE FRITH QUASI-UNIFORMITY

Let (L_0, L_1, L_2) be a strictly zero-dimensional biframe. For any $a \in L_1$ let

$$E_a := (a \oplus 1) \vee (1 \oplus \neg a).$$

This is obviously a transitive Weil entourage of L_0. It is also worth pointing out that, since $(a \oplus 1) \cup (1 \oplus \neg a)$ is already a C-ideal, E_a is simply $(a \oplus 1) \cup (1 \oplus \neg a)$. The following result, which is a particular case of Theorem 5.5 of [15], will be crucial in the sequel.

THEOREM (Hunsaker and Picado [15]). *For any strictly zero-dimensional biframe (L_0, L_1, L_2), the family $\mathcal{S} := \{E_a \mid a \in L_1\}$ is a subbase for a transitive, totally bounded quasi-uniformity $\mathcal{F}$ on L_0, for which $\mathcal{L}_i(\mathcal{F}) = L_i$ $(i = 1, 2)$.*

The quasi-uniformity $\mathcal{F}$ is called the *Frith quasi-uniformity* on L_0.

For any frame L, the Skula biframe $S(L)$ is clearly strictly zero-dimensional. Therefore, by the theorem, $\{E_{\nabla_a} \mid a \in L\}$ is a subbase for a transitive, totally bounded, quasi-uniformity $\mathcal{F}_{\mathfrak{C}L}$ on $\mathfrak{C}L$, satisfying $\mathcal{L}_1(\mathcal{F}_{\mathfrak{C}L}) = \nabla L$. This is the pointfree counterpart of the Pervin quasi-uniformity: starting with a frame L we have a quasi-uniformity on $\mathfrak{C}L$ which generates, as its first subframe, an isomorphic copy of the given frame L.

In what follows we therefore say that a quasi-uniformity $\mathcal{E}$ on a frame M is *compatible* with a given frame L if there exists a frame isomorphism $i : L \to \mathcal{L}_1(\mathcal{E})$ satisfying

$$\bigvee_{\alpha \in I} \Delta_{a_\alpha} = \bigvee_{\beta \in J} \Delta_{b_\beta} \Leftrightarrow \bigvee_{\alpha \in I} \neg i(a_\alpha) = \bigvee_{\beta \in J} \neg i(b_\beta) \tag{3.2.1}$$

for every $\{a_\alpha\}_{\alpha \in I}, \{b_\beta\}_{\beta \in J} \subseteq L$ such that $i(a_\alpha)$ and $i(b_\beta)$ are complemented elements of M, for any $\alpha \in I$, $\beta \in J$.

REMARK. It should be noted that, in the classical theory, for a quasi-uniformity to be *admissible* on a topological space means the same as being *compatible* with the topology. In the case of a quasi-uniform frame $(M, \mathcal{E})$, the structure $\mathcal{E}$, by definition – recall (2.4.1) –, has to be admissible on M but is compatible with a subframe $\mathcal{L}_1(\mathcal{E})$ of M. This is a fundamental change from quasi-uniform space theory to quasi-uniform frame theory. This is also in crucial contrast to the symmetric situation: for any uniform frame $(M, \mathcal{E})$, one has $\mathcal{L}_1(\mathcal{E}) = \mathcal{L}_2(\mathcal{E}) = M$, so the distinction between the two words does not arise and they are used interchangeably.

EXAMPLES. Obviously, any quasi-uniformity $\mathcal{E}$ on $\mathfrak{C}L$ satisfying $\mathcal{L}_1(\mathcal{E}) = \nabla L$ is compatible with L. Therefore, besides the Frith quasi-uniformity $\mathcal{F}_{\mathfrak{C}L}$, every example in [11], Section 8 (namely, the locally finite, the point-finite, the well-monotone, the fine transitive and the semicontinuous quasi-uniformities), is a quasi-uniformity on $\mathfrak{C}L$, compatible with L. In the sequel, we shall refer to these examples as *standard examples*.

In each of the standard examples, $\mathcal{L}_1(\mathcal{E}) = \nabla L$ and $\mathcal{L}_2(\mathcal{E}) = \Delta L$. This is not surprising, in view of the following result, which is the pointfree version of a classical variant of the result by Salbany ([23], pp. 37–39) referred to in the Introduction.

PROPOSITION. *For every frame L and every quasi-uniformity $\mathcal{E}$ on $\mathfrak{C}L$ satisfying $\mathcal{F}_{\mathfrak{C}L} \subseteq \mathcal{E}$ and $\mathcal{L}_1(\mathcal{E}) \subseteq \nabla L$, one has $\mathcal{L}_2(\mathcal{E}) = \Delta L$ and $\mathcal{L}_1(\mathcal{E}) = \nabla L$.*

Proof. Let $\alpha \in \mathcal{L}_2(\mathcal{E})$. Then $\alpha = \bigvee\{\beta \in \mathfrak{C}L \mid \beta \overset{\mathcal{E}}{\lhd}_2 \alpha\}$. By property (S5), $\beta \overset{\mathcal{E}}{\lhd}_2 \alpha$ implies the existence of $\nabla_a \in \mathcal{L}_1(\mathcal{E})$ satisfying $\beta \wedge \nabla_a = 0$ and $\alpha \vee \nabla_a = 1$. This means that $\beta \leq \Delta_a \leq \alpha$. Then, $\Delta_a \overset{\mathcal{E}}{\lhd}_2 \alpha$, since $st_2(\Delta_a, E_{\nabla_a}) = \Delta_a \leq \alpha$. Therefore, $\alpha \leq \bigvee\{\Delta_a \in \mathfrak{C}L \mid \Delta_a \overset{\mathcal{E}}{\lhd}_2 \alpha\} \leq \alpha$ and hence $\alpha \in \Delta L$.

To prove the reverse inclusion we only have to show that $\Delta_a \in \mathcal{L}_2(\mathcal{E})$ for every $a \in L$, which is easy: as already pointed out, $st_2(\Delta_a, E_{\nabla_a}) = \Delta_a$; since $E_{\nabla_a} \in \mathcal{F}_{\mathfrak{C}L} \subseteq \mathcal{E}$, then $\Delta_a \overset{\mathcal{E}}{\lhd}_2 \Delta_a$ and $\Delta_a \in \mathcal{L}_2(\mathcal{E})$. Similarly, one may conclude that $\nabla L \subseteq \mathcal{L}_1(\mathcal{E})$. □

3.3. THE FUNCTOR $\mathcal{C}_1^* : \mathsf{Frm} \to \mathsf{QUFrm}$

Let us show that the correspondence $L \mapsto (\mathfrak{C}L, \mathcal{F}_{\mathfrak{C}L})$ defines a functor $\mathcal{C}_1^* : \mathsf{Frm} \to \mathsf{QUFrm}$. For any frame homomorphism $h : L \to M$, take the map $\overline{h}$ given by (2.2.3). It suffices to check that $\overline{h} : (\mathfrak{C}L, \mathcal{F}_{\mathfrak{C}L}) \to (\mathfrak{C}M, \mathcal{F}_{\mathfrak{C}M})$ is a quasi-uniform homomorphism, which is easy:

$$\begin{aligned}(\overline{h} \oplus \overline{h})(E_{\nabla_a}) &= (\overline{h} \oplus \overline{h})(\nabla_a \oplus 1) \vee (\overline{h} \oplus \overline{h})(1 \oplus \Delta_a)\\ &= \big(\overline{h}(\nabla_a) \oplus \overline{h}(1)\big) \vee \big(\overline{h}(1) \oplus \overline{h}(\Delta_a)\big)\\ &= (\nabla_{h(a)} \oplus 1) \vee (1 \oplus \Delta_{h(a)}) \in \mathcal{F}_{\mathfrak{C}M}.\end{aligned}$$

In conclusion,

$$\begin{array}{llcl}\mathcal{C}_1^* : & \mathsf{Frm} & \longrightarrow & \mathsf{QUFrm}\\ & L & \longmapsto & (\mathfrak{C}L, \mathcal{F}_{\mathfrak{C}L})\\ & (h : L \to M) & \longmapsto & \big(\overline{h} : (\mathfrak{C}L, \mathcal{F}_{\mathfrak{C}L}) \to (\mathfrak{C}M, \mathcal{F}_{\mathfrak{C}M})\big)\end{array}$$

is a functor such that $T\mathcal{C}_1^*(L) = \mathcal{L}_1(\mathcal{F}_{\mathfrak{C}L}) = \nabla L$ and there is a natural isomorphism $i^{\mathcal{C}_1^*} : 1_{\mathsf{Frm}} \stackrel{\cdot}{\to} T\mathcal{C}_1^*$, given by $i_L^{\mathcal{C}_1^*} := \nabla_L : L \to \nabla L$, for every $L \in \mathsf{Frm}$ (recall diagram (2.2.3)).

3.4. T-PSEUDOSECTIONS

The functor $\mathfrak{C}_1^*$ suggests to view the type of quasi-uniformity involved here in the following way. If $T : \mathsf{QUFrm} \to \mathsf{Frm}$ is the above functor forgetting the quasi-uniformities, a functor $F : \mathsf{Frm} \to \mathsf{QUFrm}$ is a *T-pseudosection* if there exists a natural isomorphism $i^F : 1_{\mathsf{Frm}} \stackrel{\cdot}{\to} TF$; that is, for each frame L, there exists a frame isomorphism $i_L^F : L \to TF(L)$ in such a way that, for each frame homomorphism $h : L \to M$, the square

$$\begin{array}{ccc} L & \xrightarrow{i_L^F} & TF(L) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle TF(h)} \\ M & \xrightarrow[i_M^F]{} & TF(M) \end{array}$$

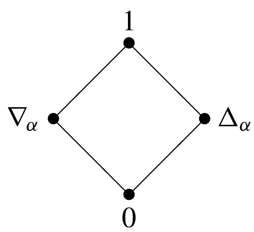
commutes. Note here another fundamental change from the classical theory [7] to quasi-uniform frame theory: TF is the identity functor up to natural isomorphism.

A T-pseudosection F is *transitive* if $F(L)$ is a transitive quasi-uniform frame for every frame L. As we shall see in the last section of this paper, each standard example in 3.2, except the point-finite quasi-uniformity, gives rise to a transitive T-pseudosection.

3.5. T-PSEUDOSECTIONS INDUCE STRICTLY ZERO-DIMENSIONAL BIFRAMES

For any T-pseudosection F, let $\mathcal{B}_F := T_b F : \mathsf{Frm} \to \mathsf{CRBiFrm}$. Clearly, one has the natural isomorphism $K_1\mathcal{B}_F \cong 1_{\mathsf{Frm}}$ given by i^F.

Let $\mathsf{3}$ denote the three-element frame $\{0 < \alpha < 1\}$. It is clear that $\mathfrak{C}\mathsf{3}$ is just the Boolean algebra with four elements



It is also an easy exercise to conclude that $\mathfrak{C}\mathsf{3}$ has a unique quasi-uniform structure, generated by the entourage E_{∇_α}. We refer to it as the *Sierpiński quasi-uniform frame*.

LEMMA. *For each T-pseudosection F, $\mathcal{B}_F(\mathsf{3}) \cong S(\mathsf{3})$.*

Proof. Clearly $\mathcal{L}_1(\mathcal{E}_{F(\mathsf{3})}) \cong \mathsf{3} \cong \nabla\mathsf{3}$. Let $x = i_{\mathsf{3}}^F(\alpha)$ denote the non-trivial element of the frame $\mathcal{L}_1(\mathcal{E}_{F(\mathsf{3})})$. Since $x = \bigvee\{y \in \mathcal{L}_1(\mathcal{E}_{F(\mathsf{3})}) \mid y \overset{\mathcal{E}_{F(\mathsf{3})}}{\lhd}_1 x\}$ and

$\mathcal{L}_1(\mathcal{E}_{F(3)}) \cong 3$, then $x \stackrel{\mathcal{E}_{F(3)}}{\lhd}_1 x$. By (S5), this means that there is some $b \in \mathcal{L}_2(\mathcal{E}_{F(3)})$ such that $b \wedge x = 0$ and $b \vee x = 1$. This shows that $\mathcal{L}_1(\mathcal{E}_{F(3)})$ is complemented by elements of $\mathcal{L}_2(\mathcal{E}_{F(3)})$.

Now consider $y \in \mathcal{L}_2(\mathcal{E}_{F(3)})$. Similarly, $y = \bigvee\{z \in \mathcal{L}_2(\mathcal{E}_{F(3)}) \mid z \stackrel{\mathcal{E}_{F(3)}}{\lhd}_2 y\}$ and, for each such z, there exists $w \in \mathcal{L}_1(\mathcal{E}_{F(3)})$ satisfying $z \wedge w = 0$ and $y \vee w = 1$. We know already that w has a complement $\neg w \in \mathcal{L}_2(\mathcal{E}_{F(3)})$. This complement satisfies $z \le \neg w \le y$ and thus y is a join of complements of members of $\mathcal{L}_1(\mathcal{E}_{F(3)})$. In conclusion, $\mathcal{B}_F(3)$ is strictly zero-dimensional. This implies that $\mathcal{L}_2(\mathcal{E}_{F(3)}) = \{0, \neg x, 1\} \cong \Delta 3$. □

PROPOSITION. *For each T-pseudosection F, $\mathcal{B}_F(L)$ is strictly zero-dimensional.*

Proof. We need to show the following:

(1) $\mathcal{L}_1(\mathcal{E}_{F(L)})$ is complemented by elements of $\mathcal{L}_2(\mathcal{E}_{F(L)})$;
(2) Every element of $\mathcal{L}_2(\mathcal{E}_{F(L)})$ is a join of complements of members of $\mathcal{L}_1(\mathcal{E}_{F(L)})$.

(1) For each $t \in \mathcal{L}_1(\mathcal{E}_{F(L)})$ let $a = (i_L^F)^{-1}(t)$ and define the frame map $f_a : 3 \to L$ by $f_a(\alpha) = a$, with α as above. Let $x = i_3^F(\alpha)$ as in the proof of the lemma. Since the diagram

$$\begin{array}{ccc} 3 & \xrightarrow{\;f_a\;} & L \\ {\scriptstyle i_3^F}\downarrow & & \downarrow{\scriptstyle i_L^F} \\ \mathcal{L}_1(\mathcal{E}_{F(3)}) & \xrightarrow[TF(f_a)]{} & \mathcal{L}_1(\mathcal{E}_{F(L)}) \end{array} \tag{3.5.1}$$

commutes, we have $t = i_L^F(f_a(\alpha)) = TF(f_a)(x)$. By the lemma, x has a complement $\neg x \in \mathcal{L}_2(\mathcal{E}_{F(3)})$. Consider the biframe map $\mathcal{B}_F(f_a) : \mathcal{B}_F(3) \to \mathcal{B}_F(L)$. Since $(K_1\mathcal{B}_F)(f_a)(x) = TF(f_a)(x) = t$, the element $(K_2\mathcal{B}_F)(f_a)(\neg x) \in \mathcal{L}_2(\mathcal{E}_{F(L)})$ is the complement of t.

(2) Let $y \in \mathcal{L}_2(\mathcal{E}_{F(L)})$. For each $z \in \mathcal{L}_2(\mathcal{E}_{F(L)})$ satisfying $z \stackrel{\mathcal{E}_{F(L)}}{\lhd}_2 y$ there exists $w \in \mathcal{L}_1(\mathcal{E}_{F(L)})$ such that $z \wedge w = 0$ and $y \vee w = 1$. By (1), w is complemented with complement in $\mathcal{L}_2(\mathcal{E}_{F(L)})$. Obviously $z \le \neg w \le y$. The conclusion now follows from the fact that $y = \bigvee\{z \in \mathcal{L}_2(\mathcal{E}_{F(L)}) \mid z \stackrel{\mathcal{E}_{F(L)}}{\lhd}_2 y\}$. □

3.6. PROPERTIES OF T-PSEUDOSECTIONS

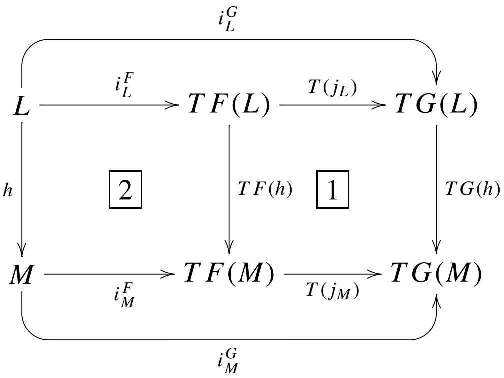
If F and G are T-pseudosections, we say that F is *coarser* than G, written $F \le G$, if there is a natural transformation $j : F \dot{\to} G$ such that $T(j_L) \circ i_L^F = i_L^G$ for every frame L. This is a reflexive and transitive relation, that is, a preorder. The natural transformation j witnessing $F \le G$ is unique:

LEMMA 1. *Given two T-pseudosections $F, G : \mathsf{Frm} \to \mathsf{QUFrm}$ with natural isomorphisms $i^F : 1_{\mathsf{Frm}} \dot{\to} TF$, $i^G : 1_{\mathsf{Frm}} \dot{\to} TG$, suppose that for each frame L we have a quasi-uniform frame morphism $j_L : F(L) \to G(L)$ with $T(j_L) \circ i^F_L = i^G_L$. Then, we have:*

(1) *Each map j_L is uniquely determined by the given conditions.*
(2) *$F \leq G$.*

Proof. (1) For each L, $T_b(F(L)) = \mathcal{B}_F(L)$ is strictly zero-dimensional by Proposition 3.5. Thus (1) follows immediately from Lemma 3.1.

(2) It suffices to show that the transformation $(j_L)_{L \in \mathsf{Frm}}$ is natural. So, consider a morphism $h : L \to M$ in Frm. We need to show that $j_M \circ F(h) = G(h) \circ j_L$. Take the image of the corresponding square under T, and appropriately join to it the arrows i^F_L, i^F_M, i^G_L, i^G_M and h, as follows:



The commutativity of square 2, of the upper and lower triangles and of the outer rectangle implies that $T(j_M) \circ TF(h) \circ i^F_L = TG(h) \circ T(j_L) \circ i^F_L$. Thus, square 1 also commutes. Now (2) follows at once from Lemma 3.1. □

For any T-pseudosection F, let $F_0 := K_0\mathcal{B}_F : \mathsf{Frm} \to \mathsf{Frm}$. By Theorem 3.2, we may endow $F_0(L)$ with the Frith quasi-uniformity $\mathcal{F}_{F_0(L)}$. This transitive quasi-uniformity is coarser than the original quasi-uniformity $\mathcal{E}_{F(L)}$:

LEMMA 2. *$\mathcal{F}_{F_0(L)} \subseteq \mathcal{E}_{F(L)}$.*

Proof. We need to show that, for each $a \in \mathcal{L}_1(\mathcal{E}_{F(L)})$, $E_a \in \mathcal{E}_{F(L)}$. For this, consider the frame homomorphism $f_a : \mathsf{3} \to L$ defined by $f_a(\alpha) = (i^F_L)^{-1}(a)$. By Lemma 3.5, $\mathcal{B}_F(\mathsf{3}) \cong S(\mathsf{3})$ so $F(\mathsf{3})$ is necessarily isomorphic to the Sierpiński quasi-uniform frame. On the other hand, $F(f_a) : F(\mathsf{3}) \to F(L)$ is a quasi-uniform homomorphism thus $(F(f_a) \oplus F(f_a))(E_{\nabla_\alpha}) \in \mathcal{E}_{F(L)}$. By the commutativity of diagram (3.5.1),

$$F(f_a)(\nabla_\alpha) = TF(f_a)(\nabla_\alpha) = TF(f_a)(i^F_{\mathsf{3}}(\alpha)) = i^F_L(f_a(\alpha)) = a.$$

Then $F(f_a)(\Delta_\alpha) = \neg a$ since $F(f_a)(\Delta_\alpha) \vee a = F(f_a)(\Delta_\alpha) \vee F(f_a)(\nabla_\alpha) = F(f_a)(1) = 1$ and $F(f_a)(\Delta_\alpha) \wedge a = F(f_a)(\Delta_\alpha) \wedge F(f_a)(\nabla_\alpha) = F(f_a)(0) = 0$. Hence

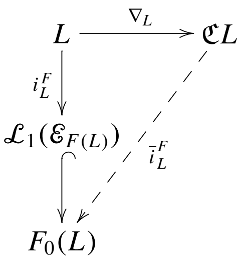
$$\begin{aligned}(F(f_a) \oplus F(f_a))(E_{\nabla_\alpha}) &= \big(F(f_a)(\nabla_\alpha) \oplus 1\big) \vee \big(1 \oplus F(f_a)(\Delta_\alpha)\big) \\ &= (a \oplus 1) \vee (1 \oplus \neg a) \\ &= E_a\end{aligned}$$

and $E_a \in \mathcal{E}_{F(L)}$, as required. □

By Lemma 2.2, for each frame L there exists a frame homomorphism

$$\bar{i}_L^F : \mathfrak{C}L \to F_0(L)$$

such that the diagram



(3.6.1)

commutes.

PROPOSITION. *Let F be a T-pseudosection. Then, for each frame L, we have:*

(1) *$\bar{i}_L^F : \mathfrak{C}_1^*(L) \to (F_0(L), \mathcal{F}_{F_0(L)})$ is a quasi-uniform homomorphism.*

(2) *$\bar{i}_L^F : \mathfrak{C}_1^*(L) \to F(L)$ is a quasi-uniform homomorphism.*

Proof. (1) For any $a \in L$, we have

$$\begin{aligned}(\bar{i}_L^F \oplus \bar{i}_L^F)(E_{\nabla_a}) &= (\bar{i}_L^F \oplus \bar{i}_L^F)(\nabla_a \oplus 1) \vee (\bar{i}_L^F \oplus \bar{i}_L^F)(1 \oplus \Delta_a) \\ &= (\bar{i}_L^F(\nabla_a) \oplus 1) \vee (1 \oplus \bar{i}_L^F(\Delta_a)) \\ &= (i_L^F(a) \oplus 1) \vee (1 \oplus \neg i_L^F(a)) \\ &= E_{i_L^F(a)} \in \mathcal{F}_{F_0(L)}.\end{aligned}$$

(2) It follows immediately from (1) and Lemma 2. □

We may now get the pointfree version of the classical result that the Pervin quasi-uniformity defines the coarsest T-section for the case of quasi-uniform spaces [5].

THEOREM. *$\mathcal{C}_1^*$ is the coarsest T-pseudosection.*

Proof. For any T-pseudosection F and any frame L, let j_L be the quasi-uniform frame homomorphism $\bar{i}_L^F : \mathcal{C}_1^*(L) \to F(L)$ given by assertion (2) of the proposition. Since $T(j_L) \circ \nabla_L = i_L^F$ (see diagram (3.6.1)), Lemma 1 applies and we may conclude that $\mathcal{C}_1^* \leq F$. □

3.7. FUNCTORIAL COMPATIBLE QUASI-UNIFORMITIES

The T-pseudosection $F := \mathcal{C}_1^*$ satisfies, moreover, the condition

$$\bigvee_{\alpha\in I}(i_L^F(a_\alpha) \wedge \neg i_L^F(b_\alpha)) = \bigvee_{\beta\in J}(i_L^F(c_\beta) \wedge \neg i_L^F(d_\beta))$$
$$\Rightarrow \bigvee_{\alpha\in I}(\nabla_{a_\alpha} \wedge \Delta_{b_\alpha}) = \bigvee_{\beta\in J}(\nabla_{c_\beta} \wedge \Delta_{d_\beta}). \tag{3.7.1}$$

Note that the reverse implication is always true for any T-pseudosection, since $\bar{i}_L^F$ is a frame map and therefore $\bigvee_{\alpha\in I}(i_L^F(a_\alpha) \wedge \neg i_L^F(b_\alpha)) = \bar{i}_L^F(\bigvee_{\alpha\in I}(\nabla_{a_\alpha} \wedge \Delta_{b_\alpha}))$. We say that a T-pseudosection F is a *functorial compatible quasi-uniformity* on frames if it satisfies condition (3.7.1). Of course, this condition implies condition (3.2.1). So, functorial compatible quasi-uniformities correspond exactly to compatible quasi-uniformities on frames which are functorial in the sense that any frame homomorphism $h : L \to M$ extends to a frame homomorphism $F(h) : K_0\mathcal{B}_F(L) \to K_0\mathcal{B}_F(M)$ which is quasi-uniform relative to the quasi-uniformities assigned to $F(L)$ and $F(M)$, respectively.

The transitive T-pseudosections given by the standard examples in 3.2 (namely, the locally finite, the well-monotone, the fine transitive and the semicontinuous quasi-uniformities) are functorial compatible quasi-uniformities.

The following result confirms that the congruence lattice is the right setting for functorial quasi-uniformities on frames.

PROPOSITION. *For every functorial compatible quasi-uniformity F and every frame L, $\mathcal{B}_F(L) \cong S(L)$.*

Proof. Since $\bar{i}_L^F(\nabla_a) = i_L^F(a) \in \mathcal{L}_1(\mathcal{E}_{F(L)})$ and $\bar{i}_L^F(\Delta_a) = \neg i_L^F(a) \in \mathcal{L}_2(\mathcal{E}_{F(L)})$, the frame map $\bar{i}_L^F$ may be seen as a biframe map $S(L) \to \mathcal{B}_F(L)$. Condition (3.7.1) means that this is an injective map. It then remains to prove surjectivity, which is a consequence of the strictly zero-dimensionality of $\mathcal{B}_F(L)$ ensured by Proposition 3.5:

For each $x \in F_0(L)$, we may write $x = \bigvee_{\alpha\in I}(y_\alpha \wedge z_\alpha)$ for some $y_\alpha \in \mathcal{L}_1(\mathcal{E}_{F(L)})$ and $z_\alpha \in \mathcal{L}_2(\mathcal{E}_{F(L)})$. Moreover, $z_\alpha = \bigvee_{\beta\in I_\alpha} \neg w_\beta$ for some $w_\beta \in \mathcal{L}_1(\mathcal{E}_{F(L)})$. Therefore, $x = \bigvee_{\alpha\in I}\bigvee_{\beta\in I_\alpha}(y_\alpha \wedge \neg w_\beta)$. Denoting $(i_L^F)^{-1}(y_\alpha)$ by a_α and $(i_L^F)^{-1}(w_\beta)$ by b_β it is clear that $\bar{i}_L^F(\theta) = x$ for $\theta := \bigvee_{\alpha\in I}\bigvee_{\beta\in I_\alpha}(\nabla_{a_\alpha} \wedge \Delta_{b_\beta})$. □

One may ask whether the above result extends to arbitrary T-pseudosections. While this may well be so, and we certainly do not have a counterexample, the above approach does not apply since we do not know whether $\overline{i}_L^F$ is injective if F is not a functorial compatible quasi-uniformity. We recall in passing that any $\mathcal{B}_F(L)$ is strictly zero-dimensional but it is not clear, however, whether this fact can be made to bear on the present question.

4. Functorial Aspects of the Fletcher Construction

4.1. INTERIOR-PRESERVING AND FLETCHER COVERS

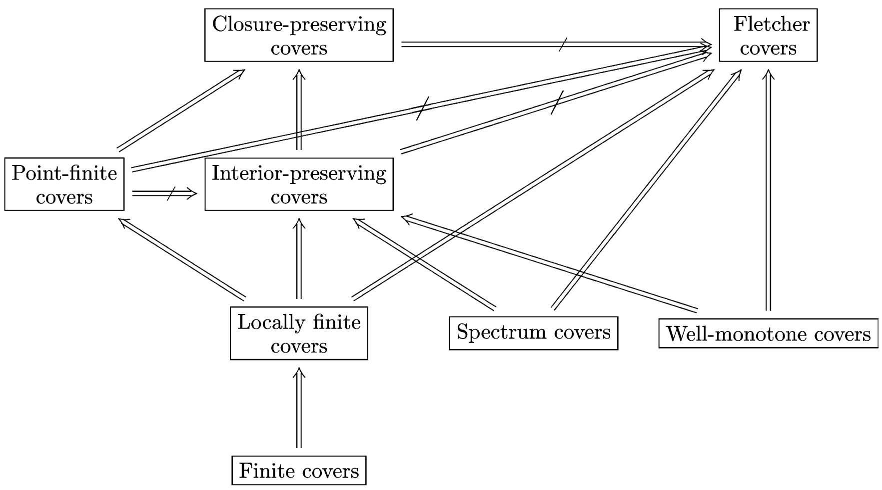
We say that a cover A of L is *interior-preserving* if, for each $B \subseteq A$, $\bigvee_{b\in B} \Delta_b = \Delta_{\bigwedge B}$.

REMARK. It should be pointed out that our present interior-preserving covers were previously called strongly interior-preserving while interior-preserving covers were defined by the weaker condition $\bigwedge_{b\in B} \nabla_b = \nabla_{\bigwedge B}$, for every $B \subseteq A$ ([11], Section 3.1). This change takes account of the fact that the stronger notion should be viewed as the primary one. We now also find more suggestive to call those covers satisfying the weaker condition *closure-preserving*, especially in view of the analysis carried out in [10].

Further, a cover A is a *Fletcher cover* whenever $R_A := \bigcap_{a\in A}(\nabla_a \oplus 1) \cup (1 \oplus \Delta_a)$ is a Weil entourage of $\mathfrak{C}L$ or, equivalently,

$$\bigvee\left\{\left(\bigwedge_{a\in A_1} \nabla_a\right) \wedge \left(\bigwedge_{a\in A_2} \Delta_a\right) \mid A_1 \cup A_2 = A\right\} = 1 \quad \text{([11], Proposition 4.2).}$$

Examples of interior-preserving Fletcher covers are finite covers, locally finite covers, spectrum covers and well-monotone covers (see [11] for the details). In summary, the situation is as follows [10]:



It might be added that, for any covers A, B of L,

$$R_A \cap R_B = R_{A\wedge B} \quad \text{([11], Lemma 4.1)} \tag{4.1.1}$$

For the remainder of the paper we shall denote the entourage $E_{\nabla_a} = (\nabla_a \oplus 1) \cup (1 \oplus \Delta_a)$ simply by H_a and, for each frame homomorphism $h : L \to M$, we denote by $\overline{h} : \mathfrak{C}L \to \mathfrak{C}M$ the morphism given by (2.2.3). Note that $(\overline{h} \oplus \overline{h})(H_a) = H_{h(a)}$.

Interior-preserving covers and Fletcher covers behave well with respect to morphisms:

PROPOSITION. *Let $h : L \to M$ be a frame homomorphism. Then:*

(1) *For every Fletcher cover A of L, $h[A]$ is a Fletcher cover of M.*
(2) *For every interior-preserving cover A of L, $h[A]$ is an interior-preserving cover of M.*

Proof. (1) Since R_A is a Weil entourage of $\mathfrak{C}L$, $(\overline{h} \oplus \overline{h})(R_A)$ is a Weil entourage of $\mathfrak{C}M$. But, clearly, $(\overline{h} \oplus \overline{h})(R_A) \subseteq \bigcap_{a\in A}(\overline{h} \oplus \overline{h})(H_a) = \bigcap_{a\in A} H_{h(a)} = R_{h[A]}$. Thus $R_{h[A]}$ is also a Weil entourage of $\mathfrak{C}M$.

(2) For each $B \subseteq A$ we have, using the hypothesis,

$$\bigvee_{b\in B} \Delta_{h(b)} = \bigvee_{b\in B} \overline{h}(\Delta_b) = \overline{h}\Big(\bigvee_{b\in B} \Delta_b\Big) = \overline{h}(\Delta_{\bigwedge B}) = \Delta_{h(\bigwedge B)} \geq \Delta_{\bigwedge h[B]}.$$

The reverse inequality $\bigvee_{b\in B} \Delta_{h(b)} \leq \Delta_{\bigwedge h[B]}$ is always true. □

In general $\overline{h}$ does not preserve arbitrary meets. But, clearly,

$$\overline{h}\Big(\bigwedge_{b\in B} \Delta_b\Big) = \bigwedge_{b\in B} \overline{h}(\Delta_b), \quad \text{for every } B \subseteq L.$$

Moreover:

LEMMA. *Let A be an interior-preserving cover of L. Then:*

(1) $\overline{h}(\bigwedge_{b\in B} \nabla_b) = \bigwedge_{b\in B} \overline{h}(\nabla_b)$ *for every $B \subseteq A$.*
(2) $(\overline{h} \oplus \overline{h})(R_A) = R_{h[A]}$.
(3) *For each $x \in L$, $st_1(\overline{h}(\nabla_x), (\overline{h} \oplus \overline{h})(R_A)) \leq \overline{h}(st_1(\nabla_x, R_A))$.*
(4) *For each $x \in L$, $st_2(\overline{h}(\Delta_x), (\overline{h} \oplus \overline{h})(R_A)) \leq \overline{h}(st_2(\Delta_x, R_A))$.*

Proof. (1) $\overline{h}(\bigwedge_{b\in B} \nabla_b) = \overline{h}(\nabla_{\bigwedge B}) = \neg\overline{h}(\Delta_{\bigwedge B}) = \neg\overline{h}(\bigvee_{b\in B} \Delta_b) = \neg(\bigvee_{b\in B} \overline{h}(\Delta_b)) = \bigwedge_{b\in B} \overline{h}(\nabla_b)$.

(2) The inclusion $(\overline{h} \oplus \overline{h})(R_A) \subseteq R_{h[A]}$ is trivial. On the other hand, let $(\alpha, \beta) \in R_{h[A]}$. This means that, for every $a \in A$, $\alpha \leq \nabla_{h(a)}$ or $\beta \leq \Delta_{h(a)}$, that is,

$\alpha \leq \bigwedge_{a\in A_1} \nabla_{h(a)}$ and $\beta \leq \bigwedge_{a\in A_2} \Delta_{h(a)}$ for some partition $A_1 \cup A_2$ of A. Consequently, by (1), $\alpha \leq \overline{h}(\bigwedge_{a\in A_1} \nabla_a)$, and, on the other hand, $\beta \leq \overline{h}(\bigwedge_{a\in A_2} \Delta_a)$. But $(\bigwedge_{a\in A_1} \nabla_a, \bigwedge_{a\in A_2} \Delta_a) \in R_A$ thus

$$(\alpha, \beta) \leq \left(\overline{h}\left(\bigwedge_{a\in A_1} \nabla_a\right), \overline{h}\left(\bigwedge_{a\in A_2} \Delta_a\right)\right) \in (\overline{h} \oplus \overline{h})(R_A).$$

(3) It suffices to check that $st_1(\nabla_{h(x)}, R_{h[A]}) \leq \overline{h}(st_1(\nabla_x, R_A))$. Let $(\alpha, \beta) \in R_{h[A]}$ with $\beta \wedge \nabla_{h(x)} \neq 0$. Then $\alpha \leq \bigwedge_{a\in A_1} \nabla_{h(a)}$ and $\beta \leq \bigwedge_{a\in A_2} \Delta_{h(a)}$, for some partition $A_1 \cup A_2$ of A. But, by (1), $\bigwedge_{a\in A_1} \nabla_{h(a)} = \bigwedge_{a\in A_1} \overline{h}(\nabla_a) = \overline{h}(\bigwedge_{a\in A_1} \nabla_a)$, so we only need to show that $\bigwedge_{a\in A_1} \nabla_a \leq st_1(\nabla_x, R_A)$, which is easy, since

$$\left(\bigwedge_{a\in A_1} \nabla_a, \bigwedge_{a\in A_2} \Delta_a\right) \in R_A$$

and $\beta \wedge \nabla_{h(x)} \neq 0$ implies $\overline{h}(\bigwedge_{a\in A_2} \Delta_a \wedge \nabla_x) = \bigwedge_{a\in A_2} \Delta_{h(a)} \wedge \nabla_{h(x)} \neq 0$, that is, $\bigwedge_{a\in A_2} \Delta_a \wedge \nabla_x \neq 0$.

(4) Similar to (3). □

4.2. THE FLETCHER CONSTRUCTION IS FUNCTORIAL

It is now our goal to study the functoriality of the pointfree version of Fletcher's construction presented by the authors in [11]. We begin with a broad outline of this method of constructing compatible quasi-uniformities for arbitrary frames (the details can be seen in [11] and [10]).

For any frame L, let $\mathcal{A}_L$ be a nonempty collection of interior-preserving Fletcher covers of L and let $\mathcal{E}_{\mathcal{A}_L}$ be the filter of $WEnt(\mathfrak{C}L)$ generated by $\{R_A \mid A \in \mathcal{A}_L\}$. In general, one has $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}) \subseteq \nabla L$ and $\mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L}) \subseteq \Delta L$ ([11, 10]). So, it may well be the case that $(\mathfrak{C}L, \mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}), \mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L}))$ is not a biframe and, consequently, that $\mathcal{E}_{\mathcal{A}_L}$ is not admissible on $\mathfrak{C}L$ (this phenomenon already presents itself in the classical setting). We show in [11] how to remedy this situation by modifying the congruence frame $\mathfrak{C}L$ to a certain subframe $\mathfrak{C}L'$, and of handling the imposed quasi-uniformity $\mathcal{E}_{\mathcal{A}_L}$ accordingly, by modifying conveniently the generating entourages R_A.

Here, in order to keep the readability of the paper and to avoid too much technicalities, we shall confine ourselves to the class of all quasi-uniformities on the congruence frame $\mathfrak{C}L$ (after all, our standard examples are all of this kind); for details about the general case $(\mathfrak{C}L', \mathcal{E}'_{\mathcal{A}_L})$ see [10], where results similar to the foregoing ones are presented.

Therefore, in what follows we assume that $\mathcal{A}_L$ is an *admissible set* of interior-preserving Fletcher covers of L, that is, a collection of interior-preserving Fletcher covers of L such that $\mathcal{E}_{\mathcal{A}_L}$ is admissible on $\mathfrak{C}L$ (in other words, $(\mathfrak{C}L, \mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}), \mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L}))$ is a biframe with $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}) \subseteq \nabla L$ and $\mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L}) \subseteq \Delta L$).

REMARKS. (1) This method of constructing compatible quasi-uniformities can be performed more generally for families $\mathcal{A}_L$ of closure-preserving Fletcher covers, as shown in [11]. Here we assume $\mathcal{A}_L$ to be a family of interior-preserving covers because we need to apply the results of the previous section to get the functoriality.

(2) Note that, if for each $a \in L$ there exists $A \in \mathcal{A}_L$ containing a, then, since $R_A \subseteq H_a$, we have the inclusion $\mathcal{F}_{\mathfrak{C}L} \subseteq \mathcal{E}_{\mathcal{A}_L}$. Therefore, by Proposition 3.2, $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}) = \nabla L$ and $\mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L}) = \Delta L$. This is what happens in each of our standard examples (cf. [11], Section 8).

(3) If $\bigcup \mathcal{A}_L$ is a subbase for L, $\mathcal{E}_{\mathcal{A}_L}$ is a transitive quasi-uniformity on $\mathfrak{C}L$, compatible with L (cf. [11], Theorem 6.7).

Following the classical terminology, we say that a *natural kind of covers* in Frm is an indexed class $\mathcal{A} = (\mathcal{A}_L)_{L \in \mathsf{Frm}}$ such that:

(1) Each $\mathcal{A}_L$ is an admissible set of interior-preserving Fletcher covers of L;
(2) For every frame homomorphism $h : L \to M$ and every $A \in \mathcal{A}_L$, $h[A] \in \mathcal{A}_M$.

The following observation will be useful later on.

LEMMA. *Let $\mathcal{A} = (\mathcal{A}_L)_{L \in \mathsf{Frm}}$ be a natural kind of covers and let $h : L \to M$ be a frame homomorphism. Then:*

(1) $\nabla_y \overset{\mathcal{E}_{\mathcal{A}_L}}{\lhd}_1 \nabla_x$ *implies* $\overline{h}(\nabla_y) \overset{\mathcal{E}_{\mathcal{A}_M}}{\lhd}_1 \overline{h}(\nabla_x)$.
(2) $\Delta_y \overset{\mathcal{E}_{\mathcal{A}_L}}{\lhd}_2 \Delta_x$ *implies* $\overline{h}(\Delta_y) \overset{\mathcal{E}_{\mathcal{A}_M}}{\lhd}_2 \overline{h}(\Delta_x)$.
(3) $\overline{h}(\mathcal{L}_i(\mathcal{E}_{\mathcal{A}_L})) \subseteq \mathcal{L}_i(\mathcal{E}_{\mathcal{A}_M})$ $(i = 1, 2)$.

Proof. (1) Consider $A_1, \ldots, A_n \in \mathcal{A}_L$ such that $st_1(\nabla_y, \bigcap_{i=1}^n R_{A_i}) \leq \nabla_x$. Then, we have $\overline{h}(st_1(\nabla_y, \bigcap_{i=1}^n R_{A_i})) \leq \overline{h}(\nabla_x)$. But, by (4.1.1), $\overline{h}(st_1(\nabla_y, \bigcap_{i=1}^n R_{A_i}))$ is equal to $\overline{h}(st_1(\nabla_y, R_{\bigwedge_{i=1}^n A_i}))$. Thus, $\overline{h}(st_1(\nabla_y, \bigcap_{i=1}^n R_{A_i})) \geq st_1(\overline{h}(\nabla_y), (\overline{h} \oplus \overline{h})(R_{\bigwedge_{i=1}^n A_i}))$, by Lemma 4.1(3). Clearly, each A_i being interior-preserving, $\bigwedge_{i=1}^n A_i$ is also interior-preserving. Therefore, by Lemma 4.1(2), we get

$$\begin{aligned}\overline{h}\left(st_1\left(\nabla_y, \bigcap_{i=1}^n R_{A_i}\right)\right) &\geq st_1(\overline{h}(\nabla_y), R_{h[\bigwedge_{i=1}^n A_i]})\\ &= st_1(\overline{h}(\nabla_y), R_{\bigwedge_{i=1}^n h[A_i]})\\ &= st_1\left(\overline{h}(\nabla_y), \bigcap_{i=1}^n R_{h[A_i]}\right).\end{aligned}$$

Hence, $st_1(\overline{h}(\nabla_y), \bigcap_{i=1}^n R_{h[A_i]}) \leq \overline{h}(\nabla_x)$, which shows that $\overline{h}(\nabla_y) \overset{\mathcal{E}_{\mathcal{A}_M}}{\lhd}_1 \overline{h}(\nabla_x)$.

(2) It can be proved in a similar way to (1).

(3) It follows immediately from (1), (2) and the inclusions $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}) \subseteq \nabla L$, $\mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L}) \subseteq \Delta L$. □

Assertion (3) of the lemma means that $\overline{h} : \mathfrak{C}L \to \mathfrak{C}M$ defines a biframe map

$$\overline{h} : \big(\mathfrak{C}L, \mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}), \mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L})\big) \to \big(\mathfrak{C}M, \mathcal{L}_1(\mathcal{E}_{\mathcal{A}_M}), \mathcal{L}_2(\mathcal{E}_{\mathcal{A}_M})\big).$$

Moreover, we have:

PROPOSITION. *$\overline{h} : (\mathfrak{C}L, \mathcal{E}_{\mathcal{A}_L}) \to (\mathfrak{C}M, \mathcal{E}_{\mathcal{A}_M})$ is a quasi-uniform homomorphism.*

Proof. Let $E \in \mathcal{E}_{\mathcal{A}_L}$. Then $\bigcap_{i=1}^n R_{A_i} \subseteq E$ for some $A_1, \ldots, A_n \in \mathcal{A}_L$, from which it follows that $(\overline{h} \oplus \overline{h})(\bigcap_{i=1}^n R_{A_i}) \subseteq (\overline{h} \oplus \overline{h})(E)$. On the other hand, by Lemma 4.1(2),

$$(\overline{h} \oplus \overline{h})\left(\bigcap_{i=1}^n R_{A_i}\right) = \bigcap_{i=1}^n (\overline{h} \oplus \overline{h})(R_{A_i}) = \bigcap_{i=1}^n R_{h[A_i]} \in \mathcal{E}_{\mathcal{A}_M}.$$

Hence $(\overline{h} \oplus \overline{h})(E) \in \mathcal{E}_{\mathcal{A}_M}$. □

Thus the correspondence $L \mapsto (\mathfrak{C}L, \mathcal{E}_{\mathcal{A}_L})$ determines a functor $Q_{\mathcal{A}} : \mathsf{Frm} \to \mathsf{QUFrm}$.

4.3. WHEN DOES THE FLETCHER CONSTRUCTION INDUCE A FUNCTORIAL COMPATIBLE QUASI-UNIFORMITY?

Of course, we are interested in the case when, for every L, $Q_{\mathcal{A}}(L) = (\mathfrak{C}L, \mathcal{E}_{\mathcal{A}_L})$ is a quasi-uniform frame compatible with L, that is, when $Q_{\mathcal{A}}$ is a functorial compatible quasi-uniformity. First, we need to recall the following from [11]:

Let $\mathcal{E}$ be a transitive quasi-uniformity on $\mathfrak{C}L$, compatible with L, and consider a transitive base $\mathcal{S}$ of $\mathcal{E}$. Since each $E \in \mathcal{S}$ is transitive,

$$st_i(\theta, E) \overset{\mathcal{E}}{\triangleleft_i} st_i(\theta, E) \quad \text{for every } \theta \in \mathfrak{C}L \ (i = 1, 2).$$

Therefore, $st_1(\theta, E) \in \mathcal{L}_1(\mathcal{E})$ and $st_2(\theta, E) \in \mathcal{L}_2(\mathcal{E})$. So, by the isomorphism $\mathcal{L}_1(\mathcal{E}) \cong \nabla L$, each $st_1(\theta, E)$ corresponds to $\nabla_{E[\theta]}$ for some element $E[\theta] \in L$. Set

$$Cov\, E := \{E[\theta] \mid (\theta, \theta) \in E\}.$$

By Propositions 7.2 and 7.3 of [11] we have:

PROPOSITION. *Let $\mathcal{E}$ be a transitive quasi-uniformity on $\mathfrak{C}L$, compatible with L, and consider a transitive base $\mathcal{S}$ of $\mathcal{E}$. Then:*

(1) *Each $Cov\, E$, for $E \in \mathcal{S}$, is an interior-preserving cover of L.*
(2) *$\bigcup_{E\in\mathcal{S}} Cov\, E$ is a subbase for L.*

For the quasi-uniformity $\mathcal{E}_{\mathcal{A}_L}$ we can say more:

LEMMA. *If* $\bigcup\{Cov\, R_A \mid A \in \mathcal{A}_L\}$ *is a subbase for* L, *then* $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}) = \nabla L$.

Proof. Let $x \in L$. By hypothesis, we may write $x = \bigvee_{i\in I}(R_{A_1^i}[\theta_1] \wedge \cdots \wedge R_{A_{n_i}^i}[\theta_{n_i}])$ for some $A_j^i \in \mathcal{A}$ and $(\theta_j, \theta_j) \in R_{A_j^i}$ ($i \in I$, $j \in \{1, \ldots, n_i\}$). Then

$$\begin{aligned}\nabla_x &= \bigvee_{i\in I}(\nabla_{R_{A_1^i}[\theta_1]} \wedge \cdots \wedge \nabla_{R_{A_{n_i}^i}[\theta_{n_i}]})\\ &= \bigvee_{i\in I}(st_1(\theta_1, R_{A_1^i}) \wedge \cdots \wedge st_1(\theta_{n_i}, R_{A_{n_i}^i})).\end{aligned}$$

So, in order to show that $\nabla_x \in \mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L})$ it suffices to check that, for every i,

$$st_1(\theta_1, R_{A_1^i}) \wedge \cdots \wedge st_1(\theta_{n_i}, R_{A_{n_i}^i}) \overset{\mathcal{E}_{\mathcal{A}_L}}{\lhd_1} \nabla_x.$$

For each i, take $\bigcap_{j=1}^{n_i} R_{A_j^i} \in \mathcal{E}_{\mathcal{A}_L}$. Then, by properties (S3) and (S4),

$$\begin{aligned}&st_1\left(st_1(\theta_1, R_{A_1^i}) \wedge \cdots \wedge st_1(\theta_{n_i}, R_{A_{n_i}^i}), \bigcap_{j=1}^{n_i} R_{A_j^i}\right)\\ &\leq \bigwedge_{j=1}^{n_i} st_1\big(st_1(\theta_1, R_{A_1^i}) \wedge \cdots \wedge st_1(\theta_{n_i}, R_{A_{n_i}^i}), R_{A_j^i}\big)\\ &\leq \bigwedge_{j=1}^{n_i} st_1\big(st_1(\theta_j, R_{A_j^i}), R_{A_j^i}\big) \leq \bigwedge_{j=1}^{n_i} st_1(\theta_j, R_{A_j^i} \circ R_{A_j^i})\\ &= \bigwedge_{j=1}^{n_i} st_1(\theta_j, R_{A_j^i}) \leq \nabla_x.\end{aligned}$$

□

THEOREM. *Let* $\mathcal{A}$ *be a natural kind of covers. The induced functor* $Q_{\mathcal{A}}$ *is a functorial compatible quasi-uniformity if and only if, for each frame* L, $\bigcup\{Cov\, R_A \mid A \in \mathcal{A}_L\}$ *is a subbase for* L.

Proof. The forward implication follows immediately from assertion (2) of the proposition, since each $Q_{\mathcal{A}}(L) = (\mathfrak{C}L, \mathcal{E}_{\mathcal{A}_L})$ is a transitive quasi-uniform frame, compatible with L, with transitive subbase $\{R_A \mid A \in \mathcal{A}_L\}$.

Conversely, by the lemma, $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}) = \nabla L$, for each L, so $Q_{\mathcal{A}}(L)$ is a quasi-uniform frame compatible with L. Since the isomorphism $i_L^{Q_{\mathcal{A}}}$, giving the compatibility with L, is simply ∇_L, it is then obvious that $Q_{\mathcal{A}}$ is a functorial compatible quasi-uniformity. □

5. The Construction of All Transitive Functorial Quasi-Uniformities

We say that a natural kind of covers $\mathcal{A} = (\mathcal{A}_L)_{L\in\mathsf{Frm}}$ is an *adequate kind of covers* if, for each frame L, $\bigcup \mathcal{A}_L$ is a subbase for L. Then, we have:

THEOREM 1. *For each adequate kind of covers $\mathcal{A}$, the induced transitive functor $Q_{\mathcal{A}}$ is a transitive functorial compatible quasi-uniformity.*

Proof. By Proposition 4.2, $Q_{\mathcal{A}}$ is a transitive functor. The fact that $\bigcup \mathcal{A}_L$ is a subbase of L, for each L, allows us to apply Remark 4.2(3) to conclude that $Q_{\mathcal{A}}(L)$ is a quasi-uniform frame compatible with L and, therefore, that $Q_{\mathcal{A}}$ is a transitive functorial compatible quasi-uniformity. □

So, by Proposition 3.7, for every adequate kind of covers $\mathcal{A}$, each $T_b Q_{\mathcal{A}}(L)$ is isomorphic to the Skula biframe.

EXAMPLES. The following are examples of adequate kinds of covers and of their induced transitive functorial compatible quasi-uniformities.

kind $\mathcal{A}$ of covers	functorial compatible quasi-uniformity $Q_{\mathcal{A}}$
Interior-preserving Fletcher covers	$\mathcal{FT}$: “Fine transitive functorial quasi-unif.”
Finite covers	$\mathcal{C}_1^*$: “Frith functorial quasi-unif.”
Locally finite covers	$\mathcal{LF}$: “Locally finite functorial quasi-unif.”
Well-monotone covers	$\mathcal{W}$: “Well-monotone functorial quasi-unif.”
Spectrum covers	$\mathcal{SC}$: “Semi-continuous functorial quasi-unif.”

In fact, they are all examples of admissible collections $\mathcal{A}_L$ of interior-preserving Fletcher covers such that $\bigcup \mathcal{A}_L$ is a subbase of L, as we proved in the last section of [11], thus adequateness follows from the following result.

PROPOSITION. *Let $h : L \to M$ be a frame homomorphism. For every locally finite* (*resp. spectrum, well-monotone*) *cover A of L, $h[A]$ is a locally finite* (*resp. spectrum, well-monotone*) *cover of M.*

Proof. (1) Let A be a locally finite cover, that is, a cover for which there exists a cover C such that $A_c := \{a \in A \mid a \wedge c \neq 0\}$ is finite for every $c \in C$. Then $h[C]$ is a cover of M and, for every $c \in C$, $h[A]_{h(c)} \subseteq \{h(a) \mid a \in A_c\}$, since $h(a) \wedge h(c) \neq 0$ implies $a \wedge c \neq 0$. Thus $h[A]$ is locally finite.

(2) In case $A = \{a_n \mid n \in \mathbb{Z}\}$ is a spectrum cover of L, that is, a cover of L satisfying $a_n \leq a_{n+1}$, for each $n \in \mathbb{Z}$, and $\bigvee_{n\in\mathbb{Z}} \Delta_{a_n} = 1$, then, immediately, $h[A]$ is a cover of M, $h(a_n) \leq h(a_{n+1})$, for each $n \in \mathbb{Z}$, and

$$\bigvee_{n\in\mathbb{Z}} \Delta_{h(a_n)} = \bigvee_{n\in\mathbb{Z}} \overline{h}(\Delta_{a_n}) = \overline{h}\Big(\bigvee_{n\in\mathbb{Z}} \Delta_{a_n}\Big) = \overline{h}(1) = 1.$$

(3) Finally, the case when A is well-monotone, that is, well-ordered by the partial order of L, is obvious. □

By Proposition 3.7, for any functorial compatible quasi-uniformity F, $\mathcal{B}_F(L) \cong S(L)$ (given by $\bar{i}_L^F$) and we may assume, to simplify notation, that $F(L) = (\mathfrak{C}L, \mathcal{E}_{F(L)})$.

Our main result is now as follows:

THEOREM 2. *Let F be a transitive functorial compatible quasi-uniformity. For each frame L, let*

$$\mathcal{A}_L := \{A \mid A \text{ is an interior-preserving cover of } L \text{ such that } R_A \in \mathcal{E}_{F(L)}\}.$$

Then $\mathcal{A} := (\mathcal{A}_L)_{L \in \mathsf{Frm}}$ is an adequate kind of covers such that $Q_{\mathcal{A}} = F$. Moreover, $\mathcal{A}$ is the largest natural kind of covers whose induced functor is the given F.

Proof. First, we prove that $\mathcal{A}$ is adequate. Trivially, each $A \in \mathcal{A}_L$ is an interior-preserving Fletcher cover of L. Let $h : L \to M$ be a frame homomorphism. Then, for each $A \in \mathcal{A}_L$, $R_A \in \mathcal{E}_{F(L)}$ thus $(\bar{h} \oplus \bar{h})(R_A) \in \mathcal{E}_{F(M)}$. By Lemma 4.1(2), this means that $R_{h[A]} \in \mathcal{E}_{F(M)}$. Consequently, $h[A] \in \mathcal{A}_M$. On the other hand, since $\{Cov\, R_A \mid A \in \mathcal{A}_L\} \subseteq \mathcal{A}_L$, it follows from Proposition 4.3(2) that $\bigcup \mathcal{A}_L$ is a subbase for L. Moreover, $\mathcal{A}_L$ is admissible on $\mathfrak{C}L$, that is, $(\mathfrak{C}L, \mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}), \mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L}))$ is a biframe. Indeed, by Lemma 4.3, $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}) = \nabla L$; on the other hand, $\mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L}) = \Delta L$, since $\Delta_a \in \mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L})$ for every $a \in L$: by Lemma 2 of 3.6, $\mathcal{F}_{\mathfrak{C}L} \subseteq \mathcal{E}_{F(L)}$; therefore, for each $a \in L$, $A := \{a, 1\} \in \mathcal{A}_L$, since $R_A = H_a \in \mathcal{F}_{\mathfrak{C}L} \subseteq \mathcal{E}_{F(L)}$; this implies that each H_a belongs to $\mathcal{E}_{\mathcal{A}_L}$. But $st_2(\Delta_a, H_a) = \Delta_a$, hence $\Delta_a \in \mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L})$.

Finally, the remaining claim follows from Theorem 7.8(a) of [11] that asserts that for any compatible transitive quasi-uniformity $\mathcal{E}$ on $\mathfrak{C}L$, $\mathcal{A}_L := \{A \mid A \in Cov\, L, R_A \in \mathcal{E}\}$ is the largest set of covers of L that induces $\mathcal{E}$. □

This is the pointfree analogue of Theorem 2.12 of [7].

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