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WKB Approximation and Krall-Type Orthogonal Polynomials

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Abstract. We give a unified approach to the Krall-type polynomials orthogonal with respect to a positive measure consisting of an absolutely continuous one *'perturbed'* by the addition of one or more Dirac delta functions. Some examples studied by different authors are considered from a unique point of view. Also some properties of the Krall-type polynomials are studied. The three-term recurrence relation is calculated explicitly, as well as some asymptotic formulas. With special emphasis will be considered the second order differential equations that such polynomials satisfy. They allow us to obtain the central moments and the WKB approximation of the distribution of zeros. Some examples coming from quadratic polynomial mappings and tridiagonal periodic matrices are also studied.

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1. Introduction

In this work, we present a survey and some new results relative to the Krall-type orthogonal polynomials, i.e., polynomials which are orthogonal with respect to an absolutely continuous measure *'perturbed'* by the addition of one or more Dirac delta functions. These polynomials were firstly studied in 1940 by H. L. Krall [31]. In his 1940's work, H. L. Krall, studied certain fourth-order differential equations satisfied by families of orthogonal polynomials. In fact, his study is related to an extension of the very well known characterization of classical orthogonal

Table I. The classical Krall-type polynomials [30, 31].

$\{P_n\}$	Measure $\mathrm{d}\mu$	$\mathrm{supp}(\mu)$
Laguerre-type	$\mathrm{e}^{-x}\,\mathrm{d}x + M\delta(x), \quad M > 0$	$[0, \infty)$
Legendre-type	$\frac{\alpha}{2}\,\mathrm{d}x + \frac{\delta(x-1)}{2} + \frac{\delta(x+1)}{2}, \quad \alpha > 0$	$[-1, 1]$
Jacobi-type	$(1-x)^{\alpha}\,\mathrm{d}x + M\delta(x), \quad M > 0, \alpha > -1$	$[0, 1]$

polynomials by S. Bochner (1929). H. L. Krall discovered that there only three extra families of orthogonal polynomials satisfying such a fourth-order differential equation which are orthogonal with respect to measures which are not absolutely continuous with respect to the Lebesgue measure. The corresponding measures are given in Table I. These polynomials were studied later by A. M. Krall [30] in 1981 and then named the Legendre-type, Laguerre-type and Jacobi-type polynomials and sometimes they are called the Krall-type polynomials. However, in the literature, the name the Krall polynomials is associated with some generalization of the Legendre-type polynomials which satisfies a sixth order differential equation of spectral type and were introduced by L. L. Littlejohn [33] in 1982.

The analysis of properties of polynomials orthogonal with respect to a perturbation of a measure via the addition of mass points was introduced by P. Nevai [38]. There the asymptotic properties of the new polynomials have been considered. In particular, he proved the dependence of such properties in terms of the location of the mass points with respect to the support of the measure. Particular emphasis was given to measures supported in $[-1, 1]$ and satisfying some extra conditions in terms of the parameters of the three-term recurrence relation that the corresponding sequence of orthogonal polynomials satisfies.

The analysis of algebraic properties for such polynomials attracted the interest of several researchers. A general analysis when a modification of a linear functional in the linear space of real polynomials with real coefficients via the addition of one Dirac delta measure was started by Chihara [13] in the positive definite case and Marcellán and Maroni [34] for quasi-definite linear functionals. From the point of view of differential equations, see [37]. For two point masses there exist very few examples in the literature (see [29, 15, 27] and [32]). In this case the difficulties increase as shows [16]. Spectral properties of the classical Krall-type polynomials [30, 31] were considered in [11].

A special emphasis was given to the modifications of classical linear functionals (Hermite, Laguerre, Jacobi and Bessel) in the framework of the so-called semi-classical orthogonal polynomials. For example in [29] the Jacobi case with two masses at points $x = \pm 1$ was considered. The hypergeometric representation of the resulting polynomials as well as the existence of a second order differential equation that such polynomials satisfy have been established. Also the particular cases of the Krall-type polynomials [30, 31] have been obtained from this general case as special cases or limit cases. In [23, 25] (see also [27]) the Laguerre case was considered in details. In particular an infinite order differential equation for these polynomials as well as their representation as hypergeometric series have been found. The case of modification of a classical symmetric functional (Hermite and Gegenbauer functionals) was considered in [6].

The modification of classical functionals have been considered also for the discrete orthogonal polynomials. In this direction Bavinck and van Haeringen [9] obtained an infinite order difference equation for generalized Meixner polynomials, i.e., polynomials orthogonal with respect to the modification of the Meixner weight with a point mass at $x = 0$. The same was found for generalized Charlier polynomials by Bavinck and Koekoek [10]. In a series of papers by Alvarez-Nodarse *et al.* [2–4] the authors have obtained the representation as hypergeometric functions for generalized Meixner, Charlier, Kravchuk and Hahn polynomials as well as the corresponding second order difference equation that such polynomials satisfy. The connection of all these discrete polynomials with the Jacobi [29] and Laguerre [23] type were studied in details in [5]. In particular, in [5] the authors proved that the Jacobi–Koornwinder polynomials [29] are a limit case of the generalized Hahn as well as the Laguerre–Koekoek [23, 25] are a limit case of the generalized Meixner polynomials.

The aim of the present contribution is to give a unified approach to this subject including the spectral properties by means of the central moments of the polynomials [12] and the WKB (Brillouin–Wentzel–Kramer method, see [19, 40]) or semiclassical approximation to the density of the distribution of zeros [8, 46, 47] and some asymptotic formulas for the polynomials. Also a new interpretation of the Krall-type polynomials in terms of special Jacobi matrices will be given.

The plan of the paper is the following. In Section 2 we give a general theory which allows us to obtain some general formulas for the Krall-type polynomials. From these formulas we obtain all the explicit formulas for the four families under consideration, i.e., the Jacobi–Koornwinder [29], the Laguerre–Koekoek [23, 25], and the Hermite–Krall-type and Gegenbauer–Krall-type [6]. Also a general algorithm is given to generate the second order differential equations that such polynomials satisfy.

In Section 3 we study the spectral properties of the Jacobi–Koornwinder [29], Laguerre–Koekoek [23, 25], Hermite–Krall-type [6] and Gegenbauer–Krall-type [6] polynomials by means of their central moments and the WKB or semiclassical approximation to the density of the distribution of zeros. Some particular cases are also included.

Finally, in Section 4 we consider some special cases of Krall-type polynomials obtained from the analysis of certain types of Jacobi matrices and quadratic polynomial mappings.

2. The Definition and the Representation

Let $\{P_n\}$ be a sequence of monic polynomials orthogonal with respect to a linear functional $\mathcal{L}$ on the linear space of polynomials $\mathbb{P}$ with real coefficients defined as (a, b can be $\mp\infty$, respectively)

$$\langle \mathcal{L}, P \rangle = \int_a^b P(x)\rho(x)\,\mathrm{d}x, \quad \rho \in \mathfrak{C}_{[a,b]},\ \rho(x) > 0 \text{ for } x \in [a, b]. \tag{1}$$

Through the paper $\mathbb{P}$ will denote the linear space of real polynomials with real coefficients.

Let us consider a new sequence $\{\tilde{P}_n\}$ of polynomials orthogonal with respect to a linear functional $\mathcal{U}$ defined on $\mathbb{P}$ which is obtained from the above functional $\mathcal{L}$ by adding Dirac delta functions at the points $x_1, x_2, \ldots, x_m$, i.e.,

$$\langle \mathcal{U}, P \rangle = \langle \mathcal{L}, P \rangle + \sum_{i=1}^{m} A_i P(x_i), \quad x_i \in \mathbb{R},\ A_i \geqslant 0. \tag{2}$$

We will determine the monic polynomials $\{\tilde{P}_n\}$ which are orthogonal with respect to the functional $\mathcal{U}$ and we will prove that they exist for all positive A_i. To obtain this, we can write the Fourier expansion of $\tilde{P}_n$ in terms of the polynomials $\{P_k\}$

$$\tilde{P}_n(x) = P_n(x) + \sum_{k=0}^{n-1} a_{n,k} P_k(x). \tag{3}$$

In order to find the unknown coefficients $a_{n,k}$ we will use the orthogonality of the polynomials $\tilde{P}_n$ with respect to $\mathcal{U}$, i.e.,

$$0 = \langle \mathcal{U}, \tilde{P}_n P_k \rangle = \langle \mathcal{L}, \tilde{P}_n P_k \rangle + \sum_{i=1}^{m} A_i \tilde{P}_n(x_i) P_k(x_i), \quad \forall k < n.$$

We get

$$a_{n,k} = -\sum_{i=1}^{m} A_i \frac{\tilde{P}_n(x_i) P_k(x_i)}{d_k^2}, \tag{4}$$

where $d_k^2 = \langle \mathcal{L}, [P_k]^2 \rangle$. Finally, (3) becomes

$$\begin{aligned} \tilde{P}_n(x) &= P_n(x) - \sum_{i=1}^{m} A_i \tilde{P}_n(x_i) \sum_{k=0}^{n-1} \frac{P_k(x_i) P_k(x)}{d_k^2} \\ &= P_n(x) - \sum_{i=1}^{m} A_i \tilde{P}_n(x_i)\, \mathrm{Ker}_{n-1}(x, x_i). \end{aligned} \tag{5}$$

In order to obtain the unknown values $\tilde{P}_n(x_i)$ for each $i = 1, 2, \ldots, m$, we evaluate (5) in x_j, $j = 1, 2, \ldots, m$. In this way, the corresponding system of linear equations

$$\tilde{P}_n(x_j) + \sum_{i=1}^{m} A_i \tilde{P}_n(x_i) \operatorname{Ker}_{n-1}(x_j, x_i) = P_n(x_j), \qquad j = 1, 2, \ldots, m, \tag{6}$$

has a unique solution if and only if the determinant

$$\begin{vmatrix} 1 + A_1 \operatorname{Ker}_{n-1}(x_1, x_1) & A_2 \operatorname{Ker}_{n-1}(x_1, x_2) & \cdots & A_m \operatorname{Ker}_{n-1}(x_1, x_m) \\ A_1 \operatorname{Ker}_{n-1}(x_2, x_1) & 1 + A_2 \operatorname{Ker}_{n-1}(x_2, x_2) & \cdots & A_m \operatorname{Ker}_{n-1}(x_2, x_m) \\ \vdots & \vdots & \ddots & \vdots \\ A_1 \operatorname{Ker}_{n-1}(x_m, x_1) & A_2 \operatorname{Ker}_{n-1}(x_m, x_2) & \cdots & 1 + A_m \operatorname{Ker}_{n-1}(x_m, x_m) \end{vmatrix} \tag{7}$$

does not vanish for all $n \in \mathbb{N}$. This is also a necessary and sufficient condition for the existence of the nth degree polynomial $\tilde{P}_n$ for all $n \in \mathbb{N}$.

In this work we will consider the particular cases when we add one or two Dirac delta functions. Let us consider these cases with more details.

2.1. THE CASE OF ONE POINT MASS AT $x = x_1$

In this case from (5) and (6) we get

$$\begin{aligned} \tilde{P}_n(x) &= P_n(x) - A \tilde{P}_n(x_1) \operatorname{Ker}_{n-1}(x, x_1), \\ \tilde{P}_n(x_1) &= \frac{P_n(x_1)}{1 + A \sum_{k=0}^{n-1} \frac{(P_k(x_1))^2}{d_k^2}}, \end{aligned} \tag{8}$$

and the condition (7) becomes

$$1 + A \sum_{k=0}^{n-1} \frac{(P_k(x_1))^2}{d_k^2} \neq 0,$$

which is always true for every $n \in \mathbb{N}$ since $A \geqslant 0$.

2.2. THE CASE OF TWO POINT MASSES AT $x = x_1$ AND x_2

Again we start from (5) and (6). Then,

$$\begin{aligned}
\tilde{P}_n(x) &= P_n(x) - A_1 \tilde{P}_n(x_1)\,\mathrm{Ker}_{n-1}(x, x_1) - A_2 \tilde{P}_n(x_2)\,\mathrm{Ker}_{n-1}(x, x_2),\\
\tilde{P}_n(x_1) &= \frac{\begin{vmatrix} P_n(x_1) & A_2\,\mathrm{Ker}_{n-1}(x_1, x_2) \\ P_n(x_2) & 1 + A_2\,\mathrm{Ker}_{n-1}(x_2, x_2) \end{vmatrix}}{\begin{vmatrix} 1 + A_1\,\mathrm{Ker}_{n-1}(x_1, x_1) & A_2\,\mathrm{Ker}_{n-1}(x_1, x_2) \\ A_1\,\mathrm{Ker}_{n-1}(x_2, x_1) & 1 + A_2\,\mathrm{Ker}_{n-1}(x_2, x_2) \end{vmatrix}},\\
\tilde{P}_n(x_2) &= \frac{\begin{vmatrix} 1 + A_1\,\mathrm{Ker}_{n-1}(x_1, x_1) & P_n(x_1) \\ A_1\,\mathrm{Ker}_{n-1}(x_2, x_1) & P_n(x_2) \end{vmatrix}}{\begin{vmatrix} 1 + A_1\,\mathrm{Ker}_{n-1}(x_1, x_1) & A_2\,\mathrm{Ker}_{n-1}(x_1, x_2) \\ A_1\,\mathrm{Ker}_{n-1}(x_2, x_1) & 1 + A_2\,\mathrm{Ker}_{n-1}(x_2, x_2) \end{vmatrix}},
\end{aligned} \tag{9}$$

and (7) becomes

$$\begin{vmatrix} 1 + A_1\,\mathrm{Ker}_{n-1}(x_1, x_1) & A_2\,\mathrm{Ker}_{n-1}(x_1, x_2) \\ A_1\,\mathrm{Ker}_{n-1}(x_2, x_1) & 1 + A_2\,\mathrm{Ker}_{n-1}(x_2, x_2) \end{vmatrix} \neq 0.$$

Moreover, if A_1 and A_2 are nonnegative real numbers then the above determinant is always positive. To prove this it is sufficient to expand the determinant and use the Cauchy–Schwarz inequality $(\sum a_k b_k)^2 \leqslant \sum a_k^2 \sum b_k^2$.

3. Applications to Classical Polynomials

In the previous section we consider the polynomials orthogonal with respect to a very general weight function $\rho \in \mathcal{C}_{[a,b]}$, $\rho(x) > 0$, $x \in [a, b]$. In this section we will consider some particular cases when ρ is one of the classical weight functions, i.e., the Jacobi, Laguerre, Hermite or Gegenbauer weight functions, respectively. Moreover, since in expressions (8) and (9) the kernel polynomials $\mathrm{Ker}_{n-1}(x, x_i)$ appear we will consider the case when we add some Dirac delta functions at the origin $x = 0$ or at the ends of the interval of orthogonality of the classical polynomials. The last consideration allows us to obtain explicit formulas for the kernel polynomials in terms of the classical polynomials and their derivatives [5, 6].

In this way, if we consider the Jacobi case and add two masses at $x = \pm 1$ we obtain the well-known Jacobi–Koornwinder polynomials [29] and for special values of the masses A_1, A_2 the classical Krall-type polynomials [30, 31]. For Laguerre case when $x = 0$ we obtain the Laguerre–Koekoek polynomials [23, 25]. Finally, for Hermite and Gegenbauer cases when $x = 0$ (the symmetric case) we obtain the Hermite–Krall-type and Gegenbauer–Krall-type polynomials introduced in [6].

The main data of the classical polynomials can be found in [18, 39, 43], for the monic polynomials see, for instance, [5, 6].

3.1. THE JACOBI–KOORNWINDER POLYNOMIALS

The Jacobi–Koornwinder orthogonal polynomials were introduced by T. H. Koornwinder [29]. They can be obtained from the generalized Hahn polynomials introduced in [4] as a limit case [5] and correspond to the case of adding two Dirac delta functions at the ends of the interval of orthogonality of the Jacobi polynomials.

DEFINITION 1. The Jacobi–Koornwinder orthogonal polynomials $P_n^{\alpha,\beta,A,B}$ are the polynomials orthogonal with respect to a linear functional $\mathcal{U}$ on $\mathbb{P}$ defined as follows ($A, B \geqslant 0, \alpha > -1, \beta > -1$)

$$\langle \mathcal{U}, P\rangle = \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P(x)\,\mathrm{d}x + AP(1) + BP(-1). \tag{10}$$

Using the expression (9) and the properties of monic Jacobi polynomials $P_n^{\alpha,\beta}$ we obtain the following representation of $P_n^{\alpha,\beta,A,B}$ in terms of the Jacobi polynomials and their derivatives [5, 29]

$$P_n^{\alpha,\beta,A,B}(x) = P_n^{\alpha,\beta}(x) + \chi_{A,B}^{n,\alpha,\beta} \frac{\mathrm{d}}{\mathrm{d}x} P_n^{\alpha-1,\beta}(x) - \chi_{B,A}^{n,\beta,\alpha} \frac{\mathrm{d}}{\mathrm{d}x} P_n^{\alpha,\beta-1}(x), \tag{11}$$

where $\chi_{A,B}^{n,\alpha,\beta} = -AP_n^{\alpha,\beta,A,B}(-1)\eta_n^{\alpha,\beta}$ and $\chi_{B,A}^{n,\beta,\alpha} = -BP_n^{B,A,\beta,\alpha}(-1)\eta_n^{\beta,\alpha}$,

$$P_n^{\alpha,\beta,A,B}(-1) = \frac{\begin{vmatrix} P_n^{\alpha,\beta}(-1) & B\,\mathrm{Ker}_{n-1}^{J,\alpha,\beta}(-1,1) \\ P_n^{\alpha,\beta}(1) & 1 + B\,\mathrm{Ker}_{n-1}^{J,\alpha,\beta}(1,1) \end{vmatrix}}{\begin{vmatrix} 1 + A\,\mathrm{Ker}_{n-1}^{J,\alpha,\beta}(-1,-1) & B\,\mathrm{Ker}_{n-1}^{J,\alpha,\beta}(-1,1) \\ A\,\mathrm{Ker}_{n-1}^{J,\alpha,\beta}(-1,1) & 1 + B\,\mathrm{Ker}_{n-1}^{J,\alpha,\beta}(1,1) \end{vmatrix}}, \tag{12}$$

and

$$P_n^{\alpha,\beta,A,B}(1) = (-1)^n P_n^{\beta,\alpha,B,A}(-1). \tag{13}$$

For the kernel polynomials we get

$$\begin{aligned}
\mathrm{Ker}_{n-1}^{J,\alpha,\beta}(-1,-1) &= \frac{\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+1)}{(n-1)!\Gamma(\beta+2)\Gamma(\alpha+n)\Gamma(\beta+1)2^{\alpha+\beta+1}},\\
\mathrm{Ker}_{n-1}^{J,\alpha,\beta}(1,1) &= \mathrm{Ker}_{n-1}^{J,\beta,\alpha}(-1,-1),\\
\mathrm{Ker}_{n-1}^{J,\alpha,\beta}(-1,1) &= \frac{(-1)^{n-1}\Gamma(\alpha+\beta+n+1)}{(n-1)!\Gamma(\alpha+1)\Gamma(\beta+1)2^{\alpha+\beta+1}},
\end{aligned}$$

and $\eta_n^{\alpha,\beta}$, $\eta_n^{\beta,\alpha}$ denote the quantities

$$\begin{aligned}
\eta_n^{\alpha,\beta} &= \frac{(-1)^{n-1}\Gamma(2n+\alpha+\beta)}{n!\Gamma(\alpha+n)\Gamma(\beta+1)2^{\alpha+\beta+n}},\\
\eta_n^{\beta,\alpha} &= \frac{(-1)^{n-1}\Gamma(2n+\alpha+\beta)}{n!\Gamma(\beta+n)\Gamma(\alpha+1)2^{\alpha+\beta+n}},
\end{aligned} \tag{14}$$

respectively.

Also the following equivalent representation, similar to the representation obtained in [29] for the monic generalized polynomials, is valid

$$\begin{aligned} P_n^{\alpha,\beta,A,B}(x) &= (1 - nJ_{A,B}^{n,\alpha,\beta} - nJ_{B,A}^{n,\beta,\alpha})P_n^{\alpha,\beta}(x) + \\ &\quad + [J_{A,B}^{n,\alpha,\beta}(x-1) + J_{B,A}^{n,\beta,\alpha}(1+x)]\frac{\mathrm{d}}{\mathrm{d}x}P_n^{\alpha,\beta}(x), \end{aligned} \tag{15}$$

where $J_{A,B}^{n,\alpha,\beta} = -AP_n^{\alpha,\beta,A,B}(-1)\tilde{\eta}_n^{\alpha,\beta}$, $J_{B,A}^{n,\beta,\alpha} = -BP_n^{B,A,\beta,\alpha}(-1)\tilde{\eta}_n^{\beta,\alpha}$ and $\tilde{\eta}_n^{\alpha,\beta}$, $\tilde{\eta}_n^{\beta,\alpha}$ denote the quantities

$$\begin{aligned} \tilde{\eta}_n^{\alpha,\beta} &= -\frac{(-1)^n\Gamma(2n+\alpha+\beta+1)}{n!\Gamma(\alpha+n+1)\Gamma(\alpha+\beta+2)\Gamma(\beta+1)2^{n+\alpha+\beta+1}}, \\ \tilde{\eta}_n^{\beta,\alpha} &= -\frac{(-1)^n\Gamma(2n+\alpha+\beta+1)}{n!\Gamma(\beta+n+1)\Gamma(\alpha+\beta+2)\Gamma(\alpha+1)2^{n+\alpha+\beta+1}}. \end{aligned} \tag{16}$$

From (15) we can obtain a lot of results, in particular the hypergeometric representation of the new polynomials [29], the second-order differential equation [29, 22, 5] (see Appendix I) and the three-term recurrence relation

$$\begin{aligned} &xP_n^{\alpha,\beta,A,B}(x) = P_{n+1}^{\alpha,\beta,A,B}(x) + \beta_n P_n^{\alpha,\beta,A,B}(x) + \gamma_n P_{n-1}^{\alpha,\beta,A,B}(x), \\ &P_{-1}^{\alpha,\beta,A,B}(x) = 0, \quad \text{and} \quad P_0^{\alpha,\beta,A,B}(x) = 1, \quad n \geqslant 0, \end{aligned} \tag{17}$$

which is a consequence of the orthogonality of the polynomials (10). The coefficients β_n can be obtained equating the coefficients of the x^n power in (17). Then,

$$\begin{aligned} \beta_n = {}& \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)} + n(J_{A,B}^{n,\alpha,\beta} - J_{B,A}^{n,\beta,\alpha}) - \\ &- (n+1)(J_{A,B}^{n+1,\alpha,\beta} - J_{B,A}^{n+1,\beta,\alpha}) + \\ &+ \frac{n(\alpha-\beta)}{2n+\alpha+\beta}(J_{A,B}^{n,\alpha,\beta} + J_{B,A}^{n,\beta,\alpha}) - \frac{(n+1)(\alpha-\beta)}{2n+\alpha+\beta+2}(J_{A,B}^{n+1,\alpha,\beta} + J_{B,A}^{n+1,\beta,\alpha}). \end{aligned}$$

To obtain γ_n we notice that $P_n^{\alpha,\beta,A,B}(1) \neq 0$ for all $n \geqslant 0$. Then, from (17)

$$\gamma_n = (1-\beta_n)\frac{P_n^{\alpha,\beta,A,B}(1)}{P_{n-1}^{\alpha,\beta,A,B}(1)} - \frac{P_{n+1}^{\alpha,\beta,A,B}(1)}{P_{n-1}^{\alpha,\beta,A,B}(1)}.$$

Also from (15) it is possible to obtain the ratio asymptotics $P_n^{\alpha,\beta,A,B}(x)/P_n^{\alpha,\beta}(x)$. Firstly, we use the asymptotic formula for the gamma function [1] to obtain

$$J_{A,B}^{n,\alpha,\beta} \sim \frac{\beta+1}{n^2}, \qquad J_{B,A}^{n,\beta,\alpha} \sim \frac{\alpha+1}{n^2}.$$

Then, the asymptotic formulas for the Jacobi–Koornwinder polynomials in and off the interval of orthogonality follow from the Darboux formula in $\theta \in [\varepsilon, \pi-\varepsilon]$,

$0 < \varepsilon \ll 1$ (see [43, Theorem 8.21.8, p. 196]) and the the Darboux formula in $\mathbb{C}\backslash[-1, 1]$ (see [43, Theorem 8.21.7, p. 196]), respectively. From the above considerations we find

$$\begin{aligned}
&n2^{n+\alpha+\beta-1}\big[P_n^{\alpha,\beta,A,B}(\cos\theta) - P_n^{\alpha,\beta}(\cos\theta)\big] \\
&\quad = \frac{-1}{\sqrt{\pi n}}\left(\sin\frac{\theta}{2}\right)^{-\alpha-3/2}\left(\cos\frac{\theta}{2}\right)^{-\beta-3/2} \times \\
&\quad \times \left[\frac{1}{2}(\alpha+\beta+2)\sin\theta\cos(N\theta+\Gamma_1) + \right. \\
&\quad \left. + \left((\beta+1)\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}(\alpha+1)\right)\sin(N\theta+\Gamma_1)\right] + \mathrm{O}\left(\frac{1}{n^{3/2}}\right),
\end{aligned}$$

where $N = n + \frac{1}{2}(\alpha+\beta+1)$, $\Gamma_1 = -(\alpha+\frac{1}{2})\frac{\pi}{2}$ and

$$\begin{aligned}
\frac{P_n^{\alpha,\beta,A,B}(z)}{P_n^{\alpha,\beta}(z)} \;=\; & 1 - \frac{\alpha+\beta+2}{n} + \\
& + \frac{1}{n}\left(\frac{(z+1)(\alpha+1)+(z-1)(\beta+1)}{\sqrt{z^2-1}}\right) + \mathrm{o}\left(\frac{1}{n}\right),
\end{aligned}$$

valid in $\theta \in [\varepsilon, \pi-\varepsilon]$, $0 < \varepsilon << 1$ and every compact subset of $\mathbb{C}\backslash[-1, 1]$, respectively. The last formula holds uniformly in the exterior of an arbitrary closed curve which enclose the segment $[-1, 1]$, moreover, if $z \in \mathbb{C}$, $z > 1$, the right-hand side expression is a real function of z.

3.2. THE LAGUERRE–KOEKOEK POLYNOMIALS

The Laguerre–Koekoek orthogonal polynomials were introduced in [29] as a limit case of the Jacobi–Koornwinder polynomials and studied with more details in several works [23, 25, 27]. They also can be obtained as a limit case of the generalized Meixner polynomials introduced in [9, 2] using an appropriate limit transition [5].

DEFINITION 2. The Laguerre–Koekoek orthogonal polynomials $L_n^{\alpha,A}$ are the polynomials orthogonal with respect to a linear functional $\mathcal{U}$ on $\mathbb{P}$ defined as follows

$$\langle \mathcal{U}, P\rangle = \int_0^\infty x^\alpha e^{-x} P(x)\,dx + AP(0), \quad A \geqslant 0,\ \alpha > -1. \tag{18}$$

Using the algorithm described before (see formula (8)) we find for the Laguerre–Koekoek polynomials the following representation formula (see [5, 27] for more details)

$$L_n^{\alpha,A}(x) = L_n^\alpha(x) + \Gamma_n\frac{d}{dx}L_n^\alpha(x), \quad \Gamma_n = \frac{A(\alpha+1)_n}{n!\big(1 + A\frac{(\alpha+2)_{n-1}}{(n-1)!}\big)\Gamma(\alpha+1)}. \tag{19}$$

From (19) we can obtain a lot of properties, for example, the hypergeometric representation of the new polynomials [27], the second-order differential equation [27] (see Appendix I) and the three-term recurrence relation

$$\begin{aligned} &xL_n^{\alpha,A}(x) = L_{n+1}^{\alpha,A}(x) + \beta_n L_n^{\alpha,A}(x) + \gamma_n L_{n-1}^{\alpha,A}(x), \quad n \geqslant 0, \\ &L_{-1}^{\alpha,A}(x) = 0, \quad \text{and} \quad L_0^{\alpha,A}(x) = 1, \end{aligned} \tag{20}$$

which is a consequence of the orthogonality of the polynomials (18). The coefficients β_n and γ_n are given by ($L_k^{\alpha,A}(0) \neq 0$ for all $n \geqslant 0$)

$$\beta_n = 2n + \alpha + 1 + \Gamma_n - \Gamma_{n+1}, \quad \gamma_n = \beta_n \frac{L_n^{\alpha,A}(0)}{L_{n-1}^{\alpha,A}(0)} - \frac{L_{n+1}^{\alpha,A}(0)}{L_{n-1}^{\alpha,A}(0)}.$$

To obtain the ratio asymptotics $L_n^{\alpha,A}(x)/L_n^{\alpha}(x)$ we use the asymptotic formula for the gamma function [1] to obtain

$$\Gamma_n \sim \frac{\alpha+1}{n}.$$

Then from (19) and by using the Perron formula for the ratio $1/\sqrt{n}\,(L_n^{\alpha})'(z)/L_n^{\alpha}(z)$ of the Laguerre polynomials (see [44, Equation (4.2.6), p. 133] or [43, Theorem 8.22.3]) we get

$$\frac{L_n^{\alpha,A}(z)}{L_n^{\alpha}(z)} = 1 + \frac{\alpha+1}{\sqrt{nz}}\left[1 - \frac{1}{4\sqrt{-nz}}(2\alpha+1-z)\right] + \mathrm{o}\left(\frac{1}{n}\right),$$

for $z \in \mathbb{C}\backslash[0,\infty)$.

3.3. THE HERMITE–KRALL-TYPE POLYNOMIALS

The Hermite–Krall-type polynomials were introduced in [6]. They can be obtained as a quadratic transformation of the Laguerre–Koekoek polynomials [6].

DEFINITION 3. The generalized monic Hermite polynomials H_n^A are the polynomials orthogonal with respect to the linear functional $\mathcal{U}$ on $\mathbb{P}$

$$\langle \mathcal{U}, P\rangle = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2} P(x)\,\mathrm{d}x + AP(0), \qquad A \geqslant 0. \tag{21}$$

Again, from formula (8) after some straightforward calculations we obtain that the Hermite–Krall-type polynomials H_n^A admit the following representations in terms of the classical polynomials

$$\begin{aligned} &H_{2m-1}^A(x) = H_{2m-1}(x), \\ &2xH_{2m}^A(x) = 2xH_{2m}(x) + B_m \frac{\mathrm{d}}{\mathrm{d}x} H_{2m}(x), \quad m = 0, 1, 2, \ldots, \end{aligned} \tag{22}$$

$$B_m = \frac{A}{\left(1 + A\frac{2\Gamma(m+\frac{1}{2})}{\pi\Gamma(m)}\right)} \frac{\Gamma(m+\frac{1}{2})}{\pi m!}.$$

Notice that the odd polynomials coincide with the classical ones. They are quadratic transformations of the Laguerre–Koekoek polynomials [6]

$$\begin{aligned} &H_{2m-1}^A(x) = xL_{m-1}^{1/2}(x^2), \\ &H_{2m}^A(x) = L_m^{-1/2,A}(x^2) = L_m^{-1/2}(x^2) + B_m \frac{\mathrm{d}}{\mathrm{d}x^2} L_m^{-1/2}(x^2), \quad m = 0, 1, 2, \ldots. \end{aligned} \tag{23}$$

Notice that from the above formula the connection with the Laguerre–Koekoek polynomials follows. Again from the above representation we can obtain the hypergeometric representation [6], the second order differential equation [6] (see Appendix I) and the three-term recurrence relation

$$\begin{aligned} &xH_n^A(x) = H_{n+1}^A(x) + \beta_n H_n^A(x) + \gamma_n H_{n-1}^A(x), \quad n \geqslant 0, \\ &H_{-1}^A(x) = 0 \quad \text{and} \quad H_0^A(x) = 1. \end{aligned} \tag{24}$$

which is a consequence of the orthogonality. The coefficient β_n is always equal to zero since the functional $\mathcal{U}$ is symmetric. For the coefficients γ_n we have [6]

$$\begin{aligned} &\gamma_{2m} = m(1 + B_m), \\ &\gamma_{2m-1} = \frac{(2m-1)}{2} \frac{1 + \frac{2A}{\pi}\frac{\Gamma(m-\frac{1}{2})}{\Gamma(m-1)}}{1 + \frac{2A}{\pi}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m)}}, \quad m = 1, 2, 3, \ldots. \end{aligned} \tag{25}$$

For the asymptotic formula we get

$$B_m \sim \frac{1}{2m}.$$

Then, for m large enough

$$\frac{H_{2m}^A(z)}{H_{2m}(z)} = 1 - \frac{1}{2\sqrt{m}iz}\left\{1 - \frac{iz}{\sqrt{m}}\right\} + \mathrm{o}\left(\frac{1}{m}\right), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{26}$$

which is a consequence of (23) and the ratio asymptotics of the Laguerre–Koekoek polynomials.

3.4. THE GEGENBAUER–KRALL-TYPE POLYNOMIALS

The Gegenbauer–Krall-type polynomials were introduced in [6]. They can be obtained as a quadratic transformation of the Jacobi–Koornwinder polynomials [6].

DEFINITION 4. The generalized monic Gegenbauer polynomials $G_n^{\lambda,A}$ are the polynomials orthogonal with respect to the linear functional $\mathcal{U}$ on $\mathbb{P}$

$$\langle \mathcal{U}, P\rangle = \int_{-1}^{1} (1-x^2)^{\lambda-1/2} P(x)\,\mathrm{d}x + AP(0), \quad A \geqslant 0,\ \lambda > -\frac{1}{2}. \tag{27}$$

From formula (8) after some straightforward calculations we obtain that the generalized Gegenbauer polynomials $G_n^{\lambda,A}$ have the following representation in terms of the classical ones

$$\begin{aligned} G_{2m+1}^{\lambda,A}(x) &= G_{2m+1}^{\lambda}(x), \quad m = 0, 1, 2, \ldots, \\ 2xG_{2m}^{\lambda,A}(x) &= 2x(1 + mW_m^A)G_{2m}^{\lambda}(x) + W_m^A(1 - x^2)\frac{\mathrm{d}}{\mathrm{d}x}G_{2m}^{\lambda}(x), \\ W_m^A &= \frac{A}{\left(1 + A\frac{2\Gamma(m+\frac{1}{2})\Gamma(m+\lambda)}{\pi(m-1)!\Gamma(m+\lambda-\frac{1}{2})}\right)}\frac{\Gamma(m+\frac{1}{2})\Gamma(m+\lambda)}{\pi m!\Gamma(m+\lambda+\frac{1}{2})}. \end{aligned} \tag{28}$$

Notice that, like in the previous case, the odd polynomials coincide with the classical ones. They are a quadratic transformation of the Jacobi–Koornwinder polynomials [6]

$$\begin{aligned} G_{2m}^{\lambda,A}(x) &= 2^{-m}P_m^{\lambda-\frac{1}{2},-\frac{1}{2},2^{\lambda}A,0}(2x^2 - 1), \\ G_{2m+1}^{\lambda,A}(x) &= 2^{-m}xP_m^{\lambda-\frac{1}{2},\frac{1}{2},}(2x^2 - 1), \quad m = 0, 1, 2, \ldots. \end{aligned} \tag{29}$$

The above formula represents the connection with the Jacobi–Koornwinder polynomials. Again from the above representations we can obtain the hypergeometric representation [6], the second order differential equation (see Appendix I), and the three-term recurrence relation

$$\begin{gathered} xG_n^{\lambda,A}(x) = G_{n+1}^{\lambda,A}(x) + \beta_n G_n^{\lambda,A}(x) + \gamma_n G_{n-1}^{\lambda,A}(x), \quad n \geqslant 0, \\ G_{-1}^{\lambda,A}(x) = 0 \quad \text{and} \quad G_0^{\lambda,A}(x) = 1. \end{gathered} \tag{30}$$

where the coefficients $\beta_n = 0$ and the γ_n are given by ($m = 1, 2, 3, \ldots$)

$$\begin{aligned} \gamma_{2m} &= \frac{m(2m + 2\lambda - 1)}{2(2m + \lambda)(2m + \lambda - 1)}\left[1 + W_m^A(m + \lambda)\right], \\ \gamma_{2m-1} &= \frac{(2m - 1)(m + \lambda - 1)}{2(2m + \lambda - 1)}\frac{1 + A\frac{2\Gamma(m-\frac{1}{2})\Gamma(m+\lambda-1)}{\pi(m-2)!\Gamma(m+\lambda-\frac{3}{2})}}{1 + A\frac{2\Gamma(m+\frac{1}{2})\Gamma(m+\lambda)}{\pi(m-1)!\Gamma(m+\lambda-\frac{1}{2})}}. \end{aligned} \tag{31}$$

Finally,

$$W_m^A \sim \frac{1}{2m^2}.$$

Then, we obtain for the generalized Gegenbauer polynomials the following asymptotic formula valid for $\theta \in [\varepsilon, \pi - \varepsilon]\backslash\{\pi/2\}$ ($0 < \varepsilon \ll 1$), $a_{2m} = (2m + 2\lambda)_{2m}/2^{2m-1}2m!$

$$\begin{aligned} &a_{2m}\cos\theta\left(G_{2m}^{\lambda,A}(\cos\theta) - G_{2m}^{\lambda}(\cos\theta)\right) \\ &\quad = \frac{1}{\sqrt{2\pi m^3}}\left(\frac{1}{2\sin\theta}\right)^{\lambda} \times \left[\cos\theta\cos(2m\theta + \lambda\theta - \frac{1}{2}\lambda\pi) + \right. \\ &\qquad \left. + \frac{1}{4}\sin\theta\sin(2m\theta + \lambda\theta - \frac{1}{2}\lambda\pi)\right] + \mathrm{O}\left(\frac{1}{m^{5/2}}\right). \end{aligned} \tag{32}$$

For $x = \cos\frac{\pi}{2} = 0$ we can use the expression

$$G_{2m}^{\lambda,A}(0) = \frac{G_{2m}^{\lambda}(0)}{1 + A\sum_{k=0}^{m-1}\left[\frac{G_{2k}^{\lambda}(0)}{d_{2k}^{G}}\right]^2},$$

where d_n^G is the norm of the Gegenbauer polynomials, which yields

$$\frac{G_{2m}^{\lambda,A}(0)}{G_{2m}^{\lambda}(0)} = \frac{\pi}{2Am} + \mathrm{O}\left(\frac{1}{m^2}\right).$$

For the ratio asymptotics off the interval of orthogonality we find

$$\frac{G_{2m}^{\lambda,A}(z)}{G_{2m}^{\lambda}(z)} = 1 + \frac{1}{2m}\left(1 - \sqrt{1 - \frac{1}{z^2}}\right) + \mathrm{o}\left(\frac{1}{m}\right). \tag{33}$$

which holds uniformly in the exterior of an arbitrary closed curve which enclose the segment $[-1, 1]$. The last expression is a consequence of the Darboux formula in $\mathbb{C}\backslash[-1, 1]$ (see [43, Theorem 8.21.7, p. 196]).

4. The Distribution of Zeros: The Moments μ_r and the WKB Density

In this section we will study the distribution of zeros of the Jacobi–Koornwinder, Laguerre–Koekoek, Hermite–Krall-type and Gegenbauer–Krall-type polynomials. We will use a general method presented in [12] for the moments of low order and the WKB approximation [8, 46, 47] in order to obtain an approximation to the density of the distribution of zeros.

First of all we point out that, since our polynomials are orthogonal with respect to a positive definite functional all their zeros are real, simple and located in the interior of the interval of orthogonality. This is a necessary condition in order to apply the next algorithms.

4.1. THE MOMENTS OF THE DISTRIBUTION OF ZEROS

The method presented in [12] allows us to compute the moments μ_r of the distribution of zeros ρ_n around the origin, i.e.,

$$\mu_r = \frac{1}{n}y_r = \frac{1}{n}\sum_{i=1}^{n} x_{n,i}^r, \quad \rho_n = \frac{1}{n}\sum_{i=1}^{n}\delta(x - x_{n,i}).$$

Buendía, Dehesa and Gálvez [12] have obtained a general formula to find these quantities (see [12, Section II, Equations (11) and (13), p. 226]). We will apply these two formulas to obtain the general expression for the moments μ_1 and μ_2, but firstly, we will introduce some notations.

We start with the second-order linear differential equation (SODE)

$$\tilde{\sigma}(x;n)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde{P}_n(x)+\tilde{\tau}(x;n)\frac{\mathrm{d}}{\mathrm{d}x}\tilde{P}_n(x)+\tilde{\lambda}(x;n)\tilde{P}_n(x)=0,$$

that such polynomials satisfy. Here

$$\tilde{\sigma}(x;n)=\sum_{k=0}^{c_2}a_k^{(2)}x^k,\quad \tilde{\tau}(x;n)=\sum_{k=0}^{c_1}a_k^{(1)}x^k,\quad \tilde{\lambda}(x;n)=\sum_{k=0}^{c_0}a_k^{(0)}x^k,\quad (34)$$

and c_2, c_1, c_0 are the degrees of the polynomials $\tilde{\sigma}$, $\tilde{\tau}$ and $\tilde{\lambda}$, respectively. The values $a_j^{(i)}$ can be found from (A.5) in a straightforward way. Let $\xi_0 = 1$ and $q = \max\{c_2-2, c_1-1, c_0\}$. Then from [12, Section II, Equations(11) and (13), p. 226]

$$\xi_1 = y_1,\qquad \xi_2=\frac{y_1^2-y_2}{2},\tag{35}$$

and

$$\xi_s=-\frac{\sum_{m=1}^{s}(-1)^m\xi_{s-m}\sum_{i=0}^{2}\frac{(n-s+m)!}{(n-s+m-i)!}a_{i+q-m}^{(i)}}{\sum_{i=0}^{2}\frac{(n-s)!}{(n-s-i)!}a_{i+q}^{(i)}}.\tag{36}$$

In general

$$\xi_k=\frac{(-1)^k}{k!}\mathcal{Y}_k(-y_1,-y_2,-2y_3,\ldots,-(k-1)!y_k),$$

where $\mathcal{Y}_k$-symbols denote the well-known Bell polynomials in number theory [42].

Let us now to apply these general formulas to obtain the first two central moments μ_1 and μ_2 of our polynomials. Equation (36) give the following values.

4.1.1. *Jacobi–Koornwinder Polynomials* $P_n^{\alpha,\beta,A,B}$

$$\xi_1=\frac{n\big(-\alpha+\beta+2\alpha J_{A,B}^{n,\alpha,\beta}-2\beta J_{B,A}^{n,\beta,\alpha}+2J_{A,B}^{n,\alpha,\beta}n-2J_{B,A}^{n,\beta,\alpha}n\big)}{\alpha+\beta+2n}.$$

Then

$$\mu_1=\frac{-\alpha+\beta+2\alpha J_{A,B}^{n,\alpha,\beta}-2\beta J_{B,A}^{n,\beta,\alpha}+2J_{A,B}^{n,\alpha,\beta}n-2J_{B,A}^{n,\beta,\alpha}n}{\alpha+\beta+2n}.$$

For the moments of second order μ_2 the expression can be found by straightforward but cumbersome calculation and we will not give it explicitly. The asymptotic behavior of the moment μ_1 is

$$\mu_1\sim\frac{\alpha-\beta}{2}+\mathrm{O}\left(\frac{1}{n}\right).$$

As particular cases we will consider the Legendre–Koornwinder polynomials $P_n^{0,0,A,B}$ and the Gegenbauer–Koornwinder polynomials $G_n^{\nu,A,B} \equiv P_n^{\nu,\nu,A,B}$. For the first ones $P_n^{0,0,A,B}$ we get

$$\begin{aligned}
\mu_1 &= -J_{A,B}^{n,0,0} + J_{B,A}^{n,0,0}, \\
\mu_2 &= \frac{1}{2n-1}\big(-1 + 2J_{A,B}^{n,0,0} + 2J_{B,A}^{n,0,0} + n - 2J_{A,B}^{n,0,0} n - J_{A,B}^{n,0,0^2} n - 2J_{B,A}^{n,0,0} n + \\
&\quad + 2J_{A,B}^{n,0,0} J_{B,A}^{n,0,0} n - J_{B,A}^{n,0,0^2} n + 2J_{A,B}^{n,0,0^2} n^2 - 4J_{A,B}^{n,0,0} J_{B,A}^{n,0,0} n^2 + 2J_{B,A}^{n,0,0^2} n^2\big),
\end{aligned}$$

and for Gegenbauer–Koornwinder polynomials

$$\begin{aligned}
\mu_1 &= -J_{A,B}^{n,\nu,\nu} + J_{B,A}^{n,\nu,\nu}, \\
\mu_2 &= \frac{1}{2n+2\nu-1}\big(-1 + 2J_{A,B}^{n,\nu,\nu} + 2J_{B,A}^{n,\nu,\nu} + n - 2J_{A,B}^{n,\nu,\nu} n - J_{A,B}^{n,\nu,\nu 2} n - \\
&\quad - 2J_{B,A}^{n,\nu,\nu} n + 2J_{A,B}^{n,\nu,\nu} J_{B,A}^{n,\nu,\nu} n - J_{B,A}^{n,\nu,\nu 2} n + 2J_{A,B}^{n,\nu,\nu 2} n^2 - 4J_{A,B}^{n,\nu,\nu} J_{B,A}^{n,\nu,\nu} n^2 + \\
&\quad + 2J_{B,A}^{n,\nu,\nu 2} n^2 + 2J_{A,B}^{n,\nu,\nu 2} n\nu - 4J_{A,B}^{n,\nu,\nu} J_{B,A}^{n,\nu,\nu} n\nu + 2J_{B,A}^{n,\nu,\nu 2} n\nu\big).
\end{aligned}$$

Notice that in both cases if $A = B$, $J_{A,B}^{n,\nu,\nu} \equiv J_{A,A}^{n,\nu,\nu} = J_{B,A}^{n,\nu,\nu}$ and then, $\mu_1 = 0$.

4.1.2. *Laguerre–Koekoek Polynomials* $L_n^{\alpha,A}$

$$\xi_1 = n(\alpha - \Gamma_n + n), \qquad \xi_2 = \frac{(1-n)(1-\alpha+2\Gamma_n - n)n(\alpha+n)}{2}.$$

Then

$$\begin{aligned}
\mu_1 &= \alpha - \Gamma_n + n, \\
\mu_2 &= -\alpha + \alpha^2 - 2\alpha\Gamma_n - n + 3\alpha n - 2n\Gamma_n + n{\Gamma_n}^2 + 2n^2.
\end{aligned}$$

The asymptotic behavior of the moments is

$$\mu_1 \sim n + \mathrm{O}(1) \quad \text{and} \quad \mu_2 \sim 2n^2 + \mathrm{O}(n).$$

4.1.3. *Hermite–Krall-type Polynomials* H_n^A

- If $n = 2m$, $m = 0, 1, 2, \ldots$, then

$$\xi_1 = 0, \qquad \xi_2 = \frac{(1 + 2B_n - 2m)m}{2},$$

and the moments are

$$\mu_1 = 0, \qquad \mu_2 = \frac{(2m - 1 - 2B_m)}{2}.$$

- If $n = 2m - 1$, $m = 1, 2, \dots$, then, $H^A_{2m-1}(x) \equiv H_{2m-1}(x)$

$$\xi_1 = 0, \qquad \xi_2 = (1 - m)m,$$

and the moments are

$$\mu_1 = 0, \qquad \mu_2 = (m - 1).$$

The asymptotic behavior of these two moments in both cases is

$$\mu_1 = 0 \quad \text{and} \quad \mu_2 \sim \frac{n}{2} + \mathrm{O}(n).$$

4.1.4. *Gegenbauer Polynomials $G_n^{\lambda,A}$*

- If $n = 2m$, $m = 0, 1, 2, \dots$, then

$$\xi_1 = 0, \qquad \xi_2 = \frac{m(-1 + 2m + W_m - n^2 W_m - 2\lambda W_m)}{2(-1 + 2m + \lambda)(-1 + 2mW_m)},$$

and the moments are

$$\mu_1 = 0, \qquad \mu_2 = \frac{1 - 2m - W_m + 4m^2 W_m + 2\lambda W_m}{2(-1 + 2m + \lambda)(-1 + 2mW_m)}.$$

- If $n = 2m - 1$, $m = 1, 2, \dots$, then, $G^{\lambda,A}_{2m-1}(x) \equiv G_{2m-1}\lambda(x)$

$$\xi_1 = 0, \qquad \xi_2 = \frac{2m(2 - 2m)}{4(-2 + 2m + \lambda)},$$

and the moments are

$$\mu_1 = 0, \qquad \mu_2 = \frac{2m - 1}{2(2m - 2 + \lambda)}.$$

The asymptotic behavior of these two moments in both cases is

$$\mu_1 = 0 \quad \text{and} \quad \mu_2 \sim \frac{1}{2} + \mathrm{O}(n^{-1}).$$

Notice that Equation (36) and relation

$$\xi_k = \frac{(-1)^k}{k!} \mathcal{Y}_k(-y_1, -y_2, -2y_3, \dots, -(k-1)! y_k)$$

provide us a general method to obtain all the moments $\mu_r = (1/n)y_r$, but it is hightly nonlinear and cumbersome. This is a reason why it is useful to analyze only for the moments of low order.

4.2. THE WKB DENSITY OF THE DISTRIBUTION OF ZEROS

Next, we will analyze the so-called semiclassical or WKB approximation of the distribution of zeros (see [8, 46, 47] and references contained therein). Denoting the zeros of $\tilde{P}_n$ by $\{x_{n,k}\}_{k=1}^{n}$ we can define its distribution function as

$$\rho_n(x) = \frac{1}{n}\sum_{k=1}^{n}\delta(x - x_{n,k}). \tag{37}$$

We will follow the method presented in [46] in order to obtain the WKB density of zeros, which is an approximate expression for the density of zeros of solutions of any second order linear differential equation with polynomial coefficients

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \tag{38}$$

The main result is established in the following theorem:

THEOREM 1. *Let S and ε be the functions*

$$S(x) = \frac{1}{4a_2^2}\{2a_2(2a_0 - a_1') + a_1(2a_2' - a_1)\}, \tag{39}$$

$$\varepsilon(x) = \frac{1}{4[S(x)]^2}\left\{\frac{5[S'(x)]^2}{4[S(x)]} - S''(x)\right\} = \frac{P(x,n)}{Q(x,n)}, \tag{40}$$

where $P(x,n)$ and $Q(x,n)$ are polynomials in x as well as in n. If the condition $\varepsilon(x) \ll 1$ holds, then the semiclassical or WKB density of zeros of the solutions of (38) *is given by*

$$\rho_{\mathrm{WKB}}(x) = \frac{1}{\pi}\sqrt{S(x)}, \quad x \in I \subseteq \mathbb{R}, \tag{41}$$

in every interval I where the function S is positive.

The proof of this theorem can be found in [8, 46].

Now we can apply this result to our differential equation (A.4)–(A.5). Using the coefficients of the equation (A.4)–(A.5) we obtain that in all cases under consideration we have for sufficiently large n, $\varepsilon(x) \sim n^{-1}$. Then, from the above theorem the corresponding WKB density of zeros of the polynomials $\{\tilde{P}_n\}$ follows. The computations are very long and cumbersome. For this reason we write a little program using MATHEMATICA [45] and provide here only some special cases and some graphics representation for the ρ_{WKB} function.

4.2.1. *Jacobi–Koornwinder Polynomials* $P_n^{\alpha,\beta,A,B}$

In this case the explicit expression of ρ_{wkb} is very large and cumbersome, so we will provide some particular cases. If we take the limit $A \to 0$, $B \to 0$ in the obtained expression we recover the classical one [46, 47]

$$\rho_{\text{wkbclas}}(x) = \frac{\sqrt{R(x)}}{2\pi(1-x^2)},$$

$$\begin{aligned} R(x) \;=\; & 4 + 2\alpha - \alpha^2 + 2\beta + 2\alpha\beta - \beta^2 + 4n + 4\alpha n + 4\beta n + 4n^2 - \\ & - 2\alpha^2 x + 2\beta^2 x - 2\alpha x^2 - \alpha^2 x^2 - 2\beta x^2 - 2\alpha\beta x^2 - \beta^2 x^2 - \\ & - 4nx^2 - 4\alpha n x^2 - 4\beta n x^2 - 4n^2 x^2. \end{aligned}$$

For the Jacobi–Koornwinder polynomials, taking the limit $\lim_{n\to\infty} \frac{1}{n} \rho_{\text{wkb}}^{\alpha=0,\beta=0}(x)$, we find

$$\rho(x) = \frac{1}{\pi\sqrt{1-x^2}}.$$

The last expression coincides with the known expression for the Legendre polynomials (see e.g. [46]), i.e., the asymptotic distribution of zeros of the Jacobi–Koornwinder is the same that the classical ones.

In Figure 1 we represent the WKB density of zeros for the Legendre–Koornwinder and Gegenbauer–Koornwinder (with $\alpha = \beta = 5$) polynomials. We have plotted the Density function for different values of n (from the top to the bottom) $n = 10^6, 10^5, 10^4$. Notice that the value of the mass doesn't play a crucial role, since for $n \gg 1$ $J_{A,B}^{n,\alpha,\beta} \sim \frac{\beta+1}{n}$, $J_{B,A}^{n,\beta,\alpha} \sim \frac{\alpha+1}{n}$, independently of the values of the masses A and B.

4.2.2. *Laguerre–Koekoek Polynomials* $L_n^{\alpha,A}$

Again the explicit expression of ρ_{wkb} is very large and cumbersome. Firstly we can convince ourselves that using (41) and taking the limit when $A \to 0$ we find

$$\rho_{\text{wkbclas}}(x) = \frac{\sqrt{(1 - \alpha^2 + 2x + 2\alpha x + 4nx - x^2)}}{2\pi x}.$$

which coincides with the classical expression [46, 47]. If we now consider the special case $\alpha = 0$ we obtain

$$\rho_{\text{wkb}}^{\alpha=0}(x) = \frac{\sqrt{R(x)}}{2\pi x^2(-\Gamma_n + {\Gamma_n}^2 n + x + \Gamma_n x)},$$

where

$$R(x) \;=\; x^2(2\Gamma_n - 2n{\Gamma_n}^2 - 5x - 4x\Gamma_n - nx{\Gamma_n}^2 - x^2 - x^2\Gamma_n) \times$$

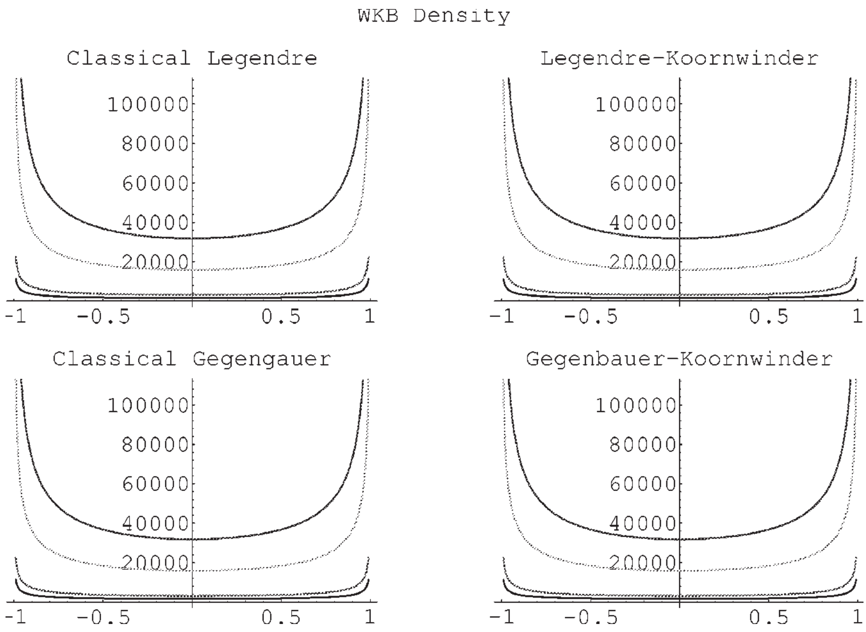

Figure 1. WKB density of zeros of $P_n^{A,B,\alpha,\beta}(x)$.

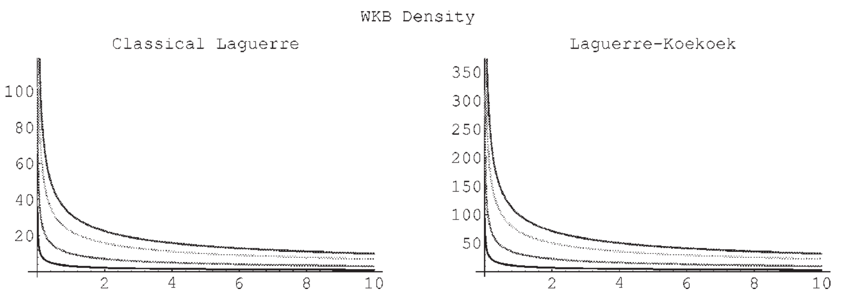

Figure 2. WKB density of zeros of $L_n^{0,A}(x)$.

$$
\begin{aligned}
&\times(2\Gamma_n - 2n{\Gamma_n}^2 - x - 2x\Gamma_n + nx\Gamma_n + x^2 + x^2\Gamma_n) + \\
&+ 2x^2(n{\Gamma_n}^2 + x - \Gamma_n + x\Gamma_n) \times \\
&\times(2\Gamma_n - 2n{\Gamma_n}^2 - 2x - 4x\Gamma_n - 4nx\Gamma_n + 2n^2x{\Gamma_n}^2 + 3x^2 + \\
&+ 3x^2\Gamma_n + 2nx^2 + 2nx^2\Gamma_n).
\end{aligned}
$$

In Figure 2 we represent the WKB density of zeros for the Laguerre–Koekoek polynomials with $\alpha = 0$. We have plotted the density function for different values of n (from the top to the bottom) $n = 10^5, 5 \times 10^4, 10^4, 10^3$. Notice that the value of the mass doesn't play a crucial role, since, for $n \gg 1$, $\Gamma_n \sim ((\alpha+1)/n)$, independently of A.

4.2.3. *Hermite–Krall-type Polynomials* H_{2m}^A

We will analyze only the polynomials of even degree, i.e., $\tilde{P}_{2m}$. In this case from (39) and (41)

$$\rho_{\text{wkbclas}}(x) = \frac{\sqrt{R(x)}}{(-B_m + 2B_m^2 m + 2x^2 + 2B_m x^2)},$$

$$\begin{aligned} R(x) = {} & -6B_m - 3B_m^2 + 24B_m^2 m + 8B_m^3 m - 32B_m^3 m^2 - 4B_m^4 m^2 + \\ & + 16B_m^4 m^3 - 8B_m x^2 - 9B_m^2 x^2 - 32B_m m x^2 - -32B_m^2 m x^2 + \\ & + 4B_m^3 m x^2 + 32B_m^2 m^2 x^2 + 32B_m^3 m^2 x^2 - 4B_m^4 m^2 x^2 + 4x^4 + \\ & + 12B_m x^4 - 8B_m^2 x^4 + 16m x^4 + 32B_m m x^4 + 8B_m^2 m x^4 - \\ & - 8B_m^3 m x^4 - 4x^6 - 8B_m x^6 - 4B_m^2 x^6. \end{aligned}$$

If we take the limit $A \to 0$, again we recover the classical expression [46, 47]

$$\rho_{\text{wkb}}^{\lambda}(x) = \frac{\sqrt{1 + 4m - x^2}}{\pi}.$$

In Figure 3 we represent the WKB density of zeros for our generalized Hermite polynomials. We have plotted the Density function for different values of n (from the top to the bottom) $n = 2 \times 10^4$, 1.5×10^4, 10^4, 10^3. Notice that the value of the mass doesn't play a crucial role since, for $n \gg 1$, $B_m \sim 1/2m$, independently of A.

4.2.4. *Gegenbauer–Krall-Type Polynomials* $G_{2m}^{\lambda,A}$

We will analyze only the polynomials of even degree, i.e., $\tilde{P}_{2m}$. In this case the expression is very large and we will provide only the limit case when $A \to 0$ which agrees with the classical expression [46, 47]

$$\rho_{\text{wkb}}^{\lambda}(x) = \frac{\sqrt{2 + 16m^2 + 4\lambda + 16m\lambda + x^2 - 16m^2x^2 - 16m\lambda x^2 - 4\lambda^2 x^2}}{2\pi(1 - x^2)}.$$

In Figure 4 we represent the WKB density of zeros for our generalized Gegenbauer polynomials. Notice that the value of the mass doesn't play a crucial role, since for $n \gg 1$, $W_m \sim \frac{1}{2m^2}$, independently of A. We have plotted the Density function for different values of the degree of the polynomials (from the top to the bottom) $n = 2 \times 10^4$, 1.5×10^4, 10^4, 10^3 for two different cases: the generalized Legendre polynomials ($\lambda = \frac{1}{2}$) and the generalized Gegenbauer with $\lambda = 5$.

5. Other Interesting Examples

In this section we will give some other examples of families of Krall-type orthogonal polynomials, obtained using quadratic transformations of the variable in a

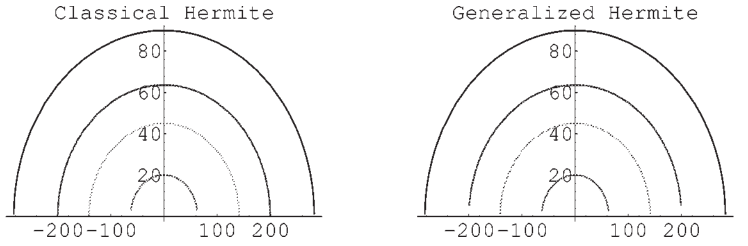

Figure 3. WKB density of zeros of the $H_n^A(x)$.

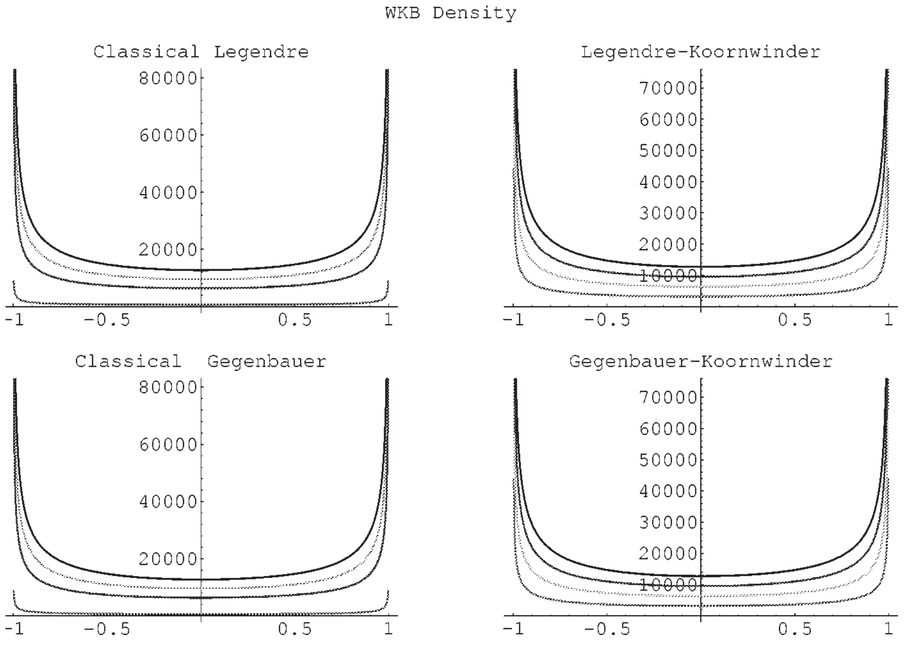

Figure 4. WKB density of zeros of the $G_n^{\lambda,A}(x)$.

given sequence of orthogonal polynomials. These examples can be obtained as an application of the following theorem [36].

THEOREM 2. *Let $\{P_n\}$ be a monic orthogonal polynomial sequence (MOPS) with respect to some uniquely determined distribution function σ and let $[\xi, \eta]$ be the true interval of orthogonality of $\{P_n\}$, with $-\infty < \xi < \eta \leqslant +\infty$. Let a and λ be fixed real numbers, $T(x) \equiv (x-a)(x-b)+c$ a real polynomial of degree two and put $\Delta = (b-a)^2 - 4c$. Let $\{Q_n\}$ be a sequence of polynomials such that*

$$Q_2(a) = \lambda, \qquad Q_{2n+1}(x) = (x-a)P_n(T(x))$$

for all $n = 0, 1, 2, \ldots$. Assume that one of the following conditions hold

(i) $\quad c \leqslant \xi + \lambda,$

$$\text{(ii)}\quad c \leqslant \xi, \quad -\infty < \lim_{n\to+\infty} \frac{P_n(\xi)}{P_{n-1}^{(1)}(\xi)} \equiv A \leqslant \lambda \leqslant B \equiv \frac{P_n(\eta)}{P_{n-1}^{(1)}(\eta)},$$

where $B \equiv +\infty$ *if* $\eta = +\infty$ *and* $\{P_n^{(1)}\}$ *denotes the sequence of the associated polynomials of the first kind* [14] *corresponding to* $\{P_n\}$. *Then,* $\{Q_n\}$ *is a MOPS with respect to a positive definite linear functional if and only if*

$$\lambda < 0, \quad Q_{2n}(x) = P_n(T(x)) - \frac{a_n(\lambda, c)}{a_{n-1}(\lambda, c)} P_{n-1}(T(x))$$

hold for all $n = 0, 1, 2, \ldots$ *and*

$$a_n(\lambda, c) = P_n(c) - \lambda P_{n-1}^{(1)}(c).$$

In these conditions, $\{Q_n\}$ *is orthogonal with respect to the uniquely determined distribution function* $\tilde{\sigma}$ *defined as*

$$\mathrm{d}\tilde{\sigma} = M\delta(x-a) - \frac{\lambda}{|x-a|}\frac{\mathrm{d}\sigma(T(x))}{T'(x)}, \quad r < \left| x - \frac{a+b}{2} \right| < s,$$

where

$$M = \mu_0 + \lambda F(c;\sigma) \geqslant 0, \quad r = \sqrt{\xi + \frac{\Delta}{4}}, \quad s = \sqrt{\eta + \frac{\Delta}{4}},$$

$F(z;\sigma) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\sigma(t)}{t-z}$ *is the Stieltjes function associated to the distribution function* σ *and* $\mu_0 = \int_{-\infty}^{\infty} \mathrm{d}\sigma(x)$.

5.1. GENERALIZED HERMITE POLYNOMIALS WITH A POINT MASS AT $x = 0$

Let $\{L_n^\alpha\}$ be the sequence of the monic Laguerre polynomials which are orthogonal with respect to the weight function $w(x) = x^\alpha \mathrm{e}^{-x}$, $x \in [0,\infty)$, $\alpha > -1$. If $\alpha > 0$, it follows from the last theorem [36] that, for each λ such that $-\alpha < \lambda < 0$, the sequence of monic polynomials defined by

$$Q_{2n+1}(x) = xL_n^\alpha(x^2), \qquad Q_{2n}(x) = L_n^\alpha(x^2) - \frac{a_n}{a_{n-1}} L_{n-1}^\alpha(x^2),$$

where

$$a_n = L_n^\alpha(0) - \lambda\big(L_{n-1}^\alpha\big)^{(1)}(0), \quad n = 0, 1, 2, \ldots,$$

and $(L_{n-1}^\alpha)^{(1)}$ denotes the associated polynomials of the first kind of the Laguerre polynomials is orthogonal with respect to the measure

$$\mathrm{d}\sigma(x) = \Gamma(\alpha+1)\left(1 + \frac{\lambda}{\alpha}\right)\delta_0(x)\,\mathrm{d}x - \lambda|x|^{2\alpha-1}\mathrm{e}^{-x^2}\,\mathrm{d}x, \quad x \in (-\infty,\infty).$$

Choosing $\lambda = -\alpha$ we deduce that, up to a constant factor, $\mathrm{d}\sigma(x) = |x|^{2\mu}\mathrm{e}^{-x^2}\,\mathrm{d}x$, with $\mu = \alpha - \frac{1}{2}$. Hence $\{Q_n\}$ is the sequence of the monic generalized Hermite polynomials $Q_n \equiv H_n^{(\mu)}$, $\mu > -\frac{1}{2}$ (cf. [14, p. 157]). However, if we choose λ such that $-\alpha < \lambda < 0$, then one can see that there is always a mass point, located at $x = 0$. This example generalizes the Hermite–Krall-type polynomials considered before.

5.2. A FINITE 2-PERIODIC JACOBI MATRIX

Let B_n be a tridiagonal 2-Toeplitz matrix, which has the general form

$$B_n = \begin{bmatrix} a_1 & b_1 & 0 & 0 & 0 & \dots \\ c_1 & a_2 & b_2 & 0 & 0 & \dots \\ 0 & c_2 & a_1 & b_1 & 0 & \dots \\ 0 & 0 & c_1 & a_2 & b_2 & \dots \\ 0 & 0 & 0 & c_2 & a_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \in \mathbb{C}^{(n,n)}, \tag{42}$$

where we assume that b_1, b_2, c_1 and c_2 are positive real numbers. This special matrix has been firstly studied from the point of view of the eigenvalue problem in [17] and more recently in [21] and also in [35].

Since $b_i > 0$ and $c_i > 0$ for $i = 1, 2$ then there exists a sequence of orthogonal polynomials $\{S_n\}$, such that B_n is the corresponding Jacobi matrix of order n. Let $\{Q_n\}$ be the corresponding MOPS. Then

$$Q_{2n}(x) = (b_1b_2)^n S_{2n}(x), \qquad Q_{2n+1}(x) = b_1(b_1b_2)^n S_{2n+1}(x).$$

Moreover, according to [35], $\{Q_n\}$ can be obtained by a quadratic polynomial mapping on a linear transformation of the monic Chebyshev polynomials of second kind $\{U_n\}$. In fact, if

$$T(x) = (x - a_1)(x - a_2), \qquad \alpha = 2\sqrt{b_1b_2c_1c_2}, \qquad \beta = b_1c_1 + b_2c_2,$$

then

$$Q_{2n+1}(x) = (x - a_1)P_n(T(x)), \qquad Q_{2n}(x) = R_n(T(x)),$$

where

$$P_n(x) = \alpha^n U_n\left(\frac{x-\beta}{\alpha}\right), \qquad R_n(x) = P_n(x) + b_2c_2P_{n-1}(x)$$

for $n = 0, 1, 2, \dots$. Notice that P_n is orthogonal with respect to the distribution function

$$\sigma_P(x) = \sigma_U\left(\frac{x-\beta}{\alpha}\right), \quad \mathrm{supp}(\sigma_P) = [\beta - \alpha, \beta + \alpha],$$

where σ_U is the distribution function of the Chebyshev polynomials [18, 43]

$$d\sigma_P(x) = \frac{2}{\pi\alpha^2}\sqrt{\alpha^2 - (x-\beta)^2}\,dx.$$

From [36] the Stieltjes function of $\{Q_n\}$ is

$$F_Q(z) = \frac{M}{a_1 - z} + \frac{b_1c_1}{z - a_1}[F_P(T(z)) - F_P(0)], \tag{43}$$

where F_P denotes the Stieltjes function associated with σ_P, $M = \mu_0 - b_1c_1F_P(0)$ and μ_0 is the first moment of σ_P, $\mu_0 = \int_{\beta-\alpha}^{\beta+\alpha} d\sigma_P(x) = 1$. Using the Stieltjes function F_U of the Chebyshev polynomials [44, p. 176]

$$F_P(z) = \frac{1}{\alpha}F_U\left(\frac{z-\beta}{\alpha}\right) = \frac{-2}{\alpha^2}\left(z - \beta - \sqrt{(z-\beta)^2 - \alpha^2}\right),$$

where the square root is such that $|z - \beta + \sqrt{(z-\beta)^2 - \alpha^2}| > \alpha$ whenever $z \notin [\beta - \alpha, \beta + \alpha]$. Since $0 \notin [\beta - \alpha, \beta + \alpha]$ for $b_1c_1 \neq b_2c_2$ (this is no restriction, because the case $b_1c_1 = b_2c_2$ corresponds to constant values along the diagonal of the corresponding Jacobi matrix), elementary computations give $F_P(0) = \min\{b_1c_1, b_2c_2\}/b_1c_1b_2c_2$. Hence

$$M = 1 - \frac{\min\{b_1c_1, b_2c_2\}}{b_2c_2}.$$

It turns out from (43) that the Stieltjes function of $\{Q_n\}$ reads as

$$\begin{aligned} F_Q(z) \;=\; & \frac{M}{a_1 - z} - \frac{(\alpha b_2c_2)^{-1}}{z - a_1}\left[\frac{T(z) - \beta - \sqrt{(T(z)-\beta)^2 - \alpha^2}}{2} + \right. \\ & \left. + \min\{b_1c_1, b_2c_2\}\right]. \end{aligned}$$

From this we deduce (see [36]) that $\{Q_n\}$ is orthogonal with respect to the distribution function

$$\begin{aligned} d\sigma_Q(x) \;&=\; M\delta(x - a_1) + \frac{b_1c_1}{|x - a_1|}\,d\sigma_P(T(x)) \\ &=\; M\delta(x - a_1) + \frac{1}{2\pi b_2c_2}\frac{1}{|x - a_1|}\times \\ &\qquad \times\sqrt{4b_1b_2c_1c_2 - (T(x) - b_1c_1 - b_2c_2)^2}\,dx. \end{aligned}$$

Its support is the union of two intervals if $M = 0$ and the union of two intervals with a singular point if $M > 0$ either

$$\operatorname{supp}(\sigma_Q) = T^{-1}(\operatorname{supp}(\sigma_P)) \quad \text{if} \quad b_1c_1 \leqslant b_2c_2,$$

or

$$\operatorname{supp}(\sigma_Q) = T^{-1}(\operatorname{supp}(\sigma_P)) \cup \{a_1\} \quad \text{if} \quad b_1c_1 > b_2c_2.$$

We notice that

$$\begin{aligned} T^{-1}(\operatorname{supp}(\sigma_P)) &= T^{-1}([\beta - \alpha, \beta + \alpha]) \\ &= \left[\frac{a_1 + a_2}{2} - s, \frac{a_1 + a_2}{2} - r\right] \cup \left[\frac{a_1 + a_2}{2} + r, \frac{a_1 + a_2}{2} + s\right], \end{aligned}$$

where

$$r = \left(\left|\sqrt{b_1c_1} - \sqrt{b_2c_2}\right|^2 + \left|\frac{a_1 - a_2}{2}\right|^2\right)^{1/2},$$

$$s = \left(\left|\sqrt{b_1c_1} + \sqrt{b_2c_2}\right|^2 + \left|\frac{a_1 - a_2}{2}\right|^2\right)^{1/2}.$$

As we can see, for the case when $b_1c_1 > b_2c_2$, we obtain a set of polynomials orthogonal with respect to a weight function of the form $\rho(x) + \delta(x - x_0)$ where ρ is a nonnegative continuous function, i.e., a Krall-type weight function appears in a very natural way.

WKB Approximation for the Distribution of Eigenvalues of a Tridiagonal Two-Periodic Symmetric Matrix. To conclude this work let us to consider an special case of a symmetric $n \times n$ matrix [7]

$$B_m = \begin{pmatrix} a & c & 0 & 0 & 0 & \dots \\ c & b & d & 0 & 0 & \dots \\ 0 & d & a & c & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{44}$$

For this matrix we will obtain the density of the distribution of eigenvalues, i.e., the WKB density of the corresponding sequence of orthogonal polynomials which are, in general, of the Krall-type. Here we want to point out that in the odd case ($m = 2n + 1$) the corresponding polynomials are $Q_{2n+1}(x) = (x - a)P_n(T(x))$, so we can consider only the distribution of zeros of P_n, since for any n, $x = a$ is always a zero of the polynomial and then an eigenvalue of B_{2n}. Furthermore, $P_n(T(x))$ is a quadratic modification of the Chebyshev polynomials U_n and then they satisfy a SODE which follows from the classical one

$$(1 - x^2)U_n''(x) - 3xU_n'(x) + n(n + 2)U_n(x) = 0, \tag{45}$$

just providing the change $x \leftrightarrow T(x)$. In fact we have that $P_n(T(x))$ satisfies a SODE (38) with the coefficients

$$\begin{aligned} a_2(x) = &\left(4c^2d^2 - (-(ab) + c^2 + d^2 + ax + bx - x^2)^2\right), \\ a_1(x) = &\ 3(-a - b + 2x)(-(ab) + c^2 + d^2 + ax + bx - x^2) - \\ &- 2\left(4c^2d^2 - (-(ab) + c^2 + d^2 + ax + bx - x^2)^2\right), \\ a_0(x) = &\ n(2 + n)(-a - b + 2x)^2. \end{aligned} \tag{46}$$

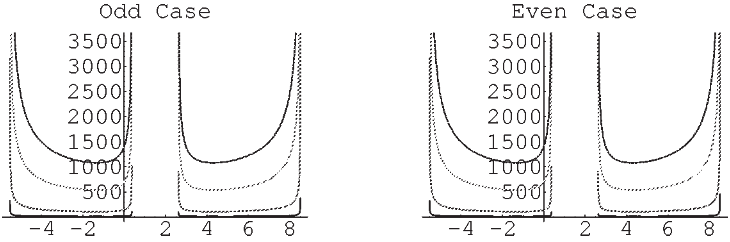

Figure 5. WKB density of the distribution of the eigenvalues of the symmetric matrix H.

For the even case the situation is more complicated since we need to calculate the SODE for the $R_n(T(x))$ polynomials. Using the symbolic program MATHEMATICA [45], as well as the package *PowerSeries* developed by Koepf [28] we obtain for the polynomials $R_n(x)$ a SODE (38) with coefficients

$$\begin{aligned} a_2(x) &= (1+n+c^4n+c^2x+2c^2nx)(-1+x^2), \\ a_1(x) &= c^2+2c^2n+3x+3nx+3c^4nx+2c^2x^2+4c^2nx^2, \\ a_0(x) &= -2n+c^4n-3n^2-n^3-c^4n^3-c^2nx-3c^2n^2x-2c^2n^3x. \end{aligned} \tag{47}$$

The change of variables $x \leftrightarrow T(x)$ $(T(x) = (x-a)(x-b))$ in the previous SODE yields

$$\tilde{a}_2(x)Q''_{2n}(x) + \tilde{a}_1(x)Q'_{2n}(x) + \tilde{a}_0(x)Q_{2n}(x) = 0,$$

where

$$\begin{aligned} \tilde{a}_2(x) &= 4c^2d^2T''(x)a_2\left(\frac{T(x)-a^2-b^2}{2cd}\right), \\ \tilde{a}_1(x) &= 2cdT'(x)^2a_1\left(\frac{T(x)-a^2-b^2}{2cd}\right) - 8c^2d^2p\left(\frac{T(x)-a^2-b^2}{2cd}\right), \\ \tilde{a}_0(x) &= T'(x)^3a_0\left(\frac{T(x)-a^2-b^2}{2cd}\right). \end{aligned} \tag{48}$$

Substituting (46), (48) in (41) we can find the WKB density of the eigenvalues of the Hamiltonian matrix B_n. A straightforward calculation shows that the conditions of the Theorem 1 are satisfied if $n \gg 1$. The expression for the ρ_{WKB}, in both cases, is very large and we will only show the typical behaviour of the WKB density (see Figure 5).

As we can see in Figure 5 we have that all eigenvalues are located inside the support of the measure up to a in the odd case. In the picture the values $a = 1$, $b = 2$, $c = 3$ and $d = 4$ are used (from the top to the bottom $n = 10000, 5000, 1000, 100$).

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Appendix: The Second-Order Differential Equation for the Krall-Type Polynomials

Table A.1. The values of q, a, b.

$\tilde{P}_n(x)$	$q(x;n)$	$a(x;n)$	$b(x;n)$
$P_n^{\alpha,\beta,A,B}(x)$	1	$1 - nJ_{A,B}^{n,\alpha,\beta} - nJ_{B,A}^{n,\beta,\alpha}$	$J_{A,B}^{n,\alpha,\beta}(x-1) + J_{B,A}^{n,\beta,\alpha}(1+x)$
$L_n^{\alpha,A}(x)$	1	1	Γ_n
$H_{2m}^{A}(x)$	$2x$	$2x$	B_m
$G_{2m}^{\lambda,A}(x)$	$2x$	$2x(1+mW_m^A)$	$W_m^A(1-x^2)$

In this section we give a general algorithm to obtain the second order linear differential equation (SODE) which satisfy the considered Krall-type polynomials, denoted here by $\tilde{P}_n$, i.e., the Jacobi–Koornwinder, the Laguerre–Koekoek, the Hermite–Krall-type and Gegenbauer–Krall-type polynomials. The main fact that we will use is that all of them can be represented in terms of the classical families $\{P_n\}$ in the form

$$q(x;n)\tilde{P}_n(x) = a(x;n)P_n(x) + b(x;n)P_n'(x), \tag{A.1}$$

where q, a, b are polynomials in x and some function on n (see formulas (15), (19), (22) and (28)). In the next table we represent q, a, b for each of the families $\{\tilde{P}_n\}$. It is known that the classical polynomials satisfy a certain SODE of hypergeometric type [39, 43]

$$\sigma(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_n(x) + \tau(x)\frac{\mathrm{d}}{\mathrm{d}x}P_n(x) + \lambda_n P_n(x) = 0, \tag{A.2}$$

where degree $(\sigma) \leqslant 2$, degree $(\tau) = 1$, degree $(\lambda_n) = 0$. Now if we take derivatives in (A.1) and use the SODE (A.2) we can obtain formulas similar to (A.1) but for the derivatives $\tilde{P}_n'(x;n)$ and $\tilde{P}_n''(x;n)$

$$r(x;n)\tilde{P}_n'(x) = c(x;n)P_n(x) + d(x;n)P_n'(x),$$

$$s(x;n)\tilde{P}_n''(x) = e(x;n)P_n(x) + f(x;n)P_n'(x),$$

where r, s, c, d, e, f are depending on x and n (they are polynomials in x of bounded degree independent of n). The above two expressions and (A.1) lead to the condition

$$\begin{vmatrix} q(x;n)\tilde{P}_n(x) & a(x;n) & b(x;n) \\ r(x;n)\tilde{P}_n'(x) & c(x;n) & d(x;n) \\ s(x;n)\tilde{P}_n''(x) & e(x;n) & f(x;n) \end{vmatrix} = 0. \tag{A.3}$$

Expanding the determinant (A.3) we get

$$\tilde{\sigma}(x;n)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde{P}_n(x) + \tilde{\tau}(x;n)\frac{\mathrm{d}}{\mathrm{d}x}\tilde{P}_n(x) + \tilde{\lambda}(x;n)\tilde{P}_n(x) = 0, \tag{A.4}$$

where

$$\begin{aligned} \tilde{\sigma}(x;n) &= s(x;n)\,[a(x;n)d(x;n) - c(x;n)b(x;n)], \\ \tilde{\tau}(x;n) &= r(x;n)[e(x;n)b(x;n) - a(x;n)f(x;n)], \\ \tilde{\lambda}(x;n) &= q(x;n)[c(x;n)f(x;n) - e(x;n)d(x;n)]. \end{aligned} \tag{A.5}$$

In some cases the coefficients can be simplified by some factor and the equation (A.4) becomes more simple. To conclude this section we will provide the SODE for the four considered polynomials. We want to remark that in order to obtain the explicit formulas of the coefficients of the SODE (A.5) we have used the symbolic package MATHEMATICA [45].

Jacobi–Koornwinder Polynomials. The existence of this SODE was proved by Koornwinder [29] and the coefficients were calculated explicitly in [22] and [5]. Using (A.5) we find

$$\begin{aligned} \tilde{\sigma}(x;n) \;=\; & (x^2-1)\Big\{2n J_{A,B}^{n,\alpha,\beta\,2}(\alpha+n)(1-x) + J_{A,B}^{n,\alpha,\beta}\big[(x-1)^2 - \\ & -\alpha(1+x)\big(1+2nJ_{B,A}^{n,\beta,\alpha} - x\big) + \beta(x-1)\big(2nJ_{B,A}^{n,\beta,\alpha} + x - 1\big) - \\ & -2n\big(1+2J_{B,A}^{n,\beta,\alpha} - x^2\big)\big] + (1+x)\times\big[1 - x + J_{B,A}^{n,\beta,\alpha}\big(1+x+ \\ & +\alpha(1+x) + 2n\big(J_{B,A}^{n,\beta,\alpha}n + x - 1\big) + \beta\big(2nJ_{B,A}^{n,\beta,\alpha} + x - 1\big)\big)\big]\Big\}, \end{aligned}$$

$$\begin{aligned} \tilde{\tau}(x;n) \;=\; & -\alpha^2\big(-1+2J_{A,B}^{n,\alpha,\beta}n - x\big)(1+x)\big(-J_{A,B}^{n,\alpha,\beta} + J_{B,A}^{n,\beta,\alpha} + \\ & +\big(J_{A,B}^{n,\alpha,\beta} + J_{B,A}^{n,\beta,\alpha}\big)x\big) + \beta^2(-1+x)\big(-1+2J_{B,A}^{n,\beta,\alpha}n + x\big)\times \\ & \times\big(-J_{A,B}^{n,\alpha,\beta} + J_{B,A}^{n,\beta,\alpha} + \big(J_{A,B}^{n,\alpha,\beta} + J_{B,A}^{n,\beta,\alpha}\big)x\big) + \\ & +\beta\Big[(x-1)^2\big(J_{A,B}^{n,\alpha,\beta}\big[3(x-1) + 2n\big(1+3J_{B,A}^{n,\beta,\alpha} + x\big)\big] - \\ & -2J_{A,B}^{n,\alpha,\beta\,2}n^2\big) + (1+x)\times\big(2nJ_{B,A}^{n,\beta,\alpha\,2}(1+n(x-1)+3x) + \end{aligned}$$

$$
\begin{aligned}
&+ (x-1)\Big[J_{B,A}^{n,\beta,\alpha}(1+2n(x-1)+3x)-x+1\big]\Big)\Big] + \\
&+ \alpha\Big\{2nJ_{A,B}^{n,\alpha,\beta^2}(1-x)(n-1-\beta+(3+\beta+n)x) + \\
&+ J_{A,B}^{n,\alpha,\beta}(1+x)\Big[1+2\beta(x-1)^2-4x+3x^2 - \\
&- 2n\big(1+3J_{B,A}^{n,\beta,\alpha}-x\big)(1+x)\Big] + (1+x)^2\Big[1-x+ \\
&+ J_{B,A}^{n,\beta,\alpha}\big(3(1+x)+2\beta\big(J_{B,A}^{n,\beta,\alpha}n+x-1\big)+ \\
&+ 2n\big(J_{B,A}^{n,\beta,\alpha}n+x-1\big)\big)\Big]\Big\} + 2\Big\{n^2J_{A,B}^{n,\alpha,\beta^2}(1-x)(3x-1)+ \\
&+ \big(1+x\big)\Big[x-x^2+n^2J_{B,A}^{n,\beta,\alpha^2}(3x+1)+J_{B,A}^{n,\beta,\alpha}(1+x(2+ \\
&+ 2n(-1+x)+x))\Big] + J_{A,B}^{n,\alpha,\beta}\Big[-1+x\big(3+(-3+x)x- \\
&- 2n\big(1+4J_{B,A}^{n,\beta,\alpha}-x^2\big)\big)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\lambda}(x;n) \;=\; & n(1+\alpha+\beta+n)\Big\{2J_{A,B}^{n,\alpha,\beta^2}(n-1)(\alpha+n)(x-1)+ \\
&+ J_{A,B}^{n,\alpha,\beta}\Big[2n-3-\beta-8J_{B,A}^{n,\beta,\alpha}-2\beta J_{B,A}^{n,\beta,\alpha}+2(4+\beta)J_{B,A}^{n,\beta,\alpha}n+ \\
&+ 4x-2\beta\big(J_{B,A}^{n,\beta,\alpha}(n-1)-1\big)x-(1+\beta+2n)x^2+ \\
&+ \alpha\big(1+2J_{B,A}^{n,\beta,\alpha}(n-1)-x\big)(1+x)\Big]-(1+x)\times \\
&\times\Big[1+2J_{B,A}^{n,\beta,\alpha^2}(n-1)(\beta+n)-x+J_{B,A}^{n,\beta,\alpha}\big(3+\alpha-\beta-2n+ \\
&+ (1+\alpha+\beta+2n)x\big)\Big]\Big\}.
\end{aligned}
$$

Laguerre–Koekoek Polynomials. The equation for the Laguerre–Koekoek polynomials was found in [27]. From (A.5) we obtain

$$
\begin{aligned}
\tilde{\sigma}(x;n) =\;& x\big(-\Gamma_n-\alpha\Gamma_n+n{\Gamma_n}^2+x+x\Gamma_n\big),\\
\tilde{\tau}(x;n) =\;& \big(-2\Gamma_n-3\alpha\Gamma_n-\alpha^2\Gamma_n+2n{\Gamma_n}^2+\alpha n{\Gamma_n}^2+x+\alpha x+\\
&+2x\Gamma_n+2\alpha x\Gamma_n-nx{\Gamma_n}^2-x^2-x^2\Gamma_n\big),\\
\tilde{\lambda}(x;n) =\;& n\big(-2\Gamma_n-\alpha\Gamma_n-{\Gamma_n}^2+n{\Gamma_n}^2+x+x\Gamma_n\big).
\end{aligned}
$$

Hermite–Krall-Type Polynomials. The equation for the Hermite–Krall-type polynomials was found in [6]. Using (A.5) we deduce

$$
\begin{aligned}
\tilde{\sigma}(x;m) &= x\big(-B_m+2B_m^2m+2x^2+2B_mx^2\big),\\
\tilde{\tau}(x;m) &= 2\big(-B_m+2B_m^2m+B_mx^2-2B_m^2mx^2-2x^4-2B_mx^4\big),\\
\tilde{\lambda}(x;m) &= 4mx\big(-3B_m-2B_m^2+2B_m^2m+2x^2+2B_mx^2\big).
\end{aligned}
$$

Gegenbauer–Krall-Type Polynomials. The equation for the Gegenbauer–Krall-type polynomials was found in [6]. Equation (A.5) yields

$$\begin{aligned}\tilde{\sigma}(x;m) &= x(1-x^2)\big(W_m^A + mW_m^{A^2} - 2m^2W_m^{A^2} - 2m\nu W_m^{A^2} - \\ &\quad - 2x^2 - 4mW_m^A x^2 - 2\nu W_m^A x^2\big),\end{aligned}$$

$$\begin{aligned}\tilde{\tau}(x;m) &= -2W_m^A - 2mW_m^{A^2} + 4m^2W_m^{A^2} + 4m\nu W_m^{A^2} + 3W_m^A x^2 + \\ &\quad + 2\nu W_m^A x^2 + 3mW_m^{A^2}x^2 - 6m^2W_m^{A^2}x^2 - 4m\nu W_m^{A^2}x^2 - \\ &\quad - 4m^2\nu W_m^{A^2}x^2 - 4m\nu^2W_m^{A^2}x^2 - 2x^4 - 4\nu x^4 - \\ &\quad - 4mW_m^A x^4 - 2\nu W_m^A x^4 - 8m\nu W_m^A x^4 - 4\nu^2 W_m^A x^4,\end{aligned}$$

$$\begin{aligned}\tilde{\lambda}(x;m) &= 4m(m+\nu)x\big(-3W_m^A + W_m^{A^2} - 3mW_m^{A^2} + 2m^2W_m^{A^2} - \\ &\quad - 2\nu W_m^{A^2} + 2m\nu W_m^{A^2} + 2x^2 + 4mW_m^A x^2 + 2\nu W_m^A x^2\big).\end{aligned}$$

In all cases if we take the limit when the masses tend to zero we obtain the SODE of the corresponding classical polynomials. Remind that all explicit expressions for the coefficients of the SODE were obtained by using the computer algebra package MATHEMATICA [45] and they will be useful to study the spectral properties of the polynomials under consideration.

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