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Topological Features of Lax Algebras

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Abstract. Having as starting point Barr's description of topological spaces as lax algebras for the ultrafilter monad, in this paper we present further topological examples of lax algebras – such as quasi-metric spaces, approach spaces and quasi-uniform spaces – and show that, in a suitable setting, the categories of lax algebras have indeed a topological nature. Furthermore, we generalize to this setting known properties of special categories of lax algebras and, extending the construction of Manes, we describe the Čech–Stone compactification of lax algebras.

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1. Introduction

Monads, or triples, and the algebras they define, proved to be very important in several fields of mathematics, homological algebra being one of the fields where they have major applications.

In this paper we explore one topological facet of monads, which leads to a unified treatment of dispersed topological structures. Indeed, guided by Barr's [2] description of topological spaces and continuous maps, we show that several *topological structures* and their corresponding *continuous maps* may be described relaxing the axioms of algebras and algebra homomorphisms for a monad. Among these we obtain the category **Ap** of approach spaces and non-expansive maps, giving a new perspective of the relationship between **Top** and **Ap**. While the structure of a topological space is defined by saying whether a point is or is not a limit point of an ultrafilter, in an approach space this information is "numerified", so that we are given a value in $[0, \infty]$ which measures how far away a point is from being a limit point of an ultrafilter. This is reflected in our setting by the choice of the 2-category: **Top** can be obtained as the category of reflexive and transitive lax algebras for the extension of the ultrafilter monad to the 2-category **Rel** of

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relations, and **Ap** can be obtained as the category of (reflexive and transitive) lax algebras of the extension of the ultrafilter monad to the 2-category **NRel** of numerical relations. One further natural example is the category of quasi-metric spaces and non-expansive maps. Indeed, Lawvere's [9] description of these spaces shows clearly that they form exactly the category of reflexive and transitive lax algebras for the identity monad in **NRel**. Also quasi-uniform spaces are described as reflexive and transitive lax algebras for the identity monad in a convenient 2-category.

The occurrence of all these topological examples of lax algebras is not a coincidence: we show that, under natural assumptions, categories of lax algebras for a lax extension of a monad in **X** are topological over **X**. In order to compensate deficits of the category **Top**, one defines several supercategories by dropping axioms, like **PsTop** or **URS**, which have better properties. The same can be done in our setting, and we show that these supercategories are nice improvements. For instance, we show that, in a convenient setting, regular epimorphisms in categories of (reflexive) lax algebras are better behaved than in categories of reflexive and transitive lax algebras: they are pullback-stable.

Finally, extending the construction of Manes [12], we generalize the Čech–Stone compactification of topological spaces; that is, we construct the reflection of a reflexive and transitive lax algebra with respect to a lax monad into the corresponding category of algebras for the monad.

2. Categories of Lax Algebras

Throughout:

- **Set** $\hookrightarrow$ **Y** is a (non-full) embedding, bijective on objects, of the category of sets into a 2-category **Y** with thin 2-cells; we will denote its objects by $X, Y, \ldots$, its morphisms by $r : X \nrightarrow Y$, and, for simplicity, for $r, s \in \mathbf{Y}(X, Y)$ we will write $r \sqsubseteq s$ if there is a 2-cell $r \to s$.
- $\mathbb{T} = (T, e, m)$ is a monad in **Set**; that is, $T : \mathbf{Set} \to \mathbf{Set}$ is a functor, and $e : 1_{\mathbf{Set}} \to T$ and $m : T^2 \to T$ are natural transformations making the following diagrams commute

$$\begin{array}{ccc} T^3 & \xrightarrow{Tm} & T^2 \\ {\scriptstyle mT}\downarrow & & \downarrow{\scriptstyle m} \\ T^2 & \xrightarrow{m} & T \end{array} \qquad \begin{array}{ccccc} T & \xrightarrow{Te} & T^2 & \xleftarrow{eT} & T \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle 1} & \\ & & T & & \end{array} \tag{1}$$

- The monad (T, e, m) can be lax extended into **Y**; that is:
  - there exists a lax functor $T : \mathbf{Y} \to \mathbf{Y}$ that extends T to **Y**, i.e., it coincides with the given functor $T : \mathbf{Set} \to \mathbf{Set}$ when restricted to **Set** and, for $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ in **Y**, there is a two-cell $Tr \cdot Ts \to T(r \cdot s)$;
  - the natural transformations e and m become now op-lax:

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ {\scriptstyle r}\downarrow & \sqsubseteq & \downarrow{\scriptstyle Tr} \\ Y & \xrightarrow[e_Y]{} & TY \end{array} \qquad\qquad \begin{array}{ccc} T^2X & \xrightarrow{m_X} & TX \\ {\scriptstyle T^2r}\downarrow & \sqsubseteq & \downarrow{\scriptstyle Tr} \\ T^2Y & \xrightarrow[m_Y]{} & TY; \end{array} \tag{2}$$

this means that there are 2-cells $e_Y \cdot r \to Tr \cdot e_X$ and $m_Y \cdot T^2 r \to Tr \cdot m_X$.

An *algebra* for the monad $\mathbb{T}$ is a pair (X, a) where X is a set and $a : TX \nrightarrow X$ is a map making the following diagram commutative

$$\begin{array}{ccccc} X & \xrightarrow{e_X} & TX & \xleftarrow{Ta} & T^2X \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle m_X} \\ & & X & \xleftarrow[a]{} & TX. \end{array} \tag{3}$$

A (homo)morphism $f : (X, a) \to (Y, b)$ of $\mathbb{T}$-algebras is a map $f : X \to Y$ such that the diagram

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array} \tag{4}$$

commutes. As usual we denote by $\mathbf{Set}^{\mathbb{T}}$ the category of $\mathbb{T}$-algebras and their morphisms. (For detailed information on monads see [11].)

A *lax algebra* for T is just a pair (X, a) where $a : TX \nrightarrow X$ is a morphism in $\mathbf{Y}$, while a *lax (homo)morphism* from (X, a) to (Y, b) is a $\mathbf{Set}$-map $f : X \to Y$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\downarrow & \sqsubseteq & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y, \end{array} \tag{5}$$

that is, there exists a 2-cell $f \cdot a \to b \cdot Tf$. This way one defines the category $\mathrm{Alg}(T)$ of lax algebras and lax (homo)morphisms.

Relaxing the axioms of algebras for a monad, among lax algebras one may consider those that are *reflexive*, that is, those (X, a) for which there is a 2-cell $1_X \to a \cdot e_X$:

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ & {\scriptstyle 1_X}\searrow \ \sqsubseteq & \downarrow{\scriptstyle a} \\ & & X, \end{array} \tag{6}$$

and those that are *transitive*, that is, those (X, a) for which there is a 2-cell $a \cdot Ta \to a \cdot m_X$:

$$\begin{array}{ccc} TX & \overset{Ta}{\longleftarrow\!\!\!\!\!\!\!\!-\!\!\!+} & T^2X \\ {\scriptstyle a}\big\downarrow & \sqsubseteq & \big\downarrow{\scriptstyle m_X} \\ X & \underset{a}{\longleftarrow\!\!\!\!\!\!\!\!-\!\!\!+} & TX. \end{array} \tag{7}$$

This way we define the (full) subcategories $\mathrm{Alg}(T, e)$ and $\mathrm{Alg}(T, e, m)$, of reflexive and of reflexive and transitive lax algebras, of $\mathrm{Alg}(T)$:

$$\mathrm{Alg}(T, e, m) \hookrightarrow \mathrm{Alg}(T, e) \hookrightarrow \mathrm{Alg}(T). \tag{8}$$

3. Examples

In this section we show that the structures described above encompass a great deal of topological life.

3.1. LAX ALGEBRAS ON **Rel**

In case that $\mathbf{Y}$ is the category **Rel** of relations, Barr [2] showed that, for the *identity monad*, $\mathrm{Alg}(\mathrm{Id}, 1, 1)$ is the category of preordered sets and monotone maps. He showed also that, for the *ultrafilter monad* (U, e, m) in **Set**, $\mathrm{Alg}(U, e, m)$ is the category of topological spaces and continuous maps.

We recall that $U : \mathbf{Set} \to \mathbf{Set}$ assigns to a set X its set of ultrafilters UX, and to each map $f : X \to Y$ the map $Uf : UX \to UY$ which assigns to an ultrafilter $\mathfrak{x} \in UX$ the ultrafilter $f(\mathfrak{x})$ generated by $\{f(A) \mid A \in \mathfrak{x}\}$. Moreover, for each set X,

$$\begin{array}{rcl} e_X : X & \to & UX \\ x & \mapsto & \dot{x} \end{array} \quad \text{and} \quad \begin{array}{rcl} m_X : UUX & \to & UX \\ \mathfrak{X} & \mapsto & m_X(\mathfrak{X}) = \bigcup\limits_{\mathcal{A} \in \mathfrak{X}} \bigcap\limits_{\mathfrak{x} \in \mathcal{A}} \mathfrak{x}, \end{array}$$

where $\dot{x}$ denotes the principal ultrafilter defined by x. It has a unique lax extension (which turns out to be strictly a functor) $U : \mathbf{Rel} \to \mathbf{Rel}$: for $r : X \nrightarrow Y$ and $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$,

$$\mathfrak{x}(Ur)\mathfrak{y} :\Leftrightarrow \forall B \in \mathfrak{y} \{x \in X \mid \exists y \in B : xry\} \in \mathfrak{x},$$

such that the natural transformations e and m become op-lax (in fact, m remains strict).

Furthermore, one can easily show that:

- If one considers the identity monad in **Rel**, an (reflexive, reflexive and transitive, respectively) algebra (X, a) is just a *relation* (*reflexive relation*, *reflexive and transitive relation*) on X, and a lax morphism is exactly a map that preserves the relation; hence (8) becomes

$$\mathbf{Prord} \hookrightarrow \mathbf{SGrph} \hookrightarrow \mathbf{Grph},$$

where **Grph** is the category of (thin) graphs, **SGrph** is the category of reflexive (thin) graphs (also called spatial graphs), and **Prord** is the category of preordered sets.

– For the lax ultrafilter monad (U, e, m) in **Rel**, an (reflexive, reflexive and transitive, respectively) algebra (X, a) is exactly an *ultrarelational* – or grizzly – *space* (see [3] and [5]) (*pseudotopological space*, *topological space*, respectively), and a map is a lax morphism if and only if it is continuous, obtaining in (8)

$$\mathbf{Top} \hookrightarrow \mathbf{PsTop} \hookrightarrow \mathbf{URS}.$$

3.2. LAX ALGEBRAS ON NUMERICAL RELATIONS

Consider now the 2-category $\mathbf{Y} = \mathbf{NRel}$ whose:

– objects are sets;
– morphisms $d : X \nrightarrow Y$ from X to Y are maps $d : X \times Y \to [0, \infty]$;
– identity of X is

$$\begin{aligned} \Delta_X : X \times X &\to [0, \infty] \\ (x, x') &\mapsto \begin{cases} 0 & \text{if } x = x', \\ \infty & \text{otherwise;} \end{cases} \end{aligned}$$

– 2-cells: for $d_1, d_2 \in \mathbf{NRel}(X, Y)$, $d_1 \sqsubseteq d_2$ if, for each pair $(x, y) \in X \times Y$, $d_2(x, y) \leq d_1(x, y)$;
– composition law is defined as follows: for $d : X \nrightarrow Y$ and $d' : Y \nrightarrow Z$, $d' \cdot d : X \nrightarrow Z$ is given by

$$(d' \cdot d)(x, z) = \inf\{d(x, y) + d'(y, z) \mid y \in Y\}.$$

The reflexive and transitive lax algebras for the identity monad in **NRel** are the *quasi-metric spaces* (where the distance can also be ∞), while, for the natural extension of the ultrafilter monad, the reflexive and transitive lax algebras are the *approach spaces* (see [10]), as we show next.

THEOREM 3.1. *For the identity monad in* **NRel**, Alg(Id, 1, 1) *is isomorphic to the category of quasi-metric spaces and non-expansive maps.*

Proof. A lax algebra $a : X \times X \to [0, \infty]$ is reflexive if and only if

$$\Delta_X \sqsubseteq a \cdot 1_X \Leftrightarrow \forall x \in X \; a(x, x) = 0.$$

It is transitive if and only if

$$\begin{aligned} a \cdot a \sqsubseteq a \cdot 1_X \;\Leftrightarrow\; & (\forall x, z \in X) \\ & a(x, z) \leq (a \cdot a)(x, z) = \inf\{a(x, y) + a(y, z) \mid y \in X\} \\ \Leftrightarrow\; & (\forall x, y, z \in X)\; a(x, z) \leq a(x, y) + a(y, z). \end{aligned}$$

Hence a map $a : X \times X \to [0, \infty]$ is a reflexive and transitive lax algebra if and only if it is a quasi-metric space (also called *quasi-pseudometric space*).

Furthermore, given two lax algebras $a : X \nrightarrow X$ and $b : Y \nrightarrow Y$, a map $f : X \to Y$ is a lax morphism if and only if it is non-expansive, since

$$\begin{aligned} f \cdot a \sqsubseteq b \cdot f \;&\Leftrightarrow\; (\forall x \in X)\ (\forall y \in Y)\ (b \cdot f)(x, y) \leq (f \cdot a)(x, y) \\ &\Leftrightarrow\; (\forall x \in X)\ (\forall y \in Y) \\ &\;\; \inf_{y' \in Y}(f(x, y') + b(y', y)) \leq \inf_{x' \in X}(a(x, x') + f(x', y)) \\ &\Leftrightarrow\; (\forall x \in X)\ (\forall y \in Y) \\ &\;\; b(f(x), y) \leq \inf\{a(x, x') \mid x' \in X,\ f(x') = y\} \\ &\Leftrightarrow\; (\forall x \in X)\ (\forall x' \in X)\ b(f(x), f(x')) \leq a(x, x'). \qquad \square \end{aligned}$$

We will now define a lax extension into **NRel** of the ultrafilter monad (U, e, m) in **Set**. In order to do this we will make use of the following result, which depends of choice (it follows in particular from the Prime Ideal Theorem: see Section 2 of [8]).

LEMMA 3.2. *If $r : X \nrightarrow Y$ is a relation, $\mathfrak{f}$ is a filter in X and $\mathfrak{y}$ is an ultrafilter in Y such that $r(\mathfrak{f}) \subseteq \mathfrak{y}$, then there exists an ultrafilter $\mathfrak{x}$ in X containing $\mathfrak{f}$ and such that $r(\mathfrak{x}) \subseteq \mathfrak{y}$.*

To define the lax functor U : **NRel** $\to$ **NRel**, for each $d : X \times Y \to [0, \infty]$ and each $\alpha \in [0, \infty]$, we consider the relation $d_\alpha : X \nrightarrow Y$ defined by $x d_\alpha y$ if $d(x, y) \leq \alpha$. As usual, for each subset A of X and each subset $\mathcal{A}$ of the powerset PX of X, we define

$$d_\alpha(A) := \{y \in Y \mid \exists x \in A : d(x, y) \leq \alpha\},$$

and $d_\alpha(\mathcal{A}) := \{d_\alpha(A) \mid A \in \mathcal{A}\}$.

PROPOSITION 3.3. *The assignment*

$$\begin{aligned} U(d) : UX \times UY &\to [0, \infty] \\ (\mathfrak{x}, \mathfrak{y}) &\mapsto \inf\{\alpha \in [0, \infty] \mid d_\alpha(\mathfrak{x}) \subseteq \mathfrak{y}\}, \end{aligned} \tag{9}$$

defines a lax extension U : **NRel** $\to$ **NRel** *of the ultrafilter monad (U, e, m) in* **Set**.

Proof. The only non-trivial part of the proof is to show that, for $d : X \times Y \to [0, \infty]$ and $d' : Y \times Z \to [0, \infty]$,

$$U(d') \cdot U(d) \sqsubseteq U(d' \cdot d);$$

that is, for $\mathfrak{x} \in UX$ and $\mathfrak{z} \in UZ$,

$$\alpha_1 := U(d' \cdot d)(\mathfrak{x}, \mathfrak{z}) \leq U(d') \cdot U(d)(\mathfrak{x}, \mathfrak{z}) =: \alpha_2.$$

Let $\varepsilon > 0$. By definition of composition, there exists $\mathfrak{y} \in UY$ such that

$$U(d)(\mathfrak{x}, \mathfrak{y}) + U(d')(\mathfrak{y}, \mathfrak{z}) \leq \alpha_2 + \varepsilon.$$

Let $\beta_1 := U(d)(\mathfrak{x}, \mathfrak{y})$ and $\beta_2 := U(d')(\mathfrak{y}, \mathfrak{z})$. By definition of U, for each $A \in \mathfrak{x}$,

$$d'_{\beta_2+\varepsilon}(d_{\beta_1+\varepsilon}(A)) \in \mathfrak{z}.$$

From

$$\begin{aligned} d'_{\beta_2+\varepsilon}(d_{\beta_1+\varepsilon}(A)) &= d'_{\beta_2+\varepsilon}(\{y \in Y \mid \exists x \in A : d(x, y) \leq \beta_1 + \varepsilon\}) \\ &= \{z \in Z \mid \exists y \in Y \exists x \in A : d(x, y) \leq \beta_1 + \varepsilon \text{ and} \\ &\quad\ d'(y, z) \leq \beta_2 + \varepsilon\} \\ &\subseteq \{z \in Z \mid \exists x \in A : (d' \cdot d)(x, z) \leq \beta_1 + \beta_2 + 2\varepsilon\} \\ &= (d' \cdot d)_{\beta_1+\beta_2+2\varepsilon}(A), \end{aligned}$$

it follows that $(d' \cdot d)_{\beta_1+\beta_2+2\varepsilon}(A) \in \mathfrak{z}$. Therefore

$$\alpha_1 \leq \beta_1 + \beta_2 + 2\varepsilon \leq \alpha_2 + 2\varepsilon,$$

hence $\alpha_1 \leq \alpha_2$. □

A lax algebra for this monad is a pair (X, a), where X is a set and $a : UX \times X \to [0, \infty]$ is a map. Moreover, (X, a) is reflective if and only if

$$\begin{aligned} \Delta_X \sqsubseteq a \cdot e_X &\Leftrightarrow (\forall x, x' \in X)\ a \cdot e_X(x, x') \leq \Delta_X(x, x') \\ &\Leftrightarrow (\forall x \in X)\ a(\dot{x}, x) = 0. \end{aligned}$$

It is transitive if and only if

$$\begin{aligned} a \cdot Ua \sqsubseteq m_X \cdot a &\Leftrightarrow (\forall x \in X)\ (\forall \mathfrak{X} \in UUX) \\ &\qquad (a \cdot m_X)\ (\mathfrak{X}, x) \leq (a \cdot Ua)(\mathfrak{X}, x) \\ &\Leftrightarrow (\forall \mathfrak{X} \in UUX)\ (\forall \mathfrak{x} \in UX)\ (\forall x \in X) \\ &\qquad a(m_X(\mathfrak{X}), x) \leq Ua(\mathfrak{X}, \mathfrak{x}) + a(\mathfrak{x}, x). \end{aligned}$$

A lax morphism $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that

$$\begin{aligned} f \cdot a \sqsubseteq b \cdot Uf &\Leftrightarrow (\forall x \in X)\ (\forall \mathfrak{x} \in UX)\ b \cdot Tf(\mathfrak{x}, x) \leq f \cdot a(\mathfrak{x}, x) \\ &\Leftrightarrow (\forall x \in X)\ (\forall \mathfrak{x} \in UX)\ b(f(\mathfrak{x}), x) \leq a(\mathfrak{x}, x). \end{aligned}$$

In order to show that $\mathrm{Alg}(U, e, m)$ is isomorphic to the category of approach spaces, we first recall the definition of the latter. (For more detailed information on approach spaces see [10].)

DEFINITION 3.4. The category **Ap** is the category with objects *approach spaces*, that is, pairs (X, δ) consisting of a set X and a map $\delta : PX \times X \to [0, \infty]$ satisfying the following axioms, for each $x \in X$, $A, B \in PX$ and $\alpha \in [0, \infty]$:

(A1) $\delta(\{x\}, x) = 0$,
(A2) $\delta(\emptyset, x) = \infty$,
(A3) $\delta(A \cup B, x) = \min\{\delta(A, x), \delta(B, x)\}$,
(A4) $\delta(A, x) \leq \delta(A^{(\alpha)}, x) + \alpha$,

where $A^{(\alpha)} = \{x \in X \mid \delta(A, x) \leq \alpha\}$, and with morphisms $f : (X, \delta) \to (Y, \gamma)$ *contrations* (=*non-expansive maps*), i.e., maps $f : X \to Y$ such that, for each subset A and each point x of X,

$$\gamma(f(A), f(x)) \leq \delta(A, x).$$

Each map $\delta : P(X) \times X \to [0, \infty]$ defines a lax algebra

$$\begin{aligned} a_\delta : UX \times X &\to [0, \infty] \\ (\mathfrak{x}, x) &\mapsto \sup\{\delta(A, x) \mid A \in \mathfrak{x}\}, \end{aligned}$$

and each lax algebra $a : UX \times X \to [0, \infty]$ induces a map

$$\begin{aligned} \delta_a : PX \times X &\to [0, \infty] \\ (A, x) &\mapsto \inf\{a(\mathfrak{x}, x) \mid A \in \mathfrak{x}\}, \end{aligned}$$

that satisfies obviously (A2) and (A3).

We will split the proof that the above assignments define an isomorphism between **Ap** and $\mathrm{Alg}(U, e, m)$ in several steps (see also [10]):

1. *If $f : (X, \delta) \to (Y, \gamma)$ is a contraction, then $f : (X, a_\delta) \to (Y, a_\gamma)$ is a lax morphism.*
   If $\mathfrak{x} \in UX$ and $x \in X$, from $\gamma(f(A), f(x)) \leq \delta(A, x)$ it follows that $a_\gamma(f(\mathfrak{x}), f(x)) \leq a_\delta(\mathfrak{x}, x)$.
2. *If $f : (X, a) \to (Y, b)$ is a lax morphism, then $f : (X, \delta_a) \to (Y, \delta_b)$ is a contraction.*
   Let $A \in PX$ and $x \in X$. For each $\mathfrak{y} \in UY$ with $f(A) \in \mathfrak{y}$ there is an ultrafilter $\mathfrak{x} \in UX$ such that $f(\mathfrak{x}) = \mathfrak{y}$ and the result follows.
3. *If $\delta : PX \times X \to [0, \infty]$ satisfies axioms* (A1) *and* (A2)*, then $\delta = \delta_{a_\delta}$.*

   (a) $\delta(A, x) \leq \delta_{a_\delta}(A, x)$: for each ultrafilter $\mathfrak{x} \in UX$ with $A \in \mathfrak{x}$, one has $a_\delta(\mathfrak{x}, x) \geq \delta(A, x)$, hence

   $$\delta(A, x) \leq \inf\{a_\delta(\mathfrak{x}, x) \mid A \in \mathfrak{x}\}.$$

   (b) $\delta_{a_\delta}(A, x) \leq \delta(A, x)$: in case $\delta(A, x) = \infty$, the inequality trivially holds; otherwise, the set $\mathcal{I} = \{B \in P(X) \mid \delta(B, x) > \delta(A, x)\}$ is an ideal disjoint from the filter $\dot{A}$, hence there exists an ultrafilter $\mathfrak{x}$ with $\dot{A} \subseteq \mathfrak{x}$

disjoint from $\mathcal{I}$; by definition of a_δ we then have that $a_\delta(\mathfrak{x}, x) \leq \delta(A, x)$, hence $\delta_{a_\delta}(A, x) \leq a_\delta(\mathfrak{x}, x) \leq \delta(A, x)$.

4. *If* $a : UX \times X \to [0, \infty]$ *is a transitive lax algebra, then* $a = a_{\delta_a}$.

   (a) $a \sqsubseteq a_{\delta_a}$: Given $\mathfrak{x} \in UX$, $A \in \mathfrak{x}$ and $x \in X$, $\delta_a(A, x) \leq a(\mathfrak{x}, x)$ by definition of δ_a; hence

$$a(\mathfrak{x}, x) \geq \sup\{\delta_a(A, x) \mid A \in \mathfrak{x}\} = a_{\delta_a}(\mathfrak{x}, x);$$

   (b) $a_{\delta_a} \sqsubseteq a$: Let $\mathfrak{x} \in UX$, $x \in X$, $\alpha = a_{\delta_a}(\mathfrak{x}, x)$ and $\varepsilon > 0$. By definition of δ_a, for each $A \in \mathfrak{x}$ there exists an ultrafilter $\mathfrak{y}_A$ in X with $A \in \mathfrak{y}_A$ and $a(\mathfrak{x}_A, x) \leq \alpha + \varepsilon$; that is, the set

$$\mathcal{A}_A := \{\mathfrak{y} \in UX \mid A \in \mathfrak{y} \text{ and } a(\mathfrak{y}, x) \leq \alpha + \varepsilon\}$$

   is non-empty. By construction, the filterbase $\mathcal{F} = \{\mathcal{A}_A \mid A \in \mathfrak{x}\}$ is such that $a_{\alpha+\varepsilon}(\mathcal{F}) \subseteq \dot{x}$. Therefore $\mathcal{F}$ can be refined to an ultrafilter $\mathfrak{X}$ on UX such that $a_{\alpha+\varepsilon}(\mathfrak{X}) \subseteq \dot{x}$, hence $Ua(\mathfrak{X}, \dot{x}) \leq \alpha + \varepsilon$. We remark that, by construction, $m_X(\mathfrak{X})$ is exactly $\mathfrak{x}$. We therefore obtain, due to the transitivity of the lax algebra a,

$$a(\mathfrak{x}, x) = a(m_X(\mathfrak{X}), x) \leq Ua(\mathfrak{X}, \dot{x}) + a(\dot{x}, x) \leq \alpha + \varepsilon.$$

5. *If* (X, δ) *is an approach space, then* (X, a_δ) *is a reflexive and transitive lax algebra.*
   It is clear that a_δ is reflexive. To show its transitivity, consider $\mathfrak{X} \in UUX$, $\mathfrak{x} \in UX$ and $x \in X$, and put $\alpha_1 := Ua_\delta(\mathfrak{X}, \mathfrak{x})$ and $\alpha_2 := a_\delta(\mathfrak{x}, x)$. Now let $B \in m_X(\mathfrak{X})$ and $\varepsilon > 0$. By definition of m_X,

$$\mathcal{B} = \{\mathfrak{y} \in UX \mid B \in \mathfrak{y}\} \in \mathfrak{X},$$

   and therefore $(a_\delta)_{\alpha_1+\varepsilon}(\mathcal{B}) \in \mathfrak{x}$. Since

$$\begin{aligned}(a_\delta)_{\alpha_1+\varepsilon}(\mathcal{B}) &= \{x \in X \mid \exists \mathfrak{y} \in \mathcal{B} : a_\delta(\mathfrak{y}, x) \leq \alpha_1 + \varepsilon\} \\ &\subseteq \{x \in X \mid \delta(B, x) \leq \alpha_1 + \varepsilon\} \\ &= B^{(\alpha_1+\varepsilon)},\end{aligned}$$

   also $B^{(\alpha_1+\varepsilon)}$ belongs to $\mathfrak{x}$. Hence $\delta(B^{(\alpha_1+\varepsilon)}, x) \leq \alpha_2$, and by (A4) we obtain

$$\delta(B, x) \leq \delta(B^{(\alpha_1+\varepsilon)}, x) + \alpha_1 + \varepsilon \leq \alpha_2 + \alpha_1 + \varepsilon.$$

   Finally, by definition of a_δ,

$$\begin{aligned} a_\delta(m_X(\mathfrak{X}), x) &= \sup\{\delta(B, x) \mid B \in m_X(\mathfrak{X})\} \\ &\leq \alpha_1 + \alpha_2 \\ &\leq Ua_\delta(\mathfrak{x}, \mathfrak{x}) + a_\delta(\mathfrak{x}, x). \end{aligned}$$

6. *If (X, a) is a reflexive and transitive algebra, then (X, δ_a) is an approach space.* Reflexivity of a clearly implies that δ_a satisfies (A1). It remains to be shown that it also satisfies (A4). To show this, let $A \subseteq X$, $x \in X$, $\alpha \in [0, \infty]$ and $\varepsilon > 0$. By definition of δ_a, there exists $\mathfrak{x}_0 \in UX$ such that $A^{(\alpha)} \in \mathfrak{x}_0$ and

$$a(\mathfrak{x}_0, x) \leq \delta_a(A^{(\alpha)}, x) + \varepsilon.$$

Now, by definition of $A^{(\alpha)}$, for each $a \in A^{(\alpha)}$ there exists an ultrafilter $\mathfrak{x}_a$ such that $A \in \mathfrak{x}_a$ and $a(\mathfrak{x}_a, a) \leq \alpha + \varepsilon$. For $\mathcal{A} = \{\mathfrak{x} \in UX \mid A \in \mathfrak{x}\}$, one has

$$a_{\alpha+\varepsilon}(\mathcal{A}) \supseteq A^{(\alpha)} \in \mathfrak{x}_0.$$

Hence, by Lemma 3.2, there exists an ultrafilter $\mathfrak{X}$ in UX such that $\mathcal{A} \in \mathfrak{X}$ and $a_{\alpha+\varepsilon}(\mathfrak{X}) \subseteq \mathfrak{x}_0$, which implies that

$$Ua(\mathfrak{X}, \mathfrak{x}_0) \leq \alpha + \varepsilon.$$

We finally have that

$$\begin{aligned} \delta_a(A, x) &\leq a(m_X(\mathfrak{X}), x) \\ &\leq Ua(\mathfrak{X}, \mathfrak{x}_0) + a(\mathfrak{x}_0, x) \\ &\leq \alpha + \varepsilon + \delta_a(A^{(\alpha)}, x) + \varepsilon \\ &\leq \delta_a(A^{(\alpha)}, x) + \alpha + 2\varepsilon. \end{aligned}$$

We have therefore shown that:

THEOREM 3.5. *For the lax extension* (9) *to* **NRel** *of the ultrafilter monad,* $\mathrm{Alg}(U, e, m)$ *is isomorphic to the category of approach spaces and contraction maps.*

3.3. "QUASI-UNIFORM" STRUCTURES AS LAX ALGEBRAS

Having as starting point the category **Rel** of relations, one can define a 2-category **Y** as follows:

– objects are sets;

– morphisms $\mathfrak{r} : X \nrightarrow Y$ are (possibly improper) filters in **Rel**(X, Y);
– identity of X is $\uparrow \mathrm{id}_X$, where id_X is the identity map on the set X;
– 2-cells: for $\mathfrak{r}, \mathfrak{r}' \in \mathbf{Y}(X, Y)$, $\mathfrak{r} \sqsubseteq \mathfrak{r}'$ if $\mathfrak{r}' \subseteq \mathfrak{r}$;
– composition law: for $\mathfrak{r} : X \nrightarrow Y$ and $\mathfrak{s} : Y \nrightarrow Z$, $\mathfrak{s} \cdot \mathfrak{r} : X \nrightarrow Z$ is the filter generated by $\{s \cdot r \mid s \in \mathfrak{s} \text{ and } r \in \mathfrak{r}\}$.

We remark that relations (in particular, maps) $r : X \nrightarrow Y$ are identified with the **Y**-morphisms

$$\uparrow r = \{s \in \mathbf{Rel}(X, Y) \mid s \geq r\}.$$

THEOREM 3.6. *If* **Y** *is the* 2*-category described above, then* Alg(Id, 1, 1) *is isomorphic to the category of quasi-uniform spaces and uniformly continuous maps.*

Proof. A lax algebra $\mathfrak{r} : X \nrightarrow X$ is reflexive if and only if

$$\begin{aligned} \uparrow \mathrm{id}_X \sqsubseteq \mathfrak{r} \;&\Leftrightarrow\; \mathfrak{r} \subseteq \uparrow \mathrm{id}_X \\ &\Leftrightarrow\; (\forall r \in \mathfrak{r})(\forall x \in X)\; xrx. \end{aligned}$$

It is transitive if and only if

$$\begin{aligned} \mathfrak{r} \cdot \mathfrak{r} \sqsubseteq \mathfrak{r} \;&\Leftrightarrow\; \mathfrak{r} \subseteq \mathfrak{r} \cdot \mathfrak{r} \\ &\Leftrightarrow\; (\forall r \in \mathfrak{r})\; (\exists s, s' \in \mathfrak{r}) : s \cdot s' \leq r \\ &\Leftrightarrow\; (\forall r \in \mathfrak{r})\; (\exists s \in \mathfrak{r}) : s \cdot s \leq r. \end{aligned}$$

That is, $\mathfrak{r} : X \nrightarrow X$ is a reflexive and transitive lax algebra if and only if it is a quasi-uniformity on X.

Given two lax algebras $\mathfrak{r} : X \nrightarrow X$ and $\mathfrak{s} : Y \nrightarrow Y$, a map $f : X \to Y$ is a lax morphism if and only if

$$\begin{aligned} \uparrow f \cdot \mathfrak{r} \sqsubseteq \mathfrak{s} \cdot \uparrow f \;&\Leftrightarrow\; \mathfrak{s} \cdot \uparrow f \subseteq \uparrow f \cdot \mathfrak{r} \\ &\Leftrightarrow\; (\forall s \in \mathfrak{s})\; (\exists r \in \mathfrak{r}) : s \cdot f \geq f \cdot r \\ &\Leftrightarrow\; (\forall s \in \mathfrak{s})\; (\exists r \in \mathfrak{r}) : (\forall x, x' \in X)\; xrx' \Rightarrow f(x)sf(x'). \end{aligned}$$

Hence, $f : (X, \mathfrak{r}) \to (Y, \mathfrak{s})$ is a lax morphism if and only if it is a uniformly continuous map. □

As in the previous examples, if we now replace the category **Rel** by the category **NRel** and proceed as above, we obtain a natural notion of *quasi-uniform approach space*. A convenient construction of quasi-uniform structures and their properties will be the subject of a subsequent paper [4].

4. Properties of the Categories of Lax Algebras

In all the previous examples we are given:

– a category **Y** such that:

(a) for every $X, Y \in \mathbf{Y}$, $\mathbf{Y}(X, Y)$ is a complete preordered set;
(b) there exists a pseudo-involution $(\,)^\circ : \mathbf{Y} \to \mathbf{Y}^{\mathrm{op}}$;

– a category **X** having as objects the objects of **Y** and as morphisms the **Y**-morphisms satisfying $f \cdot f^\circ \sqsubseteq 1_Y$ and $1_X \sqsubseteq f^\circ \cdot f$ (i.e., an **Y**-morphism belongs to **X** if and only if it is a left adjoint), and
– a lax monad $(T, e, m) : \mathbf{Y} \to \mathbf{Y}$ such that:
(c) T commutes with $(\,)^\circ$, i.e., $T(r^\circ) = (Tr)^\circ$.

Under these assumptions, one can easily show:

LEMMA 4.1. (1) *For $X, Y \in \mathbf{X}$, $\mathbf{X}(X, Y)$ is a discrete category, hence* **X** *is just an ordinary category.*
(2) *If $f \in \mathbf{X}(X, Y)$ and $r \in \mathbf{Y}(Y, Z)$, then $T(r \cdot f) = Tr \cdot Tf$.*
(3) *T maps* **X**-*morphisms into* **X**-*morphisms.*

We assume further that:

(d) The natural transformations e and m are pointwise in **X**, defining then a monad (T, e, m) in **X**.

From now on we assume we are given a 2-category **Y** and a lax monad $T : \mathbf{Y} \to \mathbf{Y}$ satisfying conditions (a)–(d), and consider the category **X** defined as above.
In order to study the properties of the categories of lax algebras of (8), we first state some auxiliary results.

LEMMA 4.2. *If $a : TX \nrightarrow X$ is a morphism in* **Y**, *then:*

(1) *a is a reflexive lax algebra if and only if $e_X^\circ \sqsubseteq a$, which in turn implies $a \sqsubseteq a \cdot Ta \cdot m_X^\circ$;*
(2) *a is a transitive lax algebra if and only if $a \cdot Ta \cdot m_X^\circ \sqsubseteq a$.*

Proof. (1) If $1_X \sqsubseteq a \cdot e_X$, then $e_X^\circ \sqsubseteq a \cdot e_X \cdot e_X^\circ \sqsubseteq a$, since e_X belongs to **X**. Conversely, if $e_X^\circ \sqsubseteq a$, then $1_X \sqsubseteq e_X^\circ \cdot e_X \sqsubseteq a \cdot e_X$.
Finally, if $e_X^\circ \sqsubseteq a$, then

$$a^\circ = m_X \cdot Te_X \cdot a^\circ \sqsubseteq m_X \cdot Ta^\circ \cdot a^\circ = (a \cdot Ta \cdot m_X^\circ)^\circ,$$

from which it follows that $a \sqsubseteq a \cdot Ta \cdot m_X^\circ$.
The proof of (2) is similar. □

Pure routine computation shows that:

LEMMA 4.3. *If, in the diagram*

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array} \tag{10}$$

$f : X \to Y$ is an $\mathbf{X}$-morphism, then the following conditions are equivalent:

(i) $f : (X, a) \to (Y, b)$ *is a lax morphism, that is* $f \cdot a \sqsubseteq b \cdot Tf$;
(ii) $a \sqsubseteq f^\circ \cdot b \cdot Tf$;
(iii) $f \cdot a \cdot Tf^\circ \sqsubseteq b$;
(iv) $Tf \cdot a^\circ \sqsubseteq b^\circ \cdot f$. □

THEOREM 4.4. *The full embeddings*

$$\mathrm{Alg}(T, e, m) \hookrightarrow \mathrm{Alg}(T, e) \hookrightarrow \mathrm{Alg}(T)$$

are reflective. Moreover, the reflections have underlying identity morphisms.

Proof. The inclusion $I : \mathrm{Alg}(T, e) \hookrightarrow \mathrm{Alg}(T)$ has as left adjoint

$$\begin{aligned} F : \mathrm{Alg}(T) &\longrightarrow \mathrm{Alg}(T, e) \\ (X, a) &\longmapsto (X, a \vee e_X^\circ) \\ f : (X, a) \to (Y, b) &\longmapsto f : (X, a \vee e_X^\circ) \to (b \vee e_Y^\circ); \end{aligned}$$

that F is well-defined follows from the observation that, if $a \sqsubseteq f^\circ \cdot b \cdot Tf$, then, since

$$f \cdot e_X^\circ \sqsubseteq e_Y^\circ \cdot e_Y \cdot f \cdot e_X^\circ = e_Y^\circ \cdot Tf \cdot e_X \cdot e_X^\circ \sqsubseteq e_Y^\circ \cdot Tf,$$

one has

$$e_X^\circ \sqsubseteq f^\circ \cdot f \cdot e_X^\circ \sqsubseteq f^\circ \cdot e_Y^\circ \cdot Tf.$$

Therefore

$$a \vee e_X^\circ \sqsubseteq f^\circ \cdot (b \vee e_Y^\circ) \cdot Tf,$$

and so F is a functor, that is obviously left adjoint to I.

The left adjoint to $\mathrm{Alg}(T, e, m) \hookrightarrow \mathrm{Alg}(T, e)$ is obtained by iteration of the (pointed) endofunctor

$$\begin{aligned} G : \mathrm{Alg}(T, e) &\longrightarrow \mathrm{Alg}(T, e) \\ (X, a) &\longmapsto (X, a \cdot Ta \cdot m_X^\circ) \\ f : (X, a) \to (Y, b) &\longmapsto f : (X, a \cdot Ta \cdot m_X^\circ) \to (Y, b \cdot Tb \cdot m_Y^\circ); \end{aligned}$$

this defines a functor, since, from $f \cdot a \sqsubseteq b \cdot Tf$ it follows that

$$\begin{aligned} f \cdot (a \cdot Ta \cdot m_X^\circ) &\sqsubseteq b \cdot Tf \cdot Ta \cdot m_X^\circ \\ &\sqsubseteq b \cdot T(f \cdot a) \cdot m_X^\circ \\ &\sqsubseteq b \cdot T(b \cdot Tf) \cdot m_X^\circ \\ &= b \cdot Tb \cdot T^2 f \cdot m_X^\circ \\ &\sqsubseteq (b \cdot Tb \cdot m_X^\circ) \cdot Tf; \end{aligned}$$

Lemma 4.2 assures that $1_X : (X, a) \to (X, a \cdot Ta \cdot m_X^\circ)$ is a lax morphism whenever (X, a) is a reflexive lax algebra, hence $\alpha = (1_X : (X, a) \to (X, a \cdot Ta \cdot m_X^\circ))_{(X,a)\in\mathrm{Alg}(T,e)} : 1 \to G$ is a natural transformation as claimed.

Finally, since there is only a set of possible lax algebra structures on each object X of $\mathbf{X}$, the iteration of G will eventually stop, defining the claimed left adjoint. □

THEOREM 4.5. *The canonical forgetful functors from* $\mathrm{Alg}(T)$, $\mathrm{Alg}(T, e)$ *and* $\mathrm{Alg}(T, e, m)$ *into* $\mathbf{X}$ *are topological functors.*

Proof. Since the following diagram is commutative

$$\begin{array}{ccccc} \mathrm{Alg}(T,e,m) & \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} & \mathrm{Alg}(T,e) & \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} & \mathrm{Alg}(T) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbf{X} & & \end{array} \tag{11}$$

it is enough to show that the functor $|\ \ | : \mathrm{Alg}(T) \to \mathbf{X}$ is topological (see [1]).

Given $(f_i : X \to (Y_i, b_i))_{i\in I}$, each of the lax algebra structures $a_i := f_i^\circ \cdot b_i \cdot Tf_i$ makes $f_i : (X, a_i) \to (Y_i, b_i)$ a morphism. Furthermore

$$a := \bigwedge_{i\in I} f_i^\circ \cdot b_i \cdot Tf_i,$$

clearly gives the $|\ \ |$-initial lifting. □

5. Pullback Stability of Regular Epimorphisms in $\mathrm{Alg}(T, e)$

One reason for considering **PsTop** a good improvement of **Top** is its pullback-stability of regular epimorphisms. We are now going to show that this property is shared by every category $\mathrm{Alg}(T, e)$ in our setting, provided that it fulfills, in addition to conditions (a)–(d) of the previous section, the following conditions:

(e) $\mathbf{X}$ has pullbacks and the functors $\mathbf{X} \hookrightarrow \mathbf{Y}$ and $T : \mathbf{X} \to \mathbf{Y}$ have *Beck–Chevalley Property* (BCP); that is, if

$$\begin{array}{ccc} Z & \xrightarrow{h} & W \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} \tag{12}$$

is a pullback diagram, then $g^\circ \cdot f = h \cdot k^\circ$ and $Tg^\circ \cdot Tf = Th \cdot Tk^\circ$;

(f) in $\mathbf{X}$ regular epimorphisms are stable under pullback, and every regular epimorphism $f : X \to Y$ satisfies the equality $f \cdot f^\circ = 1_Y$;

(g) for every diagram in $\mathbf{Y}$, with $f, g \in \mathbf{X}$,

$$\begin{array}{ccc} Z & \xrightarrow{\;g\;} & W \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ X & \xrightarrow[\;f\;]{} & Y \end{array} \tag{13}$$

$$f \cdot r \cdot g^{\circ} \wedge s \sqsubseteq f \cdot (r \wedge f^{\circ} \cdot s \cdot g) \cdot g^{\circ}.$$

We first point out that these conditions are fulfilled by the category **Set** as well as by the monads considered in the examples. (To prove this in the case of the ultrafilter monad one makes use of Lemma 3.2.)

We also remark that:

- from (BCP) it follows that every monomorphism $f : X \to Y$ in $\mathbf{X}$ satisfies $f^{\circ} \cdot f = 1_X$, hence an $\mathbf{X}$-morphism is a monomorphism in $\mathbf{X}$ if and only if it is a split monomorphism in $\mathbf{Y}$;
- condition (f) – which is in fact an equality since the other implication is trivially satisfied – follows from *Freyd Modular Laws*

$$s \cdot r \wedge t \leq s \cdot (r \wedge s^{\circ} \cdot t) \text{ and } s \cdot r \wedge t \leq (s \wedge t \cdot r^{\circ}) \cdot r \tag{FML}$$

  (see [6]), but it is not equivalent. In fact, in the category $\mathbf{Y}$ of Example 2.2, (f) holds true although (FML) fails.

PROPOSITION 5.1. *A lax morphism* $f : (X, a) \to (Y, b)$ *in* $\mathrm{Alg}(T, e)$ *is a regular epimorphism if and only if*

(1) f *is a regular epimorphism in* $\mathbf{X}$*, and*
(2) $b = f \cdot a \cdot Tf^{\circ}$.

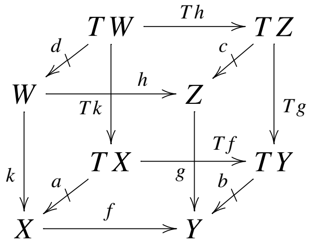
Proof. Let $f : (X, a) \to (Y, b)$ be a regular epimorphism in $\mathrm{Alg}(T, e)$. Since the forgetful functor $\mathrm{Alg}(T, e) \to \mathbf{X}$ preserves colimits, f is necessarily a regular epimorphism in $\mathbf{X}$. To prove that condition (2) is also necessary, it is enough to show that $f \cdot a \cdot Tf^{\circ} : TY \nrightarrow Y$ is a reflexive algebra provided that $a : TX \nrightarrow X$ is: if $e_X^{\circ} \sqsubseteq a$, then

$$e_Y^{\circ} = f \cdot f^{\circ} \cdot e_Y^{\circ} = f \cdot e_X^{\circ} \cdot Tf^{\circ} \sqsubseteq f \cdot a \cdot Tf^{\circ}.$$

Now, assuming (1) and (2), $b = f \cdot a \cdot Tf^{\circ}$ is clearly the final structure for $f : (X, a) \to Y$, and the result follows. □

THEOREM 5.2. *The regular epimorphisms are pullback-stable in* $\mathrm{Alg}(T, e)$.

Proof. Consider the diagram



(14)

where the front square is a pullback and $f : (X, a) \to (Y, b)$ is a regular epimorphism. By the construction of initial and final structures,

$$d = (k^\circ \cdot a \cdot Tk) \wedge (h^\circ \cdot c \cdot Th),$$

and $b = f \cdot a \cdot Tf^\circ$. We then have

$$\begin{aligned} c &\sqsubseteq g^\circ \cdot b \cdot Tg \quad \text{(because } g \text{ is a morphism)} \\ &= g^\circ \cdot f \cdot a \cdot Tf^\circ \cdot Tg \\ &= h \cdot k^\circ \cdot a \cdot Tk \cdot Th^\circ \quad \text{(by (BCP));} \end{aligned}$$

hence

$$\begin{aligned} c &\sqsubseteq h \cdot k^\circ \cdot a \cdot Tk \cdot Th^\circ \wedge c \\ &\sqsubseteq h \cdot (k^\circ \cdot a \cdot Tk \wedge h^\circ \cdot c \cdot Th) \cdot Th^\circ \quad \text{(by condition (f))} \\ &= h \cdot d \cdot Th^\circ. \end{aligned}$$

Therefore $c = h \cdot d \cdot Th^\circ$, hence $h : (W, d) \to (Z, c)$ is a regular epimorphism in $\mathrm{Alg}(T, e)$ by Proposition 5.1. □

We remark that in Proposition 5.1 and in Theorem 5.2 we can replace $\mathrm{Alg}(T, e)$ by $\mathrm{Alg}(T)$, since the entire argumentation is also valid for this category.

6. The Čech–Stone Compactification of Reflexive and Transitive Lax Algebras

It is well-known that for each topological space X there exists a compact Hausdorff space βX and a continuous map $\beta_X : X \to \beta X$ through which every continuous map from X to a compact Hausdorff space factors in a unique way. Observing that the category of compact Hausdorff spaces is isomorphic to the category of algebras for the ultrafilter monad in **Set**, and **Top** is the category of reflexive and transitive lax algebras for the same monad, one may ask whether the former result is a particular instance of a more general one. That is:

Is the category $\mathbf{X}^{\mathbb{T}}$ *a reflective subcategory of* $\mathrm{Alg}(T, e, m)$*?*

Barr proved this result in the case $\mathbf{Y} = \mathbf{Rel}$ (and hence $\mathbf{X} = \mathbf{Set}$, see [2]), while Manes constructed the corresponding reflections in [12]. The aim of this section is to show that we can extend the *Čech–Stone compactification* to our setting, provided that it satisfies some further assumptions. Therefore, throughout we will be working in the setting considered in the previous section, assuming, in addition, that $\mathbf{X}$ is a complete category. We will denote its terminal object by 1.

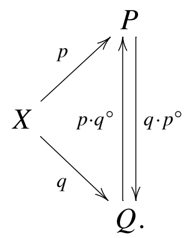
We start showing that:

PROPOSITION 6.1. *The category* $\mathbf{X}$ *is regular-cowellpowered.*

Proof. For $X \in \mathbf{X}$, to conclude that the class $\mathrm{Quot}(X)$ of regular epimorphisms with domain X has a representative set, we consider the map

$$\mathrm{Quot}(X) \longrightarrow \mathbf{Y}(X, X)$$
$$X \xrightarrow{q} Q \longmapsto X \xrightarrow{q} Q \xrightarrow{q^\circ} X$$

into the set $\mathbf{Y}(X, X)$ and show that any two elements of $\mathrm{Quot}(X)$ with the same image are isomorphic. For $p, q \in \mathrm{Quot}(X)$ with $p^\circ \cdot p = q^\circ \cdot q$, the following diagram commutes



(15)

Hence, if $f := q \cdot p^\circ$, then $f^\circ = p \cdot q^\circ$. Since p and q are epimorphisms in $\mathbf{Y}$, we obtain

$$f \cdot f^\circ = 1_P \quad \text{and} \quad f^\circ \cdot f = 1_Q,$$

hence, as a monic regular epimorphism in $\mathbf{X}$, f is an $\mathbf{X}$-isomorphism.

A necessary condition for the existence of a left adjoint of the functor $\mathbf{X}^{\mathbb{T}} \hookrightarrow \mathrm{Alg}(T, e, m)$ is its preservation of limits. The preservation of equalizers is guaranteed by (BCP), since, for every monomorphism $m : (X, a) \to (Y, b)$ in $\mathbf{X}^{\mathbb{T}}$, one has $a = m^\circ \cdot m \cdot a = m^\circ \cdot b \cdot Tm$, that is a is the initial structure in $\mathrm{Alg}(T, e, m)$. Next we analyze the closedness under products of $\mathbf{X}^{\mathbb{T}}$ in $\mathrm{Alg}(T, e, m)$.

LEMMA 6.2. *The following conditions are equivalent:*

(i) $\mathbf{X}^{\mathbb{T}}$ *is closed under products in* $\mathrm{Alg}(T, e, m)$*;*
(ii) *For each product* $(X, (\pi_i : X \to X_i)_{i \in I})$ *in* $\mathbf{X}$*,*

$$\bigwedge_{i \in I} (\pi_i^\circ \cdot \pi_i) = \mathrm{id}_X;$$

(iii) *If* $(f_i : X \to X_i)_{i\in I}$ *is a (small) monosource in* **X**, $r : Y \to X$ *is an* **X**-*morphism and* $s : Y \nrightarrow X$ *is an* **Y**-*morphism, then*

$$(\forall i \in I f_i \cdot s \sqsubseteq f_i \cdot r) \Rightarrow s \sqsubseteq r.$$

Proof. (i) ⇔ (ii) A product $((X, a), (\pi_i : (X, a) \to (X_i, a_i))_{i\in I})$ in $\mathbf{X}^{\mathbb{T}}$ is still a product in $\mathrm{Alg}(T, e, m)$ if and only if a is the initial structure for π_i; that is,

$$a = \bigwedge_{i\in I}(\pi_i^\circ \cdot a_i \cdot T\pi_i).$$

Since a is a map

$$\bigwedge_{i\in I}(\pi_i^\circ \cdot a_i \cdot T\pi_i) = \bigwedge_{i\in I}(\pi_i^\circ \cdot \pi_i \cdot a) = \left(\bigwedge_{i\in I}\pi_i^\circ \cdot \pi_i\right) \cdot a \sqsupseteq a,$$

hence we have that

$$\begin{aligned} a = \bigwedge_{i\in I}(\pi_i^\circ \cdot a_i \cdot T\pi_i) \;&\Leftrightarrow\; a = \left(\bigwedge_{i\in I}\pi_i^\circ \cdot \pi_i\right) \cdot a \\ &\Leftrightarrow\; \mathrm{id}_X = \bigwedge_{i\in I}(\pi_i^\circ \cdot \pi_i) \\ &\qquad \text{(since } a \text{ is a split epimorphism).} \end{aligned}$$

(ii) ⇔ (iii) Condition (ii) follows from (iii) in the case $m_i = \pi_i$, $r = \mathrm{id}_X$ and $s = \bigwedge_{i\in I}\pi_i^\circ \cdot \pi_i$.

The converse follows from the fact that every (small) monosource factors through a product source followed by a monomorphism. □

We remark that, in the lemma above, only the first condition depends on $\mathbb{T}$, and that in fact the second one is a particular case of the former one for $\mathbb{T}$ the identity monad.

LEMMA 6.3. *Each of the (equivalent) conditions of the lemma above implies that, for each object* X *of* **X**, *the* **X**-*morphism* $(!_X : X \to 1)$ *is the top element of the preordered set* $\mathbf{Y}(X, 1)$.

Proof. We first remark that condition (ii) of Lemma 6.2 means $\mathrm{id}_1 = \max \mathbf{Y}(1, 1)$ in case $I = \emptyset$. Now, if $a : X \to 1$ is a morphism in **Y** such that $a \sqsupseteq !_X$, then $a^\circ \sqsupseteq !_X^\circ$, and therefore $a^\circ \cdot a \sqsupseteq !_X^\circ \cdot !_X \sqsupseteq \mathrm{id}_X$. Furthermore, $a \cdot a^\circ \sqsubseteq \max \mathbf{Y}(1, 1) = \mathrm{id}_1$. Hence a, as a map from X to 1, must coincide with $!_X$.

THEOREM 6.4. *Let* **X** *have (regular epi, mono)-factorizations. Given a monad* $\mathbb{T} = (T, e, m)$ *in* **X** *such that* T *preserves regular epimorphisms, the following conditions are equivalent:*

(i) $\mathbf{X}^{\mathbb{T}}$ *is a reflective subcategory of* $\mathrm{Alg}(T, e, m)$;

(ii) *for every product* $(X, (\pi_i : X \to X_i)_{i\in I})$, $\bigwedge_{i\in I}(\pi_i^\circ \cdot \pi_i) = \mathrm{id}_X$.

Proof. We first point out that the preservation of regular epimorphisms by T assures that the (regular epi, mono)-factorization system in $\mathbf{X}$ as well as the regular-cowellpoweredness of $\mathbf{X}$ can be lifted to $\mathbf{X}^{\mathbb{T}}$.

Let (X, a) be a reflexive and transitive lax algebra. Since (TX, m_X) is a free $\mathbb{T}$-algebra, although $e_X : X \to TX$ is not a lax morphism, every lax morphism $f : (X, a) \to (Y, b)$ into a $\mathbb{T}$-algebra (Y, b) factors uniquely through e_X.

Now, a morphism $h : (TX, m_X) \to (Y, b)$ in $\mathbf{X}^{\mathbb{T}}$ makes $h \cdot e_X : (X, a) \to (Y, b)$ a lax morphism if and only if $h \cdot a^\circ \cdot a \sqsubseteq h$. In fact, it is a lax morphism if and only if

$$h \cdot e_X \cdot a \sqsubseteq b \cdot Th \cdot Te_X = h \cdot m_X \cdot Te_X = h;$$

hence, from $h \cdot a^\circ \cdot a \sqsubseteq h$ it follows that $h \cdot e_X \cdot a \sqsubseteq h$ since $e_X \cdot a \sqsubseteq a^\circ \cdot a$, by the fact that (X, a) is a reflexive algebra. Conversely, if $h \cdot e_X \cdot a \sqsubseteq h$, then

$$\begin{aligned} h \cdot a^\circ \cdot a &= h \cdot m_X \cdot Te_X \cdot a^\circ \cdot a \\ &= b \cdot Th \cdot Te_X \cdot a^\circ \cdot a \sqsubseteq b \cdot b^\circ \cdot h \cdot e_X \cdot a \sqsubseteq h \cdot e_X \cdot a \sqsubseteq h. \end{aligned}$$

Furthermore, if

$$(TX, m_X) \xrightarrow{h} (Y, b) = (TX, m_X) \xrightarrow{q} (Z, c) \xrightarrow{u} (Y, b)$$

is the (regular epi, mono)-factorization of h in $\mathbf{X}^{\mathbb{T}}$, then $h \cdot a^\circ \cdot a \sqsubseteq h$ if and only if $q \cdot a^\circ \cdot a \sqsubseteq q$. Hence, in order to construct the reflection of (X, a) into $\mathbf{X}^{\mathbb{T}}$, we just have to consider the family of regular epimorphisms $(q_i : (TX, m_X) \to (Y_i, b_i))_{i\in I}$ in $\mathbf{X}^{\mathbb{T}}$ such that $q_i \cdot a^\circ \cdot a \sqsubseteq a$ for every $i \in I$. By the regular-cowellpoweredness of $\mathbf{X}^{\mathbb{T}}$ we may assume that I is a set. It is even non-empty since $!_{TX} : (TX, m_X) \to (1, !_{T1})$ verifies $!_X \cdot a^\circ \cdot a \sqsubseteq !_X$ by Lemma 6.3. Let

$$(TX, m_X) \xrightarrow{q_i} (Y, b) = (TX, m_X) \xrightarrow{q} (Q, d) \xrightarrow{f_i} (Y_i, b_i),$$

be the (regular epi, mono)-factorization of $(q_i)_{i\in I}$. Since $(f_i)_{i\in I}$ is a monosource, from $q_i \cdot a^\circ \cdot a \sqsubseteq q_i$ for every $i \in I$ it follows that $q \cdot a^\circ \cdot a \sqsubseteq q$. Therefore, $q \cdot e_X : (X, a) \to (Q, d)$ is a lax morphism and by construction it is the desired reflection. □

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