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COMPOUND POISSON INTEGER-VALUED GARCH PROCESSES

Tese de Doutoramento do Programa Inter-Universitário de Doutoramento em Matemática, orientada pela Professora Doutora Maria de Nazaré Mendes Lopes e pela Professora Doutora Maria Esmeralda Gonçalves e apresentada ao Departamento de Matemática da Faculdade de Ciências e Tecnologia da Universidade de Coimbra.

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Abstract

Count time series modeling has drawn much attention and considerable development in the recent decades since many of the observed stochastic systems in various contexts and scientific fields are driven by such kind of data. The first modelings, with linear character and essentially inspired by the classic ARMA models, are proved to be insufficient to give an adequate answer for some empirical characteristics, also observed in this type of data, such as the conditional heteroscedasticity. In order to capture such kind of characteristics several models for nonnegative integer-valued time series arise in literature inspired by the classic GARCH model of Bollerslev [10], among which is highlighted the integer-valued GARCH model with conditional Poisson distribution (briefly INGARCH model), proposed in 2006 by Ferland, Latour and Oraichi [25].

The aim of this thesis is to introduce and analyze a new class of integer-valued models having an analogous evolution as considered in [25] for the conditional mean, but with an associated comprehensive family of conditional distributions, namely the family of infinitely divisible discrete laws with support in $\mathbb{N}_0$, inflated (or not) in zero. So, we consider a family of conditional distributions that in its more general form can be interpreted as a mixture of a Dirac law at zero with any discrete infinitely divisible law, whose specification is made by means of the corresponding characteristic function. Taking into account the equivalence, in the set of the discrete laws with support $\mathbb{N}_0$, between infinitely divisible and compound Poisson distributions, this new model is designated as zero-inflated compound Poisson integer-valued GARCH model (briefly ZICP-INGARCH model).

We point out that the model is not limited to a specific conditional distribution; moreover, this model has as main advantage to unify and enlarge substantially the family of integer-valued stochastic processes. It is stressed that it is possible to present new models with conditional distributions with interest in practical applications as, in particular, the zero-inflated geometric Poisson INGARCH and the zero-inflated Neyman type-A INGARCH models, and also recover recent contributions such as the (zero-inflated) negative binomial INGARCH [81, 84], (zero-inflated) INGARCH [25, 84] and (zero-inflated) generalized Poisson INGARCH [52, 82] models. In addition to having the ability to describe different distributional behaviors and consequently, different kinds of conditional heteroscedasticity, the ZICP-INGARCH model is able to incorporate simultaneously other stylized facts that have been recorded in real count data, in particular overdispersion and high occurrence of zeros.

The probabilistic analysis of these models, concerning in particular the development of necessary and sufficient conditions of different kinds of stationarity (first-order, weak and strict) as well as the property of ergodicity and also the existence of higher order moments, is the main goal of this study. It is still derived estimates for the parameters of the model using a two-step approach which is based on the conditional least squares and moments methods.

Resumo

A modelação de séries temporais de contagem conheceu nas últimas décadas grande impulso e desenvolvimento, devido sobretudo ao fato de muitos dos sistemas estocásticos observados, nos mais diversos contextos e áreas científicas, terem como resposta tal tipo de dados. As primeiras modelações, de caráter linear e essencialmente inspiradas nos clássicos modelos ARMA, revelaram-se insuficientes para dar resposta a algumas características empíricas, também observadas neste tipo de dados, como a heteroscedasticidade condicional. De modo a ter em conta tal tipo de características, surgiram na literatura vários modelos para séries temporais de valores inteiros não negativos inspirados nos GARCH clássicos de Bollerslev [10], entre os quais se destacam os modelos GARCH de valores inteiros com distribuição condicional de Poisson (designados modelos INGARCH), propostos em 2006 por Ferland, Latour e Oraichi [25].

O objetivo fundamental deste trabalho é introduzir e analisar uma nova classe de modelos de valores inteiros com evolução para a média condicional análoga à considerada em [25] mas em que se considera associada uma família abrangente de leis condicionais, nomeadamente a das leis infinitamente divisíveis discretas com suporte em $\mathbb{N}_0$, inflacionadas (ou não) em zero. Consideramos então uma família de leis condicionais que, na sua forma mais geral, podem ser interpretadas como misturas de uma lei de Dirac com uma qualquer lei discreta infinitamente divisível, sendo a sua especificação feita através da função característica. Em consequência da equivalência, no conjunto das leis discretas com suporte $\mathbb{N}_0$, entre leis infinitamente divisíveis e leis de Poisson compostas, este novo modelo denomina-se modelo GARCH de valor inteiro Poisson Composto inflacionado em zero (abreviadamente ZICP-INGARCH).

Para além de não se limitar a considerar como lei condicional uma lei específica, este modelo tem como principal vantagem unificar e alargar significativamente a família de processos estocásticos de valores inteiros. Destaca-se que é possível evidenciar novos modelos com leis condicionais com interesse nas aplicações práticas como, em particular, os modelos INGARCH Poisson geométrico e INGARCH Neyman tipo-A eventualmente inflacionados em zero, e também reencontrar contribuições recentes como os modelos INGARCH binomial negativo [81, 84], INGARCH Poisson [25, 84] e INGARCH Poisson generalizado [52, 82] eventualmente inflacionados em zero. Para além de ter a capacidade de descrever diferentes comportamentos distribucionais e, consequentemente, diferentes tipos de heteroscedasticidade condicional, o modelo ZICP-INGARCH consegue incorporar outros factos estilizados muito associados a séries de contagem, nomeadamente a sobredispersão e a elevada ocorrência de zeros. A análise probabilista destes modelos, no que diz respeito em particular ao desenvolvimento de condições necessárias e suficientes de estacionaridade (de primeira ordem, forte e fraca) e ergodicidade e também de existência de momentos de ordem elevada, é o objeto principal deste estudo. São ainda determinados estimadores para os parâmetros do modelo seguindo uma metodologia em duas etapas que envolve o método dos mínimos quadrados e o dos momentos.

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Chapter 1

Introduction

"*God made the integers, all the rest is the work of man.*" - Kronecker

A time series is a collection of observations made sequentially through time. The list of areas in which time series are studied is endless. Examples include meteorology (e.g., the temperature at a particular location at noon on successive days), electricity (e.g., electricity prices in a particular country for successive one-hour periods), or tourism (e.g., the monthly number of tourist arrivals in a certain city). The main goal of time series analysis is to develop mathematical models that enable plausible description of the phenomena, allowing to understand its past and to predict its future behavior.

1.1 Count time series

Until the end of the seventies, the studies of time series models were dominated by real-valued stochastic processes. However, many authors have underlined that such models do not give an adequate answer for integer-valued time series. For instance, when we deal with low dimension samples disregarding the nature of the data leads, in general, to senseless results as the asymptotic behavior of the corresponding statistical parameters or distributions is not available ([27]).

Thus, the investigation of appropriate methodologies for integer-valued time series has attracted much attention in the last years, also motivated by its importance and common occurrence in various contexts and scientific fields. In particular, the interest in nonnegative integer-valued time series, or count time series, has been growing since integer-valued time series often arise as counts of events. The hourly number of visits to a web site, the daily number of hospital patient admissions, the monthly number of claims reported by an insurance company, the yearly number of plants in a region or the annual number of couples who marry in Portugal are some examples of count time series.

This type of data exhibits certain empirical characteristics which are crucial for correct model specification and consequent estimation and forecasting. The variance greater than the mean (or overdispersion) and an excess of zeros (or zero inflation) are commonly observed in count time series being its modeling of great interest for many researchers. The reasons frequently reported in literature for such overdispersion are the presence of positive correlation between the monitored events or a variation in the probability of the monitored events (see Weiß [77] and references therein).

The observed overdispersion may also be the result of excess of zeros in the count data. Such data sets are abundant in many disciplines, including econometrics, environmental sciences, species abundance, medical, and manufacturing applications ([67]). This potential cause of overdispersion is of great interest because zero counts frequently have special status. First, the zeros may be true values concerning the absence of the event of interest (e.g. no pregnancies, no diseases, no alcohol consumption and no victimizations). These are called expected or sampling zeros. Second, some of the zeros may reflect those individuals who produce always a zero in the event of interest. For example, it is reasonable to assume that among the women who have not been pregnant, at least a few of them are simply unable to get pregnant; an individual may have no disease response because of immunity or resistance to the disease; a student may never drink alcohol for health, religious, or legal reasons. These zeros are inevitable and are called structural zeros. Finally, the zeros may be the result of underreporting of the occurrence of the event or they may be due to design, survey or observer errors. For example, respondents may not report victimizations due to forgetfulness or social desirability. It has also been noted that certain crimes go unreported to the police such as victimless crimes (e.g. drug offenses, gambling, and prostitution), intimate partner violence, and minor offenses in general ([62]). Zuur et al. [87], about the bird abundances in forest patches, referred that a design error can be obtained from sampling for a very short time period, sampling in the wrong season, or sampling in a small area. Sometimes these zeros are called false zeros. The production of structural zeros or the underreporting events can be conceptualized as sources of zero inflation. Perumean-Chaney et al. [62] used simulations to assess the importance of accounting for zero inflation and the consequences of its misspecifying in the statistical model.

To illustrate the importance of taking into account the characteristics referred above we consider a time series that represents counts of hours in a day in which the prices of electricity for Portugal and Spain are different. OMIE is the company that manages the wholesale electricity market (referred to as cash or "spot") on the Iberian Peninsula. Electricity prices in Europe are set on a daily basis (every day of the year) at 12 noon, for the twenty-four hours of the following day, known as daily market. The market splitting is the mechanism used for setting the price of electricity on the daily market. When the price of electricity is the same in Portugal and Spain, which corresponds to the desired situation, it means that the integration of the Iberian market is working properly. ([1])

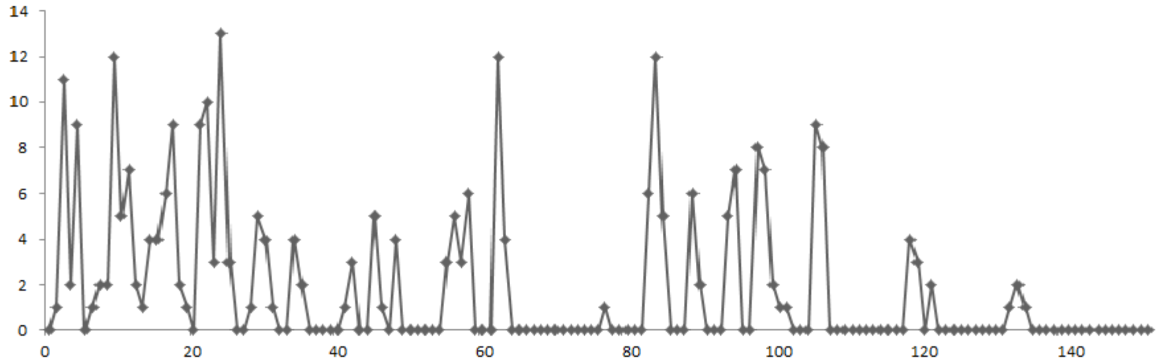

Fig. 1.1 Daily number of hours in which the price of electricity of Portugal and Spain are different.

[1]This information was taken from the OMIE site (http://www.omie.es).

The data presented in Figure 1.1 consists of 153 observations, starting from April 2013 and ending in August 2013. Empirical mean and variance of the data are 1.8039 and 9.4481, respectively. There are 92 zeros which corresponds to 60.13% of the series. Thus, this series presents a large proportion of zeros, as well as evidence of overdispersion. Let us observe that this series exhibits also characteristics of heteroscedasticity.

1.2 A review in count data literature

It is not possible to list here all the integer-valued models available in the literature. We present a sample of what has been developed hoping to illustrate the great interest in this subject in recent years. Many of the proposed integer-valued models take as reference the modeling by the real-valued stochastic processes, namely the autoregressive moving average (or briefly, ARMA) evolution.

One of these approaches was proposed by Jacobs and Lewis [43, 44] developing a discrete ARMA (DARMA) model using a mixture of a sequence of independent and identically distributed (i.i.d.) discrete random variables. Another way to obtain models for integer-valued data consists in replacing the usual multiplication in the standard ARMA models by a random operator, which preserves the discreteness of the process, denominated thinning operator. This operator was introduced as the binomial thinning and leads to the family of integer-valued ARMA (INARMA) models. Although originally introduced by Steutel and van Harn [72] for theoretical purposes, adding to its intuitive interpretation and mathematical elegance the fact that it has similar properties with the scalar multiplication turned it quite popular. The first INARMA model, the INAR(p) model, was proposed by McKenzie [54] and Al-Osh and Alzaid [4] for the case $p = 1$, and it has been developed by several authors (e.g., [6], [20], [70] and [71]). Therefore, several alternative thinning concepts have been developed as the signed thinning or the generalized thinning, yielding the SINAR(p) model [49] and the GINAR(p) model [30], respectively. The first INMA(q) models have been introduced by Al-Osh and Alzaid [5] and McKenzie [55] and more recent approaches may be found in [11] and [76]. For a recent review of a broad variety of such thinning operations as well as how they can be applied to define ARMA-like processes we refer, for example, Weiß [76] and Turkman et al. [74, chapter 5].

Alternatively to the thinning operation, Kachour and Yao [48] used the rounding operator to the nearest integer to introduce the called p-th order rounded integer-valued autoregressive (RINAR(p)) model. One of the advantages of this model is the fact that it can be used to analyse a time series with negative values, a situation also covered by the SINAR(p) model. Recently, Jadi et al. [45] studied the INAR(1) process with zero-inflated Poisson innovations and Kachour [47] proposed a modified and generalized version of the RINAR(p) model. Some integer-valued bilinear models have been also introduced by Doukhan et al. [18] and Drost et al. [19].

As in real-valued modeling, the introduction of conditionally heteroscedastic models seems to be very useful in many important situations. To take into account this feature, Heinen [40] defined an autoregressive conditional Poisson (ACP) model by adapting the autoregressive conditional duration model of Engle and Russell [23] to the integer-valued case, assuming a conditional Poisson distribution. Because of its analogy with the standard GARCH model introduced by Bollerslev [10] in 1986, Ferland et al. [25] suggested to denominate these models as integer-valued GARCH (INGARCH, hereafter) processes. Specifically, they defined the INGARCH model as a process $X = (X_t, t \in \mathbb{Z})$ such that

$$\begin{cases} X_t|\underline{X}_{t-1} : \mathscr{P}(\lambda_t), & \forall t \in \mathbb{Z}, \\ \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k}, \end{cases} \tag{1.1}$$

with $\alpha_0 > 0$, $\alpha_j \geq 0$, $\beta_k \geq 0$, $j = 1,...,p$, $k = 1,...,q$, $p \geq 1$, $q \geq 1$, $\underline{X}_{t-1}$ the $\sigma-$field generated by $\{X_{t-1}, X_{t-2}, \ldots\}$ and where $\mathscr{P}(\lambda)$ represents the Poisson distribution with parameter λ. If $q = 1$ and $\beta_1 = 0$, the INGARCH(p,q) model is simply denoted by INARCH(p).

This model has already received considerable study in the literature. In particular, it has been presented by Heinen [40] a first-order stationarity condition of the model for any orders p and q, and the corresponding variance and autocorrelation function for the particular case $p = q = 1$. Ferland et al. [25] extended the studies of this model establishing a condition for the existence of a strictly stationary process which has finite first and second-order moments and deduced the maximum likelihood parameters estimators. They also stated a condition under which all moments of an INGARCH$(1,1)$ model are finite. Fokianos et al. [26] considered likelihood-based inference when $p = q = 1$ using a perturbed version of the model. Weiß [77] derived a set of equations from which the variance and the autocorrelation function of the general model can be obtained. Neumann [56], Davis and Liu [16] and Christou and Fokianos [13] discussed some aspects related to the ergodicity. For the INARCH(p) model, Zhu and Wang [86] derived conditional weighted least squares estimators of the parameters and presented a test for conditional heteroscedasticity. Given the simple structure and the practical relevance of the INARCH(1) process, Weiß [77, 78, 79] studied its properties in more detail. He characterized the stationary marginal distribution in terms of its cumulants, showed how to approximate its marginal process distribution via the Poisson-Charlier expansion and calculated its higher-order moments and jumps. He also provided a conditional least squares approach for the estimation of its two parameters and constructed various simultaneous confidence regions.

Although the INGARCH model had been applied to several fields and appears to provide an adequate framework for modeling overdispersed count time series data with conditional heteroscedasticity, some authors pointed out that one of its limitations is to have the conditional mean equal to the conditional variance. For instance, Zhu [81] referred that this restriction of the model can lead to poor performance in the existence of potential extreme observations. In order to address this issue and to improve the model, some authors proposed to replace the Poisson distribution by other discrete ones.

Based on the double-Poisson (DP) distribution ([21]), Heinen [40] introduced two versions of an INGARCH$(1,1)$ model and Grahramani and Thavaneswaran [37] extended its results to higher orders. This DP-INGARCH(p,q) model is difficult to be utilized because of the intractability of a normalizing constant and moments. On the other hand, Zhu [81] used the negative binomial distribution instead of the Poisson to introduce the NB-INGARCH(p,q) model. The numerical results obtained in the study of the monthly counts of poliomyelitis cases in the United States from 1970 to 1983 indicated that the proposed approach performs better than the previously referred Poisson and double-Poisson model-based methods. Other alternatives proposed by Zhu were INGARCH models based on the generalized Poisson and the Conway-Maxwell Poisson ([69]) distributions namely, the GP-INGARCH

([82]) and the COM-Poisson INGARCH ([83]) models. Motivated by the zero inflation phenomenon, Zhu [84] also recently introduced the ZIP-INGARCH and the ZINB-INGARCH models and Lee et al. [52] the ZIGP AR model, replacing the conditional Poisson distribution by the zero-inflated Poisson, zero-inflated negative binomial and zero-inflated generalized Poisson distributions, respectively. The analysis of the weekly dengue cases in Singapore from year 2001 to 2010 encouraged Xu et al. [80] to propose a more general model, the DINARCH (which includes as special cases the INARCH, DP-INARCH and GP-INARCH models), where the conditional mean of X_t given its past is assumed to satisfy the second equation of (1.1) with $q = 1$, $\beta_1 = 0$ and the ratio between the conditional variance and the conditional mean is constant. Let us observe that the DP-INGARCH, the GP-INGARCH, the DINARCH and the COM-Poisson INGARCH models referred above were proposed with the aim of capturing overdispersion and underdispersion in the same framework. In fact, the opposite phenomenon to the overdispersion, that is, the underdispersion (which means variance less than the mean) occurs less frequently but it may be encountered in some real situations (see [66] and references therein for some examples).

1.3 Overview of the Thesis

In this thesis, instead of specifying the discrete conditional distribution, we propose a wide class of integer-valued GARCH models which includes, as particular cases, some of the recent contributions referred above as well as new interesting models with practical potential. The study of the above INGARCH-type models, especially the Poisson INGARCH, NB-INGARCH, GP-INGARCH and NB-DINARCH models, showed us that there was a common fundamental basis between some of them: the conditional distribution is nonnegative integer-valued infinitely divisible and the evolution of the conditional mean satisfies, unless a scale factor, the second equation of (1.1).

The family of infinitely divisible distributions is huge and particularly important. Thus, the introduction of an INGARCH model with a conditional nonnegative integer-valued infinitely divisible distribution seems to be natural since it unifies the study of several models already introduced in literature and assures the enlargement of the class of the INGARCH models. The equivalence, in the discrete case, between the infinitely divisible distributions and the compound Poisson ones ([24]) allows us to define easily this new general model. Given the importance of this equivalence we decide to denominate the model as compound Poisson INGARCH (CP-INGARCH hereafter).

Due to the recent enthusiasm to the zero-inflated INGARCH models and in order to add the characteristic of zero inflation to the general class of models introduced, we extend it and we propose the Zero-Inflated Compound Poisson INGARCH model, denoted ZICP-INGARCH. This model is able to capture in the same framework characteristics of zero inflation and, in a general distributional context, different kinds of overdispersion and conditional heteroscedasticity.

After the Introduction, this Thesis is organized as follows:

In Chapter 2 we introduce the compound Poisson INGARCH model by means of the conditional characteristic function, as it is a closed-form of characterizing the class of discrete infinitely divisible laws. The wide range of this proposal is stressed referring the most important models recently studied and also presenting a general procedure to obtain new ones. In fact, we show the main nature of the processes that are solution of the model equations, namely the fact that they may be expressed as a Poissonian random sum of independent random variables with common discrete distribution. Among the discrete infinitely divisible laws with support in $\mathbb{N}_0$, we highlight the geometric Poisson and the Neyman type-A ones which allow us the introduction of new models: the geometric Poisson INGARCH and the Neyman type-A INGARCH. We point out the practical interest of these models as the associated conditional laws are particularly useful in various areas of application. The geometric Poisson distribution is, for instance, useful in the study of the traffic accident data and the Neyman type-A law is widely used in describing populations under the influence of contagion ([46]).

Chapter 3 is dedicated to the properties of stationarity and ergodicity of this new class of processes which leads us to impose some hypotheses which, as we will see, are not too restrictive. A very simple necessary and sufficient condition on the model coefficients of first-order stationarity is given. Imposing an assumption on the family of the conditional distributions, which do not exclude any of the particular important cases referred above, we also state a necessary and sufficient condition of weak stationarity by a new approach based on a vectorial state space representation of the process. This condition is illustrated by the study of some particular cases. The autocorrelation function of the CP-INGARCH(p,q) is deduced and, from the general closed-form expression of the m-th moment of a compound Poisson random variable, a necessary and sufficient condition ensuring finiteness of its moments is established in the case $p=q=1$. Finally, we finish the chapter presenting a strictly stationary and ergodic solution of the model in a wide subclass. The existence of such solution is guaranteed under the same simple condition of first-order stationarity.

Chapter 4 is focused on the CP-INARCH(1) model. Its importance and great practical relevance is supported by the applications already studied and reported using the model with particular conditional distributions: the monthly claims counts of workers in the heavy manufacturing industry [77], the weakly number of dengue cases in Singapore [80] or the monthly counts of poliomyelitis cases in the U.S. [81] are some examples of real data where the model performs well. We determine its moments and cumulants up to order 4 and deduce its skewness and kurtosis. A procedure to estimate the model parameters, without specifying the conditional law by its density probability function, is presented based on a two-step approach using the conditional least squares and moments estimation methods. We finish presenting a simulation study to examine its performance.

In Chapter 5 we add the characteristic of zero inflation to the family of conditional distributions defining the ZICP-INGARCH model. The chapter is dedicated to generalize some of the results stated previously in what concerns the properties of stationarity, the autocorrelation function, expressions for moments and cumulants and a condition ensuring the finiteness of the moments of the process.

Finally, conclusions and some suggestions for future research are presented.

Chapter 2

The compound Poisson integer-valued GARCH model

The aim of this chapter is to introduce a new class of integer-valued processes namely the compound Poisson integer-valued GARCH model. We start, in Section 2.1, by reviewing general concepts and results directly related to the infinitely divisible laws. In particular, we present a relation between discrete infinitely divisible and compound Poisson distributions. Using this relation, we define a new integer-valued GARCH model in Section 2.2, making explicit the conditional distribution by using the characteristic function of a compound Poisson law. In Section 2.3 we make an overview of important examples that can be included in this new framework.

2.1 Infinitely divisible distributions and their fundamental properties

Infinitely divisible distributions play an important role in varied problems of probability theory.

This concept was introduced by de Finetti [17] in 1929 in the context of the processes with stationary independent increments, and the most fundamental results were developed by Kolmogorov, Lévy and Khintchine in the thirties. In this section we define infinitely divisible distributions and describe their main properties. Then we present compound Poisson distributions and from the relation between them we introduce, in the next section, an integer-valued GARCH model with a discrete infinitely divisible conditional distribution with support in $\mathbb{N}_0$.

Definition 2.1 (Infinite divisibility) *A random variable X (or equivalently, the corresponding distribution function) is said to be infinitely divisible if for any positive integer n, there are i.i.d. random variables $Y_{n,j}$, $j = 1,...,n$, such that $X \stackrel{d}{=} Y_{n,1} + \ldots + Y_{n,n}$, where $\stackrel{d}{=}$ means "equal in distribution".*

The notion of infinite divisibility can also be introduced by means of the characteristic function. In fact, for an infinitely divisible distribution its characteristic function φ_X turns out to be, for every positive integer n, the n-th power of some characteristic function. This means that there exists, for every positive integer n, a characteristic function φ_n such that $\varphi_X(u) = [\varphi_n(u)]^n$, $u \in \mathbb{R}$. In this case, we say that φ_X is an infinitely divisible characteristic function. The function φ_n is uniquely determined by φ_X provided that one selects the principal branch of the n-th root.

Most well-known distributions are infinitely divisible. We give some examples in the following.

Example 2.1 **a)** *Let $X = a \in \mathbb{R}$, with probability* 1. *For every $n \in \mathbb{N}$, there are i.i.d. random variables $Y_{n,j}$, $j = 1,...,n$, with the distribution of $Y_{n,j}$ concentrated at $\frac{a}{n}$, such that X has the same distribution as $Y_{n,1} + \ldots + Y_{n,n}$. Therefore a degenerate distribution is infinitely divisible.*

b) *Let X have a Poisson distribution with mean $\lambda > 0$. In this case the characteristic function is $\varphi_X(u) = \exp\{\lambda(e^{iu} - 1)\}$, $u \in \mathbb{R}$, which is infinitely divisible since $\varphi_n(u) = \exp\{\lambda/n(e^{iu} - 1)\}$ is the characteristic function of the Poisson distribution with mean λ/n and $\varphi_X(u) = (\varphi_n(u))^n$.*

c) *Let X have the gamma distribution with parameters $(\alpha, \lambda) \in]0, +\infty[\times]0, +\infty[$. In this case the characteristic function is given by $\varphi_X(u) = (\lambda/(\lambda - iu))^{\alpha}$, $u \in \mathbb{R}$. Thus X is infinitely divisible since $\varphi_n(u) = (\lambda/(\lambda - iu))^{\alpha/n}$ is the characteristic function of the gamma distribution with parameters $(\frac{\alpha}{n}, \lambda)$ and $\varphi_X(u) = (\varphi_n(u))^n$. In particular, the exponential distribution with parameter λ (recovered when $\alpha = 1$) is also infinitely divisible.*

d) *Stable laws form a subclass of infinitely divisible distributions. A random variable X is called stable (or said to have a stable law) if for every $n \in \mathbb{N}$, there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that $X_1 + ... + X_n \overset{d}{=} a_n X + b_n$, where X_j, $j = 1,...,n$, are i.i.d. random variables with the same distribution as X. Examples of such laws are the normal, the Cauchy and the Levy distributions ([1]). By the definition of stability, for every $n \in \mathbb{N}$, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that $\frac{1}{a_n}(X_1 + ... + X_n - b_n) = \sum_{j=1}^{n} \frac{1}{a_n}\left(X_j - \frac{b_n}{n}\right)$ has the same law as X. But $\frac{1}{a_n}\left(X_j - \frac{b_n}{n}\right)$, $j = 1,...,n$, are i.i.d. random variables, and hence the stable law of X is infinitely divisible.*

e) *The class of the compound Poisson distributions is also infinitely divisible. Because of its importance in the study of infinite divisibility we discuss them in detail later.*

A random variable with a bounded support cannot be infinitely divisible unless it is a constant. This fact, proved in the following theorem, immediately excludes the uniform, the binomial and the beta distributions from the class of infinitely divisible distributions.

Theorem 2.1 *A non-degenerate bounded random variable is not infinitely divisible.*

Proof: We make the proof by contradiction. Let us suppose that X is an infinitely divisible random variable such that $|X| \leq a < \infty$, with probability 1 and non-degenerate. Then for every positive integer n, by Definition 2.1, there exist i.i.d. random variables $Y_{n,j}$, $j = 1,...,n$, with some distribution F_n such that X has the same distribution as $Y_{n,1} + ... + Y_{n,n}$. Since X takes values in the interval $[-a, a]$, the supremum of the support of F_n is at most a/n. This implies $V(Y_{n,1}) \leq E(Y_{n,1}^2) \leq (a/n)^2$ and hence

$$0 \leq V(X) = n\,V(Y_{n,1}) \leq n(a/n)^2 = a^2/n, \quad \forall n \in \mathbb{N},$$

that is, $0 \leq V(X) \leq \inf_{n \in \mathbb{N}} a^2/n = 0$. So X is a constant, which is the contradiction. ■

Also the laws with characteristic functions having zeros cannot be infinitely divisible. The next theorem shows this result and some other important properties of infinitely divisible distributions.

[1] For more information on stable laws see [58] and [73].

Theorem 2.2 *We have the following properties:*

(i) *The product of a finite number of infinitely divisible characteristic functions is also an infinitely divisible characteristic function. In particular, if φ is an infinitely divisible characteristic function then $\psi = |\varphi|^2$ is also infinitely divisible;*

(ii) *The characteristic function of an infinitely divisible distribution never vanishes;*

(iii) *The distribution function of the sum of a finite number of independent infinitely divisible random variables is itself infinitely divisible;*

(iv) *A distribution function which is the limit, in the sense of weak convergence, of a sequence of infinitely divisible distributions functions is itself infinitely divisible.*

Proof:

(i) Regarding the recurrence property, it is sufficient to consider the case of two characteristic functions. So, let θ and ϕ be infinitely divisible characteristic functions with $\theta(u) = [\theta_n(u)]^n$ and $\phi(u) = [\phi_n(u)]^n$, $u \in \mathbb{R}$, $\forall n \in \mathbb{N}$. The function $\varphi = \theta\phi$ is a characteristic function and

$$\varphi(u) = [\theta_n(u)]^n[\phi_n(u)]^n = [\theta_n(u)\phi_n(u)]^n = [\varphi_n(u)]^n, \quad u \in \mathbb{R},$$

for every $n \in \mathbb{N}$, where $\varphi_n = \theta_n\phi_n$ is a characteristic function. Hence, φ is infinitely divisible.

(ii) Let φ be an infinitely divisible characteristic function. From (i), $|\varphi|^2$ is also infinitely divisible and then, for any $n \in \mathbb{N}$, $\theta_n = |\varphi|^{\frac{2}{n}}$ is a characteristic function. But

$$\forall u \in \mathbb{R}, \quad \lim_{n\to\infty} \theta_n(u) = \lim_{n\to\infty} |\varphi(u)|^{\frac{2}{n}} = \begin{cases} 0, & \text{for } \{u: \ \varphi(u) = 0\} \\ 1, & \text{for } \{u: \ \varphi(u) \neq 0\}. \end{cases}$$

Since φ is uniformly continuous in $\mathbb{R}$ and $\varphi(0) = 1$, then $\varphi(u) \neq 0$ in a neighborhood of 0. Hence $\lim_{n\to\infty} \theta_n(u) = 1$ in a neighborhood of 0. By the Lévy Continuity Theorem ([2]), $\lim_{n\to\infty} \theta_n(u)$ is a characteristic function. Since its only possible values are 0 and 1, and since all characteristic functions are uniformly continuous in $\mathbb{R}$, it cannot be zero at any value of u.

(iii) As in (i), we just have to prove the statement for two random variables. Let φ_X and φ_Y be characteristic functions of X and Y, respectively, with X and Y independent infinitely divisible random variables. For each positive integer n, let θ_n and ϕ_n be characteristic functions which satisfy $\varphi_X(u) = [\theta_n(u)]^n$ and $\varphi_Y(u) = [\phi_n(u)]^n$, $u \in \mathbb{R}$. Then, from the independence,

$$\varphi_{X+Y}(u) = \varphi_X(u) \cdot \varphi_Y(u) = [\theta_n(u) \cdot \phi_n(u)]^n, \quad n \geq 1.$$

Since $\theta_n \cdot \phi_n$ is a characteristic function, the result follows.

[2] Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables with X_n having characteristic function ϕ_n. If X_n converges weakly (or in law) to X then $\lim_{n\to\infty} \phi_n(u) = \phi_X(u)$, $u \in \mathbb{R}$. Conversely if ϕ_n converges pointwise to a function ϕ which is continuous at 0, then ϕ is a characteristic function of a random variable X, and $X_n \to X$. For a proof see, e.g., [65, p. 304].

(iv) Let us suppose that the sequence $\left(F^{[k]}\right)$ of infinitely divisible distribution functions converges to the distribution function F, as $k \to \infty$. If $\varphi^{[k]}$ and φ are the characteristic functions of $F^{[k]}$ and F, respectively, then from the Lévy Continuity Theorem, $\lim_{k\to\infty} \varphi^{[k]}(u) = \varphi(u)$, $u \in \mathbb{R}$. By the condition of infinite divisibility, for every $n \in \mathbb{N}$, $\varphi_n^{[k]}(u) = [\varphi^{[k]}(u)]^{1/n}$ is a characteristic function and never vanishes for any $u \in \mathbb{R}$ (from (ii)). Thus, for any $n \in \mathbb{N}$,

$$\lim_{k\to\infty} \varphi_n^{[k]}(u) = \lim_{k\to\infty} [\varphi^{[k]}(u)]^{1/n} = [\varphi(u)]^{1/n} = \varphi_n(u), \ \ \forall u \in \mathbb{R}.$$

As φ is a characteristic function it follows from the Bochner's Theorem ([3]) that φ_n is a characteristic function. Since $\varphi(u) = [\varphi_n(u)]^n$, $u \in \mathbb{R}$, for every $n \in \mathbb{N}$, with φ_n a characteristic function, the proof is complete. ■

Remark 2.1 *We note that, generally, the converse of the statements of Theorem 2.2 are not true.*

*For instance, in **(ii)** let us consider the Bernoulli distribution with parameter* $p \neq \{0, \frac{1}{2}, 1\}$. *It is not infinitely divisible because its support is* $\{0,1\}$, *so bounded (Theorem 2.1). Its characteristic function is given by* $\varphi(u) = 1 - p + pe^{iu}$ *and has no real roots, except when* $p = \frac{1}{2}$. *In fact,*

$$\varphi(u) = 0 \ \Leftrightarrow \ \cos(u) + i\sin(u) = (p-1)/p$$

$$\Rightarrow \ \sin(u) = 0 \ \Leftrightarrow \ u = k\pi, \quad k \in \mathbb{Z},$$

$$\Rightarrow \ \cos(k\pi) = \begin{cases} -1, & \text{if } k \text{ odd} \\ \ \ 1, & \text{if } k \text{ even.} \end{cases}$$

Then $\varphi(u) = 0$ *only when* $p = \frac{1}{2}$, $u = k\pi$, *and k odd. So we have an example of a distribution with a characteristic function that has no real roots but it is not infinitely divisible.*

*For the statement **(i)** and **(iii)**, Gnedenko and Kolmogorov [31, p. 81] proved that* φ *such that*

$$\forall u \in \mathbb{R}, \quad \varphi(u) = \frac{1-\beta}{1+\alpha}\frac{1+\alpha e^{-iu}}{1-\beta e^{-iu}}, \ \ 0 < \alpha \leq \beta < 1,$$

and $\overline{\varphi}$ *are not infinitely divisible characteristic functions whereas* $|\varphi|^2$ *is the characteristic function of an infinitely divisible law, and so the converse of **(i)** fails. The same example illustrates the falseness of the converse of **(iii)**. In fact, considering X and Y i.i.d. random variables with characteristic function* φ, *the function* $\psi = \varphi_{X-Y} = |\varphi|^2$ *is infinitely divisible.*

*Finally, the Poisson distribution is the limit, in the sense of weak convergence, of a sequence of binomial distributions which proves that the converse of **(iv)** is false.*

Theorem 2.3 *If* φ *is the characteristic function of an infinitely divisible distribution function, then for every* $c > 0$ *the function* φ^c *is also a characteristic function.*

Proof: For $c = 1/n$, $n \in \mathbb{N}$, the result follows from the definition of infinite divisibility. Since the product of characteristic functions is again a characteristic function the statement holds for any rational

[3][53, p. 60]: A continuous function $\phi : \mathbb{R} \to \mathbb{C}$ with $\phi(0) = 1$ is a characteristic function if and only if ϕ is positive definite, i.e., for all $n \in \mathbb{N}$, $\sum_{j=1}^{n}\sum_{k=1}^{n} \phi(u_j - u_k) z_j \bar{z}_k \ \geq 0, \quad u_j \in \mathbb{R},\ z_k \in \mathbb{C}$.

number $c > 0$. Finally, for an irrational number $c > 0$ the function $[\varphi]^c$ can be approximated uniformly in every finite interval by the function $[\varphi]^{c_1^{(n)}}$, where $c_1^{(n)}$ is a sequence of rational numbers that goes to c. Then from the statement **(iv)** of Theorem 2.2 the result holds. ■

Now, we present the notion of compound Poisson random variable and some of its properties. This class of distributions is known in the literature under a wide variety of names such as stuttering Poisson or Poisson-stopped sum distributions [46, sections 4.11 and 9.3] and it includes several well-known laws as Poisson, negative binomial or generalized Poisson.

Definition 2.2 (Compound Poisson distribution) *A random variable X is said to be compound Poisson if $X \stackrel{d}{=} Y_1 + ... + Y_N$, where N is a Poisson random variable with parameter $\lambda > 0$, and $\{Y_j\}_{j\geq 1}$ are i.i.d. random variables which are also independent of N. The distribution of Y_j is called compounding or secondary distribution and we assume X equal to zero if $N = 0$ ([4]).*

The following theorem provides a characterization of random variables with compound Poisson distribution via their characteristic functions. This is a simple, but useful result.

Theorem 2.4 *Let Y_1, Y_2, ... be i.i.d. random variables having common characteristic function φ_Y and let N be a Poisson random variable with parameter λ, independent of $\{Y_j\}_{j\geq 1}$. The characteristic function of the compound Poisson random variable $X = Y_1 + ... + Y_N$ is given by*

$$\varphi_X(u) = \exp\{\lambda(\varphi_Y(u) - 1)\}, \quad u \in \mathbb{R}.$$

Proof: Let us establish the result in a more general setting. Let N be a non-negative integer-valued random variable independent of $\{Y_j\}_{j\geq 1}$ and having characteristic function φ_N. We have

$$\varphi_X(u) = \sum_{n=0}^{\infty} E\left[e^{iuX}\,|N=n\right]P(N=n) = \sum_{n=0}^{\infty} E\left[e^{iu(Y_1+...+Y_N)}\,|N=n\right]P(N=n)$$

$$= \sum_{n=0}^{\infty}\varphi_Y^n(u)P(N=n) = \sum_{n=0}^{\infty} e^{in(-i\ln\varphi_Y(u))}P(N=n) = E[e^{iN(-i\ln\varphi_Y(u))}] = \varphi_N(-i\ln\varphi_Y(u)),$$

in view of the assumptions of independence. Therefore, when N is a Poisson random variable with parameter λ, $\varphi_N(v) = \exp\{\lambda(e^{iv} - 1)\}$, $v \in \mathbb{R}$, and the result holds. ■

Following this approach we may show that the Poisson distribution with parameter λ is itself a compound Poisson law and arises considering the random variables Y_1, Y_2,... Dirac distributed on $\{1\}$.

Alternatively, when X is a nonnegative integer-valued discrete random variable, we can characterize the distribution of X using its probability generating function, namely $g_X(z) = E(z^X) = e^{\lambda(g(z)-1)}$, for any $z \in \mathbb{C}$ such that the expectation is finite, using the same conditioning technique as used in Theorem 2.4, where g represents the common probability generating function of Y_j, $j = 1, 2, ...$

[4] We note that the distribution of Y_j may be continuous. Despite this, we will see in the following that we are only interested in discrete compound Poisson distributions so Y_j will be a discrete random variable. A more general definition is given when N is a nonnegative integer-valued random variable. In this case, X is said to have a compound distribution and the distribution of N is called counting or compounded distribution.

Remark 2.2 *From Theorem* 2.4*, we can deduce that when $E(Y_1^2) < \infty$, the mean and the variance of X are given by $E(X) = -i\varphi_X'(0) = \lambda E(Y_1)$ and $V(X) = -\varphi_X''(0) - \lambda^2[E(Y_1)]^2 = \lambda E(Y_1^2)$. This means that, except when the compounding distribution is the Dirac law on $\{0\}$ or the Dirac law on $\{1\}$ (hence X is Poisson distributed), all the nonnegative integer-valued compound Poisson distributions are overdispersed, i.e., have variance larger than the mean, since*

$$\frac{V(X)}{E(X)} = \frac{E(Y_1^2)}{E(Y_1)} = 1 + \frac{E[Y_1(Y_1-1)]}{E(Y_1)} > 1.$$

Remark 2.3 *The moments of any order $m \geq 1$ of a compound Poisson distribution can be calculated using the closed-form formulae provided by Grubbström and Tang [39]. They stated that for a compound Poisson random variable X its m-th moment is given by*

$$E(X^m) = \sum_{r=0}^{m} \frac{1}{r!} E\left[\prod_{k=0}^{r-1}(N-k)\right] \left\{ \sum_{k=0}^{r} \begin{pmatrix} r \\ k \end{pmatrix} (-1)^{r-k} E\left[\left(\sum_{j=1}^{k} Y_j\right)^m\right]\right\}, \qquad (2.1)$$

interpreting $\sum_{j=1}^{k} Y_j$ to be zero for $k = 0$. Since N follows a Poisson distribution with parameter λ its r-th descending factorial moment is $E[\prod_{k=0}^{r-1}(N-k)] = \lambda^r$, for $r \geq 1$ [46, p. 161]. ([5])

We now give some examples of compound Poisson laws which are relevant in the following: the negative binomial, the generalized Poisson, the Neyman type-A and the geometric Poisson distributions. More examples can be found in [46, chap. 9].

Example 2.2 (Negative binomial distribution) *Given $r \in \mathbb{N}$ and $p \in]0,1[$, let $\{Y_j\}_{j\geq 1}$ be a sequence of i.i.d. logarithmic random variables with parameter $1-p$, i.e., with probability mass function*

$$P(Y_j = y) = -\frac{(1-p)^y}{y \, \ln p}, \quad y = 1, 2, \ldots,$$

and let N be a random variable independent of $\{Y_j\}_{j\geq 1}$ and having Poisson law with mean $-r\ln p$. Then $X = Y_1 + \ldots + Y_N$ follows a negative binomial (NB for brevity) law with parameters (r, p), i.e.,

$$P(X = x) = \begin{pmatrix} x+r-1 \\ r-1 \end{pmatrix} p^r (1-p)^x, \quad x = 0, 1, \ldots$$

Indeed, from Theorem 2.4*, the characteristic function of X has the form*

$$\varphi_X(u) = \exp\{-r[\ln(1-(1-p)e^{iu}) - \ln p]\} = \left(\frac{p}{1-(1-p)e^{iu}}\right)^r,$$

since $\lambda = -r\ln p$ and $\varphi(u) = \ln(1-(1-p)e^{iu})/\ln p$ (which is the characteristic function of the logarithmic distribution). So, the NB(r,p) distribution (and also the geometric(p) as particular case when $r = 1$) belongs to the class of compound Poisson distributions. We observe that

$$E(Y_1) = -\frac{1-p}{p\ln p}, \quad E(Y_1^2) = -\frac{1-p}{p^2\ln p}, \quad E(X) = \frac{r(1-p)}{p}, \quad and \ V(X) = \frac{r(1-p)}{p^2}.$$

[5] In fact, Grubbström and Tang [39] proved that formula (2.1) is valid for any compound distribution.

Example 2.3 (Generalized Poisson distribution) *For $0 < \kappa < 1$ and $\lambda > 0$, let $\{Y_j\}_{j\geq 1}$ be a sequence of i.i.d. random variables distributed according to the Borel law with parameter κ, i.e.,*

$$P(Y_j = y) = \frac{(y\kappa)^{y-1}e^{-\kappa y}}{y!}, \quad y = 1, 2, \ldots,$$

and let N be a Poisson random variable independent of $\{Y_j\}_{j\geq 1}$ with mean λ. Then $X = Y_1 + ... + Y_N$ has a generalized Poisson (GP for brevity) distribution with parameters (λ, κ), i.e.,

$$P(X = x) = \frac{\lambda(\lambda + \kappa x)^{x-1}e^{-(\lambda+\kappa x)}}{x!}, \quad x = 0, 1, ...$$

In fact, for $0 < \kappa < 1$, the probability generating function of X is given by $g_X(z) = e^{\lambda(z-1)}$ where $z = ue^{\kappa(z-1)}$ represents the probability generating function of the Borel law ([6]). In this case we have

$$E(Y_1) = \frac{1}{1-\kappa}, \quad E(Y_1^2) = \frac{1}{(1-\kappa)^3}, \quad E(X) = \frac{\lambda}{1-\kappa}, \quad \text{and} \quad V(X) = \frac{\lambda}{(1-\kappa)^3}.$$

The next two examples will be fundamental in Section 2.3 because they allow us to introduce some new particular integer-valued GARCH models. Because of their importance in our study and since they involve less well-known distributions we give them more attention.

Example 2.4 (Neyman type-A or Poisson Poisson distribution) *The Neyman type-A law was developed by Neyman [57] to describe the distribution of larvae in experimental field plots. It is widely used in order to describe populations under the influence of contagion, e.g., entomology, accidents, and bacteriology. This is a compound Poisson law with a Poisson compounding distribution.*

In fact, given $\lambda > 0$, $\phi > 0$, let $\{Y_j\}_{j\geq 1}$ be a sequence of i.i.d. Poisson random variables with mean ϕ and let N be a Poisson random variable with mean λ and independent of $\{Y_j\}_{j\geq 1}$. The random variable $X = Y_1 + ... + Y_N$ follows a Neyman type-A (NTA for brevity) distribution with parameters (λ, ϕ), i.e., its characteristic function is given by

$$\varphi_X(u) = \exp\left\{\lambda\left[\exp\left(\phi(e^{iu} - 1)\right) - 1\right]\right\},$$

and its probability mass function can be expressed as

$$P(X = x) = \frac{\exp\left\{-\lambda + \lambda e^{-\phi}\right\}\phi^x}{x!}\sum_{j=1}^{x} S(x, j)\lambda^j e^{-j\phi}, \quad x = 0, 1, ...,$$

where the coefficient $S(x, j)$ represents the Stirling number of the second kind ([7]).

We observe that $E(X) = \lambda\phi$ and $V(X) = \lambda\phi(1+\phi)$.

[6] We note that the general definition of the GP distribution allows the parameter κ to take negative values, namely $\max(-1, -\lambda/m) \leq \kappa < 1$, with m (≥ 4) the largest positive integer for which $\lambda + \kappa m > 0$. Despite of this, only for nonnegative values of κ (when $\kappa = 0$, the GP law reduces to the usual Poisson distribution with mean λ) we can include this distribution in the class of the compound Poisson distributions. For more details concerning the Borel and the generalized Poisson distributions see, e.g., [14], pp. 158-160 and Chap. 9.

[7] We note that $S(0,0) = 1$, $S(x,0) = 0$, for $x \neq 0$, $S(x,1) = S(x,x) = 1$, $S(x, j) = 0$ if $j > x$ and these numbers satisfy the recurrence relation $S(x, j) = S(x-1, j-1) + jS(x-1, j)$. For details, see e.g. [1].

From [46, Section 9.6], when ϕ is small X is approximately distributed as a Poisson variable with mean $\lambda\phi$ and if λ is small X is approximately distributed as a zero-inflated Poisson (ZIP for brevity, see Section 5.1) variable with parameters $(\phi, 1-\lambda)$. We illustrate these properties with the graph present in Figure 2.1, where we represent the probability mass function (p.m.f.) of the NTA distribution considering different values for the parameters (λ, ϕ).

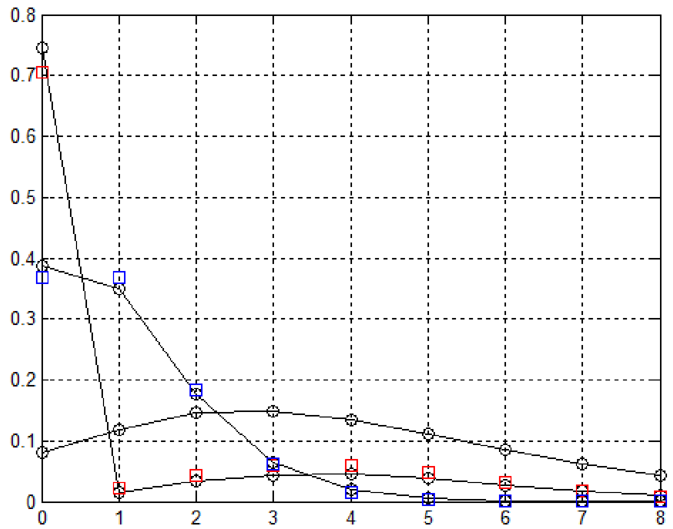

Fig. 2.1 Probability mass function of $X \sim NTA(\lambda, \phi)$. From the top to the bottom in abscissa $x = 2$, $(\lambda, \phi) = (10, 0.1)$ (approximately Poisson(1), with p.m.f. represented in blue), $(4, 1)$, $(0.3, 4)$ (approximately ZIP$(4, 0.7)$, with p.m.f. represented in red).

Example 2.5 (Geometric Poisson or Pólya-Aeppli distribution) *The geometric Poisson law was described by Pólya [63] and has been applied to a variety of biological data, in the control of defects in software or in traffic accident data ([8]). This is an example of a compound Poisson law with a geometric compounding distribution. Indeed, given $\lambda > 0$ and $p \in]0, 1[$ let $\{Y_j\}_{j\geq 1}$ be a sequence of i.i.d. geometric random variables with parameter p, i.e., with probability mass function*

$$P(Y_j = y) = p(1-p)^y, \quad y = 0, 1, \ldots,$$

and let N be a Poisson random variable with parameter $\frac{\lambda}{1-p}$ and independent of $\{Y_j\}_{j\geq 1}$. The random variable $X = Y_1 + \ldots + Y_N$ follows a geometric Poisson (GEOMP for brevity) distribution with parameters (λ, p), i.e., its characteristic function is given by

$$\varphi_X(u) = \exp\left\{\lambda\left(\frac{e^{iu}-1}{1-(1-p)e^{iu}}\right)\right\},$$

and its probability mass function can be expressed as

$$P(X = 0) = e^{-\lambda},$$

[8]For more details see, e.g., [46, Section 9.7] or [59].

$$P(X=x)=\sum_{n=1}^{x} e^{-\lambda}\frac{\lambda^n}{n!}\left(\begin{array}{c} x-1 \\ n-1 \end{array}\right) p^n(1-p)^{x-n}, \quad x=1,2,...$$

We observe that $E(X)=\lambda/p$, $V(X)=\lambda(2-p)/p^2$ and when $p=1$ the geometric Poisson distribution reduces to the Poisson law with parameter λ. In Figure 2.2, we represent the probability mass function of the GEOMP distribution considering different values for the parameters (λ,p).

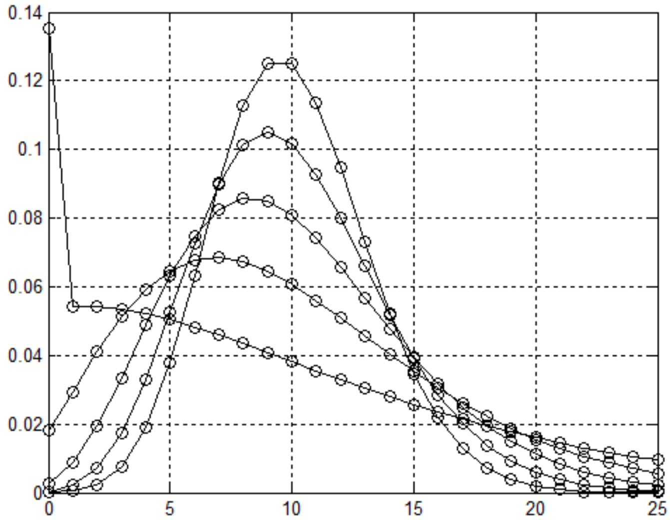

Fig. 2.2 Probability mass function of $X \sim GEOMP(\lambda,p)$ with $\lambda/p=10$ and p taking several values. From the top to the bottom in abscissa $x=10$, p equals 1, 0.8, 0.6, 0.4, 0.2.

Let us now prove that a compound Poisson distribution is infinitely divisible. From Theorem 2.4, we know that if X is a compound Poisson random variable then its characteristic function has the form $\varphi_X(u)=\exp\{\lambda(\varphi(u)-1)\}$, $u\in\mathbb{R}$, for some $\lambda>0$ and φ a characteristic function. Then, for any $n\in\mathbb{N}$, φ_X can be represented as $\varphi_X(u)=[\varphi_n(u)]^n$ with $\varphi_n(u)=\exp\left\{\frac{\lambda}{n}(\varphi(u)-1)\right\}$ the characteristic function of the random variable $Y_1+...+Y_{N_n}$, where the random variable N_n has the Poisson distribution with parameter λ/n and is independent of the random variables Y_1, Y_2,... Therefore compound Poisson distributions belong to the class of infinitely divisible distributions.

Although the converse of this statement is not true, all the infinitely divisible distributions may be obtained from the family of compound Poisson as stated in the following theorem whose proof can be found in Gnedenko and Kolmogorov [31, p. 74].

Theorem 2.5 *The class of infinitely divisible distributions coincides with the class of the compound Poisson laws and of limits of these laws in the sense of the weak convergence.*

As, in this work, we are developing models for the study of time series of counts, we now concentrate our attention on the set of the nonnegative integer-valued discrete infinitely divisible laws.

Generally a nonnegative integer-valued discrete random variable X is called infinitely divisible when, according to Definition 2.1, $Y_{n,j}$, $j=1,...,n$, are i.i.d. nonnegative integer-valued discrete random variables as well. Sometimes, to make a clear distinction, X is said to be "discretely infinite divisible" ([38, p. 26]). For instance, the discrete random variable $X = 1$, with probability 1, is infinitely divisible (Example 2.1a)) but not discretely infinite divisible since, for any $n \in \mathbb{N}$, $Y_{n,j} = \frac{1}{n}$ with probability 1. In the next theorem, we present necessary and sufficient conditions that allow us to distinguish the class of the infinitely divisible laws on $\mathbb{N}_0$ whose components are integer-valued from those whose components are real-valued. We will see that the extra requirement of integer-valued components is equivalent to the condition that $P(X = 0) > 0$, which is also equivalent to the statement that the support of X coincides with that of its components $Y_{n,j}$, $j = 1,...,n$.

Theorem 2.6 *Let X be a nonnegative integer-valued infinitely divisible random variable.*
The following statements are equivalent:

(i) *The support of X coincides with that of each component $Y_{n,j}$, $j = 1,...,n$, for every $n \in \mathbb{N}$;*

(ii) *For every $n \in \mathbb{N}$, $Y_{n,j}$, $j = 1,...,n$, are nonnegative integer-valued;*

(iii) $P(X = 0) > 0$.

Proof:

(i) ⇒ (ii) It is obvious from the definition of X.

(ii) ⇒ (iii) Under the assumptions on X, for all $n \in \mathbb{N}$, there are i.i.d. nonnegative integer-valued random variables $Y_{n,j}$, $j = 1,...,n$, such that $X \stackrel{d}{=} Y_{n,1} + \ldots + Y_{n,n}$. We make the proof by contradiction. So, let us suppose that $P(X = 0) = 0$ and let $k > 0$ be the smallest integer such that $P(X = k) > 0$. Since $P(X = 0) = [P(Y_{n,j} = 0)]^n$, for every $n \in \mathbb{N}$ then $P(Y_{n,j} = 0) = 0$ which implies $Y_{n,j} \geq 1$ almost surely (a.s.). So, for a fixed n such that $n < k$ we cannot have the representation $X \stackrel{d}{=} Y_{n,1} + \ldots + Y_{n,n}$ with $Y_{n,j}$, $j = 1,...,n$, nonnegative integer-valued, which is the contradiction.

(iii) ⇒ (i) For any $x > 0$,

$$P(X = x) \geq P\left(\bigcup_{j=1}^{n} \{Y_{n,1} = 0, \ldots,\ Y_{n,j-1} = 0,\ Y_{n,j} = x,\ Y_{n,j+1} = 0, \ldots,\ Y_{n,n} = 0\}\right)$$

$$= nP(Y_{n,j} = x)[P(Y_{n,j} = 0)]^{n-1} \geq nP(Y_{n,j} = x)P(X = 0), \quad \forall n,$$

since $[P(Y_{n,j} = 0)]^{n-1} \geq [P(Y_{n,j} = 0)]^n = P(X = 0)$. So in the presence of the hypothesis $P(X = 0) > 0$ any possible value of $Y_{n,j}$, i.e., a value taken with positive probability, is also a possible value of X. To prove the converse inclusion see, for instance, [42, Lemma 1]. ■

The following theorem characterizes all nonnegative integer-valued discrete infinite divisible random variables and is the key for the construction of the new class of models that we propose. To prove it we need an auxiliary lemma whose proof can be found at [28, p. 73].

Lemma 2.1 (Characterization of probability generating functions) *A real-valued function g defined on $[0,1]$ is a probability generating function of a nonnegative integer-valued random variable if and only if $g(1) = 1$, $g(1^-) \leq 1$, $g(0) \geq 0$, and all derivatives of g are finite and nonnegative on $[0,1[$.*

Theorem 2.7 *Let X be a nonnegative integer-valued random variable such that* $0 < P(X=0) < 1$. *Then, X is infinitely divisible if and only if X has a compound Poisson distribution.*

Proof: We need only to prove the necessary condition of infinite divisibility. So, let g be the probability generating function of X which is infinitely divisible. We note that $0 < g(0) < 1$. Then, for every $n \in \mathbb{N}$, $g_n = g^{1/n}$ is a probability generating function. Let us define the function h in the form

$$h_n(z) = \frac{g_n(z) - g_n(0)}{1 - g_n(0)} = 1 - \frac{1 - g(z)^{1/n}}{1 - g(0)^{1/n}}.$$

Since $h_n(0) = 0$, $h_n(1) = 1$ (because $g_n(1) = 1$), $h_n(1^-) \leq 1$ and there exist all the derivatives of h_n, namely $h_n^{(k)}(z) = \frac{g_n^{(k)}(z)}{1-g_n(0)}$, which are nonnegative in $[0,1[$ because $1 - g_n(0) > 0$, then from the previous lemma h_n is a probability generating function.

Letting $n \to \infty$ and using the fact that $\ln \alpha = \lim_{n\to\infty} n(\alpha^{1/n} - 1)$ for $\alpha > 0$, we have

$$\lim_{n\to\infty} h_n(z) = 1 - \lim_{n\to\infty} \frac{n(-1+g(z)^{1/n})}{n(-1+g(0)^{1/n})} = 1 - \frac{\ln g(z)}{\ln g(0)} = h(z),$$

and then from the continuity theorem ([9]) we conclude that the function h is a probability generating function. Then, it follows that $g(z) = \exp\{\lambda(h(z)-1)\}$ with $\lambda = -\ln g(0)$, i.e., g is a probability generating function of a compound Poisson distribution, which concludes the proof. ∎

Let us note that any nonnegative integer-valued compound Poisson random variable X assumes the value zero with positive probability, namely, $P(X=0) = g_X(0) = e^{-\lambda(1-g(0))}$, which is in accordance with the hypothesis of the previous theorems.

2.2 The definition of the CP-INGARCH model

Let $X = (X_t, t \in \mathbb{Z})$ be a nonnegative integer-valued stochastic process and, for any $t \in \mathbb{Z}$, let $\underline{X}_{t-1}$ be the σ-field generated by $\{X_{t-s}, s \geq 1\}$.

Definition 2.3 (CP-INGARCH(p, q) model) *The process X is said to follow a compound Poisson integer-valued GARCH model with orders p and q (where $p,q \in \mathbb{N}$), briefly a CP-INGARCH(p,q), if, for all $t \in \mathbb{Z}$, the characteristic function of X_t conditioned on $\underline{X}_{t-1}$ is given by*

$$\Phi_{X_t|\underline{X}_{t-1}}(u) = \exp\left\{ i \frac{\lambda_t}{\varphi_t'(0)} [\varphi_t(u) - 1] \right\}, \quad u \in \mathbb{R}, \tag{2.2}$$

with

$$E(X_t|\underline{X}_{t-1}) = \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k}, \tag{2.3}$$

[9] Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative integer-valued random variables with X_n having probability generating function g_n. If X_n converges weakly to X then $\lim_{n\to\infty} g_n(z) = g_X(z)$ for $0 \leq z \leq 1$. Conversely, if $\lim_{n\to\infty} g_n(z) = g(z)$ for $0 \leq z \leq 1$ with g a function that is (left-) continuous at one, then g is the probability generating function of a random variable X and X_n converges weakly to X. See, for example, [73, p. 489].

for some constants $\alpha_0 > 0$, $\alpha_j \geq 0$ $(j = 1, ..., p)$, $\beta_k \geq 0$ $(k = 1, ..., q)$, *and where* $(\varphi_t, t \in \mathbb{Z})$ *is a family of characteristic functions on* $\mathbb{R}$, $\underline{X}_{t-1}$*-measurables, associated to a family of discrete laws with support in* $\mathbb{N}_0$ *and finite mean* ([10]). *i represents the imaginary unit.*

If $q = 1$ *and* $\beta_1 = 0$, *the CP-INGARCH*(p,q) *model is simply denoted by CP-INARCH*(p).

Assuming that the functions $(\varphi_t, t \in \mathbb{Z})$ are twice differentiable at zero, in addition to the conditional expectation λ_t we can also specify the evolution of the conditional variance of X as

$$V(X_t|\underline{X}_{t-1}) = -\Phi''_{X_t|\underline{X}_{t-1}}(0) - \lambda_t^2 = -i\frac{\varphi''_t(0)}{\varphi'_t(0)}\lambda_t. \quad (^{11})$$

Remark 2.4 *The CP-INGARCH model is able to capture different kinds of overdispersion. This results from the fact that whenever the conditional distribution is overdispersed we deduce from well-known properties on conditional moments* ([12]) *that*

$$\frac{V(X_t)}{E(X_t)} \geq \frac{E(V(X_t|\underline{X}_{t-1}))}{E(X_t)} > \frac{E(E(X_t|\underline{X}_{t-1}))}{E(X_t)} = 1;$$

moreover, if we have a conditional Poisson distribution the corresponding unconditional law is overdispersed as $V(X_t) = E(\lambda_t) + V(\lambda_t) > E(X_t)$, *whenever we have conditional heteroscedasticity.*

Similarly to what was established by Bollerslev [10] for the GARCH model, it is possible, in some cases, to state a CP-INARCH(∞) representation of the CP-INGARCH(p,q) process, i.e., X_t may be written explicitly as a function of its infinite past. With this goal, let us consider the polynomials A and B of degrees p and q given, respectively, by

$$A(L) = \alpha_1 L + ... + \alpha_p L^p,$$

$$B(L) = 1 - \beta_1 L - ... - \beta_q L^q,$$

whose coefficients are those presented in equation (2.3) and L is the backshift operator ([13]). Furthermore, to ensure the existence of the inverse B^{-1} of B, let us suppose that the roots of $B(z) = 0$ lie outside the unit circle. In fact, under this assumption, we can write

$$B(L) = 1 - \sum_{j=1}^{q} \beta_j L^j = \prod_{j=1}^{q} \left(1 - \frac{L}{z_j}\right)$$

[10] We note that, as φ_t is the characteristic function of a discrete distribution with support in $\mathbb{N}_0$ and finite mean, the derivative of $\varphi_t(u)$ at $u = 0$, $\varphi'_t(0)$, exists and is nonzero.

[11] We observe that

$$\Phi'_{X_t|\underline{X}_{t-1}}(u) = i\frac{\varphi'_t(u)\lambda_t}{\varphi'_t(0)} \exp\left\{i\frac{\lambda_t}{\varphi'_t(0)}\left[\varphi_t(u) - 1\right]\right\},$$

$$\Phi''_{X_t|\underline{X}_{t-1}}(u) = \left[i\frac{\varphi''_t(u)\lambda_t}{\varphi'_t(0)} - \left(\frac{\varphi'_t(u)\lambda_t}{\varphi'_t(0)}\right)^2\right] \exp\left\{i\frac{\lambda_t}{\varphi'_t(0)}\left[\varphi_t(u) - 1\right]\right\}, \quad u \in \mathbb{R}.$$

[12] $E(X_t) = E[E(X_t|\underline{X}_{t-1})]$ and $V(X_t) = E[V(X_t|\underline{X}_{t-1})] + V[E(X_t|\underline{X}_{t-1})]$.

[13] For any integer j, $L^j X_t = X_{t-j}$.

where $z_1,...,z_q$ are the roots of the polynomial $B(L)$. From this equality it follows that $B(L)$ will be invertible if the polynomial $1-\frac{L}{z_j}$ is invertible, for all $j\in\{1,...,q\}$. But $1-\theta L$ is invertible if and only if $|\theta|\neq 1$ ([36]) and thus, if we assume $|z_j|>1$, for all $j\in\{1,...,q\}$, then $B(L)$ is invertible ([14]).

Let us consider in the following the **Hypothesis H1**: $\sum\limits_{j=1}^{q}\beta_j<1$.

Lemma 2.2 *The roots of the polynomial* $B(z)=1-\beta_1 z-...-\beta_q z^q$*, with nonnegative* β_j*,* $j=1,...,q$*, lie outside the unit circle if and only if the coefficients* β_j *satisfy the hypothesis* **H1**.

Proof: If $\sum_{j=1}^{q}\beta_j\geq 1$, then $B(1)\leq 0$. As $B(0)=1>0$ and $B(z)$ is a continuous function in $[0,1]$, then there is a real root of $B(z)$ in the interval $]0,1]$. On the other hand, if $\sum_{j=1}^{q}\beta_j<1$, let us suppose by contradiction that there is at least a root z_0 of $B(z)$ such that $|z_0|\leq 1$. Under these conditions,

$$B(z_0)=0\Leftrightarrow 1-\sum_{i=1}^{q}\beta_j z_0^j=0\Leftrightarrow 1=\sum_{j=1}^{q}\beta_j z_0^j=\left|\sum_{j=1}^{q}\beta_j z_0^j\right|\leq\sum_{j=1}^{q}\beta_j|z_0|^j\leq\sum_{j=1}^{q}\beta_j,$$

so $1\leq\sum_{j=1}^{q}\beta_j<1$, which is a contradiction. ■

So, given the polynomials $A(L)$ and $B(L)$, and assuming the hypothesis **H1**, we can rewrite the conditional expectation (2.3) in the form

$$B(L)\lambda_t=\alpha_0+A(L)X_t \;\Leftrightarrow\; \lambda_t=B^{-1}(L)[\alpha_0+A(L)X_t]$$
$$\Leftrightarrow\; \lambda_t=\alpha_0 B^{-1}(1)+H(L)X_t,$$

with $H(L)=B^{-1}(L)A(L)=\sum_{j=1}^{\infty}\psi_j L^j$, where ψ_j is the coefficient of z^j in the Maclaurin expansion of the rational function $A(z)/B(z)$, that is,

$$\psi_j=\begin{cases}\alpha_1, & \text{if } j=1,\\ \alpha_j+\sum\limits_{k=1}^{j-1}\beta_k\psi_{j-k}, & \text{if } 2\leq j\leq p,\\ \sum\limits_{k=1}^{q}\beta_k\psi_{j-k}, & \text{if } j\geq p+1,\end{cases}$$

and then, denoting $\alpha_0 B^{-1}(1)$ as ψ_0, we get

$$\lambda_t=\psi_0+\sum_{j=1}^{\infty}\psi_j X_{t-j}, \tag{2.4}$$

which together with (2.2) expresses a CP-INARCH(∞) representation of the model in study. This representation will be useful in the construction of a solution of the model presented in Section 3.4.

[14]We note that for $B(L)$ to have inverse it is sufficient that the roots of $B(z)=0$ are, in module, different from 1. However, in what follows we will consider them outside the unit circle, since this condition will allow us to express the conditional expectation λ_t only in terms of the past information of X_t.

2.3 Important cases - known and new models

The functional form of the conditional characteristic function (2.2) allows a wide flexibility of the class of compound Poisson INGARCH models. In fact, as we assume that the family of discrete characteristic functions $(\varphi_t, t \in \mathbb{Z})$ (respectively, the associated laws of probability) is $\underline{X}_{t-1}$-measurable it means that its elements may be random functions (respectively, random measures) or deterministic ones. So, it is not surprising that this new model includes a lot of recent contributions on integer-valued time series modeling as well as several new processes.

To illustrate the large class of models enclosed in this framework, let us recall that since the conditional distribution of X_t is a discrete compound Poisson law with support in $\mathbb{N}_0$ then, for all $t \in \mathbb{Z}$ and conditioned on $\underline{X}_{t-1}$, X_t can be identified in distribution with the random sum

$$X_t \stackrel{d}{=} X_{t,1} + X_{t,2} + \ldots + X_{t,N_t}, \tag{2.5}$$

where N_t is a random variable following a Poisson distribution with parameter $\lambda_t^* = \lambda_t / E(X_{t,j})$, and $X_{t,1}, X_{t,2}$, ... are discrete and independent random variables, with support contained in $\mathbb{N}_0$, independent of N_t and having common characteristic function φ_t, with finite mean.

Some concrete examples that fall in the preceding framework are discussed in the following.

1. The INGARCH model [25] corresponds to a CP-INGARCH model considering $\lambda_t^* = \lambda_t$ and φ_t the characteristic function of the Dirac's law concentrated in $\{1\}$, i.e., $\varphi_t(u) = e^{iu}$, $u \in \mathbb{R}$.

2. Inspired by the INGARCH model, Zhu [81] proposed the negative binomial INGARCH(p,q) process (NB-INGARCH for brevity), defined as

$$X_t \mid \underline{X}_{t-1} \sim \mathbf{NB}\left(r, \frac{1}{1+\lambda_t}\right), \quad \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k},$$

   with $r \in \mathbb{N}$, $\alpha_0 > 0$, $\alpha_j \geq 0$, $\beta_k \geq 0$, $j = 1,...,p$, $k = 1,...,q$. We observe that in this model $E(X_t|\underline{X}_{t-1}) = r\lambda_t$ and $V(X_t|\underline{X}_{t-1}) = r\lambda_t(1+\lambda_t)$.

   Considering in the representation (2.5), the random variables $X_{t,j}$, $j = 1,2,...$, having a logarithmic distribution with parameter $\frac{\lambda_t}{1+\lambda_t}$ and $\lambda_t^* = r\ln(1+\lambda_t)$ (see Example 2.2) we recover, unless a scale factor, the NB-INGARCH(p,q) model.

3. To handle both conditional over-, equi- and underdispersion, Zhu [82] introduced a generalized Poisson INGARCH(p,q) process (GP-INGARCH for brevity) by considering

$$X_t \mid \underline{X}_{t-1} \sim \mathbf{GP}\left((1-\kappa)\lambda_t, \kappa\right), \quad \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k},$$

   where $\alpha_0 > 0$, $\alpha_j \geq 0$, $\beta_k \geq 0$, $j = 1,...,p$, $k = 1,...,q$ and $\max\{-1, -(1-\kappa)\lambda_t/4\} < \kappa < 1$. In this case we have $E(X_t|\underline{X}_{t-1}) = \lambda_t$ and $V(X_t|\underline{X}_{t-1}) = \lambda_t/(1-\kappa)^2$.

   For $0 < \kappa < 1$, we recover the GP-INGARCH(p,q) model from the CP-INGARCH(p,q) considering that in the representation (2.5) the common distribution of the random variables $X_{t,j}$, $j = 1,2,...$, is the Borel law with parameter κ and $\lambda_t^* = (1-\kappa)\lambda_t$ (see Example 2.3).

4. Xu et al. [80] recently proposed the family of dispersed INARCH models (DINARCH for brevity) to deal with different types of conditional dispersion assuming that the conditional variance is equal to the conditional expectation multiplied by a constant $\alpha > 0$. As a particular case of this model, they present the NB-DINARCH(p) for which the conditional law is a negative binomial one, where the random parameter is the order of occurrences and not its probability as in the NB-INARCH(p) of Zhu [81]; namely they considered

$$X_t \mid \underline{X}_{t-1} \sim \mathbf{NB}\left(\frac{\lambda_t}{\alpha-1}, \frac{1}{\alpha}\right), \quad \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j},$$

with $\alpha > 1$, $\alpha_0 > 0$, $\alpha_j \geq 0$, $j = 1, ..., p$. We note that $E(X_t|\underline{X}_{t-1}) = \lambda_t$ and $V(X_t|\underline{X}_{t-1}) = \alpha\lambda_t$. This process results from the CP-INARCH(p) model considering in the representation (2.5) the random variables $X_{t,j}$ with a logarithmic law with parameter $\frac{\alpha-1}{\alpha}$ and $\lambda_t^* = -\frac{\lambda_t}{\alpha-1} \ln\left(\frac{1}{\alpha}\right)$.

In the NB-INGARCH model (case 2) the parameter involved in the distribution of the random variables $X_{t,j}$, $j = 1, 2, ...$, depends on λ_t, and thus depends on the previous observations of the process; so, it is a clear example of a CP-INGARCH model where the characteristic function φ_t is a random function. In the other particular CP-INGARCH models presented (cases 1, 3 and 4) the law of the random variables $X_{t,j}$ have the same parameter for every $t \in \mathbb{Z}$ (1, κ and $\frac{\alpha-1}{\alpha}$, respectively). So, in the INGARCH, GP-INGARCH and NB-DINARCH models, the characteristic function φ_t is deterministic and independent of t. For that reason, in such cases we will refer these functions simply as φ.

Specifying the distribution of the random variables $X_{t,j}$, $j = 1, 2, ...$, in representation (2.5) enables us to find new interesting models as, for instance, the GEOMP-INGARCH and the NTA-INGARCH ones. We note that these new models are naturally interesting in practice as the associated conditional distributions, namely the geometric Poisson and the Neyman type-A, explain phenomena in various areas of application (recall Examples 2.4 and 2.5).

In the following examples we present some of these new models in which we also find situations where $(\varphi_t, t \in \mathbb{Z})$ is a family of dependent on t deterministic characteristic functions (namely case 7).

5. Let us define a geometric Poisson INGARCH(p,q) model (GEOMP-INGARCH) as

$$X_t \mid \underline{X}_{t-1} \sim \mathbf{GEOMP}\left(\frac{r\lambda_t}{\lambda_t + r}, \frac{r}{\lambda_t + r}\right), \quad \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k},$$

with $r > 0$, $\alpha_0 > 0$, $\alpha_j \geq 0$, $\beta_k \geq 0$, $j = 1, ..., p$, $k = 1, ..., q$. Let us note that $E(X_t|\underline{X}_{t-1}) = \lambda_t$ and $V(X_t|\underline{X}_{t-1}) = \lambda_t\left(1 + \frac{2}{r}\lambda_t\right)$. Thus, if we consider in representation (2.5) the random variables $X_{t,j}$, $j = 1, 2, ...$, following the geometric distribution with parameter $\frac{r}{r+\lambda_t}$ and $\lambda_t^* = r$ we recover this model, which means that it satisfies a CP-INGARCH model.

6. Let us consider independent random variables $(X_{t,j}, t \in \mathbb{Z})$ following the same discrete distribution with constant parameters, finite mean and support contained in $\mathbb{N}_0$. The process X defined by (2.5) with N_t independent of each $X_{t,j}$, $j = 1, 2, ...$, and having a Poisson distribution with parameter $\lambda_t / E(X_{t,j})$ satisfies a CP-INGARCH model.

For instance, if we consider

(a) the geometric distribution with parameter $p^* \in]0,1[$ for $X_{t,j}$, $j=1,2,...$, and the parameter $\frac{p^*\lambda_t}{1-p^*}$ in the Poisson law of the variable N_t we obtain as conditional distribution

$$X_t \mid \underline{X}_{t-1} \sim \textbf{GEOMP}(p^*\lambda_t, p^*), \quad \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k},$$

with $\alpha_0 > 0$, $\alpha_j \geq 0$, $\beta_k \geq 0$, $j=1,...,p$, $k=1,...,q$. We observe that $E(X_t|\underline{X}_{t-1}) = \lambda_t$ and $V(X_t|\underline{X}_{t-1}) = \lambda_t\left(\frac{2-p^*}{p^*}\right)$. This model will be denoted by GEOMP2-INGARCH(p,q) to distinguish it from the model presented in case 5. In fact, although these two models have as conditional distribution the geometric Poisson law, the difference between them results from the parameter of the geometric distribution which involves or not λ_t.

(b) the Poisson distribution with parameter $\phi > 0$ for $X_{t,j}$, $j=1,2,...$, and the parameter λ_t/ϕ in the Poisson distribution of N_t, we define the Neyman type-A INGARCH(p,q) model (NTA-INGARCH for brevity), that is,

$$X_t \mid \underline{X}_{t-1} \sim \textbf{NTA}\left(\frac{\lambda_t}{\phi}, \phi\right), \quad \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k},$$

with $\alpha_0 > 0$, $\alpha_j \geq 0$, $\beta_k \geq 0$, $j=1,...,p$, $k=1,...,q$.
We note that $E(X_t|\underline{X}_{t-1}) = \lambda_t$ and $V(X_t|\underline{X}_{t-1}) = \lambda_t(1+\phi)$.

7. (a) Let $(X_{t,j}, t \in \mathbb{Z})$ be independent random variables following a binomial distribution with parameters $r \in \mathbb{N}$ and $e^{-|t|}$, that is, $\varphi_t(u) = \left(e^{iu-|t|} + 1 - e^{-|t|}\right)^r$, $u \in \mathbb{R}$, $t \in \mathbb{Z}$, and let N_t be an independent of $X_{t,j}$ random variable following a Poisson distribution with parameter $\lambda_t / re^{-|t|}$. The process X defined by (2.5) satisfies a CP-INGARCH model.

   (b) Changing the success probability of the binomial distribution by $(t^2+1)^{-1}$, that is, considering the characteristic function $\varphi_t(u) = (\frac{e^{iu}+t^2}{t^2+1})^r$, $u \in \mathbb{R}$, $t \in \mathbb{Z}$, and the mean of the Poisson law equal to $\lambda_t(t^2+1)/r$, the process X still satisfies a CP-INGARCH model.

Remark 2.5 *We note that the pair (λ_t^*, φ_t) is not uniquely determined by $\Phi_{X_t|\underline{X}_{t-1}}$, but it will happen, if we choose φ_t such that $P(X_{t,j}=0)=0$, what can always be done. In fact, modifying the probability at zero in the distribution of $X_{t,j}$, $j=1,2,...$, does not add a new conditional distribution because it is equivalent to modifying the parameter λ_t^* in the Poisson distribution, [60, Theorem 5.11].*

For example, let us consider the GEOMP2-INGARCH model. This process was obtained taking φ_t the characteristic function of a geometric law with parameter p^* and then $\lambda_t^* = \frac{p^*\lambda_t}{1-p^*}$. But we can also obtain the same process considering φ_t the characteristic function of a shifted geometric law ([15]) with parameter p^* and in this case $\lambda_t^* = p^*\lambda_t$. Nevertheless, when we refer to the GEOMP2-INGARCH model it must be considered as it was initially defined. The same for the other models.

[15] The probability mass function of a random variable Y following a shifted geometric distribution with parameter $p \in]0,1[$ is given by $P(Y=y) = p(1-p)^{y-1}$, for $y=1,2,\ldots$ So, $E(Y)=1/p$ and $V(Y)=(1-p)/p^2$.

Using the methodology here proposed we generate some CP-INGARCH$(1,1)$ processes by considering Poisson (Figure 2.3), Neyman type-A (Figure 2.4) geometric Poisson (Figure 2.5 and Figure 2.7), binomial (Figure 2.6) and generalized Poisson deviates (Figure 2.8) ([16]). Let us note that all the possibilities for the nature of the characteristic functions φ_t are illustrated in the figures. The trajectories of these series as well as their basic descriptives are presented in the following.

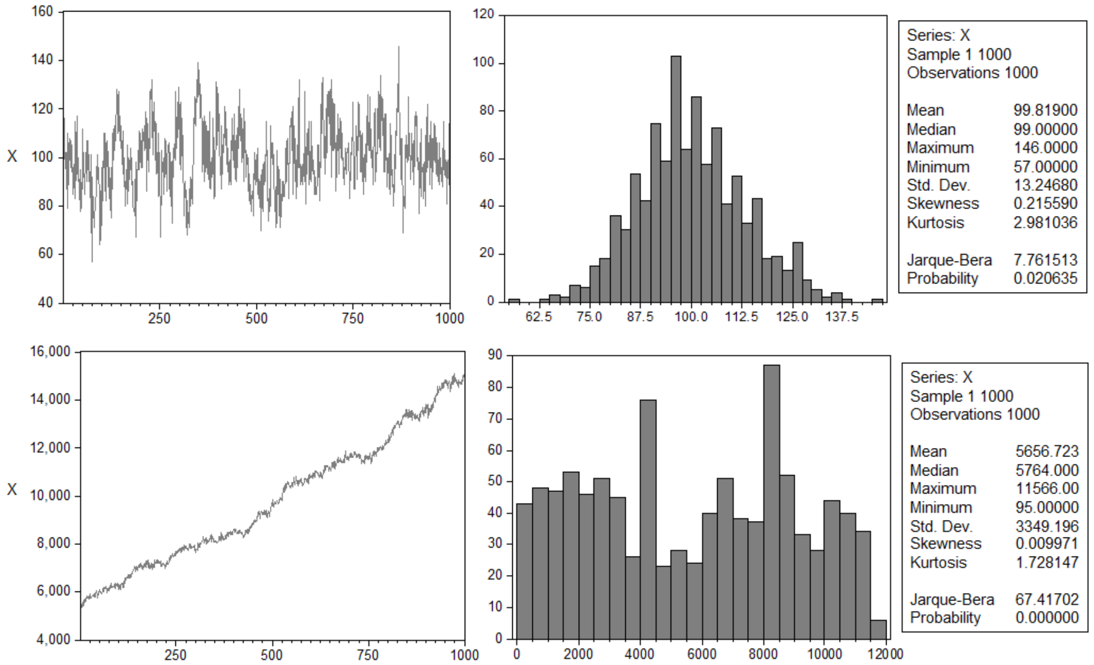

Fig. 2.3 Trajectories and descriptives of INGARCH$(1,1)$ models with $\alpha_0 = 10$, $\alpha_1 = 0.4$ and $\beta_1 = 0.5$ (on top); $\alpha_0 = 10$, $\alpha_1 = 0.5$ and $\beta_1 = 0.5$ (below).

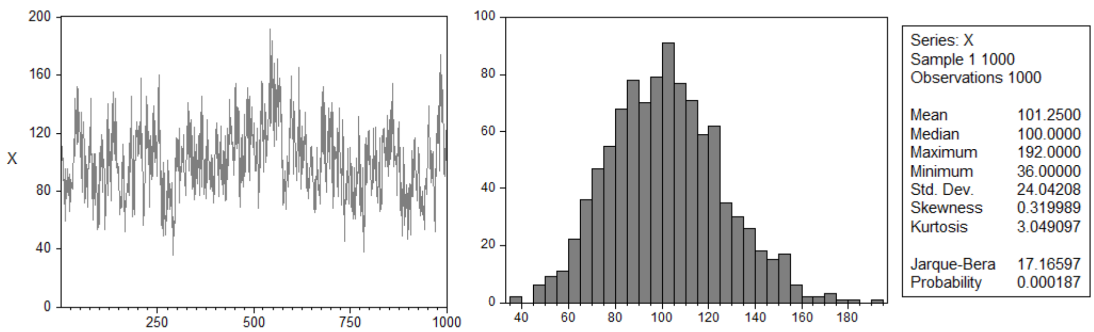

Fig. 2.4 Trajectory and descriptives of a NTA-INGARCH$(1,1)$ model with $\alpha_0 = 10$, $\alpha_1 = 0.4$, $\beta_1 = 0.5$ and $\phi = 2$.

[16]The programs used in the simulation of these trajectories were developed by us with the aid of the software Eviews and can be found in Appendix $D.1$. For each trajectory, the first 100 observations were discarded to eliminate the effect of choosing the values of the first observations.

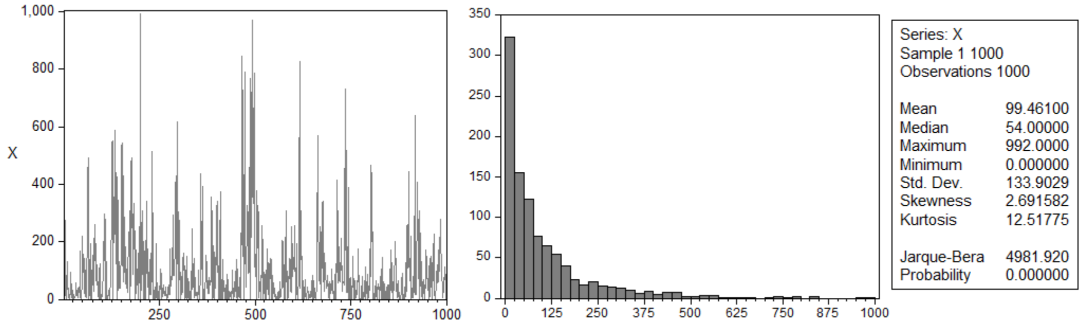

Fig. 2.5 Trajectory and descriptives of a GEOMP-INGARCH$(1,1)$ model with $\alpha_0 = 10$, $\alpha_1 = 0.4$, $\beta_1 = 0.5$ and $r = 2$.

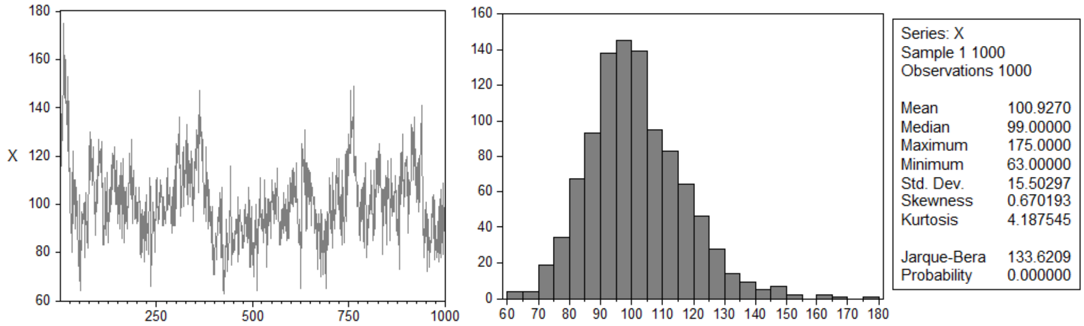

Fig. 2.6 Trajectory and descriptives of a CP-INGARCH$(1,1)$ model with φ_t the characteristic function of a binomial$(5000, \frac{1}{t^2+1})$ distribution considering $\alpha_0 = 10$, $\alpha_1 = 0.4$, and $\beta_1 = 0.5$.

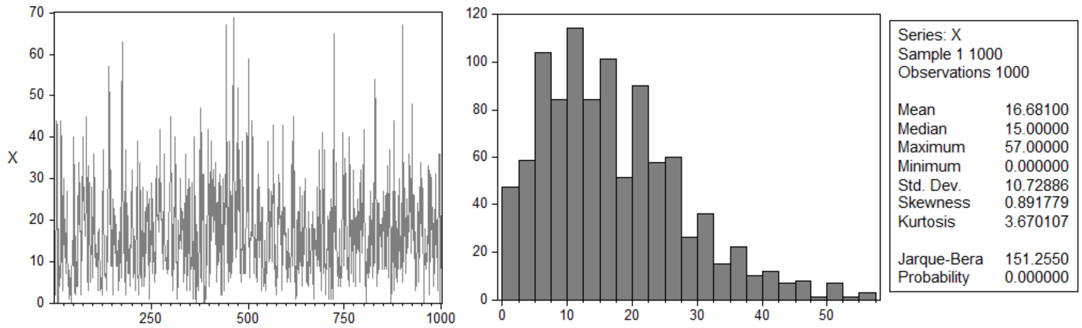

Fig. 2.7 Trajectory and descriptives of a GEOMP2-INARCH(1) model with $\alpha_0 = 10$, $\alpha_1 = 0.4$, and $p^* = 0.3$.

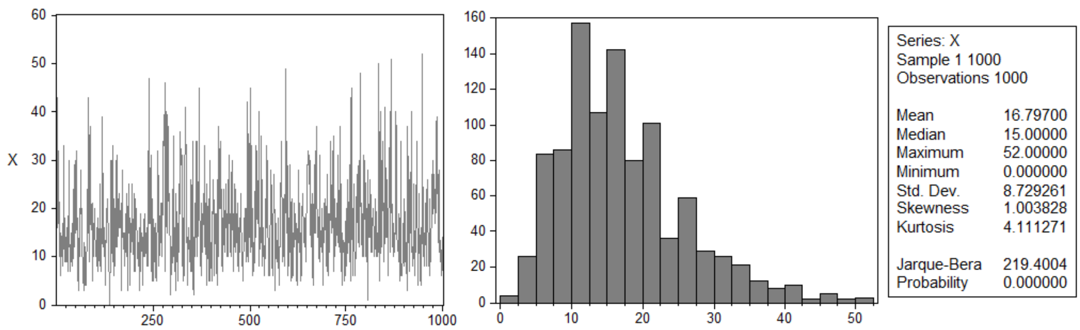

Fig. 2.8 Trajectory and descriptives of a GP-INARCH(1) model with $\alpha_0 = 10$, $\alpha_1 = 0.4$, and $\kappa = 0.5$.

Comparing the two trajectories presented in Figure 2.3 it is evident that the change made in the coefficient α_1 results in a non-stationary process, namely, in mean. So, it seems natural to think that the stationarity of the CP-INGARCH model is strongly related to the coefficients involved in the evolution of the conditional mean λ_t. The unconditional mean, when we consider the same coefficients α_0, α_1 and β_1, does not seem to be affected by the conditional distribution nor by the nature of the characteristic functions φ_t. In fact, it presents similar values when we compare the processes with the same orders namely 99.819, 101.25, 99.461, 100.93 (figures 2.3 on top, 2.4, 2.5 and 2.6, respectively) and 17.183, 16.797 (figures 2.7 and 2.8, respectively).

The CP-INGARCH trajectories represented seem to concern processes with characteristics as overdispersion, leptokurtosis (figures 2.5, 2.6, 2.7 and 2.8) and asymmetry around the mean. We point out also the strong volatility in all the cases but especially in the NTA-INGARCH and GEOMP-INGARCH models.

The aim of the next chapter is to develop a unified and enlarged study on the probabilistic properties within the class of the integer-valued GARCH models.

Chapter 3

Stationarity and Ergodicity in the CP-INGARCH process

An important property of a process is its stationarity or, in other words, its invariance under translation in time. In time series modeling, to evaluate stability properties over time is important particularly in statistical developments for instance to reach good forecasts. Moreover, we are in conditions to analyse all the characteristics of a process from a single (infinitely long) realisation when we have an ergodic process. This is very important, since in the study of a time series only a single realization from the series is available. Stationarity and ergodicity are then the two cornerstones on which the time series analysis rests and they will be the subject of study in this chapter.

Necessary and sufficient conditions of first and second-order stationarity expressed in terms of the coefficients of a CP-INGARCH(p,q) process are discussed respectively in Section 3.1 and 3.2, and illustrated by the study of some particular cases. Then, in Section 3.3, we obtain its autocorrelation function and we investigate the existence of higher-order moments when $p=q=1$. We finish the chapter establishing a necessary and sufficient condition to ensure the strict stationarity and the ergodicity of the CP-INGARCH(p,q) process. We should remark that the assumptions considered on the family of characteristic functions to establish these properties concern a huge class of processes.

3.1 First-order stationarity

The following theorem is the starting point for establishing the weak stationarity for the CP-INGARCH process in the next section. We notice that the result obtained is not affected by the form of the conditional distribution but mainly by the evolution of λ_t specified in (2.3).

Theorem 3.1 *Let X be a process satisfying the CP-INGARCH*(p,q) *model as specified in* (2.2) *and* (2.3). *This process is first-order stationary if and only if*

$$\sum_{j=1}^{p}\alpha_j+\sum_{k=1}^{q}\beta_k<1.$$

Proof: To establish the first-order stationarity we should prove that $E(X_t)$ exists and is independent of t, for any $t \in \mathbb{Z}$. As X_t is a positive measurable function, we can write formally

$$\mu_t = E(X_t) = E(E(X_t|\underline{X}_{t-1})) = E(\lambda_t) = E\left(\alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k}\right)$$

$$\Leftrightarrow \ \mu_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j \mu_{t-j} + \sum_{k=1}^{q} \beta_k \mu_{t-k},$$

taking into account that the involved sums exist although they may be non finite. This non-homogeneous difference equation has a stable solution, which is finite and independent of t, if and only if all roots z_1, ..., $z_{\max(p,q)}$ of the equation $1-\sum_{j=1}^{p} \alpha_j z^j - \sum_{k=1}^{q} \beta_k z^k = 0$ lie outside the unit circle, that is, if and only if $\sum_{j=1}^{p} \alpha_j + \sum_{k=1}^{q} \beta_k < 1$. ■

Remark 3.1 *As a consequence of the Theorem* 3.1*, provided that* $\sum_{j=1}^{p} \alpha_j + \sum_{k=1}^{q} \beta_k < 1$*, the processes* $\lambda = (\lambda_t, t \in \mathbb{Z})$ *and* X *are both first-order stationary and its common unconditional mean is*

$$E(X_t) = E(\lambda_t) = \mu = \frac{\alpha_0}{1-\sum_{j=1}^{p} \alpha_j - \sum_{k=1}^{q} \beta_k}.$$

From Theorem 3.1 we have assured that if the sum of the parameters α_j's and β_k's, $j = 1, ..., p$, $k = 1, ..., q$, is greater than or equal to 1 then the process X is not first-order stationary and, obviously, it will also not be a weakly stationary process. The second trajectory presented in Figure 2.3 is an INGARCH$(1,1)$ process where the sum of the parameters α_1 and β_1 is equal to 1 and, as referred before, it is an example of a non-stationary process.

Moreover, we will see later that the condition of first-order stationarity displayed is also a necessary and sufficient condition to obtain a strictly stationary solution of this model.

3.2 Weak stationarity

In this section, we analyse weak stationarity conditions for the CP-INGARCH(p,q) model. With this goal and in order to assure the existence of the corresponding distribution variance, we assume in what follows that the family of characteristic functions $(\varphi_t, t \in \mathbb{Z})$ is twice differentiable.

The general class of models considered and the complexity in the study of the weak stationarity in this class leads us to fix ourselves in the subclass of the CP-INGARCH(p,q) models for which the characteristic functions φ_t satisfy the following condition:

$$\textbf{Hypothesis H2}: \ -i\frac{\varphi_t''(0)}{\varphi_t'(0)} = v_0 + v_1 \lambda_t,$$

with $v_0 \geq 0$, $v_1 \geq 0$, not simultaneously zero. We note that $\dfrac{V(X_t|\underline{X}_{t-1})}{E(X_t|\underline{X}_{t-1})} = -i\dfrac{\varphi_t''(0)}{\varphi_t'(0)}$.

Despite the restriction, a quite general subclass is considered containing both random and deterministic characteristic functions φ_t. Recalling the variance of the involved conditional distributions, many of the examples presented in Section 2.3 can be included in this subclass. Namely,

- INGARCH(p,q) model with $v_0 = 1$;
- GP-INGARCH(p,q) model with $v_0 = \dfrac{1}{(1-\kappa)^2}$;
- NB-DINARCH(p) model with $v_0 = \alpha$;
- GEOMP2-INGARCH(p,q) model with $v_0 = \dfrac{2-p^*}{p^*}$;
- NTA-INGARCH(p,q) model with $v_0 = 1+\phi$,

all of them with $v_1 = 0$, and also the

- NB-INGARCH(p,q) model with $v_0 = v_1 = 1$;
- GEOMP-INGARCH(p,q) model with $v_0 = 1$ and $v_1 = \dfrac{2}{r}$.

Let us note that all the characteristic functions φ_t which are deterministic and independent of t satisfy the hypothesis **H2** since they imply $v_1 = 0$.

Remark 3.2 *Let us consider a real function k such that $k'(0)/k(0) = 1$. Considering*

$$-i\frac{\varphi_t''(u)}{\varphi_t'(u)} = (v_0 + v_1\lambda_t)\frac{k'(u)}{k(u)}, \qquad u \in \mathbb{R},$$

we may write

$$\frac{d}{du}\left(-i\ln\left(\varphi_t'(u)\right) + \theta(t)\right) = (v_0 + v_1\lambda_t)\,\frac{k'(u)}{k(u)}$$

which implies

$$-i\ln\left(\varphi_t'(u)\right) + \theta(t) = (v_0 + v_1\lambda_t)\ln\left(k(u)\right) + \widetilde{\zeta}(t)$$

and so

$$\varphi_t'(u) = \exp\left\{i\,(v_0 + v_1\lambda_t)\ln\left(k(u)\right) + \zeta(t)\right\}, \qquad u \in \mathbb{R}.$$

Thus the following general class of characteristic functions φ_t such that

$$\varphi_t(u) = e^{\zeta(t)}\int k(u)^{i(v_0+v_1\lambda_t)}du + \delta(t), \quad u \in \mathbb{R},$$

is solution of **H2**. *For instance, the characteristic function $\varphi(u) = e^{iu}$, $u \in \mathbb{R}$, can be written in this form with $v_0 = 1$, $v_1 = 0$ and considering $k(u) = e^u$, $\delta(t) = 0$ and $\zeta(t) = \ln i$.*

Remark 3.3 *As noted in Remark 2.2, the family of conditional distributions considered can be equi- or overdispersed. Despite this, some of the next results are not restricted to these cases. In fact, we observe that when $v_1 = 0$ the hypothesis* **H2** *allows conditional underdispersion since the ratio between the conditional variance and the conditional mean, which corresponds to v_0, can be in $]0,1[$. Thus, the DINARCH(p) model proposed by Xu et al. [80] is also included in this study.*

In order to obtain a necessary and sufficient condition of weak stationarity we begin by establishing a vectorial state space representation of X. To accomplish this let us observe that

$$E(X_{t-j}\lambda_{t-k}) = E\left[E(X_{t-j}|\underline{X}_{t-j-1})\lambda_{t-k}\right] = E(\lambda_{t-j}\lambda_{t-k}), \text{ if } k \geq j, \tag{3.1}$$

$$E(X_{t-j}\lambda_{t-k}) = E\left[X_{t-j}E(X_{t-k}|\underline{X}_{t-k-1})\right] = E(X_{t-j}X_{t-k}), \text{ if } k < j, \tag{3.2}$$

from which we can deduce the expressions

$$\begin{aligned}
&E(X_tX_{t-h}) = E\left[E(X_t|\underline{X}_{t-1})X_{t-h}\right] = E(\lambda_tX_{t-h})\\
&= E\left(\left[\alpha_0 + \sum_{j=1}^{p}\alpha_jX_{t-j} + \sum_{k=1}^{q}\beta_k\lambda_{t-k}\right]X_{t-h}\right)\\
&= \alpha_0E\left(X_{t-h}\right) + \sum_{j=1}^{p}\alpha_jE\left(X_{t-j}X_{t-h}\right) + \sum_{k=1}^{q}\beta_kE\left(\lambda_{t-k}X_{t-h}\right)\\
&= \alpha_0E\left(X_{t-h}\right) + \sum_{j=1}^{p}\alpha_jE\left(X_{t-j}X_{t-h}\right) + \sum_{k=1}^{h-1}\beta_kE(X_{t-k}X_{t-h}) + \sum_{k=h}^{q}\beta_kE(\lambda_{t-k}\lambda_{t-h}), \qquad h \geq 1,
\end{aligned} \tag{3.3}$$

and in a similar way

$$\begin{aligned}
&E(\lambda_t\lambda_{t-h}) = E\left(\left[\alpha_0 + \sum_{j=1}^{p}\alpha_jX_{t-j} + \sum_{k=1}^{q}\beta_k\lambda_{t-k}\right]\lambda_{t-h}\right)\\
&= \alpha_0E\left(\lambda_{t-h}\right) + \sum_{j=1}^{p}\alpha_jE\left(X_{t-j}\lambda_{t-h}\right) + \sum_{k=1}^{q}\beta_kE\left(\lambda_{t-k}\lambda_{t-h}\right)\\
&= \alpha_0E\left(\lambda_{t-h}\right) + \sum_{j=1}^{h}\alpha_jE\left(\lambda_{t-j}\lambda_{t-h}\right) + \sum_{j=h+1}^{p}\alpha_jE(X_{t-j}X_{t-h}) + \sum_{k=1}^{q}\beta_kE\left(\lambda_{t-k}\lambda_{t-h}\right), \qquad h \geq 0.
\end{aligned} \tag{3.4}$$

Proposition 3.1 *Let X be a first-order stationary process following a CP-INGARCH(p,q) model such that* **H2** *is satisfied. The vector W_t, $t \in \mathbb{Z}$, of dimension $p+q-1$ given by*

$$W_t = \begin{bmatrix} E(X_t^2) \\ E(X_tX_{t-1}) \\ \cdots \\ E(X_tX_{t-(p-1)}) \\ E(\lambda_t\lambda_{t-1}) \\ \cdots \\ E(\lambda_t\lambda_{t-(q-1)}) \end{bmatrix}$$

satisfies an autoregressive equation of order $\max(p,q)$*:*

$$W_t = B_0 + \sum_{k=1}^{\max(p,q)} B_kW_{t-k}, \tag{3.5}$$

where B_0 is a real vector of dimension $p+q-1$ and B_k ($k = 1,...,\max(p,q)$) are real squared matrices of order $p+q-1$.

Proof: We begin noting that $E(X_t^2)$, $E(X_tX_{t-k})$ and $E(\lambda_t\lambda_{t-k})$ are not necessarily finite but, as we have positive and measurable functions, we may apply the operator expectation E. For sake of simplicity, we focus the proof on the case $p=q$ since the other cases can be obtained from this one by setting additional parameters to 0.

Let us start by calculating $E(X_t^2)$ for any $t\in\mathbb{Z}$. We have $E(X_t^2)=E[E(X_t^2|\underline{X}_{t-1})]$ and

$$E(X_t^2|\underline{X}_{t-1})=V(X_t|\underline{X}_{t-1})+[E(X_t|\underline{X}_{t-1})]^2=-i\frac{\varphi_t''(0)}{\varphi_t'(0)}\lambda_t+\lambda_t^2=v_0\lambda_t+(1+v_1)\lambda_t^2$$

$$=v_0\left[\alpha_0+\sum_{j=1}^{p}\alpha_jX_{t-j}+\sum_{k=1}^{p}\beta_k\lambda_{t-k}\right]+(1+v_1)\left[\alpha_0^2+2\alpha_0\left(\sum_{j=1}^{p}\alpha_jX_{t-j}+\sum_{k=1}^{p}\beta_k\lambda_{t-k}\right)\right.$$
$$\left.+\left(\sum_{j=1}^{p}\alpha_jX_{t-j}\right)^2+2\sum_{j=1}^{p}\alpha_jX_{t-j}\sum_{k=1}^{p}\beta_k\lambda_{t-k}+\left(\sum_{k=1}^{p}\beta_k\lambda_{t-k}\right)^2\right]$$

$$=v_0\alpha_0+(1+v_1)\alpha_0^2+[v_0+2\alpha_0(1+v_1)]\left(\sum_{j=1}^{p}\alpha_jX_{t-j}+\sum_{k=1}^{p}\beta_k\lambda_{t-k}\right)+(1+v_1)\left[\sum_{j=1}^{p}\alpha_j^2X_{t-j}^2\right.$$
$$\left.+\sum_{\substack{j,k=1\\ j\neq k}}^{p}\alpha_j\alpha_kX_{t-j}X_{t-k}+2\sum_{j=1}^{p}\sum_{k=1}^{p}\alpha_j\beta_k\lambda_{t-k}X_{t-j}+\sum_{j=1}^{p}\beta_j^2\lambda_{t-j}^2+\sum_{\substack{j,k=1\\ j\neq k}}^{p}\beta_j\beta_k\lambda_{t-j}\lambda_{t-k}\right].$$

So, using the first-order stationary hypothesis, we conclude

$$E(X_t^2)=\widetilde{C}+(1+v_1)\left[\sum_{j=1}^{p}\alpha_j^2E(X_{t-j}^2)+\sum_{\substack{j,k=1\\ j\neq k}}^{p}\alpha_j\alpha_kE(X_{t-j}X_{t-k})\right.$$
$$\left.+2\sum_{j,k=1}^{p}\alpha_j\beta_kE\left(X_{t-j}\lambda_{t-k}\right)+\sum_{j=1}^{p}\beta_j^2E(\lambda_{t-j}^2)+\sum_{\substack{j,k=1\\ j\neq k}}^{p}\beta_j\beta_kE(\lambda_{t-j}\lambda_{t-k})\right],\tag{3.6}$$

where $\widetilde{C}=v_0\mu+(1+v_1)\left[2\alpha_0\mu-\alpha_0^2\right]$ since $\mu-\alpha_0=\mu\sum_{j=1}^{p}(\alpha_j+\beta_j)$. We note that $\widetilde{C}$ is a positive constant independent of t. Now, let us take into account the fact:

$$E(X_t^2)=v_0E(\lambda_t)+(1+v_1)E(\lambda_t^2)\;\Leftrightarrow\;E(\lambda_t^2)=\frac{E(X_t^2)-v_0\mu}{1+v_1},\tag{3.7}$$

from where we deduce that X is a second-order process if and only if the same happens to the process λ. Applying the expressions (3.1), (3.2) and (3.7) in (3.6), we finally obtain

$$E(X_t^2)=\widetilde{C}+(1+v_1)\left[\sum_{j=1}^{p}\alpha_j^2E(X_{t-j}^2)+\sum_{j=1}^{p}\beta_j^2\frac{E(X_{t-j}^2)-v_0\mu}{1+v_1}+2\sum_{\substack{j,k=1\\ j<k}}^{p}\alpha_j\alpha_kE(X_{t-j}X_{t-k})\right.$$
$$\left.+2\sum_{\substack{j,k=1\\ j\leq k}}^{p}\alpha_j\beta_kE(\lambda_{t-j}\lambda_{t-k})+2\sum_{\substack{j,k=1\\ j<k}}^{p}\alpha_k\beta_jE(X_{t-j}X_{t-k})+2\sum_{\substack{j,k=1\\ j<k}}^{p}\beta_j\beta_kE(\lambda_{t-j}\lambda_{t-k})\right]$$

$$= C + (1+v_1)\left[\sum_{j=1}^{p}\left(\alpha_j^2 + \frac{2\alpha_j\beta_j + \beta_j^2}{1+v_1}\right)E(X_{t-j}^2)\right.$$

$$\left. +2\sum_{j=1}^{p-1}\sum_{k=j+1}^{p}\alpha_k(\alpha_j+\beta_j)E(X_{t-j}X_{t-k}) + 2\sum_{j=1}^{p-1}\sum_{k=j+1}^{p}\beta_k(\alpha_j+\beta_j)E(\lambda_{t-j}\lambda_{t-k})\right], \qquad (3.8)$$

where $C = \widetilde{C} - v_0\mu\sum_{j=1}^{p}(2\alpha_j\beta_j + \beta_j^2)$ is a positive constant independent of t. We observe that the positivity of the constant C is a consequence of the first-order stationarity of X since from it we have $1 > (\sum_{j=1}^{p}(\alpha_j+\beta_j))^2 > \sum_{j=1}^{p}(\alpha_j+\beta_j)^2 > \sum_{j=1}^{p}(2\alpha_j\beta_j+\beta_j^2)$.

Using again the hypothesis of first-order stationarity and the expression (3.7) stated above, we have from (3.3) and (3.4),

$$\begin{aligned}
E(X_tX_{t-h}) &= \alpha_0\mu + \alpha_h E(X_{t-h}^2) + \sum_{\substack{j=1\\ j\neq h}}^{p}\alpha_j E\left(X_{t-j}X_{t-h}\right) + \sum_{k=1}^{h-1}\beta_k E(X_{t-k}X_{t-h})\\
&\quad +\beta_h E(\lambda_{t-h}^2) + \sum_{k=h+1}^{p}\beta_k E(\lambda_{t-k}\lambda_{t-h})\\
&= \left[\alpha_0 - \frac{v_0\beta_h}{1+v_1}\right]\mu + \left[\alpha_h + \frac{\beta_h}{1+v_1}\right]E(X_{t-h}^2) + \sum_{j=h+1}^{p}\beta_j E(\lambda_{t-j}\lambda_{t-h})\\
&\quad + \sum_{j=1}^{h-1}(\alpha_j+\beta_j)E(X_{t-j}X_{t-h}) + \sum_{j=h+1}^{p}\alpha_j E(X_{t-j}X_{t-h}), \qquad h \geq 1, \qquad (3.9)
\end{aligned}$$

$$\begin{aligned}
E(\lambda_t\lambda_{t-h}) &= \alpha_0\mu + \sum_{j=1}^{h-1}\alpha_j E\left(\lambda_{t-j}\lambda_{t-h}\right) + \sum_{j=h+1}^{p}\alpha_j E(X_{t-j}X_{t-h})\\
&\quad +(\alpha_h+\beta_h)E(\lambda_{t-h}^2) + \sum_{\substack{k=1\\ k\neq h}}^{p}\beta_k E(\lambda_{t-k}\lambda_{t-h})\\
&= \left[\alpha_0 - \frac{v_0(\alpha_h+\beta_h)}{1+v_1}\right]\mu + \frac{\alpha_h+\beta_h}{1+v_1}E(X_{t-h}^2) + \sum_{j=h+1}^{p}\alpha_j E(X_{t-j}X_{t-h})\\
&\quad + \sum_{j=1}^{h-1}(\alpha_j+\beta_j)E(\lambda_{t-j}\lambda_{t-h}) + \sum_{j=h+1}^{p}\beta_j E(\lambda_{t-j}\lambda_{t-h}), \qquad h \geq 1. \qquad (3.10)
\end{aligned}$$

From expressions (3.8), (3.9) and (3.10) it is clear now that the vector W_t satisfies the autoregressive equation of order p, $W_t = B_0 + \sum_{k=1}^{p}B_kW_{t-k}$, with $B_0 = (b_j)$ the vector such that

$$b_j = \begin{cases} C, & j=1\\ \mu\alpha_0 - \mu\,\dfrac{v_0\beta_{j-1}}{1+v_1}, & j=2,\dots,p\\ \mu\alpha_0 - \mu\,\dfrac{v_0(\alpha_{j-p}+\beta_{j-p})}{1+v_1}, & j=p+1,\dots,2p-1 \end{cases}$$

and B_k ($k=1,\dots,p$) the squared matrices having generic element $b_{ij}^{(k)}$ given by

- row $i=1$:

$$b_{1j}^{(k)} = \begin{cases} (1+v_1)\alpha_k^2 + 2\alpha_k\beta_k + \beta_k^2, & \text{if } j=1 \\ 2(1+v_1)(\alpha_k+\beta_k)\alpha_{j+k-1}, & \text{if } j=2,...,p \\ 2(1+v_1)(\alpha_k+\beta_k)\beta_{j+k-p}, & \text{if } j=p+1,...,2p-1 \end{cases}$$

- row $i=k+1$, $(k \neq p)$:

$$b_{k+1,j}^{(k)} = \begin{cases} \alpha_k + \dfrac{\beta_k}{1+v_1}, & \text{if } j=1 \\ \alpha_{j+k-1}, & \text{if } j=2,...,p \\ \beta_{j+k-p}, & \text{if } j=p+1,...,2p-1 \end{cases}$$

- row $i=k+p$:

$$b_{k+p,j}^{(k)} = \begin{cases} \dfrac{\alpha_k+\beta_k}{1+v_1}, & \text{if } j=1 \\ \alpha_{j+k-1}, & \text{if } j=2,...,p \\ \beta_{j+k-p}, & \text{if } j=p+1,...,2p-1 \end{cases}$$

- row $i=k+j$:

$$b_{k+j,j}^{(k)} = \begin{cases} \alpha_k+\beta_k, & \text{if } j=2,...,p-k,p+1,...,2p-1-k \\ 0 & \text{if } j=p-k+1,...,p \end{cases}$$

and for any other case $b_{ij}^{(k)} = 0$, where we consider $\alpha_j = \beta_j = 0$, for $i > p$. The general form of these matrices B_k can be found in Appendix $A.1$. ■

For a CP-INARCH(p) model the entries of the vector B_0 and of the matrices B_k, $(k=1,...,p)$ become quite simpler and the previous result assumes the form presented in the following corollary.

Corollary 3.1 *Let X be a first-order stationary process following a CP-INARCH(p) model such that* **H2** *is satisfied. The vector W_t, $t \in \mathbb{Z}$, of dimension p given by*

$$W_t = \begin{bmatrix} E(X_t^2) \\ E(X_t X_{t-1}) \\ \cdots \\ E(X_t X_{t-(p-1)}) \end{bmatrix}$$

satisfies an autoregressive equation of order p:

$$W_t = B_0 + \sum_{k=1}^{p} B_k W_{t-k},$$

where B_0 is the vector of dimension p given by $B_0 = (v_0\mu + \alpha_0(1+v_1)(2\mu-\alpha_0), \alpha_0\mu, ..., \alpha_0\mu)$ and B_k $(k=1,...,p)$ are the squared matrices of order p with generic element $b_{ij}^{(k)}$ given by:

- *row $i=1$:*

$$b_{1j}^{(k)} = \begin{cases} (1+v_1)\alpha_k^2, & \textit{if } j=1 \\ 2(1+v_1)\alpha_k\alpha_{j+k-1}, & \textit{if } j=2,...,p \end{cases}$$

- *row* $i \neq 1$*:*

$$b_{ij}^{(k)} = \begin{cases} \alpha_{j+k-1}, & \text{if } i = k+1,\ j = 1,...,p \\ \alpha_k, & \text{if } i = k+j,\ j = 2,...,p \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha_j = 0$ *for* $j > p$*.*

Now we state a necessary and sufficient condition of weak stationarity of the process under study.

Theorem 3.2 *Let X be a first-order stationary process following a CP-INGARCH*(p,q) *model such that* **H2** *is satisfied. This process is weakly stationary if and only if*

$$P(L) = I_{p+q-1} - \sum_{k=1}^{\max(p,q)} B_k L^k$$

is a polynomial matrix such that $\det P(z)$ *has all its roots outside the unit circle, where* I_{p+q-1} *is the identity matrix of order* $p+q-1$ *and* B_k $(k = 1,...,\max(p,q))$ *are the squared matrices of the autoregressive equation* (3.5)*. Moreover, denoting the covariance function of X and* λ *by respectively* $\Gamma(j) = Cov(X_t, X_{t-j})$ *and* $\widetilde{\Gamma}(j) = Cov(\lambda_t, \lambda_{t-j})$*, we have under the weak stationarity of X*

$$\Gamma(j) = e_{j+1}[P(1)]^{-1}B_0 - \mu^2, \quad j = 0,...,p-1,$$

$$\widetilde{\Gamma}(j) = e_{p+j}[P(1)]^{-1}B_0 - \mu^2, \quad j = 1,...,q-1,$$

with e_j *denoting the* j*-th row of the identity matrix.*

Proof: Let us consider $C_0 = I_{p+q-1}$ and $C_k = B_k$, $k \geq 1$. Since $C_k = 0$ when $k > \max(p,q)$, the autoregressive equation $W_t = B_0 + \sum_{k=1}^{\max(p,q)} B_k W_{t-k}$ can be rewritten in the form

$$W_t = B_0 + \sum_{k=1}^{\max(p,q)} C_k W_{t-k} \Leftrightarrow W_t = B_0 + \sum_{k=0}^{t} C_{t-k} W_k - W_t, \quad \text{when } t \geq \max(p,q). \tag{3.11}$$

Introducing the z-transform ([1]) of W_t and that of C_t, namely $\widetilde{W}(z) = \sum_{k=0}^{\infty} W_k z^{-k}$ and $\widetilde{C}(z) = C_0 + \sum_{k=1}^{\max(p,q)} C_k z^{-k}$, and taking the z-transform of both sides of equation (3.11) we get

$$\widetilde{W}(z) = B_0 + \widetilde{C}(z)\widetilde{W}(z) - \widetilde{W}(z) \ \Leftrightarrow \ \left(I_{p+q-1} - \widetilde{C}(z) + I_{p+q-1}\right)\widetilde{W}(z) = B_0.$$

So, according to [22, p. 299], a necessary and sufficient condition for weak stationarity is

$$\det\left(I_{p+q-1} - \widetilde{C}(z) + I_{p+q-1}\right) \neq 0, \quad \text{for all } z \text{ such that } |z| \geq 1,$$

that is, $\det\left(I_{p+q-1} - \sum_{k=1}^{\max(p,q)} B_k z^{-k}\right) = \det P\left(\frac{1}{z}\right)$ has all its roots inside the unit circle.

[1]Let $x(n)$ be a sequence which is identically zero for negative integers n, i.e., with $x(n) = 0$ for $n < 0$. The z-transform of $x(n)$ is defined as $\widetilde{x}(z) = Z(x(n)) = \sum_{j=0}^{\infty} x(j) \cdot z^{-j}$, for $z \in \mathbb{C}$. See, e.g., [22, Section 6.1].

From the weak stationarity and since $P(1)$ is an invertible matrix ([2]), the autoregressive equation (3.5) reduces to $\left(I_{p+q-1}-\sum_{k=1}^{\max(p,q)}B_k\right)W_t=B_0 \Leftrightarrow W_t=[P(1)]^{-1}B_0$. So, in what concerns the values of the autocovariances as, for $j=0,...,p-1$, the order j entry of the vector W_t is $E(X_tX_{t-j})$, we obtain $Cov(X_t,X_{t-j})=E(X_tX_{t-j})-\mu^2=e_{j+1}[P(1)]^{-1}B_0-\mu^2$. The expression of the autocorrelation function of λ follows similarly, which completes the proof. ■

Remark 3.4 *We provide now an alternative proof for the sufficient condition of weak stationarity. If all the roots of* $\det P(z)$ *are outside the unit circle then* $P(1)$ *is invertible and we have*

$$W_t=B_0+\sum_{k=1}^{\max(p,q)}B_kW_{t-k} \Leftrightarrow \left(I_{p+q-1}-\sum_{k=1}^{\max(p,q)}B_kL^k\right)W_t=B_0$$

$$\Leftrightarrow \left(I_{p+q-1}-\sum_{k=1}^{\max(p,q)}B_kL^k\right)W_t=\left(I_{p+q-1}-\sum_{k=1}^{\max(p,q)}B_k\right)[P(1)]^{-1}B_0$$

$$\Leftrightarrow W_t-[P(1)]^{-1}B_0=\sum_{k=1}^{\max(p,q)}B_k\left(W_{t-k}-[P(1)]^{-1}B_0\right),$$

which means that $\{W_t-[P(1)]^{-1}B_0\}_{t\in\mathbb{Z}}$ *satisfies an homogeneous linear recurrence equation. The solution of this equation is asymptotically independent of* t *since* $\det P(z)$ *has all roots outside the unit circle, and then from the definition of* W_t*, the weak stationarity of* X *and* λ *follows.*

Hereafter we present some examples to illustrate the condition of weak stationarity displayed.

Example 3.1 *Let us consider a CP-INGARCH*(p,p) *model satisfying* **H2** *with* $\alpha_1=...=\alpha_{p-1}=\beta_1=...=\beta_{p-1}=0$ *and such that* $\alpha_p+\beta_p<1$*. To analyze the necessary and sufficient condition for weak stationarity of* X *given by Theorem* 3.2*, we consider the polynomial matrix*

$$P(z)=I_{2p-1}-B_1z-...-B_pz^p=\begin{bmatrix} 1-[(\alpha_p+\beta_p)^2+v_1\alpha_p^2]z^p & \boldsymbol{0}_{1\times(p-1)} & \boldsymbol{0}_{1\times(p-1)} \\ \boldsymbol{0}_{(p-1)\times 1} & A & B \\ \boldsymbol{0}_{(p-1)\times 1} & C & D \end{bmatrix}$$

where A*,* B*,* C *and* D *are squared matrices of order* $p-1$ *given by*

$$B=\begin{bmatrix} 0 & 0 & \cdots & 0 & -\beta_pz \\ 0 & 0 & \cdots & -\beta_pz^2 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & -\beta_pz^{p-2} & \cdots & 0 & 0 \\ -\beta_pz^{p-1} & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$C=\begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_pz \\ 0 & 0 & \cdots & -\alpha_pz^2 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & -\alpha_pz^{p-2} & \cdots & 0 & 0 \\ -\alpha_pz^{p-1} & 0 & \cdots & 0 & 0 \end{bmatrix},$$

[2] Under the hypothesis of weak stationarity, $\det P(z)$ has all roots outside the unit circle. So, $P(1)$ is an invertible matrix since $\det P(1)=\det\left(I_{p+q-1}-\sum_{k=1}^{p}B_k\right)\neq 0$.

when p is odd

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 & -\alpha_p z \\ 0 & 1 & \cdots & -\alpha_p z^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -\alpha_p z^{p-2} & \cdots & 1 & 0 \\ -\alpha_p z^{p-1} & 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & \cdots & 0 & -\beta_p z \\ 0 & 1 & \cdots & -\beta_p z^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -\beta_p z^{p-2} & \cdots & 1 & 0 \\ -\beta_p z^{p-1} & 0 & \cdots & 0 & 1 \end{bmatrix},$$

and when p is even

$$A = \begin{bmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & -\alpha_p z \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ 0 & \cdots & 1 & 0 & -\alpha_p z^{\left[\frac{p-1}{2}\right]} & \cdots & 0 \\ 0 & \cdots & 0 & 1-\alpha_p z^{\left[\frac{p-1}{2}\right]+1} & 0 & \cdots & 0 \\ 0 & \cdots & -\alpha_p z^{\left[\frac{p-1}{2}\right]+2} & 0 & 1 & \cdots & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_p z^{p-1} & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & -\beta_p z \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ 0 & \cdots & 1 & 0 & -\beta_p z^{\left[\frac{p-1}{2}\right]} & \cdots & 0 \\ 0 & \cdots & 0 & 1-\beta_p z^{\left[\frac{p-1}{2}\right]+1} & 0 & \cdots & 0 \\ 0 & \cdots & -\beta_p z^{\left[\frac{p-1}{2}\right]+2} & 0 & 1 & \cdots & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_p z^{p-1} & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where $[x]$ represents the greatest integer less than or equal to x.

In what follows we denote by $P_{11}(z)$ the submatrix of $P(z)$ obtained by deleting the row 1 and the column 1. Applying Laplace theorem to the first row of the matrix $P(z)$ we deduce

$$\det P(z) = [1-((\alpha_p+\beta_p)^2+v_1\alpha_p^2)z^p]\cdot \det P_{11}(z)$$

$$= \begin{cases} (1-((\alpha_p+\beta_p)^2+v_1\alpha_p^2)z^p)\cdot(1-(\alpha_p+\beta_p)^2 z^p)^{\frac{p-1}{2}}, & \textit{for p odd,} \\ \\ (1-((\alpha_p+\beta_p)^2+v_1\alpha_p^2)z^p)\times \\ (1-(\alpha_p+\beta_p)z^{\left[\frac{p-1}{2}\right]+1})\cdot(1-(\alpha_p+\beta_p)^2 z^p]^{\frac{p-2}{2}}), & \textit{for p even.} \end{cases}$$

In fact, applying the formula of Schur ([3]) to the 2×2 block matrix $P_{11}(z)$ we deduce that $\det P_{11}(z)=\det(AD-CB)$*, since $AC=CA$ for every p:*

- *for p odd*

$$AC=\begin{bmatrix} \alpha_p^2 z^p & 0 & \cdots & 0 & -\alpha_p z \\ 0 & \alpha_p^2 z^p & \cdots & -\alpha_p z^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -\alpha_p z^{p-2} & \cdots & \alpha_p^2 z^p & 0 \\ -\alpha_p z^{p-1} & 0 & \cdots & 0 & \alpha_p^2 z^p \end{bmatrix}$$

- *when p is even*

$$AC=\begin{bmatrix} \alpha_p^2 z^p & \cdots & 0 & 0 & 0 & \cdots & -\alpha_p z \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ 0 & \cdots & \alpha_p^2 z^p & 0 & -\alpha_p z^{\left[\frac{p-1}{2}\right]} & \cdots & 0 \\ 0 & \cdots & 0 & -\alpha_p z^{\left[\frac{p-1}{2}\right]+1}\left(1-\alpha_p z^{\left[\frac{p-1}{2}\right]+1}\right) & 0 & \cdots & 0 \\ 0 & \cdots & -\alpha_p z^{\left[\frac{p-1}{2}\right]+2} & 0 & \alpha_p^2 z^p & \cdots & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_p z^{p-1} & \cdots & 0 & 0 & 0 & \cdots & \alpha_p^2 z^p \end{bmatrix}.$$

Let us prove that the determinant of $P_{11}(z)$ is that given above. We have $CB=\alpha_p\beta_p z^p\cdot I_{p-1}$, and the generic element of $AD=(a_{ij})$, $i,j=1,\dots,p-1$, when p is odd, is given by

$$a_{ij}=\begin{cases} 1+\alpha_p\beta_p z^p, & \text{if } i=j, \\ -(\alpha_p+\beta_p)z^i, & \text{if } j=p-i, \\ 0, & \text{otherwise}, \end{cases}$$

and when p is even is given by

$$a_{ij}=\begin{cases} 1+\alpha_p\beta_p z^p, & \text{if } i=j \text{ and } i\neq\left[\frac{p-1}{2}\right]+1, \\ -(\alpha_p+\beta_p)z^i, & \text{if } j=p-i \text{ and } i\neq\left[\frac{p-1}{2}\right]+1, \\ \left(1-\alpha_p z^i\right)\left(1-\beta_p z^i\right), & \text{if } i=j=\left[\frac{p-1}{2}\right]+1, \\ 0, & \text{otherwise}. \end{cases}$$

Thus when p is odd, and using the formula of Schur in the matrix $AD-CB$, we get

$$\det(AD-CB)=\begin{vmatrix} 1 & 0 & \cdots & 0 & -(\alpha_p+\beta_p)z \\ 0 & 1 & \cdots & -(\alpha_p+\beta_p)z^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -(\alpha_p+\beta_p)z^{p-2} & \cdots & 1 & 0 \\ -(\alpha_p+\beta_p)z^{p-1} & 0 & \cdots & 0 & 1 \end{vmatrix}$$

[3] [29, p. 46]: Let A, B, C, D be squared matrices of order n. If $AC=CA$, then

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix}=\det(AD-CB).$$

$$= \det \left(I_{\frac{p-1}{2}} - \begin{bmatrix} (\alpha_p+\beta_p)^2 z^p & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (\alpha_p+\beta_p)^2 z^p \end{bmatrix} \right)$$

$$= \begin{vmatrix} 1-(\alpha_p+\beta_p)^2 z^p & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1-(\alpha_p+\beta_p)^2 z^p \end{vmatrix} = [1-(\alpha_p+\beta_p)^2 z^p]^{\frac{p-1}{2}}.$$

On the other hand, when p is even, applying the Laplace theorem to the $([\frac{p-1}{2}]+1)$-th row of the matrix $AD-CB$ we obtain that its determinant is given by

$$\begin{vmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & -(\alpha_p+\beta_p)z \\ \vdots & \ddots & \vdots & \vdots & \vdots & ⋰ & \vdots \\ 0 & \cdots & 1 & 0 & -(\alpha_p+\beta_p)z^{[\frac{p-1}{2}]} & \cdots & 0 \\ 0 & \cdots & 0 & 1-(\alpha_p+\beta_p)z^{[\frac{p-1}{2}]+1} & 0 & \cdots & 0 \\ 0 & \cdots & -(\alpha_p+\beta_p)z^{[\frac{p-1}{2}]+2} & 0 & 1 & \cdots & 0 \\ \vdots & ⋰ & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(\alpha_p+\beta_p)z^{p-1} & \cdots & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= [1-(\alpha_p+\beta_p)z^{[\frac{p-1}{2}]+1}] \begin{vmatrix} 1 & \cdots & 0 & 0 & \cdots & -(\alpha_p+\beta_p)z \\ \vdots & \ddots & \vdots & \vdots & ⋰ & \vdots \\ 0 & \cdots & 1 & -(\alpha_p+\beta_p)z^{[\frac{p-1}{2}]} & \cdots & 0 \\ 0 & \cdots & -(\alpha_p+\beta_p)z^{[\frac{p-1}{2}]+2} & 1 & \cdots & 0 \\ \vdots & ⋰ & \vdots & \vdots & \ddots & \vdots \\ -(\alpha_p+\beta_p)z^{p-1} & \cdots & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= [1-(\alpha_p+\beta_p)z^{[\frac{p-1}{2}]+1}][1-(\alpha_p+\beta_p)^2 z^p]^{\frac{p-2}{2}},$$

where the last equality follows using the same strategy as in the case of the p being odd.

So, the necessary and sufficient condition for weak stationarity of the considered model is given by

$$(\alpha_p+\beta_p)^2 + v_1\alpha_p^2 < 1,$$

under $\alpha_p+\beta_p<1$, and plotted in Figure 3.1 considering different values of v_1. We note that, contrary to what happens in the necessary and sufficient condition of first-order stationarity (Theorem 3.1), in this case the condition is affected by the nature of the characteristic functions φ_t, via parameter v_1.

Let us remember that the case $v_1=0$ contains all the deterministic and independent of t characteristic functions φ_t. For instance, the dark to lightest gray region represented in Figure 3.1 corresponds, in particular, to the weak stationarity region for the INGARCH$(1,1)$, GP-INGARCH$(1,1)$, GEOMP2-INGARCH$(1,1)$ and NTA-INGARCH$(1,1)$ processes under the condition $\alpha_1+\beta_1<1$. We can also conclude, for example, that the weak stationarity region of the NTA-INGARCH$(1,1)$ process, for any value of parameter ϕ (dark to lightest gray) is larger than the weak stationarity region of the GEOMP-INGARCH$(1,1)$ process (recall in this case $v_1=2/r$) considering $r=0.4$ (darkest gray). These regions get closer when we increase the value of the parameter r.

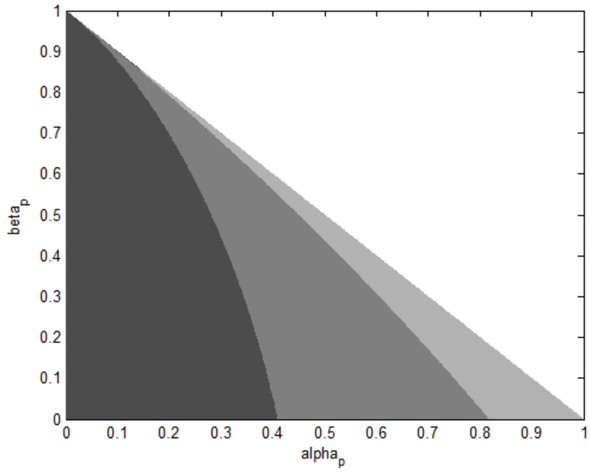

Fig. 3.1 Weak stationarity regions of a CP-INGARCH(p,p) model under the condition $\alpha_p+\beta_p<1$, with the coefficients $\alpha_1=...=\alpha_{p-1}=\beta_1=...=\beta_{p-1}=0$ and considering $v_1=5$ (darkest gray), 0.5 (darkest and medium gray) and 0 (dark to lightest gray).

Example 3.2 *Let us consider a CP-INGARCH$(2,2)$ model satisfying the hypothesis* **H2** *and such that $\sum_{j=1}^{2}(\alpha_j+\beta_j)<1$. To examine the necessary and sufficient condition of weak stationarity we consider the polynomial matrix $P(z)=I_3-B_1z-B_2z^2$, with B_1 and B_2 the 3×3 matrices given by*

$$B_1=\begin{bmatrix}(\alpha_1+\beta_1)^2+v_1\alpha_1^2 & 2(1+v_1)\alpha_2(\alpha_1+\beta_1) & 2(1+v_1)\beta_2(\alpha_1+\beta_1)\\ \alpha_1+\frac{\beta_1}{1+v_1} & \alpha_2 & \beta_2\\ \frac{\alpha_1+\beta_1}{1+v_1} & \alpha_2 & \beta_2\end{bmatrix},$$

$$B_2=\begin{bmatrix}(\alpha_2+\beta_2)^2+v_1\alpha_2^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Thus the determinant of this polynomial is

$$\begin{aligned}\det P(z) &= \left(1-[(\alpha_1+\beta_1)^2+v_1\alpha_1^2]z-[(\alpha_2+\beta_2)^2+v_1\alpha_2^2]z^2\right)(1-\alpha_2z)(1-\beta_2z)\\ &\quad -2(1+v_1)\left(\alpha_1+\frac{\beta_1}{1+v_1}\right)\alpha_2\beta_2(\alpha_1+\beta_1)z^3-2\alpha_2\beta_2(\alpha_1+\beta_1)^2z^3\\ &\quad -2(1+v_1)\beta_2(\alpha_1+\beta_1)^2(1-\alpha_2z)z^2\\ &\quad -2(1+v_1)\alpha_2(\alpha_1+\beta_1)(1-\beta_2z)\left(\alpha_1+\frac{\beta_1}{1+v_1}\right)z^2\\ &\quad -\alpha_2\beta_2\left(1-[(\alpha_1+\beta_1)^2+v_1\alpha_1^2]z-[(\alpha_2+\beta_2)^2+v_1\alpha_2^2]z^2\right)z^2\\ &= 1-\left[(\alpha_1+\beta_1)^2+\alpha_2+\beta_2+v_1\alpha_1^2\right]z\\ &\quad -\left[(\alpha_1+\beta_1)^2(\alpha_2+\beta_2)+(\alpha_2+\beta_2)^2+v_1(\alpha_2^2-\alpha_1^2\beta_2+\alpha_1^2\alpha_2+2\alpha_1\alpha_2\beta_1)\right]z^2\\ &\quad -\left[-(\alpha_2+\beta_2)^3-v_1\alpha_2^2(\alpha_2+\beta_2)\right]z^3.\end{aligned}$$

Using, for instance, the Matlab software we can exhibit the roots of $\det P(z)$ *in some particular cases. For example, if we consider $\alpha_1=\alpha_2=\beta_1$, $\beta_2=0$ and $v_1=0$ we obtain the roots*

$$z_1 = -\frac{1}{\alpha_1}, \quad z_2 = \frac{2\alpha_1 - 2\alpha_1^2\left(\sqrt{\frac{1+\alpha_1}{\alpha_1^3}}+1\right)}{\alpha_1}, \quad z_3 = \frac{2\alpha_1 + 2\alpha_1^2\left(\sqrt{\frac{1+\alpha_1}{\alpha_1^3}}+1\right)}{\alpha_1},$$

which are always outside the unit circle under the first-order stationarity condition $3\alpha_1 < 1$. *On the other hand, when we consider* $\alpha_1 = \beta_1 = 0$ *we get the roots*

$$z_1 = \frac{1}{\alpha_2+\beta_2}, \quad z_2 = \frac{1}{\sqrt{(\alpha_2+\beta_2)^2+v_1\alpha_2^2}}, \quad z_3 = -\frac{1}{\sqrt{(\alpha_2+\beta_2)^2+v_1\alpha_2^2}},$$

from where we deduce the condition $(\alpha_2+\beta_2)^2+v_1\alpha_2^2 < 1$ *already stated in Example* 3.1.

Taking $\beta_2 = 0$, *we plot in Figure* 3.2 *the weak stationarity regions of a CP-INGARCH*$(2,1)$ *process considering different values for* v_1, *namely,* $v_1 = 0$, 0.5 *and* 5 *(*[4]*).*

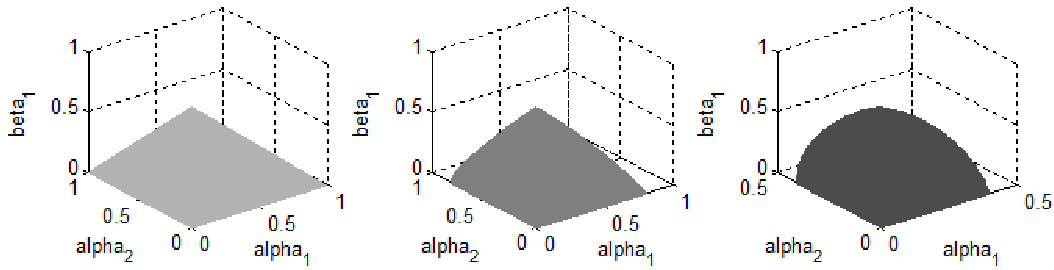

Fig. 3.2 Weak stationarity regions of a CP-INGARCH$(2,1)$ model with $\alpha_1+\alpha_2+\beta_1<1$, considering $v_1 = 0$ (lightest gray), 0.5 (medium gray) and 5 (darkest gray).

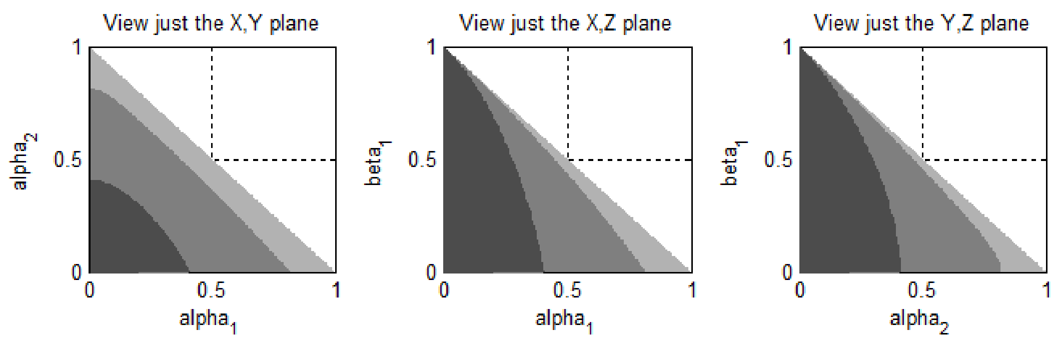

Fig. 3.3 The three planes that define the weak stationarity regions of Figure 3.2.

We conclude that, as in Example 3.1, *when we increase the value of* v_1 *the weak stationarity region becomes smaller. To better view, we represent in Figure* 3.3 *the three planes that define the mentioned regions where it is now clear that the weak stationarity region with* $v_1 = 5$ *(darkest gray) is contained in the weak stationarity region with* $v_1 = 0.5$ *(darkest and medium gray) and this one is contained in the weak stationarity region with* $v_1 = 0$ *(dark to lightest gray). Let us note that the first plot of the Figure* 3.3 *corresponds to the CP-INARCH(*2*) weak stationarity regions.*

In Appendix *B* we establish another necessary condition for weak stationarity which coincides with the previous one in some particular cases, as we will prove.

[4]The program in MATLAB code can be found in Appendix *D*.2.

3.3 Moments structure

The autocorrelation function for the CP-INGARCH(p,q) model is derived in this section and illustrated when $p=q=1$. Moreover, in this particular case, we deduce a necessary and sufficient condition for the existence of unconditional moments of any order of the process.

3.3.1 The autocovariance function

We recall that in Theorem 3.2 we derive some values for the autocovariance function. Despite its simplicity, especially when the orders p and q are low, these expressions are quite insufficient to describe the autocovariance function. In fact, for example, for a CP-INGARCH$(1,1)$ such expressions allow us only to determine the value of $V(X_t)$. In the next theorem we derive a set of equations from which the autocovariance function of the general CP-INGARCH(p,q) model can be obtained.

Theorem 3.3 *Let X be a weakly stationary CP-INGARCH(p,q) process. The autocovariances of the processes X and λ, respectively Γ and $\widetilde{\Gamma}$, satisfy the linear equations*

$$\Gamma(h)=\sum_{j=1}^{p}\alpha_j\cdot\Gamma(h-j)+\sum_{k=1}^{\min(h-1,q)}\beta_k\cdot\Gamma(h-k)+\sum_{k=h}^{q}\beta_k\cdot\widetilde{\Gamma}(k-h),\qquad h\geq 1,$$

$$\widetilde{\Gamma}(h)=\sum_{j=1}^{\min(h,p)}\alpha_j\cdot\widetilde{\Gamma}(h-j)+\sum_{j=h+1}^{p}\alpha_j\cdot\Gamma(j-h)+\sum_{k=1}^{q}\beta_k\cdot\widetilde{\Gamma}(h-k),\qquad h\geq 0,$$

assuming that $\sum_{k=h}^{q}\beta_k\cdot\widetilde{\Gamma}(k-h)=0$ if $h>q$ and $\sum_{j=h+1}^{p}\alpha_j\cdot\Gamma(j-h)=0$ if $h\geq p$.

Proof: From expressions (3.3) and (3.4) we have

$$E\left(X_tX_{t-h}\right)=\alpha_0\mu+\sum_{j=1}^{p}\alpha_jE\left(X_{t-j}X_{t-h}\right)+\sum_{k=1}^{\min(h-1,q)}\beta_kE\left(X_{t-k}X_{t-h}\right)+\sum_{k=h}^{q}\beta_kE\left(\lambda_{t-k}\lambda_{t-h}\right),\quad h\geq 1,$$

$$E\left(\lambda_t\lambda_{t-h}\right)=\alpha_0\mu+\sum_{j=1}^{\min(h,p)}\alpha_jE\left(\lambda_{t-j}\lambda_{t-h}\right)+\sum_{j=h+1}^{p}\alpha_jE\left(X_{t-j}X_{t-h}\right)+\sum_{k=1}^{q}\beta_kE\left(\lambda_{t-k}\lambda_{t-h}\right),\quad h\geq 0.$$

Thus, for $h\geq 1$, we obtain

$$\Gamma(h)=E[(X_t-\mu)(X_{t-h}-\mu)]=E\left(X_tX_{t-h}\right)-\mu^2$$

$$=\alpha_0\mu+\sum_{j=1}^{p}\alpha_j\cdot\Gamma(h-j)+\sum_{k=1}^{\min(h-1,q)}\beta_k\cdot\Gamma(h-k)+\sum_{k=h}^{q}\beta_k\cdot\widetilde{\Gamma}(k-h)+\mu^2\left(\sum_{j=1}^{p}\alpha_j+\sum_{k=1}^{q}\beta_k\right)-\mu^2$$

$$=\sum_{j=1}^{p}\alpha_j\cdot\Gamma(h-j)+\sum_{k=1}^{\min(h-1,q)}\beta_k\cdot\Gamma(h-k)+\sum_{k=h}^{q}\beta_k\cdot\widetilde{\Gamma}(k-h),$$

and, for $h\geq 0$, proceeding in an analogous way

$$\widetilde{\Gamma}(k)=E[(\lambda_t-\mu)(\lambda_{t-h}-\mu)]=E(\lambda_t\lambda_{t-h})-\mu^2$$

$$= \sum_{j=1}^{\min(h,p)} \alpha_j \cdot \widetilde{\Gamma}(h-j) + \sum_{j=h+1}^{p} \alpha_j \cdot \Gamma(j-h) + \sum_{k=1}^{q} \beta_k \cdot \widetilde{\Gamma}(h-k),$$

which completes the proof. ■

We point out that this general result includes those of Weiß [77], Zhu [81, 82] and Xu et al. [80] on INGARCH, NB-INGARCH, GP-INGARCH and NB-DINARCH models as special cases.

The autocovariance function of the CP-INGARCH(p,q) process indicates that it can be represented as an ARMA process. In fact, as a consequence of Theorem 3.3 we have

$$\Gamma(h) = \sum_{j=1}^{p} \alpha_j \cdot \Gamma(h-j) + \sum_{k=1}^{\min(h-1,q)} \beta_k \cdot \Gamma(h-k) + \sum_{k=h}^{q} \beta_k \cdot \widetilde{\Gamma}(k-h), \quad h \geq 1,$$

$$= \begin{cases} \sum_{j=1}^{p} \alpha_j \cdot \Gamma(h-j) + \sum_{k=1}^{h-1} \beta_k \cdot \Gamma(h-k) + \sum_{k=h}^{q} \beta_k \cdot \widetilde{\Gamma}(k-h), & 1 \leq h \leq q, \\ \sum_{j=1}^{p} \alpha_j \cdot \Gamma(h-j) + \sum_{k=1}^{q} \beta_k \cdot \Gamma(h-k), & h \geq q+1, \end{cases}$$

which means that $\{X_t - \mu\}_{t \in \mathbb{Z}}$ has an ARMA$(\max(p,q),q)$ representation [12, p. 90]. We point out that this result may be useful in the identification of the model, i.e., in the choice of the orders p and q.

For a CP-INGARCH$(1,1)$ model we are able to present explicitly the autocovariance function.

Corollary 3.2 *Let X be a weakly stationary CP-INGARCH*$(1,1)$ *process.*

The autocovariances of X are given by

$$\Gamma(h) = \frac{\alpha_1(1-\beta_1(\alpha_1+\beta_1))(\alpha_1+\beta_1)^{h-1}}{1-(\alpha_1+\beta_1)^2+\alpha_1^2}\,\Gamma(0), \qquad h \geq 1.$$

Proof: From Theorem 3.3, we obtain for $h \geq 2$ that

$$\Gamma(h) = \alpha_1 \cdot \Gamma(h-1) + \beta_1 \cdot \Gamma(h-1) = (\alpha_1+\beta_1) \cdot \Gamma(h-1) = \ \ldots$$

$$= (\alpha_1+\beta_1)^{h-1} \cdot \Gamma(1) = (\alpha_1+\beta_1)^{h-1}[\alpha_1 \cdot \Gamma(0) + \beta_1 \cdot \widetilde{\Gamma}(0)]. \tag{3.12}$$

To determine an expression for $V(\lambda_t) = \widetilde{\Gamma}(0)$, we note first that for $h \geq 1$,

$$\widetilde{\Gamma}(h) = \alpha_1 \cdot \widetilde{\Gamma}(h-1) + \beta_1 \cdot \widetilde{\Gamma}(h-1) = \ldots \ = (\alpha_1+\beta_1)^h \cdot \widetilde{\Gamma}(0),$$

$$\begin{aligned} \widetilde{\Gamma}(0) &= \alpha_1 \cdot \Gamma(1) + \beta_1 \cdot \widetilde{\Gamma}(1) \\ &= \alpha_1^2 \cdot \Gamma(0) + \alpha_1\beta_1 \cdot \widetilde{\Gamma}(0) + \beta_1(\alpha_1+\beta_1) \cdot \widetilde{\Gamma}(0) \\ &= \alpha_1^2 \cdot \Gamma(0) + [(\alpha_1+\beta_1)^2 - \alpha_1^2] \cdot \widetilde{\Gamma}(0) \\ \Leftrightarrow \widetilde{\Gamma}(0) &= \frac{\alpha_1^2 \cdot \Gamma(0)}{1-(\alpha_1+\beta_1)^2+\alpha_1^2}. \end{aligned}$$

Therefore, replacing the previous expression in equation (3.12) we obtain

$$\begin{aligned}\Gamma(h) &= (\alpha_1+\beta_1)^{h-1}\cdot\left[\alpha_1\cdot\Gamma(0)+\frac{\alpha_1^2\beta_1\cdot\Gamma(0)}{1-(\alpha_1+\beta_1)^2+\alpha_1^2}\right]\\ &= (\alpha_1+\beta_1)^{h-1}\left[\frac{\alpha_1-\alpha_1^3-2\alpha_1^2\beta_1-\alpha_1\beta_1^2+\alpha_1^3+\alpha_1^2\beta_1}{1-(\alpha_1+\beta_1)^2+\alpha_1^2}\right]\Gamma(0)\\ &= (\alpha_1+\beta_1)^{h-1}\,\frac{\alpha_1(1-\beta_1(\alpha_1+\beta_1))}{1-(\alpha_1+\beta_1)^2+\alpha_1^2}\,\Gamma(0), \qquad h\geq 1. \quad \blacksquare\end{aligned}$$

So in this case the autocorrelations of X and λ, under the weak stationarity, are respectively given by

$$\rho(h)=\frac{\Gamma(h)}{\Gamma(0)}=\frac{\alpha_1(1-\beta_1(\alpha_1+\beta_1))}{1-(\alpha_1+\beta_1)^2+\alpha_1^2}\,(\alpha_1+\beta_1)^{h-1}, \qquad h\geq 1,$$

$$\widetilde{\rho}(h)=(\alpha_1+\beta_1)^h, \qquad h\geq 0.$$

We underline that for any CP-INGARCH$(1,1)$ process the correlation between the values at different times is independent of its conditional distribution. To illustrate these expressions we plot in Figure 3.4 the empirical autocorrelation function of the INGARCH$(1,1)$ and NTA-INGARCH$(1,1)$ models corresponding to the trajectories of the Figures 2.3 and 2.4, respectively ([5]). We observe that these processes are weakly stationary since the parameters $\alpha_1=0.4$ and $\beta_1=0.5$ belongs to the weak stationarity region (see Figure 3.1). Some theoretical values according to the above formulas are, for instance, $\rho(1)\simeq 0.629$, $\rho(2)\simeq 0.566$, $\rho(3)\simeq 0.509$, $\rho(6)\simeq 0.371$, $\rho(10)\simeq 0.244$, $\rho(16)\simeq 0.129$, from which some closeness with the empirical autocorrelation values is evident.

Autocorrelation		AC
	1	0.612
	2	0.550
	3	0.508
	4	0.460
	5	0.377
	6	0.341
	7	0.281
	8	0.264
	9	0.211
	10	0.185
	11	0.170
	12	0.162
	13	0.112
	14	0.114
	15	0.106
	16	0.077

Autocorrelation		AC
	1	0.632
	2	0.581
	3	0.514
	4	0.437
	5	0.410
	6	0.321
	7	0.305
	8	0.273
	9	0.246
	10	0.194
	11	0.204
	12	0.196
	13	0.180
	14	0.161
	15	0.165
	16	0.163

Fig. 3.4 Autocorrelation function of an INGARCH$(1,1)$ (on the left) and NTA-INGARCH$(1,1)$ model with $\phi=2$ (on the right): $\alpha_0=10$, $\alpha_1=0.4$ and $\beta_1=0.5$.

[5]These correlograms are obtained with the aid of the software EViews when we generated the trajectories of Section 2.3.

Remark 3.5 *If the hypothesis* **H2** *is satisfied then the variance of X_t is given by*

$$\Gamma(0)=\mu\ \frac{(v_0+v_1\mu)[1-(\alpha_1+\beta_1)^2+\alpha_1^2]}{1-(\alpha_1+\beta_1)^2-v_1\alpha_1^2},$$

where $\mu=\alpha_0/(1-\alpha_1-\beta_1)$. In fact, the variance of X_t can be obtained from

$$V(X_t)=E[(v_0+v_1\lambda_t)\lambda_t]+V(\lambda_t)=v_0\mu+v_1E(\lambda_t^2)+V(\lambda_t)$$

$$=v_0\mu+(1+v_1)V(\lambda_t)+v_1\mu^2$$

$$\Leftrightarrow\ \Gamma(0)=v_0\mu+\frac{(1+v_1)\alpha_1^2\cdot\Gamma(0)}{1-(\alpha_1+\beta_1)^2+\alpha_1^2}+v_1\mu^2$$

$$\Leftrightarrow\ \left[1-(\alpha_1+\beta_1)^2-v_1\alpha_1^2\right]\Gamma(0)=\mu(v_0+v_1\mu)[1-(\alpha_1+\beta_1)^2+\alpha_1^2]$$

$$\Leftrightarrow\ \Gamma(0)=\mu\ \frac{(v_0+v_1\mu)[1-(\alpha_1+\beta_1)^2+\alpha_1^2]}{1-(\alpha_1+\beta_1)^2-v_1\alpha_1^2}.$$

We observe that the variance $\Gamma(0)$ can also be obtained by applying the formula stated in Theorem 3.2 *since, in this case, we have*

$$P(1)=1-B_1=1-(\alpha_1+\beta_1)^2-v_1\alpha_1^2,$$

$$B_0=v_0\mu+(1+v_1)[2\alpha_0\mu-\alpha_0^2]-v_0\mu(2\alpha_1\beta_1+\beta_1^2).$$

Therefore, if we consider $\alpha_0=10$, $\alpha_1=0.4$ and $\beta_1=0.5$ then $\Gamma(0)=\dfrac{35(v_0+100v_1)}{0.19-0.16v_1}$.

In particular, for the INGARCH$(1,1)$ process we get $\sqrt{\Gamma(0)}\simeq 13.572$ ($v_0=1$ and $v_1=0$), for the NTA-INGARCH$(1,1)$ process with $\phi=2$ we get $\sqrt{\Gamma(0)}\simeq 23.508$ ($v_0=1+\phi$ and $v_1=0$), and for the GEOMP-INGARCH$(1,1)$ process with $r=2$ we get $\sqrt{\Gamma(0)}\simeq 343.269$ ($v_0=1$ and $v_1=2/r$). Comparing these theoretical values, respectively, with those presented in the simulated trajectories of Figures 2.3 *(= 13.247), 2.4 (= 24.042) and 2.5 (= 133.902), we notice a discrepancy in the GEOMP-INGARCH which may be a consequence from its hight variability.*

3.3.2 Moments of a CP-INGARCH(1,1)

In this section we give a necessary and sufficient condition for the existence of all moments of a CP-INGARCH$(1,1)$. The study undertaken allows us to establish this result in a subclass of models for which the characteristic functions φ_t satisfy

Hypothesis H3 : φ_t is deterministic.

We underline that this particular case still includes a wide class of models, many of them here introduced, as the GEOMP2-INGARCH or the NTA-INGARCH processes.

We start by presenting the next lemma which proof is in Appendix C.1.

Lemma 3.1 *For $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$, the m-th derivative of the function $\varphi_t^k = \prod_{j=1}^k \varphi_t$ is given by*

$$\left(\varphi_t^k\right)^{(m)}(u) = \sum_{n=\max\{0,m-k\}}^{m-1} \frac{k!}{(k-m+n)!}\, \varphi_t^{k-m+n}(u) \times$$

$$\sum_{\substack{k_1+\ldots+k_m=m-n\\ k_1+2k_2+\ldots+mk_m=m\\ k_r\in\mathbb{N}_0}} (m;k_1,\ldots,k_m)\left[\varphi_t'(u)\right]^{k_1}\ldots[\varphi_t^{(m)}(u)]^{k_m}, \qquad u\in\mathbb{R}, \tag{3.13}$$

where $(m;k_1,\ldots,k_m)$ represents the number of ways of partitioning a set of $m = k_1+2k_2+\ldots+mk_m$ different objects into k_r subsets containing r objects for $r = 1,2,\ldots,m$, i.e., (see, e.g., [2, p. 823])

$$(m;k_1,\ldots,k_m) = \frac{m!}{(1!)^{k_1}k_1!(2!)^{k_2}k_2!\ldots(m!)^{k_m}k_m!}.$$

Theorem 3.4 *Let X be a CP-INGARCH$(1,1)$ model such that the hypothesis* **H3** *is satisfied. The moments of X are all finite if and only if $\alpha_1+\beta_1<1$.*

Proof: Let us recall the representation (2.5) stated in Section 2.3. Since the $X_{t,j}$, $j=1,\ldots,N_t$, are i.i.d. random variables with common characteristic function φ_t, the characteristic function of the sum $\sum_{j=1}^k X_{t,j}$ is $\prod_{j=1}^k \varphi_t(u) = \varphi_t^k(u)$, $u\in\mathbb{R}$. As $X_t|\underline{X}_{t-1}$ is a compound Poisson random variable, we have, according to Remark 2.3,

$$\begin{aligned} E[X_t^m|\underline{X}_{t-1}] &= \sum_{r=0}^m \frac{(\lambda_t^*)^r}{r!}\left\{\sum_{k=0}^r \binom{r}{k}(-1)^{r-k}E\left[\left(\sum_{j=1}^k X_{t,j}\right)^m|\underline{X}_{t-1}\right]\right\}\\ &= \sum_{r=0}^m \frac{1}{r!}\frac{\lambda_t^r}{(\varphi_t'(0))^r}\sum_{k=0}^r\binom{r}{k}\frac{(-1)^{r-k}}{i^{m-r}}\left(\varphi_t^k\right)^{(m)}(0), \qquad m\geq 1,\end{aligned}$$

with $\left(\varphi_t^k\right)^{(m)}$ given by expression (3.13). Thus,

$$E[X_t^m] = \sum_{r=0}^m\sum_{k=0}^r \frac{1}{r!}\binom{r}{k}\frac{(-1)^{r-k}\left(\varphi_t^k\right)^{(m)}(0)}{i^{m-r}(\varphi_t'(0))^r}E[\lambda_t^r]. \tag{3.14}$$

The binomial formula yields

$$\lambda_t^r = (\alpha_0+\alpha_1X_{t-1}+\beta_1\lambda_{t-1})^r = \sum_{n=0}^r\binom{r}{n}\alpha_0^{r-n}\sum_{l=0}^n\binom{n}{l}\alpha_1^l\beta_1^{n-l}X_{t-1}^l\lambda_{t-1}^{n-l}$$

$$= \alpha_0^r + \sum_{n=1}^r\binom{r}{n}\alpha_0^{r-n}\left[\beta_1^n\lambda_{t-1}^n + \sum_{l=1}^n\binom{n}{l}\alpha_1^l\beta_1^{n-l}X_{t-1}^l\lambda_{t-1}^{n-l}\right].$$

As λ_{t-1}^{n-l} is $\underline{X}_{t-2}$-measurable we obtain

$$E[X_{t-1}^l\lambda_{t-1}^{n-l}|\underline{X}_{t-2}] = \sum_{v=0}^l \frac{1}{v!}\frac{\lambda_{t-1}^{v+n-l}}{(\varphi_{t-1}'(0))^v}\sum_{x=0}^v\binom{v}{x}\frac{(-1)^{v-x}}{i^{l-v}}\left(\varphi_{t-1}^x\right)^{(l)}(0), \qquad l\geq 1,$$

and consequently,

$$\begin{aligned} E[\lambda_t^r|\underline{X}_{t-2}] &= \alpha_0^r+\sum_{n=1}^{r}\binom{r}{n}\alpha_0^{r-n}\left[\beta_1^n\lambda_{t-1}^n+\sum_{l=1}^{n}\binom{n}{l}\alpha_1^l\beta_1^{n-l}\lambda_{t-1}^{n-l}\,E[X_{t-1}^l|\underline{X}_{t-2}]\right] \\ &= \alpha_0^r+\sum_{n=1}^{r}\binom{r}{n}\alpha_0^{r-n}\beta_1^n\lambda_{t-1}^n+\sum_{n=1}^{r}\binom{r}{n}\alpha_0^{r-n}\sum_{l=1}^{n}\binom{n}{l}\times \\ &\quad\times\sum_{v=0}^{l}\frac{\alpha_1^l\beta_1^{n-l}}{v!(\varphi_{t-1}'(0))^v}\sum_{x=0}^{v}\binom{v}{x}\frac{(-1)^{v-x}}{i^{l-v}}\left(\varphi_{t-1}^x\right)^{(l)}(0)\;\lambda_{t-1}^{v+n-l}. \end{aligned} \tag{3.15}$$

Let $\Lambda_t=(\lambda_t^m,...,\lambda_t)^\top$. In the algebraic expression of $E[\lambda_t^r|\underline{X}_{t-2}]$, for $r=1,...,m$, all the powers of λ_{t-1} are less or equal to r. Therefore, a constant vector $\mathbf{d}$ and an upper triangular matrix $\mathbf{D}=(d_{ij})$, $i,j=1,...,m$, exist such that the following equation is satisfied:

$$E[\Lambda_t|\underline{X}_{t-2}] \;=\; \mathbf{d}+\mathbf{D}\Lambda_{t-1}$$

$$\Leftrightarrow \begin{bmatrix} E[\lambda_t^m|\underline{X}_{t-2}] \\ \vdots \\ E[\lambda_t^2|\underline{X}_{t-2}] \\ E[\lambda_t|\underline{X}_{t-2}] \end{bmatrix} = \begin{bmatrix} \alpha_0^m \\ \vdots \\ \alpha_0^2 \\ \alpha_0 \end{bmatrix} + \begin{bmatrix} (\alpha_1+\beta_1)^m & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & (\alpha_1+\beta_1)^2 & * \\ 0 & \cdots & 0 & \alpha_1+\beta_1 \end{bmatrix} \begin{bmatrix} \lambda_{t-1}^m \\ \vdots \\ \lambda_{t-1}^2 \\ \lambda_{t-1} \end{bmatrix}.$$

Indeed, let us prove that the diagonal entries of the matrix $\mathbf{D}$ are those given above.

The k-th diagonal entry of the matrix $\mathbf{D}$ corresponds to the case where in equation (3.15), we consider $r=m-k+1$. Thus, to obtain the coefficient of λ_{t-1}^{m-k+1}, we look at the terms corresponding to $n=m-k+1$ and $l=v$. Then, we get

$$\begin{aligned} d_{kk} &= \beta_1^{m-k+1}+\sum_{l=1}^{m-k+1}\binom{m-k+1}{l}\frac{\alpha_1^l\beta_1^{m-k+1-l}}{l!(\varphi_{t-1}'(0))^l}\sum_{x=0}^{l}\binom{l}{x}(-1)^{l-x}\left(\varphi_{t-1}^x\right)^{(l)}(0) \\ &= \beta_1^{m-k+1}+\sum_{l=1}^{m-k+1}\binom{m-k+1}{l}\alpha_1^l\beta_1^{m-k+1-l} \\ &= (\alpha_1+\beta_1)^{m-k+1}, \qquad\qquad k=1,...,m, \end{aligned}$$

since it can be proved that

$$\sum_{x=0}^{l}\binom{l}{x}(-1)^{l-x}\left(\varphi_{t-1}^x\right)^{(l)}(0)=l!\;[\varphi_{t-1}'(0)]^l. \tag{3.16}$$

In fact, using the expression (3.13) we obtain

$$\begin{aligned} \sum_{x=0}^{l}\binom{l}{x}(-1)^{l-x}\left(\varphi_{t-1}^x\right)^{(l)}(0) &= \sum_{x=0}^{l}\binom{l}{x}(-1)^{l-x}\sum_{j=l-x}^{l-1}\frac{x!}{(x-l+j)!}\,\varphi_{t-1}^{x-l+j}(0) \\ &\quad\times\sum_{\substack{k_1+...+k_l=l-j \\ k_1+2k_2+...+lk_l=l \\ k_r\in\mathbb{N}_0}}(l;k_1,...,k_l)\left[\varphi_{t-1}'(0)\right]^{k_1}...\,[\varphi_{t-1}^{(l)}(0)]^{k_l}, \end{aligned}$$

and thus, for any arbitrarily fixed $k_1,...,k_l \in \mathbb{N}_0$ such that $k_1+...+k_l = l-j$ and $k_1+2k_2+...+lk_l = l$, the coefficient of $\left[\varphi'_{t-1}(0)\right]^{k_1} \dots [\varphi^{(l)}_{t-1}(0)]^{k_l}$ is given by

$$\left[\sum_{x=0}^{l}\begin{pmatrix} l \\ x \end{pmatrix}(-1)^{l-x}\frac{x!}{(x-(k_1+...+k_l))!}\right](l;k_1,...,k_l)$$

$$=\left[\sum_{x=k_1+...+k_l}^{l}\frac{(-1)^{l-x}}{(l-x)!(x-(k_1+...+k_l))!}\right]l!(l;k_1,...,k_l)$$

$$=\frac{(-1)^{l-(k_1+...+k_l)}}{(l-(k_1+...+k_l))!}\left[\sum_{m=0}^{l-(k_1+...+k_l)}\begin{pmatrix} l-(k_1+...+k_l) \\ m \end{pmatrix}(-1)^{-m}\right]l!(l;k_1,...,k_l).$$

When $k_1 = l$, $k_2 = \ldots = k_l = 0$, we obtain the coefficient $l!(l;l,0,...,0) = l!$. Otherwise, the coefficient equals zero. Therefore, we finally conclude the equality (3.16).

Since $\underline{X}_{t-3} \subset \underline{X}_{t-2}$, we obtain

$$E[\Lambda_t|\underline{X}_{t-3}] = E\left[E(\Lambda_t|\underline{X}_{t-2})|\underline{X}_{t-3}\right] = E[\mathbf{d}+\mathbf{D}\Lambda_{t-1}|\underline{X}_{t-3}]$$

$$= \mathbf{d}+\mathbf{D}\,E[\Lambda_{t-1}|\underline{X}_{t-3}] = \mathbf{d}+\mathbf{D}\left(\mathbf{d}+\mathbf{D}\Lambda_{t-2}\right) = \mathbf{d}+\mathbf{Dd}+\mathbf{D}^2\Lambda_{t-2}.$$

Iterating this recurrence l times gives:

$$E[\Lambda_t|\underline{X}_{t-2-l}] = \left(I_m+\mathbf{D}+\mathbf{D}^2+\ldots+\mathbf{D}^l\right)\mathbf{d}+\mathbf{D}^{l+1}\Lambda_{t-(l+1)}.$$

Substituting $k = l+2$ leads to:

$$E[\Lambda_t|\underline{X}_{t-k}] = \left(\sum_{r=0}^{k-2}D^r\right)\mathbf{d}+\mathbf{D}^{k-1}\Lambda_{t-(k-1)}.$$

If the eigenvalues of $\mathbf{D}$ are inside the unit circle, i.e., if for any eigenvalue λ of $\mathbf{D}$ we have $|\lambda| < 1$ then there is a norm $\|.\|$ under $\mathbb{R}^m$ such that $\|\mathbf{D}\| < 1$ [51, Theorem 3.32] and then the matrix $I_m - \mathbf{D}$, where I_m represents the identity matrix of order m, is invertible and we can write

$$\sum_{r=0}^{k-2}D^r = \sum_{r=0}^{\infty}D^r - \sum_{r=0}^{\infty}D^{r+k-1} = \sum_{r=0}^{\infty}D^r - D^{k-1}\sum_{r=0}^{\infty}D^r = (I_m-\mathbf{D})^{-1}(I_m-\mathbf{D}^{k-1}),$$

that is, we get

$$E[\Lambda_t|\underline{X}_{t-k}] = (\mathbf{I}_m-\mathbf{D})^{-1}(I_m-\mathbf{D}^{k-1})\mathbf{d}+\mathbf{D}^{k-1}\Lambda_{t-(k-1)}.$$

So, we conclude that the eigenvalues of $\mathbf{D}$ (which coincide with its diagonal entries because it is an upper triangular matrix) are inside the unit circle if and only if $\alpha_1+\beta_1 < 1$. Consequently, since $\mathbf{D}^{k-1} \to 0$ when $k \to \infty$, we have

$$\lim_{k\to\infty} E[\Lambda_t|\underline{X}_{t-k}] = (I_m-\mathbf{D})^{-1}\mathbf{d} = E[\Lambda_t],$$

and then from (3.14) all the moments of X_t of order $\leq m$ are finite. ■

As in the particular case studied by Ferland et al. [25] we assure the existence of all moments of this process under a very simple condition on the model coefficients. In accordance to these authors, we point out that it is an unexpected result taking into consideration what is known on the complexity analysis and on the conditions of moments existence for conditional heteroscedastic models in general.

In the following we illustrate the expressions stated above in some particular cases.

Remark 3.6 *Let us consider the*

- *INGARCH model:* $\left(\varphi^k\right)^{(m)}(u) = (ik)^m e^{iku}$ *and* $\left(\varphi^k\right)^{(m)}(0) = (ik)^m$, $u \in \mathbb{R}$, $m \geq 1$. *Thus*

$$E[X_t^m] \;=\; \sum_{r=0}^{m}\sum_{k=0}^{r}\frac{1}{r!}\left(\begin{array}{c} r \\ k \end{array}\right)(-1)^{r-k}k^m\, E[\lambda_t^r] = \sum_{r=0}^{m} S(m,r)E[\lambda_t^r]. \quad (^6)$$

  A complete proof for this particular case can be found in [25].

- *NTA-INGARCH model: We have*

$$\begin{aligned} \varphi^k(u) &= \exp\left(k\phi\left(e^{iu}-1\right)\right), \\ \left(\varphi^k\right)^{(m)}(u) &= i^m\sum_{j=1}^{m} S(m,j)(k\phi)^j \exp\left(k\phi\left(e^{iu}-1\right)+iju\right), \qquad u\in\mathbb{R}. \end{aligned} \tag{3.17}$$

  In fact, for $m = 1$, *we get*

$$\left(\varphi^k(u)\right)' \;=\; ik\phi\exp\left(k\phi\left(e^{iu}-1\right)+iu\right),$$

  and, by induction,

$$\begin{aligned} \left(\varphi^k\right)^{(m+1)}(u) &= \frac{d}{du}\left[i^m\sum_{j=1}^{m} S(m,j)(k\phi)^j\exp\left(k\phi\left(e^{iu}-1\right)+iju\right)\right] \\ &= i^m\sum_{j=1}^{m} S(m,j)(k\phi)^j\left(ik\phi e^{iu}+ij\right)\exp\left(k\phi\left(e^{iu}-1\right)+iju\right) \\ &= i^{m+1}\sum_{j=1}^{m} S(m,j)\left[(k\phi)^{j+1}\exp\left(k\phi\left(e^{iu}-1\right)+i(j+1)u\right)\right. \\ &\quad \left.+j(k\phi)^j\exp\left(k\phi\left(e^{iu}-1\right)+iju\right)\right] \\ &= i^{m+1}\sum_{j=2}^{m}(k\phi)^j[S(m,j-1)+jS(m,j)]\exp\left(k\phi\left(e^{iu}-1\right)+iju\right) \\ &\quad +i^{m+1}k\phi\exp\left(k\phi\left(e^{iu}-1\right)+iu\right) \\ &\quad +i^{m+1}(k\phi)^{m+1}\exp\left(k\phi\left(e^{iu}-1\right)+i(m+1)u\right) \\ &= i^{m+1}\sum_{j=1}^{m+1} S(m+1,j)(k\phi)^j\exp\left(k\phi\left(e^{iu}-1\right)+iju\right), \end{aligned}$$

---

[6] [1, p. 835]: A closed form for the Stirling numbers of the second kind is

$$S(m,r) = \frac{1}{r!}\sum_{k=0}^{r}(-1)^{r-k}\left(\begin{array}{c} r \\ k \end{array}\right)k^m.$$

from the recurrence relation of the Stirling numbers of the second kind and so (3.17) *is proved.*

Taking $u = 0$, we conclude

$$\left(\varphi^k\right)^{(m)}(0) = i^m \sum_{j=1}^{m} S(m,j)(k\phi)^j.$$

Thus,

$$\begin{aligned} E[X_t^m] &= \sum_{r=0}^{m} \sum_{k=0}^{r} \frac{1}{r!} \binom{r}{k} \frac{(-1)^{r-k}}{\phi^r} \sum_{j=1}^{m} S(m,j)(k\phi)^j \, E[\lambda_t^r], && (\textit{recall } (3.14)) \\ E[\lambda_t^r | \underline{X}_{t-2}] &= \alpha_0^r + \sum_{n=1}^{r} \binom{r}{n} \alpha_0^{r-n} \beta_1^n \lambda_{t-1}^n + \sum_{n=1}^{r} \binom{r}{n} \alpha_0^{r-n} \sum_{l=1}^{n} \binom{n}{l} \times \\ &\times \sum_{v=0}^{l} \frac{\alpha_1^l \beta_1^{n-l}}{v! \, \phi^v} \sum_{k=1}^{l} \sum_{j=0}^{v} (-1)^{v-j} S(l,k) \binom{v}{j} (j\phi)^k \lambda_{t-1}^{v+n-l}, && (\textit{recall } (3.15)) \end{aligned}$$

and the diagonal entries of the matrix **D** *are given by*

$$\begin{aligned} d_{ii} &= \sum_{l=0}^{m-i+1} \binom{m-i+1}{l} \frac{\alpha_1^l \beta_1^{m-i+1-l}}{l! \phi^l} \sum_{k=1}^{l} \sum_{j=0}^{l} (-1)^{l-j} S(l,k) \binom{l}{j} (j\phi)^k \\ &= \sum_{l=0}^{m-i+1} \binom{m-i+1}{l} \alpha_1^l \beta_1^{m-i+1-l} \sum_{k=1}^{l} S(l,k) S(k,l) \phi^{k-l} \\ &= (\alpha_1 + \beta_1)^{m-i+1}, \qquad i = 1, \ldots, m, \end{aligned}$$

because $S(k,l) \neq 0$ only when $k = l$.

3.4 Strict stationarity and Ergodicity

In this section we study the existence of strictly stationary solutions for the class of models previously introduced. We begin by building, recursively, a first-order stationary process solution of the model following the theory presented by Ferland et al. [25]. This solution will be, under certain conditions, strictly stationary and ergodic.

Let us consider the CP-INGARCH model as specified in (2.2) and (2.3) associated to a given family of characteristic functions $(\varphi_t, t \in \mathbb{Z})$ such that **H1** and **H3** are satisfied and let $\{\psi_j\}_{j \in \mathbb{N}_0}$ be the sequence of coefficients associated to the CP-INARCH(∞) representation of the model.

Let $\{U_t\}_{t \in \mathbb{Z}}$ be a sequence of independent real random variables distributed according to a discrete compound Poisson distribution with characteristic function

$$\Phi_{U_t}(u) = exp\left\{\psi_0 \frac{i}{\varphi_t'(0)} \left[\varphi_t(u) - 1\right]\right\}.$$

For each $t \in \mathbb{Z}$ and $k \in \mathbb{N}$, let $\mathscr{Z}_{t,k} = \{Z_{t,k,j}\}_{j\in\mathbb{N}}$ be a sequence of independent discrete compound Poisson random variables with characteristic function

$$\Phi_{Z_{t,k,j}}(u) = exp\left\{\psi_k \frac{i}{\varphi'_{t+k}(0)}\left[\varphi_{t+k}(u)-1\right]\right\}.$$

We note that $E(U_t) = \psi_0$, $E(Z_{t,k,j}) = \psi_k$ and that $Z_{t,k,j}$ are identically distributed for each $(t,k) \in \mathbb{Z}\times\mathbb{N}$. We also assume that all the variables U_s, $Z_{t,k,j}$, ($s,t \in \mathbb{Z}$ and $k, j \in \mathbb{N}$), are mutually independent.

Based on these random variables, we define the sequence $X_t^{(n)}$ as follows:

$$X_t^{(n)} = \begin{cases} 0, & n<0 \\ U_t, & n=0 \\ U_t + \sum_{k=1}^{n} \sum_{j=1}^{X_{t-k}^{(n-k)}} Z_{t-k,k,j}, & n>0 \end{cases} \tag{3.18}$$

where it is assumed that $\sum_{j=1}^{0} Z_{t-k,k,j} = 0$. Using the notion of thinning operation ([7]), $X_t^{(n)}$ admits, for $n>0$, the representation

$$X_t^{(n)} = U_t + \sum_{k=1}^{n} \psi_k^{(t-k)} \circ X_{t-k}^{(n-k)}, \tag{3.19}$$

where the notation ($\psi_k^{(\tau)}\circ$) means that the sequence of random variables of mean ψ_k involved in the thinning operation corresponds to time τ, i.e., the sequence $\mathscr{Z}_{\tau,k} = \{Z_{\tau,k,j}\}_{j\in\mathbb{N}}$. Note that we can really use the thinning operation since $\mathscr{Z}_{t-k,k}$ is a sequence of i.i.d. random variables and $X_{t-k}^{(n-k)}$ and $Z_{t-k,k,j}$ are independent, for each $k \in \{1,...,n\}$. Indeed, for a fixed k, we have

$$X_{t-k}^{(n-k)} = U_{t-k} + \sum_{r=1}^{n-k} \sum_{j=1}^{X_{t-k-r}^{(n-k-r)}} Z_{t-k-r,r,j}$$

$$= f_n\left(U_{t-k},\, X_{t-k-1}^{(n-k-1)},\, X_{t-k-2}^{(n-k-2)},\, \ldots,\, X_{t-n+1}^{(1)},\, U_{t-n},\, Z_{t-k-1,1,j},\, \ldots,\, Z_{t-n+1,n-k-1,j},\, Z_{t-n,n-k,j},\; j\in\mathbb{N}\right)$$

$$= f_n\Big(U_{t-k},\, U_{t-k-1},\, X_{t-k-2}^{(n-k-2)},\, \ldots,\, X_{t-n+1}^{(1)},\, U_{t-n},\, Z_{t-k-1,1,j},\, Z_{t-k-2,2,j},\, \ldots,\, Z_{t-n+1,n-k-1,j},$$

$$Z_{t-n,n-k,j},\, Z_{t-k-2,1,j},\, Z_{t-k-3,2,j},\, \ldots,\, Z_{t-n+1,n-k-2,j},\, Z_{t-n,n-k-1,j},\; j\in\mathbb{N}\Big)$$

$$= \ldots = f_n\left(U_{t-k},\, \ldots,\, U_{t-n},\, Z_{t-k-r,s,j},\; r=1,...,n-k,\; s=1,...,r,\; j\in\mathbb{N}\right),$$

and the required independence holds from the construction of the variables.

The representation (3.19) shows that $X_t^{(n)}$ is obtained through a cascade of thinning operations along the sequence $\{U_t\}_{t\in\mathbb{Z}}$. Indeed, using recursively the thinning operator, we have

[7] [30]: Let X be a nonnegative integer-valued random variable. For any $\alpha \geq 0$ the thinning operation is defined by

$$\alpha \circ X = \begin{cases} \sum_{j=1}^{X} Y_j, & \text{if } X>0, \\ 0, & \text{otherwise,} \end{cases}$$

where '$\circ$' denotes the thinning operator, $\{Y_j\}$ is a sequence of i.i.d. nonnegative integer-valued random variables, independent of X, and such that $E(Y_j) = \alpha$. The sequence $\{Y_j\}$ is known as counting series. In the binomial thinning operation the counting series is considered to be a sequence of i.i.d. Bernoulli random variables with parameter α.

$$X_t^{(1)} = U_t + \psi_1^{(t-1)} \circ X_{t-1}^{(0)} = U_t + \psi_1^{(t-1)} \circ U_{t-1},$$

$$\begin{aligned} X_t^{(2)} &= U_t + \psi_1^{(t-1)} \circ X_{t-1}^{(1)} + \psi_2^{(t-2)} \circ X_{t-2}^{(0)} \\ &= U_t + \psi_1^{(t-1)} \circ \left(U_{t-1} + \psi_1^{(t-2)} \circ U_{t-2}\right) + \psi_2^{(t-2)} \circ U_{t-2}, \end{aligned}$$

$$\begin{aligned} X_t^{(3)} = & \; U_t + \psi_1^{(t-1)} \circ \left[U_{t-1} + \psi_2^{(t-3)} \circ U_{t-3} + \psi_1^{(t-2)} \circ \left(U_{t-2} + \psi_1^{(t-3)} \circ U_{t-3}\right)\right] \\ & + \psi_2^{(t-2)} \circ \left(U_{t-2} + \psi_1^{(t-3)} \circ U_{t-3}\right) + \psi_3^{(t-3)} \circ U_{t-3}, \end{aligned}$$

and so on. For any value n, $X_t^{(n)}$ can be expanded in that way, which is useful on the study of its strict stationarity and ergodicity.

In what follows we present some properties of the sequence $\{X_t^{(n)}\}_{n\in\mathbb{N}}$, useful on the analysis of its probabilistic behavior. In fact, we will prove that $\{(X_t^{(n)}, t \in \mathbb{Z}),\ n \in \mathbb{Z}\}$ is a non-decreasing sequence of first-order stationary processes that converges almost surely, in L^1 and in L^2.

Theorem 3.5 *If* $\sum_{j=1}^{p} \alpha_j + \sum_{k=1}^{q} \beta_k < 1$ *then* $\{(X_t^{(n)}, t \in \mathbb{Z}),\ n \in \mathbb{Z}\}$ *is a sequence of first-order stationary processes such that* $\mu_n = E(X_t^{(n)}) \longrightarrow \mu$*, as* $n \to \infty$.

Proof: We start by noting that the expectation of $X_t^{(n)}$ is well defined because $X_t^{(n)}$ is a finite sum of independent compound Poisson random variables. We will denote this expectation by μ_n and we prove in the following by induction, with respect to $n \in \mathbb{Z}$, that it does not depend on t.

For $n = 0$ we obtain $E(X_t^{(0)}) = E(U_t) = \psi_0$, which is independent of t (for $n < 0$ the result is trivial). Now let us consider as induction hypothesis, that for $n > 0$ and for any fixed value of t, $E(X_t^{(n)})$ is independent of t. Thus, from the mean of the thinning random variable ([8].), we obtain

$$\begin{aligned} E\left(X_t^{(n+1)}\right) &= E\left(U_t + \sum_{k=1}^{n+1} \psi_k^{(t-k)} \circ X_{t-k}^{(n+1-k)}\right) = \psi_0 + \sum_{k=1}^{n+1} \psi_k E\left(X_{t-k}^{(n+1-k)}\right) \\ &= g\left(E\left(X_{t-n-1}^{(0)}\right), \ldots, E\left(X_{t-1}^{(n)}\right)\right), \end{aligned}$$

which is, by the induction hypothesis, a function independent of t. So

$$\mu_n = E\left(X_t^{(n)}\right) = \begin{cases} 0, & n < 0 \\ \psi_0, & n = 0 \\ \psi_0 + \sum_{k=1}^{n} \psi_k \mu_{n-k}, & n > 0 \end{cases},$$

and using the fact that $\mu_{n-k} = 0$ if $k > n$, we can write

$$\begin{aligned} \mu_n = \sum_{k=1}^{\infty} \psi_k \mu_{n-k} + \psi_0 = B^{-1}(L)\left[A(L)\mu_n + \alpha_0\right] \;&\Leftrightarrow\; B(L)\mu_n = A(L)\mu_n + \alpha_0 \\ &\Leftrightarrow\; K(L)\mu_n = \alpha_0, \end{aligned}$$

[8] [30, p. 52]: $E[\alpha \circ X] = \alpha E[X]$.
More properties of the thinning operation can be found in [30] and [71]

where $K(L) = B(L) - A(L)$ with $A(L)$ and $B(L)$ the polynomials introduced in Section 2.2.

The last equation indicates that the sequence $\{\mu_n\}_{n\in\mathbb{Z}}$ satisfies a finite difference equation of degree $\max(p,q)$ with constant coefficients, namely the equation $\mu_n - \sum_{j=1}^{p}\alpha_j\mu_{n-j} - \sum_{k=1}^{q}\beta_k\mu_{n-k} = \alpha_0$. The characteristic polynomial of this equation, $K(z)$, has all its roots outside the unit circle since $\sum_{j=1}^{p}\alpha_j + \sum_{k=1}^{q}\beta_k < 1$ and so, $\{(X_t^{(n)}, t\in\mathbb{Z}),\ n\in\mathbb{Z}\}$ is a sequence of first-order stationary processes. From this stationarity, we deduce that $\mu_n = \psi_0 + \sum_{k=1}^{n}\psi_k\mu_n$, $n > 0$, and then

$$\lim_{n\to\infty}\mu_n = \frac{\psi_0}{1-\sum_{k=1}^{\infty}\psi_k} = \frac{\alpha_0 B^{-1}(1)}{1-A(1)B^{-1}(1)} = \frac{\alpha_0}{K(1)} = \frac{\alpha_0}{1-\sum_{j=1}^{p}\alpha_j - \sum_{k=1}^{q}\beta_k} = \mu. \ \blacksquare$$

Theorem 3.6 *For a fixed value of t, the sequence $\{(X_t^{(n)}, t\in\mathbb{Z}),\ n\in\mathbb{Z}\}$ is a non-decreasing sequence of nonnegative integer-valued random variables.*

Proof: This result is stated by induction with respect to n, for any fixed value of t. In fact, we have

$$X_t^{(1)} - X_t^{(0)} = U_t + \sum_{j=1}^{X_{t-1}^{(0)}} Z_{t-1,1,j} - U_t = \sum_{j=1}^{U_{t-1}} Z_{t-1,1,j},$$

which is nonnegative because this is a random sum of nonnegative integer-valued random variables.

Let us suppose now that, for $n > 0$, $X_t^{(n+1)} - X_t^{(n)} \geq 0$. So,

$$X_t^{(n+2)} - X_t^{(n+1)} = U_t + \sum_{k=1}^{n+2}\sum_{j=1}^{X_{t-k}^{(n+2-k)}} Z_{t-k,k,j} - U_t - \sum_{k=1}^{n+1}\sum_{j=1}^{X_{t-k}^{(n+1-k)}} Z_{t-k,k,j}$$

$$= \sum_{j=1}^{X_{t-1}^{(n+1)}} Z_{t-1,1,j} + \sum_{j=1}^{X_{t-2}^{(n)}} Z_{t-2,2,j} + \ \ldots\ + \sum_{j=1}^{X_{t-n-1}^{(1)}} Z_{t-n-1,n+1,j} + \sum_{j=1}^{U_{t-n-2}} Z_{t-n-2,n+2,j}$$

$$- \sum_{j=1}^{X_{t-1}^{(n)}} Z_{t-1,1,j} - \sum_{j=1}^{X_{t-2}^{(n-1)}} Z_{t-2,2,j} - \ \ldots\ - \sum_{j=1}^{X_{t-n-1}^{(0)}} Z_{t-n-1,n+1,j}.$$

By the induction hypothesis it follows that $X_{t-1}^{(n+1)} \geq X_{t-1}^{(n)}$, $X_{t-2}^{(n)} \geq X_{t-2}^{(n-1)}$, ..., $X_{t-n-1}^{(1)} \geq X_{t-n-1}^{(0)}$. Using that, we can rewrite the above equality in the form

$$X_t^{(n+2)} - X_t^{(n+1)} = \sum_{j=X_{t-1}^{(n)}+1}^{X_{t-1}^{(n+1)}} Z_{t-1,1,j} + \ \ldots\ + \sum_{j=X_{t-n-1}^{(0)}+1}^{X_{t-n-1}^{(1)}} Z_{t-n-1,n+1,j} + \sum_{j=1}^{U_{t-n-2}} Z_{t-n-2,n+2,j}$$

$$= \sum_{k=1}^{n+1}\sum_{j=X_{t-k}^{(n+1-k)}+1}^{X_{t-k}^{(n+2-k)}} Z_{t-k,k,j} + \sum_{j=1}^{U_{t-n-2}} Z_{t-n-2,n+2,j},$$

which is once again a nonnegative integer-valued random variable.

Then, $\{(X_t^{(n)}, t\in\mathbb{Z}), n\in\mathbb{Z}\}$ is, for each t, a non-decreasing sequence. $\blacksquare$

Theorem 3.7 *If* $\sum_{j=1}^{p}\alpha_j+\sum_{k=1}^{q}\beta_k<1$ *then the sequence* $\{(X_t^{(n)},t\in\mathbb{Z}),\ n\in\mathbb{Z}\}$ *converges almost surely and in* L^1 *(or in mean) to a process* $X^*=(X_t^*,t\in\mathbb{Z})$.

Proof: Let $(\Omega,\mathscr{A},P)$ be the common probability space on which the relevant random variables are defined. Since $X_t^{(n)}$ is a non-decreasing sequence of nonnegative integers we have

$$\forall\ \omega\in\Omega,\ \ \lim_{n\to\infty}X_t^{(n)}(\omega)=X_t^*(\omega),$$

where the limit $X_t^*(\omega)$ is not necessarily finite. To prove the almost sure convergence of this sequence we will use the Borel-Cantelli Lemma ([9]). In fact, we need to prove that $X_t^*(\omega)$ is finite with probability one which is equivalent to say that the set $A_\infty=\{\omega:\ X_t^*(\omega)=\infty\}$ is of probability zero, i.e., $P(A_\infty)=0$. But it is equivalent to prove that the events

$$A_n=\left\{\omega:\ X_t^{(n)}(\omega)-X_t^{(n-1)}(\omega)>0\right\}$$

occur infinitely often with probability zero, i.e., $P(\limsup_n A_n)=0$.

In fact, we have

$$\begin{aligned} E\left[X_t^{(n)}-X_t^{(n-1)}\right] &= \sum_{k=1}^{+\infty}kP\left(\left\{\omega:X_t^{(n)}(\omega)-X_t^{(n-1)}(\omega)=k\right\}\right)\\ &\geq \sum_{k=1}^{+\infty}P\left(\left\{\omega:X_t^{(n)}(\omega)-X_t^{(n-1)}(\omega)=k\right\}\right)=P(A_n). \end{aligned} \tag{3.20}$$

On the other hand

$$\begin{aligned} E\left[X_t^{(n)}-X_t^{(n-1)}\right] &= \mu_n-\mu_{n-1}=\psi_0+\sum_{k=1}^{n}\psi_k\mu_{n-k}-\psi_0-\sum_{k=1}^{n-1}\psi_k\mu_{n-1-k}=\nu_n\\ &\Leftrightarrow\ \psi_n\mu_0+\sum_{k=1}^{n-1}\psi_k\left(\mu_{n-k}-\mu_{n-k-1}\right)=\nu_n\ \Leftrightarrow\ \nu_n=\sum_{k=1}^{n}\psi_k\nu_{n-k}. \end{aligned}$$

Since $\nu_{n-k}=0$ if $k>n$, we conclude

$$\nu_n=\sum_{k=1}^{\infty}\psi_k\nu_{n-k}\ \Leftrightarrow\ \nu_n=B^{-1}(L)A(L)\nu_n\ \Leftrightarrow\ K(L)\nu_n=0, \tag{3.21}$$

where, as in the proof of Theorem 3.5, $K(L)=B(L)-A(L)$. So, the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ satisfies a homogeneous finite difference equation with characteristic polynomial $K(z)$ that has all roots outside the unit circle because $\sum_{j=1}^{p}\alpha_j+\sum_{k=1}^{q}\beta_k<1$. From Section 3.6 of [12], we know that the general solution of the equation (3.21) has the form

$$\nu_n=\sum_{j=1}^{k}\sum_{s=0}^{r_j-1}c_{js}n^s\xi_j^{-n},$$

where ξ_j, $j=1,...,k$ are the distinct roots of $K(z)$, r_j is the multiplicity of the root ξ_j and $\{c_{js}\}$ a set of coefficients determined by the initial conditions. Thus, the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ tends towards zero

[9] [65, p. 102]: Let $\{A_n\}$ be any events. If $\sum_n P(A_n)<\infty$ then $P(\limsup_{n\to\infty}A_n)=0$.

with a geometric rate as $n \to \infty$. In other words, there exist constants $M \geq 0$ and $0 < c < 1$ such that $\nu_n \leq Mc^n$. Using (3.20) and (3.21) we get $P(A_n) \leq \nu_n$ and then

$$\sum_{n=1}^{\infty} P(A_n) \leq M \sum_{n=1}^{\infty} c^n < \infty.$$

Finally, from the Borel-Cantelli lemma, we obtain $P(\limsup_n A_n) = 0$, which allow us to conclude the almost sure convergence of the sequence $\{(X_t^{(n)}, t \in \mathbb{Z}), n \in \mathbb{Z}\}$ to a process $\{X_t^*\}_{t\in\mathbb{Z}}$.

For the L^1 convergence, recall that we have nonnegative integer-valued random variables. Applying Beppo Lévi's Theorem and then Theorem 3.5 we conclude that the first moment of $\{X_t^*\}_{t\in\mathbb{Z}}$ is finite, namely $E(X_t^*) = \lim_{n\to\infty} E(X_t^{(n)}) = \lim_{n\to\infty} \mu_n = \mu$. Consequently $\lim_{n\to\infty} E(|X_t^{(n)} - X_t^*|) = 0$. ■

Theorem 3.8 *If* $\sum_{j=1}^{p} \alpha_j + \sum_{k=1}^{q} \beta_k < 1$ *and* $(\varphi_t, t \in \mathbb{Z})$ *is derivable at zero up to order* 2 *then the sequence* $\{(X_t^{(n)}, t \in \mathbb{Z}),\ n \in \mathbb{Z}\}$ *converges in* L^2 *(or in quadratic mean) to the process* $\{X_t^*\}_{t\in\mathbb{Z}}$.

Proof: To prove the convergence in L^2 we start by showing that the second moment of X_t^* is finite.

In fact, for $n > 0$, we obtain

$$E\left[\left(X_t^{(n)}\right)^2\right] = E\left[\left(U_t + \sum_{k=1}^{n} \sum_{j=1}^{X_{t-k}^{(n-k)}} Z_{t-k,k,j}\right)^2\right]$$

$$= E[U_t^2] + 2E\left[U_t \sum_{k=1}^{n} \sum_{j=1}^{X_{t-k}^{(n-k)}} Z_{t-k,k,j}\right] + E\left[\left(\sum_{k=1}^{n} \sum_{j=1}^{X_{t-k}^{(n-k)}} Z_{t-k,k,j}\right)^2\right] \tag{3.22}$$

By construction, U_s and $Z_{t,k,j}$ ($s, t \in \mathbb{Z}$ and $k, j \in \mathbb{N}$) are mutually independent and so

$$E\left[U_t \sum_{k=1}^{n} \sum_{j=1}^{X_{t-k}^{(n-k)}} Z_{t-k,k,j}\right] = E(U_t)\, E\left(\sum_{k=1}^{n} \sum_{j=1}^{X_{t-k}^{(n-k)}} Z_{t-k,k,j}\right) = \psi_0 \sum_{k=1}^{n} \psi_k E\left(X_t^{(n-k)}\right) \leq \psi_0 E(X_t^*) \sum_{k=1}^{n} \psi_k,$$

using the first-order stationarity of the process $X_t^{(n)}$ and the fact that $E(X_t^{(n)}) \leq E(X_t^*)$ since $X_t^{(n)}$ is a non-decreasing sequence in n that converges almost surely to X_t^*. We have also

$$E\left[\left(\sum_{k=1}^{n} \sum_{j=1}^{X_{t-k}^{(n-k)}} Z_{t-k,k,j}\right)^2\right] = \sum_{x=0}^{\infty} E\left[\left(\sum_{k=1}^{n} \sum_{j=1}^{X_{t-k}^{(n-k)}} Z_{t-k,k,j}\right)^2 \mid X_{t-k}^{(n-k)} = x\right] P\left(X_{t-k}^{(n-k)} = x\right)$$

$$= \sum_{x=0}^{\infty} \left[V\left(\sum_{k=1}^{n} \sum_{j=1}^{x} Z_{t-k,k,j}\right) + \left(E\left[\sum_{k=1}^{n} \sum_{j=1}^{x} Z_{t-k,k,j}\right]\right)^2\right] P\left(X_{t-k}^{(n-k)} = x\right)$$

$$= E\left(X_{t-k}^{(n-k)}\right) \sum_{k=1}^{n} V\left(Z_{t-k,k,j}\right) + E\left[\left(X_{t-k}^{(n-k)}\right)^2\right] \left(\sum_{k=1}^{n} \psi_k\right)^2 \tag{3.23}$$

$$\leq E\left(X_t^{(n)}\right)\sum_{k=1}^{n}V\left(Z_{t-k,k,j}\right)+E\left[\left(X_t^{(n)}\right)^2\right]\left(\sum_{k=1}^{n}\psi_k\right)^2,$$

where to obtain the equality (3.23) we use the independence of the random variables $Z_{t,k,j}$ ($t\in\mathbb{Z}$ and $k,j\in\mathbb{N}$) and the fact that $E[\sum_{k=1}^{n}\sum_{j=1}^{x}Z_{t-k,k,j}]=\sum_{k=1}^{n}\sum_{j=1}^{x}E[Z_{t-k,k,j}]=x\sum_{k=1}^{n}\psi_k$.

Using the hypothesis **H3** and the relation between the characteristic function of $Z_{t-k,k,j}$ and its moments we get

$$V\left(Z_{t-k,k,j}\right)=-\Phi''_{Z_{t-k,k,j}}(0)-\psi_k^2=-i\frac{\varphi_t''(0)}{\varphi_t'(0)}\psi_k=R_t(\psi_k)<\infty,$$

since

$$\Phi'_{Z_{t-k,k,j}}(u)=i\psi_k\frac{\varphi_t'(u)}{\varphi_t'(0)}\exp\left\{\psi_k\frac{i}{\varphi_t'(0)}[\varphi_t(u)-1]\right\},$$

$$\Phi''_{Z_{t-k,k,j}}(u)=\left[i\psi_k\frac{\varphi_t''(u)}{\varphi_t'(0)}+\left(i\psi_k\frac{\varphi_t'(u)}{\varphi_t'(0)}\right)^2\right]\exp\left\{\psi_k\frac{i}{\varphi_t'(0)}[\varphi_t(u)-1]\right\}.$$

So, we obtain

$$E\left[\left(\sum_{k=1}^{n}\sum_{j=1}^{X_{t-k}^{(n-k)}}Z_{t-k,k,j}\right)^2\right]\leq\sum_{k=1}^{n}R_t(\psi_k)E\left(X_t^{(n)}\right)+\left(\sum_{k=1}^{n}\psi_k\right)^2E\left[\left(X_t^{(n)}\right)^2\right]$$

$$\leq\sum_{k=1}^{n}R_t(\psi_k)E\left(X_t^*\right)+\left(\sum_{k=1}^{n}\psi_k\right)^2E\left[\left(X_t^{(n)}\right)^2\right],$$

and finally replacing in (3.22) we obtain

$$E\left[\left(X_t^{(n)}\right)^2\right]\leq E(U_t^2)+2\psi_0E\left(X_t^*\right)\sum_{k=1}^{n}\psi_k+\sum_{k=1}^{n}R_t(\psi_k)E\left(X_t^*\right)+\left(\sum_{k=1}^{n}\psi_k\right)^2E\left[\left(X_t^{(n)}\right)^2\right].$$

Hence,

$$E\left[\left(X_t^{(n)}\right)^2\right]\leq\frac{E(X_t^*)\left(\sum_{k=1}^{n}R_t(\psi_k)+2\psi_0\sum_{k=1}^{n}\psi_k\right)+E(U_t^2)}{1-\left(\sum_{k=1}^{n}\psi_k\right)^2}$$

$$\leq\frac{E(X_t^*)\left(\sum_{k=1}^{\infty}R_t(\psi_k)+2\psi_0\sum_{k=1}^{\infty}\psi_k\right)+E(U_t^2)}{1-\left(\sum_{k=1}^{\infty}\psi_k\right)^2}=C_t,$$

with C_t a constant dependent of t. By Lebesgue's dominated convergence theorem ([10]), we conclude that $E[(X_t^*)^2]$ is finite. Since the first two moments of $\{X_t^{(n)}\}_{n\in\mathbb{N}}$ are finite consequently the random variables $X_t^{(n)}$ are in $L^2(\Omega,\mathscr{A},P)$. Let us define $V_t^{(n)}=(X_t^{(n)}-X_t^*)^2$. The sequence $\{V_t^{(n)}\}_{n\in\mathbb{N}}$ is decreasing, bounded below by 0 and it satisfies $E(V_t^{(0)})=E[(U_t-X_t^*)^2]<\infty$. Consequently,

$$\lim_{n\to\infty}E[(X_t^{(n)}-X_t^*)^2]=\lim_{n\to\infty}E\left[V_t^{(n)}\right]=E\left[\lim_{n\to\infty}(a.s.)V_t^{(n)}\right]=0,$$

[10][65, p. 133]: Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables which converges a.s. or in probability to a random variable X. If there exists a random variable $Y\in L^1$ such that $|X_n|\leq Y$, then $\lim_{n\to\infty}E(X_n)=E(X)$.

by Lebesgue's dominated convergence theorem because $\{X_t^{(n)}\}_{n\in\mathbb{N}}$ converges almost surely to X_t^*. Hence, we deduce that $\{X_t^{(n)}\}_{n\in\mathbb{N}}$ also converges to X_t^* in $L^2(\Omega,\mathscr{A},P)$. ■

We have defined a sequence that has an almost sure and a mean-squared limit, but we still have to verify that this is a solution of the CP-INGARCH model. This is the goal of the next theorem.

Theorem 3.9 *Under the hypothesis* **H3***, if* $\sum_{j=1}^{p}\alpha_j+\sum_{k=1}^{q}\beta_k<1$ *the process* X^* *is a first-order stationary solution of the CP-INGARCH*(p,q) *model.*

Proof: We shall prove the theorem in two steps. First we prove that, for all $t\in\mathbb{Z}$, the characteristic function of X_t^* conditioned on $\underline{X}_{t-1}^*$, denoted by $\Phi_{X_t^*|\underline{X}_{t-1}^*}$, is equal to $\lim_{n\to+\infty}\Phi_n$ with Φ_n the characteristic function of the sequence $\{r_t^{(n)}|\underline{X}_{t-1}^*\}_{n\in\mathbb{N}}$, where

$$r_t^{(n)}=U_t+\sum_{k=1}^{n}\sum_{j=1}^{X_{t-k}^*}Z_{t-k,k,j}.$$

This equality follows from the Lévy Continuity Theorem since, for a fixed t, $\{r_t^{(n)}|\underline{X}_{t-1}^*\}_{n\in\mathbb{N}}$ converges in law to $X_t^*|\underline{X}_{t-1}^*$. Indeed, let us denote by $Y_t^{(n)}$ the sequence $r_t^{(n)}-X_t^{(n)}$. Then

$$Y_t^{(n)}=U_t+\sum_{k=1}^{n}\sum_{j=1}^{X_{t-k}^*}Z_{t-k,k,j}-U_t-\sum_{k=1}^{n}\sum_{j=1}^{X_{t-k}^{(n-k)}}Z_{t-k,k,j}=\sum_{k=1}^{n}\sum_{j=X_{t-k}^{(n-k)}+1}^{X_{t-k}^*}Z_{t-k,k,j}\geq 0,$$

because $\{X_{t-k}^{(n-k)}\}_{n\in\mathbb{N}}$ is a non-decreasing sequence that converges almost surely to X_{t-k}^*. Additionally,

$$\begin{aligned}E\left[Y_t^{(n)}\right]&=E\left[\sum_{k=1}^{n}\sum_{j=X_{t-k}^{(n-k)}+1}^{X_{t-k}^*}Z_{t-k,k,j}\right]=\sum_{k=1}^{n}E\left[\sum_{j=X_{t-k}^{(n-k)}+1}^{X_{t-k}^*}Z_{t-k,k,j}\right]\\&=\sum_{k=1}^{n}\left[\sum_{x=0}^{\infty}x\,E\left(Z_{t-k,k,j}\right)P\left(X_{t-k}^*-X_{t-k}^{(n-k)}=x\right)\right]\\&=\sum_{k=1}^{n}\psi_k\,E\left[X_{t-k}^*-X_{t-k}^{(n-k)}\right]=\mu\sum_{k=1}^{n}\psi_k-\sum_{k=1}^{n}\psi_k\,\mu_{n-k},\end{aligned}$$

which allows us to conclude

$$\begin{aligned}\lim_{n\to\infty}E\left[Y_t^{(n)}\right]&=\lim_{n\to\infty}\left(\mu\sum_{k=1}^{n}\psi_k-\sum_{k=1}^{n}\psi_k\,\mu_{n-k}\right)=\mu\,H(1)-\lim_{n\to\infty}\sum_{k=1}^{n}\psi_k\,\mu_{n-k}\\&=\frac{\alpha_0\,A(1)}{B(1)\,[B(1)-A(1)]}-\lim_{n\to\infty}\left(\mu_n+\frac{\alpha_0}{B(1)}\right),\qquad\text{from Theorem 3.5,}\\&=\frac{\alpha_0A(1)}{B(1)\,[B(1)-A(1)]}-\frac{\alpha_0}{B(1)-A(1)}+\frac{\alpha_0}{B(1)}=0.\end{aligned}$$

This means that, when $n\to\infty$, the sequence $\{Y_t^{(n)}\}_{n\in\mathbb{N}}$ converges to zero in L^1 because $Y_t^{(n)}$ is nonnegative. So, from the relation between the different types of convergence, $\{Y_t^{(n)}\}_{n\in\mathbb{N}}$ also converges in probability to zero and the same happens to $X_t^* - X_t^{(n)}$. Therefore,

$$X_t^* - r_t^{(n)} = (X_t^* - X_t^{(n)}) + (X_t^{(n)} - r_t^{(n)}) = (X_t^* - X_t^{(n)}) - Y_t^{(n)},$$

allows us to conclude that $\{r_t^{(n)}\}_{n\in\mathbb{N}}$ converges in probability to X_t^* and then $\{r_t^{(n)}|\underline{X}_{t-1}^*\}_{n\in\mathbb{N}}$ converges in law to $X_t^*|\underline{X}_{t-1}^*$. From that, we deduce finally $\Phi_{X_t^*|\underline{X}_{t-1}^*}(u) = \lim_{n\to+\infty}\Phi_n(u)$, $u\in\mathbb{R}$.

The second step reduces to the calculation of Φ_n and then realize that $\Phi_n(u)$, $u\in\mathbb{R}$, equals $\exp\{i\frac{\lambda_t}{\varphi'(0)}[\varphi(u)-1]\}$, when $n\to\infty$. So, let us obtain Φ_n. Conditionally to $\underline{X}_{t-1}^*$, we have

$$\Phi_{\sum_{j=1}^{X_{t-k}^*} Z_{t-k,k,j}}(u) = \prod_{j=1}^{X_{t-k}^*}\Phi_{Z_{t-k,k,j}}(u) = \exp\left\{\sum_{j=1}^{X_{t-k}^*}\psi_k\,\frac{i}{\varphi_t'(0)}\,[\varphi_t(u)-1]\right\}$$

$$= \exp\left\{\psi_k\,X_{t-k}^*\frac{i}{\varphi_t'(0)}\,[\varphi_t(u)-1]\right\},\quad u\in\mathbb{R}.$$

From the independence of the variables involved in the definition of $r_t^{(n)}$, we get

$$\Phi_n(u) = \exp\left(\psi_0\,\frac{i}{\varphi_t'(0)}\,[\varphi_t(u)-1] + \sum_{k=1}^{n}\psi_k\,X_{t-k}^*\frac{i}{\varphi_t'(0)}\,[\varphi_t(u)-1]\right)$$

$$= \exp\left\{\left(\psi_0 + \sum_{k=1}^{n}\psi_k\,X_{t-k}^*\right)\frac{i}{\varphi_t'(0)}\,[\varphi_t(u)-1]\right\},\quad u\in\mathbb{R},$$

and thus, we have

$$\lim_{n\to+\infty}\Phi_n(u) = \exp\left\{i\frac{\lambda_t}{\varphi_t'(0)}\,[\varphi_t(u)-1]\right\},\quad u\in\mathbb{R}.$$

These two steps together enable us to conclude that the almost sure limit of the sequence $\{X_t^{(n)}\}_{n\in\mathbb{N}}$ is a solution of the model. The first-order stationarity of this solution is a consequence of Theorem 3.1. We observe that from the unicity of the limit follows that this solution is uniquely defined. ■

Remark 3.7 *From Theorems* 3.7 *and* 3.8*, we conclude that the condition* $\sum_{j=1}^{p}\alpha_j + \sum_{k=1}^{q}\beta_k < 1$ *guarantees the existence of a second-order solution of a CP-INGARCH*(p,q) *process.*

Now let us consider, additionally to the hypothesis **H3**, that φ_t is independent of t, i.e.,

Hypothesis H4 : $\varphi_t = \varphi$ and φ deterministic.

In this subclass, which still includes, among others, the INGARCH, the NB-DINARCH, the NTA-INGARCH and the GEOMP2-INGARCH models, it is possible to set the strict stationarity and the ergodicity of $\{X_t^{(n)}\}_{n\in\mathbb{N}}$ and the same for the process $\{X_t^*\}_{t\in\mathbb{Z}}$.

Theorem 3.10 *Under the hypothesis* **H4**, $\{(X_t^{(n)}, t\in\mathbb{Z}),\ n\in\mathbb{Z}\}$ *is a sequence of strictly stationary and ergodic processes.*

Proof: Let us start by proving the strict stationarity. To simplify the notation we will prove the statement only for a two-dimensional random vector, i.e., for any positive integer h we show that

$$\mathbf{X}_{12}^{(n)}=(X_1^{(n)},X_2^{(n)})^\top \quad \text{and} \quad \mathbf{X}_{1+h,2+h}^{(n)}=(X_{1+h}^{(n)},X_{2+h}^{(n)})^\top$$

have the same joint distribution when $n>0$. As the probability generating function characterizes the distribution, let us prove that the probability generating functions of these random vectors ([11]) are the same. Since $X_1^{(n)}=f\,(U_{1-n},...,U_1)$ and $X_2^{(n)}=f^*\,(U_{2-n},...,U_2)$,

$$\begin{aligned} g_{\mathbf{X}_{12}^{(n)}}(\mathbf{s}) &= E\left[s_1^{X_1^{(n)}}s_2^{X_2^{(n)}}\right]=E\left(E\left[s_1^{X_1^{(n)}}s_2^{X_2^{(n)}}|\mathbf{U}_{1-n...2}\right]\right)\\ &= \sum_{\mathbf{x}\in\mathbb{N}^{2+n}}E\left[s_1^{X_1^{(n)}}s_2^{X_2^{(n)}}|\mathbf{U}_{1-n...2}=\mathbf{x}\right]P(\mathbf{U}_{1-n...2}=\mathbf{x}),\end{aligned}$$

where $\mathbf{U}_{1-n...2}$ represents the vector $(U_{1-n},...,U_2)^\top$. For a given value $\mathbf{u}_{t-n...t+1}=(u_{t-n},...,u_{t+1})^\top$ of the random vector $\mathbf{U}_{t-n...t+1}$, the components of the vector $(X_t^{(n)},X_{t+1}^{(n)})^\top$ are computed using a set of well-determined variables coming from the sequences $\mathscr{Z}_{\tau,\eta}$, $\tau=t-n,...,t$ and $\eta=1,...,n$. It follows that if $\mathbf{U}_{1-n...2}$ and $\mathbf{U}_{1-n+h...2+h}$ are fixed to the same value $\mathbf{x}$, the conditional distributions of $\mathbf{X}_{12}^{(n)}$ and $\mathbf{X}_{1+h,2+h}^{(n)}$ given $\mathbf{U}_{1-n...2}$ and $\mathbf{U}_{1-n+h...2+h}$, respectively, are the same and hence

$$E\left[s_1^{X_{1+h}^{(n)}}s_2^{X_{2+h}^{(n)}}|\mathbf{U}_{1-n+h...2+h}=\mathbf{x}\right]=E\left[s_1^{X_1^{(n)}}s_2^{X_2^{(n)}}|\mathbf{U}_{1-n...2}=\mathbf{x}\right].$$

As under the hypothesis **H4** the vectors $\mathbf{U}_{1-n...2}$ and $\mathbf{U}_{1-n+h...2+h}$ are of components identically distributed we have $P(\mathbf{U}_{1-n+h...2+h}=\mathbf{x})=P(\mathbf{U}_{1-n...2}=\mathbf{x})$, and so

$$g_{\mathbf{X}_{12}^{(n)}}(\mathbf{s})=\sum_{\mathbf{x}\in\mathbb{N}^{2+n}}E\left[s_1^{X_{1+h}^{(n)}}s_k^{X_{2+h}^{(n)}}|\mathbf{U}_{1-n+h...2+h}=\mathbf{x}\right]P(\mathbf{U}_{1-n+h...2+h}=\mathbf{x})=g_{\mathbf{X}_{1+h,2+h}^{(n)}}(\mathbf{s}),$$

which allows to conclude that $\mathbf{X}_{12}^{(n)}$ and $\mathbf{X}_{1+h,2+h}^{(n)}$ have the same joint distribution for any $h\in\mathbb{Z}$. Analogously, it can be proved that for any $k,h\in\mathbb{Z}$ the random vectors $\mathbf{X}_{1...k}^{(n)}$ and $\mathbf{X}_{1+h...k+h}^{(n)}$, have the same joint distribution and then the strict stationarity of the process $\{X_t^{(n)}\}_{t\in\mathbb{Z}}$, for each n, is deduced.

Regarding the ergodicity, under **H4**, the sequences $(U_t,t\in\mathbb{Z})$ and $(\mathscr{Z}_{t,k},t\in\mathbb{Z},k\in\mathbb{N})$ previously introduced are of i.i.d. random variables. Then, $\{(X_t^{(n)},t\in\mathbb{Z}),\ n\in\mathbb{Z}\}$ is a sequence of ergodic processes, because it is a measurable function of the sequence of i.i.d. random variables $\{(U_t,\mathscr{Z}_{t,j}),t\in\mathbb{Z}, j\in\mathbb{N}\}$, say $X_t^{(n)}=T_n\,(U_t,\ldots,U_{t-n},Z_{t-r,s,k},\ r=1,\ldots,n,\ s=1,\ldots,r,\ k\in\mathbb{N})$ ([12]). ■

[11] Let $\mathbf{W}$ be a random vector with nonnegative integer-valued entries and let $p(\mathbf{W})$ denotes $P(\mathbf{W}=(w_1,\ldots,w_k)^\top)$. The probability generating function of $\mathbf{W}$, for $\mathbf{s}=(s_1,\ldots,s_k)^\top\in[-1,1]^k$, is given by

$$g_{\mathbf{W}}(\mathbf{s})=E[s_1^{W_1}\ldots s_k^{W_k}]=\sum_{w\in\mathbb{N}^k}p(\mathbf{W})\prod_{j=1}^k s_j^{w_j}.$$

[12] [7, p. 32]: A sequence X of i.i.d. random variables is an ergodic process.

[68, p. 33]: Let $g:\mathbb{R}^{\mathbb{Z}}\to\mathbb{R}$ be a measurable function and X be an ergodic process. Then the process $Y=(Y_t,t\in\mathbb{Z})$ with $Y_t=g(...,X_{t-1},X_t,X_{t+1},...)$ is also ergodic.

Theorem 3.11 *Let us consider the model CP-INGARCH*(p,q) *such that* **H4** *is satisfied.*

There is a strictly stationary and ergodic process that satisfies the model if $\sum_{j=1}^{p}\alpha_j+\sum_{k=1}^{q}\beta_k<1$. *Moreover, the first two moments of this process are finite.*

Proof: In Theorem 3.9 we proved that $\{X_t^*\}_{t\in\mathbb{Z}}$ is a solution of the CP-INGARCH model. So, it is enough to prove that under the hypothesis **H4**, the process $\{X_t^*\}_{t\in\mathbb{Z}}$ is strictly stationary and ergodic.

Let us start with the strict stationarity. From Theorem 3.10, $\{X_t^{(n)}\}_{n\in\mathbb{N}}$ is a sequence of strictly stationary processes. Otherwise, $\{X_t^{(n)}\}_{n\in\mathbb{N}}$ converges almost surely to X_t^* when $\sum_{j=1}^{p}\alpha_j+\sum_{k=1}^{q}\beta_k<1$. So, considering without loss of generality, the indexes $\{1,...,k\}$, we have, for any $h\in\mathbb{Z}$, the following almost sure convergence

$$(X_1^{(n)},...,X_k^{(n)}) \xrightarrow{a.s.}_{n\to+\infty} (X_1^*,...,X_k^*),$$

$$(X_{1+h}^{(n)},...,X_{k+h}^{(n)}) \xrightarrow{a.s.}_{n\to+\infty} (X_{1+h}^*,...,X_{k+h}^*),$$

and consequently, also the convergence in law. Considering the strict stationarity of $\{X_t^{(n)}\}_{n\in\mathbb{N}}$ and the unicity of the limit, we conclude that $\{X_t^*\}_{t\in\mathbb{Z}}$ is a strictly stationary process.

To prove the ergodicity, we note that $\{X_t^*\}_{t\in\mathbb{Z}}$ may be written as $X_t^*=T\left(\{(U_t,\mathscr{X}_{t,j}),t\in\mathbb{Z},j\in\mathbb{N}\}\right)$ where T is the almost sure limit of the sequence of the measurable functions T_n. T is then a measurable function, and so the process $\{X_t^*\}_{t\in\mathbb{Z}}$ is ergodic as it is a measurable function of an ergodic process.

The existence of the two first moments of $\{X_t^*\}_{t\in\mathbb{Z}}$ is ensured by Theorems 3.7 and 3.8. ■

Remark 3.8 **(a)** *Under the conditions of the previous theorem it follows that* $\{X_t^*\}_{t\in\mathbb{Z}}$ *is also a weakly stationary solution of the model because it is a strictly stationary second-order process.*

(b) *In the set of the* L^1 *processes, the condition of the previous theorem is necessary and sufficient for the existence of a strictly stationary and ergodic process solution of the CP-INGARCH*(p,q).

We finish the section by stating that, to the best of our knowledge, the general technique here proposed to show the ergodicity is different from those existing in the literature for the integer-valued models [13, 16, 52, 56] since we established the ergodicity of the strict stationarity solution displayed.

Chapter 4

CP-INARCH(1) process: Moments and Estimation Procedure

The CP-INARCH(1) process has a simple structure and a great potential for applications in practice. This simple model with its AR(1)-like serial dependence structure has already proved to be of great practical relevance, with particular applications namely in monthly claims counts of workers in the heavy manufacturing industry ([77]), daily download counts of the program CW β TeXpert ([85]), monthly strike data published by the U.S. Bureau of Labor Statistics ([79]), weakly number of dengue cases in Singapore ([80]), or monthly counts of poliomyelitis cases in the U.S. ([81]).

Regarding the importance of developing a statistical analysis of such general model, a contribution to the model estimation independent of the specific conditional law is analysed in this chapter.

Let us consider, in what follows, the CP-INARCH(1) model defined as

$$\Phi_{X_t|\underline{X}_{t-1}}(u) = \exp\left\{ i\frac{\lambda_t}{\varphi_t'(0)}\left[\varphi_t(u)-1\right]\right\}, \quad u\in\mathbb{R}, \qquad \lambda_t = \alpha_0+\alpha_1 X_{t-1}, \tag{4.1}$$

with $\alpha_0 > 0$, $0 < \alpha_1 < 1$ and where $(\varphi_t, t\in\mathbb{Z})$ is a family of deterministic characteristic functions on $\mathbb{R}$, $\underline{X}_{t-1}$-measurables, associated to a family of discrete laws with support in $\mathbb{N}_0$ and finite mean.

This chapter consists of two sections. The section 4.1 is dedicated to present closed-form expressions for joint (central) moments and cumulants of the CP-INARCH(1) model up to order 4. These results are applied in Section 4.2 to determine an explicit expression for the asymptotic distribution of the conditional least square (CLS) estimator of the model. In fact, in the final section of the chapter we discuss the CLS approach for the estimation of the parameters α_0 and α_1 of the model and, for the additional parameters associated to its conditional distribution, an approach based on the moment estimation method is developed. In the sequel, we present results from a simulation study, where we investigate the finite-sample performance of the confidence region based on the CLS estimators.

4.1 Moments and Cumulants

We have seen that, under quite simple conditions, a CP-INGARCH(p,q) model admits a strictly stationary and ergodic solution X. However, the marginal distribution of the process X is not known explicitly. The aim of this section is to highlight some properties of this distribution through some of its moments. We start by computing the first four cumulants of the process and then, as a consequence, the corresponding skewness and kurtosis are deduced ([1]).

Let us consider the CP-INARCH(1) process defined in (4.1), which is first-order stationary (Theorem 3.1) and admits moments of all orders (Theorem 3.4). In addition, let us assume φ_t derivable as many times as necessary. From the conditional characteristic function we have

$$\begin{aligned}\Phi_{X_t}(z) &= E\left(e^{izX_t}\right) = E\left[E\left(e^{izX_t}|\underline{X}_{t-1}\right)\right] = E\left[\exp\left(i\frac{\lambda_t}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)\right]\\ &= E\left[\exp\left(\frac{i\alpha_0}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)\cdot\exp\left(\frac{i\alpha_1 X_{t-1}}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)\right]\\ &= \exp\left(\frac{i\alpha_0}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)\cdot\Phi_{X_{t-1}}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right), \quad z\in\mathbb{R},\end{aligned}$$

and hence, the cumulant generating function is given by

$$\kappa_{X_t}(z) = \ln\left(\Phi_{X_t}(z)\right) = \frac{i\alpha_0}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]+\kappa_{X_{t-1}}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right).$$

Taking derivatives on both sides of the previous equality, it follows that

$$\kappa_{X_t}'(z) = \frac{i\alpha_0\varphi_t'(z)}{\varphi_t'(0)}+\frac{\alpha_1}{\varphi_t'(0)}\varphi_t'(z)\cdot\kappa_{X_{t-1}}'\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right), \tag{4.2}$$

$$\begin{aligned}\kappa_{X_t}^{(n)}(z) &= \frac{i\alpha_0\varphi_t^{(n)}(z)}{\varphi_t'(0)}+\sum_{j=1}^{n-1}a_{n-1,j}(z)\cdot\kappa_{X_{t-1}}^{(j)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)\\ &\quad+\left[\frac{\alpha_1\varphi_t'(z)}{\varphi_t'(0)}\right]^n\cdot\kappa_{X_{t-1}}^{(n)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right), \quad n\in\mathbb{N},\end{aligned} \tag{4.3}$$

where the coefficients $a_{n-1,j}$ are given by

$$a_{n-1,j}(z) = \left[\frac{\alpha_1}{\varphi_t'(0)}\right]^j \sum_{\substack{k_1+\ldots+k_n=j\\ k_1+2k_2+\ldots+nk_n=n\\ k_r\in\mathbb{N}_0}} (n;k_1,\ldots,k_n)\left[\varphi_t'(z)\right]^{k_1}\ldots[\varphi_t^{(n)}(z)]^{k_n}, \quad j\geq 1,$$

[1] The cumulants are defined via the cumulant generating function which is simply the natural logarithm of the characteristic function. In fact, if Φ_X denotes the characteristic function of X, its cumulant generating function is given by

$$\kappa_X(z)=\ln\left(\Phi_X(z)\right)=\sum_{j=1}^{\infty}\kappa_j(X)\cdot(iz)^j/j!, \quad z\in\mathbb{R},$$

where the coefficient of the series expansion, $\kappa_j(X)$, is the j-th cumulant [15, p. 185].

with $(n;k_1,...,k_n)$ denoting $n!/[(1!)^{k_1}k_1!(2!)^{k_2}k_2!\dots(n!)^{k_n}k_n!]$ as before. The proof of formula (4.3) is made by induction and it is provided in Appendix $C.2$. In particular, we get

$$a_{n-1,1}(z) = \frac{\alpha_1\varphi_t^{(n)}(z)}{\varphi_t'(0)},$$

$$a_{n-1,n-2}(z) = \frac{n(n-1)(n-2)}{24}\left(\frac{\alpha_1}{\varphi_t'(0)}\right)^{n-2}\left[4(\varphi_t'(z))^{n-3}\varphi_t'''(z)+3(n-3)(\varphi_t'(z))^{n-4}(\varphi_t''(z))^2\right],$$

$$a_{n-1,n-1}(z) = \frac{n(n-1)}{2}\left(\frac{\alpha_1}{\varphi_t'(0)}\right)^{n-1}(\varphi_t'(z))^{n-2}\varphi_t''(z).$$

We note that in order to obtain $a_{n-1,1}$ we take $k_1=...=k_{n-1}=0$ and $k_n=1$ since it is the only way to get $k_1+...+k_n=1$ with $k_1+2k_2+...+nk_n=n$. To obtain $a_{n-1,n-1}$ we take into account that $k_1+...+k_n=n-1$ with $k_1+2k_2+...+nk_n=n$ if and only if $k_1=n-2$, $k_2=1$ and $k_3=...=k_n=0$. For $a_{n-1,n-2}$ we get $k_1+...+k_n=n-2$ with $k_1+2k_2+...+nk_n=n$ if and only if we take $k_1=n-3$, $k_3=1$ and $k_2=k_4=...=k_n=0$ or, if we take $k_1=n-4$, $k_2=2$ and $k_3=...=k_n=0$.

Inserting $z=0$ into the equations (4.2) and (4.3) we obtain

$$\kappa_{X_t}'(0) = i\alpha_0+\alpha_1\cdot\kappa_{X_{t-1}}'(0), \tag{4.4}$$

$$\kappa_{X_t}^{(n)}(0) = \frac{i\alpha_0\varphi_t^{(n)}(0)}{\varphi_t'(0)}+\sum_{j=1}^{n-1}a_{n-1,j}(0)\cdot\kappa_{X_{t-1}}^{(j)}(0)+\alpha_1^n\cdot\kappa_{X_{t-1}}^{(n)}(0), \quad n=2,3,4. \tag{4.5}$$

Let us fix now some notation:

$$v_{0,t} = -i\,\frac{\varphi_t''(0)}{\varphi_t'(0)} = \frac{E(X_{t,1}^2)}{E(X_{t,1})}, \qquad d_{0,t} = -\frac{\varphi_t'''(0)}{\varphi_t'(0)} = \frac{E(X_{t,1}^3)}{E(X_{t,1})},$$

$$c_{0,t} = i\frac{\varphi_t^{(iv)}(0)}{\varphi_t'(0)} = \frac{E(X_{t,1}^4)}{E(X_{t,1})}, \qquad f_k = \frac{\alpha_0}{\prod_{j=1}^k(1-\alpha_1^j)}, \quad k\in\mathbb{N}. \tag{4.6}$$

Theorem 4.1 (Marginal Cumulants of CP-INARCH(1) process) *Let X be a first-order stationary CP-INARCH(1) process such that the hypothesis* **H3** *is satisfied with φ_t derivable up to order 4. Then, the first four cumulants of X_t are given, respectively, by*

$$\kappa_1(X_t)=\mu, \qquad \kappa_2(X_t)=v_{0,t}f_2, \qquad \kappa_3(X_t)=f_3[d_{0,t}(1-\alpha_1^2)+3v_{0,t}^2\alpha_1^2],$$

$$\kappa_4(X_t)=f_4[c_{0,t}(1-\alpha_1^2)(1-\alpha_1^3)+v_{0,t}^3(3\alpha_1^2+15\alpha_1^5)+v_{0,t}d_{0,t}(4\alpha_1^2+6\alpha_1^3-10\alpha_1^5)].$$

Proof: Since the j-th cumulant of X_t can be obtained as $\kappa_j(X_t)=(-i)^j\kappa_{X_t}^{(j)}(0)$ ([15]), then from expressions (4.4) and (4.5) we get, using the hypothesis of first-order stationarity,

$$\kappa_1(X_t) = \alpha_0+\alpha_1\cdot\kappa_1(X_{t-1}) \Rightarrow \kappa_1(X_t)=\frac{\alpha_0}{1-\alpha_1}=\mu,$$

$$\kappa_n(X_t) \;=\; \sum_{j=1}^{n-1} b_{n-1,j}\cdot\kappa_j(X_{t-1}) + \alpha_1^n\cdot\kappa_n(X_{t-1}), \quad n=2,3,4,$$

where the coefficients $b_{n-1,j}$ are given by

$$b_{n-1,1} \;=\; (-i)^{n-1}\frac{\varphi_t^{(n)}(0)}{\varphi_t'(0)},$$

$$\begin{aligned} b_{n-1,n-2} \;&=\; -\;\frac{n(n-1)(n-2)}{24}\left[4\frac{\varphi_t'''(0)}{\varphi_t'(0)}+3(n-3)\left(\frac{\varphi_t''(0)}{\varphi_t'(0)}\right)^2\right]\alpha_1^{n-2} \\ &=\; \frac{n(n-1)(n-2)}{24}\left[4d_{0,t}+3(n-3)v_{0,t}^2\right]\alpha_1^{n-2}, \end{aligned}$$

$$b_{n-1,n-1} \;=\; -i\,\frac{n(n-1)}{2}\frac{\varphi_t''(0)}{\varphi_t'(0)}\,\alpha_1^{n-1} \;=\; \frac{n(n-1)}{2}\,v_{0,t}\alpha_1^{n-1},$$

$$b_{n-1,j} \;=\; (-i)^{n-j}\left[\frac{\alpha_1}{\varphi_t'(0)}\right]^j \sum_{\substack{k_1+\ldots+k_n=j\\ k_1+2k_2+\ldots+nk_n=n\\ k_r\in\mathbb{N}_0}} (n;k_1,\ldots,k_n)\left[\varphi_t'(0)\right]^{k_1}\ldots[\varphi_t^{(n)}(0)]^{k_n}, \qquad 2\le j\le n-3.$$

So, we deduce that the second, third and fourth cumulant of X_t are given by

$$\begin{aligned} \kappa_2(X_t) \;&=\; b_{1,1}\cdot\kappa_1(X_{t-1})+\alpha_1^2\cdot\kappa_2(X_{t-1}) \\ &=\; v_{0,t}\cdot\kappa_1(X_{t-1})+\alpha_1^2\cdot\kappa_2(X_{t-1}) \\ \Rightarrow\;\kappa_2(X_t) \;&=\; v_{0,t}f_2, \qquad \text{(as stated in Remark 3.5 under the hypothesis } \mathbf{H2}\text{)} \end{aligned}$$

$$\begin{aligned} \kappa_3(X_t) \;&=\; b_{2,1}\cdot\kappa_1(X_{t-1})+b_{2,2}\cdot\kappa_2(X_{t-1})+\alpha_1^3\cdot\kappa_3(X_{t-1}) \\ &=\; d_{0,t}\cdot\kappa_1(X_{t-1})+3v_{0,t}\alpha_1^2\cdot\kappa_2(X_{t-1})+\alpha_1^3\cdot\kappa_3(X_{t-1}) \\ \Rightarrow\;\kappa_3(X_t) \;&=\; f_3[d_{0,t}(1-\alpha_1^2)+3v_{0,t}^2\alpha_1^2], \end{aligned}$$

$$\begin{aligned} \kappa_4(X_t) \;&=\; b_{3,1}\cdot\kappa_1(X_{t-1})+b_{3,2}\cdot\kappa_2(X_{t-1})+b_{3,3}\cdot\kappa_3(X_{t-1})+\alpha_1^4\cdot\kappa_4(X_{t-1}) \\ &=\; c_{0,t}\cdot\kappa_1(X_{t-1})+\left[4d_{0,t}+3v_{0,t}^2\right]\alpha_1^2\cdot\kappa_2(X_{t-1}) \\ &\quad +6v_{0,t}\alpha_1^3\cdot\kappa_3(X_{t-1})+\alpha_1^4\cdot\kappa_4(X_{t-1}) \\ \Rightarrow\;\kappa_4(X_t) \;&=\; f_4[c_{0,t}(1-\alpha_1^2)(1-\alpha_1^3)+v_{0,t}^3(3\alpha_1^2+15\alpha_1^5) \\ &\quad +v_{0,t}d_{0,t}(4\alpha_1^2+6\alpha_1^3-10\alpha_1^5)]. \;\blacksquare \end{aligned}$$

Skewness and kurtosis are moment parameters that give relevant information on the symmetry and shape of the related distribution. As a consequence of Theorem 4.1 and adding the hypothesis **H2**, X is an asymmetric process around the mean and it is leptokurtic.

Corollary 4.1 *The skewness and kurtosis of X_t, respectively S_{X_t} and K_{X_t}, are given by*

$$S_{X_t} = \frac{\kappa_3(X_t)}{\kappa_2^{3/2}(X_t)} = \frac{d_{0,t}(1-\alpha_1^2)+3\alpha_1^2 v_{0,t}^2}{v_{0,t}(1+\alpha_1+\alpha_1^2)}\sqrt{\frac{1+\alpha_1}{v_{0,t}\alpha_0}},$$

$$K_{X_t} = 3 + \frac{\kappa_4(X_t)}{\kappa_2^2(X_t)} = 3 + \frac{c_{0,t}(1-\alpha_1^2)(1-\alpha_1^3) + v_{0,t}^3(3\alpha_1^2 + 15\alpha_1^5) + v_{0,t}d_{0,t}(4\alpha_1^2 + 6\alpha_1^3 - 10\alpha_1^5)}{\alpha_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)v_{0,t}^2}.$$

In the following we illustrate the expressions displayed above for the cumulants, skewness and kurtosis of a CP-INARCH(1) process considering some particular compound Poisson distributions. In Example 4.5 the dependence of the time t in these expressions is evident.

Example 4.1 (INARCH(1) process) *We have* $\varphi(u) = e^{iu}$ *and* $\varphi^{(n)}(0) = i^n$. *So* $v_{0,t} = d_{0,t} = c_{0,t} = 1$, *which are independent of* t. *Then the first four cumulants of* X_t *are equal to*

$$\kappa_1(X_t) = \frac{\alpha_0}{1-\alpha_1}, \qquad \kappa_2(X_t) = \frac{\alpha_0}{(1-\alpha_1)(1-\alpha_1^2)},$$

$$\kappa_3(X_t) = \frac{\alpha_0(1+2\alpha_1^2)}{(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)}, \quad \kappa_4(X_t) = \frac{\alpha_0(1+6\alpha_1^2+5\alpha_1^3+6\alpha_1^5)}{(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)(1-\alpha_1^4)},$$

and the skewness and the kurtosis are given by

$$S_{X_t} = \frac{1+2\alpha_1^2}{1+\alpha_1+\alpha_1^2}\sqrt{\frac{1+\alpha_1}{\alpha_0}}, \quad K_{X_t} = 3 + \frac{1+6\alpha_1^2+5\alpha_1^3+6\alpha_1^5}{\alpha_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)}.$$

In this case the cumulants of X_t *can be determined recursively from*

$$\kappa_1(X_t) = \frac{\alpha_0}{1-\alpha_1}, \qquad \kappa_n(X_t) = (1-\alpha_1^n)^{-1} \cdot \sum_{j=1}^{n-1} b_{n-1,j} \cdot \kappa_j(X_t), \quad n \geq 2,$$

where the coefficients $b_{n-1,j}$ *are given by*

$$b_{n-1,1} = 1, \qquad b_{n-1,n-1} = \frac{n(n-1)}{2}\,\alpha_1^{n-1},$$

$$b_{n-1,j} = \alpha_1^j \cdot \sum_{\substack{k_1+\ldots+k_n=j \\ k_1+2k_2+\ldots+nk_n=n}} (n;k_1,\ldots,k_n) = \alpha_1^j \cdot S(n,j), \quad 2 \leq j \leq n-2,$$

from the relation between $(n;k_1,\ldots,k_n)$ *and Stirling numbers of the second kind stated in [2, p. 823].*

Example 4.2 (GP-INARCH(1) process) *In this case,* φ *is the characteristic function of the i.i.d. random variables* $X_{t,j}$, $j = 1,\ldots,N_t$, *having the Borel distribution with parameter* κ.

All the moments exist for $0 < \kappa < 1$; *in particular, we have* ([2])

$$E(X_{t,1}) = \frac{1}{1-\kappa}, \quad E(X_{t,1}^2) = \frac{1}{(1-\kappa)^3}, \quad E(X_{t,1}^3) = \frac{2\kappa+1}{(1-\kappa)^5}, \quad E(X_{t,1}^4) = \frac{6\kappa^2+8\kappa+1}{(1-\kappa)^7},$$

$$v_{0,t} = \frac{1}{(1-\kappa)^2}, \qquad d_{0,t} = \frac{2\kappa+1}{(1-\kappa)^4}, \qquad c_{0,t} = \frac{6\kappa^2+8\kappa+1}{(1-\kappa)^6}.$$

[2]Using the recurrence relation [14, p. 159]:

$$E(X) = \frac{1}{1-\kappa}, \qquad E(X^{r+1}) = \frac{1}{1-\kappa}\left[\kappa\frac{dE(X^r)}{d\kappa} + E(X^r)\right], \quad r = 1,2,\ldots$$

Thus, we obtain the cumulants

$$\kappa_2(X_t) = \frac{\alpha_0}{(1-\kappa)^2(1-\alpha_1)(1-\alpha_1^2)}, \qquad \kappa_3(X_t) = \frac{\alpha_0(1-\alpha_1^2)(2\kappa+1)+3\alpha_0\alpha_1^2}{(1-\kappa)^4(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)},$$

$$\kappa_4(X_t) = \alpha_0\,\frac{6\kappa^2+8\kappa+1-6\alpha_1^2(\kappa^2+1)-\alpha_1^3(6\kappa^2-4\kappa-5)+6\alpha_1^5(\kappa^2-2\kappa+1)}{(1-\kappa)^6(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)(1-\alpha_1^4)},$$

and the skewness and the kurtosis

$$S_{X_t} = \frac{(1-\alpha_1^2)(2\kappa+1)+3\alpha_1^2}{(1-\kappa)(1+\alpha_1+\alpha_1^2)}\sqrt{\frac{1+\alpha_1}{\alpha_0}},$$

$$K_{X_t} = 3+\frac{6\kappa^2+8\kappa+1-6\alpha_1^2(\kappa^2+1)-\alpha_1^3(6\kappa^2-4\kappa-5)+6\alpha_1^5(\kappa^2-2\kappa+1)}{\alpha_0(1-\kappa)^2(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)}.$$

Let us take into account Figure 2.8 *where we present the trajectory of a GP-INARCH*(1) *process with* $\alpha_0 = 10$, $\alpha_1 = 0.4$, $\kappa = 0.5$. *We notice evident closeness between the theoretical values, namely* $S_{X_t} \simeq 1.0362$ *and* $K_{X_t} = 4.2527$ *and the empirical ones, respectively,* 1.004 *and* 4.111.

We stress that taking into consideration the fact that the generalized Poisson distribution is compound Poisson allowed us to generalize the results of Zhu [82] and also to present a much more prompt deduction of the first four cumulants.

Example 4.3 (GEOMP2-INARCH(1) process) *For the geometric law present in the GEOMP2-INARCH*(1) *model we have*

$$\varphi'(0) = i\,\frac{1-p^*}{p^*}, \qquad \varphi''(0) = \frac{(1-p^*)(p^*-2)}{(p^*)^2}, \qquad \varphi'''(0) = -i\,\frac{(1-p^*)[6-6p^*+(p^*)^2]}{(p^*)^3},$$

$$\varphi^{(iv)}(0) = \frac{(1-p^*)(2-p^*)[12-12p^*+(p^*)^2]}{(p^*)^4},$$

$$v_{0,t} = \frac{2-p^*}{p^*}, \qquad d_{0,t} = \frac{6-6p^*+(p^*)^2}{(p^*)^2}, \qquad c_{0,t} = \frac{(2-p^*)[12-12p^*+(p^*)^2]}{(p^*)^3},$$

from where we deduce the skewness and the kurtosis of X_t*, respectively,*

$$S_{X_t} = \frac{6-6p^*+6(p^*)^2+2(p^*)^2\alpha_1^2}{(2p^*-(p^*)^2)(1+\alpha_1+\alpha_1^2)}\sqrt{\frac{p^*(1+\alpha_1)}{\alpha_0(2-p^*)}},$$

$$K_{X_t} = 3+\frac{(1-\alpha_1^2)(1-\alpha_1)(2-p^*)[12-12p^*+(p^*)^2]}{\alpha_0 p^*(2-p^*)^2(1+\alpha_1^2)}+\frac{\alpha_1^2(3+15\alpha_1^3)(2-p^*)}{\alpha_0 p^*(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)}$$
$$-\frac{2\alpha_1^2(1-\alpha_1)(5\alpha_1^2+5\alpha_1+2)[6-6p^*+(p^*)^2]}{\alpha_0 p^*(2-p^*)(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)}.$$

Let us remember Figure 2.7 *where we present the trajectory of a GEOMP-INARCH*(1) *process with* $\alpha_0 = 10$, $\alpha_1 = 0.4$, $p^* = 0.3$. *Also here the theoretical values, namely* $S_{X_t} \simeq 0.942$ *and* $K_{X_t} = 3.493$ *and the empirical ones, respectively,* 0.892 *and* 3.670 *are quite similar.*

Example 4.4 (NTA-INARCH(1) process) *In this process the skewness and kurtosis are given by*

$$S_{X_t} = \frac{\phi(1-\alpha_1^2)+(1+\phi)^2(1+2\alpha_1^2)}{(1+\phi)(1+\alpha_1+\alpha_1^2)}\sqrt{\frac{1+\alpha_1}{\alpha_0(1+\phi)}},$$

$$K_{X_t} = 3+\frac{(1-\alpha_1)(1-\alpha_1^2)\left((1+\phi)^3+3\phi^2+4\phi\right)}{\alpha_0(1+\alpha_1^2)(1+\phi)^2}+\frac{(7\alpha_1^2+6\alpha_1^3+5\alpha_1^5)(1+\phi)}{\alpha_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)}$$
$$+\frac{\phi(4\alpha_1^2+6\alpha_1^3-10\alpha_1^5)}{\alpha_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)(1+\phi)},$$

since $v_{0,t} = 1+\phi$, $d_{0,t} = 1+3\phi+\phi^2$ *and* $c_{0,t} = 1+7\phi+6\phi^2+\phi^3$.

Example 4.5 *Let us recall the model 7(b) defined in Section* 2.3 *where the conditional distribution is the binomial Poisson law with parameters* $(r, \frac{1}{t^2+1})$, $r \in \mathbb{N}$, $t \in \mathbb{Z}$, *which is deterministic and dependent of* t. *In this case we have* $\varphi_t(u) = \left(\frac{t^2+e^{iu}}{t^2+1}\right)^r$, $u \in \mathbb{R}$, *and we can prove*

$$\varphi_t^{(n)}(u) = i^n\sum_{k=1}^{n} S(n,k)\frac{r!}{(r-k)!}\left(\frac{t^2+e^{iu}}{t^2+1}\right)^{r-k}\left(\frac{e^{iu}}{t^2+1}\right)^k, \quad n\in\mathbb{N},\ u\in\mathbb{R}.$$

In fact, by induction with respect to n, we have

$$\begin{aligned}
\varphi_t^{(n+1)}(u) &= \frac{d}{du}\left(i^n\sum_{k=1}^{n} S(n,k)\frac{r!}{(r-k)!}\left(\frac{t^2+e^{iu}}{t^2+1}\right)^{r-k}\left(\frac{e^{iu}}{t^2+1}\right)^k\right)\\
&= i^{n+1}\sum_{k=1}^{n} S(n,k)\frac{r!}{(r-k-1)!}\left(\frac{t^2+e^{iu}}{t^2+1}\right)^{r-k-1}\left(\frac{e^{iu}}{t^2+1}\right)^{k+1}\\
&\quad +i^{n+1}\sum_{k=1}^{n} kS(n,k)\frac{r!}{(r-k)!}\left(\frac{t^2+e^{iu}}{t^2+1}\right)^{r-k}\left(\frac{e^{iu}}{t^2+1}\right)^k\\
&= i^{n+1}\sum_{j=2}^{n+1}[S(n,j-1)+jS(n,j)]\frac{r!}{(r-j)!}\left(\frac{t^2+e^{iu}}{t^2+1}\right)^{r-j}\left(\frac{e^{iu}}{t^2+1}\right)^j\\
&\quad +S(n,1)\frac{r!}{(r-1)!}\left(\frac{t^2+e^{iu}}{t^2+1}\right)^{r-1}\frac{e^{iu}}{t^2+1}\\
&= i^{n+1}\sum_{k=1}^{n+1} S(n+1,k)\frac{r!}{(r-k)!}\left(\frac{t^2+e^{iu}}{t^2+1}\right)^{r-k}\left(\frac{e^{iu}}{t^2+1}\right)^k, \quad u\in\mathbb{R},
\end{aligned}$$

from the recurrence relation of the Stirling numbers of the second kind and since $S(n,1) = S(n+1,1)$ *([3]). Then we deduce*

$$\varphi_t^{(n)}(0) = i^n\sum_{k=1}^{n} S(n,k)\frac{r!}{(r-k)!(t^2+1)^k},$$

$$\varphi_t'(0) = i\,\frac{r}{t^2+1}, \qquad \varphi_t''(0) = -\frac{r(t^2+r)}{(t^2+1)^2}, \qquad \varphi_t'''(0) = -i\,\frac{r[(t^2+3r-1)t^2+r^2]}{(t^2+1)^3},$$

[3] We note that here we use this relation only for the first three moments of the binomial distribution. However, we stated a general expression from which we can obtain the moment of any order.

$$v_{0,t} = \frac{t^2+r}{t^2+1}, \qquad d_{0,t} = \frac{(t^2+3r-1)t^2+r^2}{(t^2+1)^2},$$

and finally we obtain the first three cumulants

$$\kappa_1(X_t) = \frac{\alpha_0}{1-\alpha_1}, \qquad \kappa_2(X_t) = \frac{\alpha_0(t^2+r)}{(t^2+1)(1-\alpha_1)(1-\alpha_1^2)},$$

$$\kappa_3(X_t) = \frac{\alpha_0(1-\alpha_1^2)([t^2+3r-1]t^2+r^2)+3\alpha_0\alpha_1^2(t^2+r)^2}{(t^2+1)^2(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)},$$

and the skewness

$$S_{X_t} = \frac{(1-\alpha_1^2)([t^2+3r-1]t^2+r^2)(t^2+1)^{3/2}+3\alpha_1^2(t^2+r)^2}{(t^2+r)^3(1+\alpha_1+\alpha_1^2)}\sqrt{\frac{1+\alpha_1}{\alpha_0(t^2+1)(t^2+r)}}.$$

In the following theorem we provide closed-form expressions for the joint (central) moments and cumulants of the CP-INARCH(1) model up to order 4 when the family of the characteristic functions $(\varphi_t, t\in\mathbb{Z})$ is deterministic and independent of t. For this purpose, we simply denote the previous $v_{0,t}$, $c_{0,t}$ and $d_{0,t}$ respectively as v_0, c_0 and d_0, and we introduce the following notations:

$$\begin{aligned}
\mu(s_1,\dots,s_{r-1}) &= E\left(X_t X_{t+s_1}\dots X_{t+s_{r-1}}\right),\\
\widetilde{\mu}(s_1,\dots,s_{r-1}) &= E\left((X_t-\mu)(X_{t+s_1}-\mu)\dots(X_{t+s_{r-1}}-\mu)\right),\\
\kappa(s_1,\dots,s_{r-1}) &= Cum\left[X_t, X_{t+s_1},\dots,X_{t+s_{r-1}}\right],
\end{aligned} \tag{4.7}$$

with $r=2,3,4$ and $0\le s_1\le\dots\le s_{r-1}$.

Theorem 4.2 (Moments of a CP-INARCH(1) process) *Let X be a first-order stationary process following a CP-INARCH(1) model such that* **H4** *is satisfied.*

(a) *For any $k\ge 0$, we have*

$$\mu(k) = f_2(v_0\alpha_1^k+\alpha_0(1+\alpha_1)).$$

(b) *For any $l\ge k\ge 0$, we have*

$$\begin{aligned}
\mu(k,l) &= [d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{l+k}\\
&\quad +\frac{v_0(\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^l+v_0f_1f_2\alpha_1^{l-k}+f_1\mu(k).
\end{aligned}$$

(c) *For any $m\ge l\ge k\ge 0$, we have*

$$\begin{aligned}
\mu(k,l,m) &= \alpha_1^{m-l}\left[\left\{(c_0-4v_0d_0+3v_0^3)+3v_0(v_0^2-d_0)\alpha_1+(3v_0d_0-c_0)\alpha_1^2\right.\right.\\
&\quad \left.+(7v_0d_0-6v_0^3-c_0)\alpha_1^3+3v_0(d_0-2v_0^2)\alpha_1^4+(6v_0^3-6v_0d_0+c_0)\alpha_1^5\right\}f_4\alpha_1^{2l+k}\\
&\quad +\frac{2v_0+\alpha_0}{1-\alpha_1}f_3\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]\alpha_1^{2l}
\end{aligned}$$

$$+\frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)} f_2 \left[2v_0\alpha_0 + d_0(1-\alpha_1) + v_0^2(2\alpha_1 - 1)\right] \alpha_1^{2l-k}$$
$$+\frac{\alpha_0 f_3}{1-\alpha_1}\left\{d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1 - 2\alpha_1^2)\right\}\alpha_1^{2(l-k)} + \frac{v_0+\alpha_0}{1-\alpha_1}\mu(k,l)$$
$$-f_2\mu(k)[\alpha_0 + (v_0+\alpha_0)\alpha_1]] + f_1\mu(k,l).$$

The proof of Theorem 4.2 is provided in Appendix C.3. These expressions are particularly important to deduce an explicit expression for the asymptotic distribution of the conditional least squares estimators of the parameters α_0 and α_1 in Section 4.2.

From Theorem 4.2 we derive analogous expressions for the joint central moments and cumulants of a CP-INARCH(**1**) process. Bakouch [8, p. 5] stated the following general relations between joint moments and joint cumulants of stationary processes:

$$\kappa(s) = \mu(s) - \mu^2, \tag{4.8}$$
$$\kappa(s,l) = \mu(s,l) - \mu\mu(s) - \mu\left[\mu(l-s) + \mu(l) - 2\mu^2\right], \tag{4.9}$$
$$\begin{aligned}\kappa(s,l,m) &= \mu(s,l,m) - \mu\mu(s,l)\\ &\quad -\mu[\mu(l-s,m-s) - \mu\mu(l-s) - \mu(l,m) - \mu\mu(l) + \mu(s,m) - \mu\mu(s)]\\ &\quad -(\mu(s)-\mu^2)(\mu(m-l)-\mu^2) - (\mu(l)-\mu^2)(\mu(m-s)-\mu^2)\\ &\quad -(\mu(l-s)-\mu^2)(\mu(m)-\mu^2)\\ &\quad +\mu^2(\mu(m)+\mu(m-s)+\mu(m-l)-3\mu^2),\end{aligned} \tag{4.10}$$

where $m \geq l \geq s \geq 0$. Furthermore, it follows that the joint central moments are given by

$$\widetilde{\mu}(s) = E((X_t-\mu)(X_{t+s}-\mu)) = E(X_tX_{t+s}) - \mu^2 = \kappa(s), \tag{4.11}$$
$$\begin{aligned}\widetilde{\mu}(s,l) &= E[(X_t-\mu)(X_{t+s}-\mu)(X_{t+l}-\mu)]\\ &= E[(X_tX_{t+s} - \mu X_t - \mu X_{t+s} + \mu^2)X_{t+l}] - \mu\widetilde{\mu}(s)\\ &= \mu(s,l) - \mu\mu(l) - \mu\mu(s) + \mu^3 - \mu\mu(s) + \mu^3 = \kappa(s,l),\end{aligned} \tag{4.12}$$
$$\begin{aligned}\widetilde{\mu}(s,l,m) &= E[(X_t-\mu)(X_{t+s}-\mu)(X_{t+l}-\mu)(X_{t+m}-\mu)]\\ &= E[(X_tX_{t+s}X_{t+l} - \mu X_tX_{t+l} - \mu X_{t+s}X_{t+l} + \mu^2X_{t+l})X_{t+m}]\\ &\quad -E[(\mu X_tX_{t+s} - \mu^2X_t - \mu^2X_{t+s} + \mu^3)X_{t+m}] - \mu\widetilde{\mu}(s,l)\\ &= \mu(s,l,m) + \mu^2[\mu(l-s)+\mu(m-s)+\mu(m-l)+\mu(s)+\mu(l)+\mu(m)]\\ &\quad -\mu[\mu(l-s,m-s)+\mu(l,m)+\mu(s,l)+\mu(s,m)] - 3\mu^4\\ &= \kappa(s,l,m) + (\mu(s)-\mu^2)(\mu(m-l)-\mu^2) + (\mu(l)-\mu^2)(\mu(m-s)-\mu^2)\\ &\quad +(\mu(l-s)-\mu^2)(\mu(m)-\mu^2),\end{aligned} \tag{4.13}$$

where $m \geq l \geq s \geq 0$.

The following corollary may now be enounced.

Corollary 4.2 (Central Moments and Cumulants of a CP-INARCH(1) process) *Let X be a first-order stationary process following a CP-INARCH(1) model such that* **H4** *is satisfied.*

(a) *For any* $s \geq 0$, *we have*

$$\widetilde{\mu}(s) = \kappa(s) = v_0 \alpha_1^s f_2.$$

(b) *For any* $l \geq s \geq 0$, *we have*

$$\begin{aligned}\widetilde{\mu}(s,l) &= \kappa(s,l) \\ &= f_3 \alpha_1^l [v_0^2(1+\alpha_1+\alpha_1^2) - \{v_0^2(1+\alpha_1-2\alpha_1^2) - d_0(1-\alpha_1^2)\}\alpha_1^s].\end{aligned}$$

(c) *For any* $m \geq l \geq s \geq 0$, *we have*

$$\begin{aligned}\kappa(s,l,m) &= \alpha_1^m f_4 \left[\{c_0 + 3v_0^3 - 4v_0 d_0 + 3v_0(v_0^2 - d_0)\alpha_1 + (3\alpha_0 d_0 - c_0)\alpha_1^2\right. \\ &\quad + (7v_0 d_0 - 6v_0^3 - c_0)\alpha_1^3 + 3v_0(d_0 - 2v_0^2)\alpha_1^4 + (6v_0^3 - 6v_0 d_0 + c_0)\alpha_1^5\} \alpha_1^{l+s} \\ &\quad + v_0(1+\alpha_1+\alpha_1^2+\alpha_1^3)[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)](2\alpha_1^l + \alpha_1^s) \\ &\quad \left. + v_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)[(1+\alpha_1)v_0^2 + (d_0(1-\alpha_1) + v_0^2(2\alpha_1-1))\alpha_1^{l-s}]\right], \\ \widetilde{\mu}(s,l,m) &= \kappa(s,l,m) + v_0^2 f_2^2 \alpha_1^{m-l+s} + 2v_0^2 f_2^2 \alpha_1^{m+l-s}.\end{aligned}$$

The proof of Corollary 4.2 is presented in Appendix C.4. Considering v_0, c_0 and d_0 computed in Examples 4.1–4.4 we obtain the particular expressions of the central moments and cumulants for the INARCH(1) (already stated by Weiß [78]), GP-INARCH(1), GEOMP2-INARCH(1) and NTA-INARCH(1) processes.

4.2 Two-step estimation method based on Conditional Least Squares Approach

This section is intended to derive estimates for the parameters of the CP-INARCH(1) process defined in (4.1) when the family of the characteristic functions $(\varphi_t, t \in \mathbb{Z})$ is independent of t, with special attention to the recent NTA-INARCH(1) and GEOMP2-INARCH(1) models.

Let X be the referred CP-INARCH(1) process whose distribution depends on a vector $\theta = (\alpha^\top, b)$ of unknown parameters, with $\alpha^\top = (\alpha_0, \alpha_1)$. Let us observe that, for example, $b = \phi$ in the NTA-INARCH(1) model and $b = p^*$ in the GEOMP2-INARCH(1) model.

Since the conditional law is not specified by its density probability function, we apply a two-step estimation procedure using the Conditional Least Squares (CLS) and the moments estimation methods: we start by estimating α from the CLS estimator assuming that b is known and then holding that estimate fixed, we estimate b from the expression of the empirical variance. We shall see that the CLS estimator is computed easily and have an explicit, data-independent expression for the asymptotic law.

Given a set of observations x_t, $t = 1, ..., n$, from a CP-INARCH(1) process according to the above conditions, the CLS estimator of α is obtained by minimizing the conditional sum of squares

$$Q_n(\alpha) = \sum_{t=2}^{n} [x_t - E(X_t | X_{t-1} = x_{t-1})]^2 = \sum_{t=2}^{n} [x_t - \alpha_0 - \alpha_1 x_{t-1}]^2,$$

with respect to α. Solving the least squares equations

$$\begin{cases} \dfrac{\partial Q_n(\alpha)}{\partial \alpha_0} = -2\sum_{t=2}^{n} (x_t - \alpha_0 - \alpha_1 x_{t-1}) = 0 \\[2ex] \dfrac{\partial Q_n(\alpha)}{\partial \alpha_1} = -2\sum_{t=2}^{n} x_{t-1} (x_t - \alpha_0 - \alpha_1 x_{t-1}) = 0, \end{cases}$$

yields the following explicit expressions for the CLS estimator $\widehat{\alpha}_n = (\widehat{\alpha}_{0,n}, \widehat{\alpha}_{1,n})$:

$$\widehat{\alpha}_{1,n} = \frac{\sum_{t=2}^{n} X_t X_{t-1} - \frac{1}{n-1} \cdot \sum_{t=2}^{n} X_t \cdot \sum_{s=2}^{n} X_{s-1}}{\sum_{t=2}^{n} X_{t-1}^2 - \frac{1}{n-1} \left(\sum_{t=2}^{n} X_{t-1}\right)^2}, \qquad \widehat{\alpha}_{0,n} = \frac{\sum_{t=2}^{n} X_t - \widehat{\alpha}_{1,n} \sum_{t=2}^{n} X_{t-1}}{n-1}. \tag{4.14}$$

The consistency and the asymptotic distribution of the CLS estimator $\widehat{\alpha}_n$ can be stated by applying the results of Klimko and Nelson [50, Section 3]. We summarize the main ideas in the next lemma.

Lemma 4.1 *Let X be a stationary and ergodic Markov sequence with finite third order moments. Let us assume that the function $g = g(\alpha; X_{t-1}) = E\left(X_t | \underline{X}_{t-1}\right)$ satisfies the following regularity conditions:*

(i) *for $i, j, k \in \{0, 1\}$, $\dfrac{\partial g}{\partial \alpha_i}$, $\dfrac{\partial^2 g}{\partial \alpha_i \partial \alpha_j}$ and $\dfrac{\partial^3 g}{\partial \alpha_i \partial \alpha_j \partial \alpha_k}$ exist and are continuous for all α;*

(ii) *for $i, j \in \{0, 1\}$, $E\left|(X_t - g)\frac{\partial g}{\partial \alpha_i}\right| < \infty$, $E\left|(X_t - g)\frac{\partial^2 g}{\partial \alpha_i \partial \alpha_j}\right| < \infty$ and $E\left|\frac{\partial g}{\partial \alpha_i} \cdot \frac{\partial g}{\partial \alpha_j}\right| < \infty$, where g and its partial derivatives are evaluated at true value of parameter α and X_{t-1};*

(iii) *for $i, j, k \in \{0, 1\}$, there exist functions*

$$H^{(0)} \equiv H^{(0)}(X_{t-1}, \ldots), \qquad H_i^{(1)} \equiv H_i^{(1)}(X_{t-1}, \ldots),$$

$$H_{ij}^{(2)} \equiv H_{ij}^{(2)}(X_{t-1}, \ldots), \qquad H_{ijk}^{(3)} \equiv H_{ijk}^{(3)}(X_{t-1}, \ldots),$$

such that, for all α,

$$|g| \leq H^{(0)}, \qquad \left|\frac{\partial g}{\partial \alpha_i}\right| \leq H_i^{(1)}, \qquad \left|\frac{\partial^2 g}{\partial \alpha_i \partial \alpha_j}\right| \leq H_{ij}^{(2)}, \qquad \left|\frac{\partial^3 g}{\partial \alpha_i \partial \alpha_j \partial \alpha_k}\right| \leq H_{ijk}^{(3)},$$

$$E|X_t \cdot H_{ijk}^{(3)}| < \infty, \qquad E\left(H^{(0)} \cdot H_{ijk}^{(3)}\right) < \infty, \qquad E\left(H_i^{(1)} \cdot H_{jk}^{(2)}\right) < \infty.$$

Let V be the 2×2 matrix whose (i, j)-th elements are respectively given by

$$V_{ij} = E\left(\frac{\partial g(\alpha; X_{t-1})}{\partial \alpha_i} \frac{\partial g(\alpha; X_{t-1})}{\partial \alpha_j}\right), \qquad i, j = 1, 2.$$

Then, the CLS estimator $\widehat{\alpha}_n$ of α is consistent and asymptotically normal, i.e., as $n \to \infty$

$$\sqrt{n}(\widehat{\alpha}_n - \alpha) \xrightarrow{d} N(\boldsymbol{0}_{2\times 1}, \boldsymbol{V}^{-1}\boldsymbol{W}\boldsymbol{V}^{-1}),$$

whenever V is invertible, where "$\xrightarrow{d}$" means convergence in distribution, $N(\boldsymbol{0}_{2\times 1}, \boldsymbol{V}^{-1}\boldsymbol{W}\boldsymbol{V}^{-1})$ denotes a bivariate normal law with mean vector $\boldsymbol{0}_{2\times 1}$ and covariance matrix $\boldsymbol{V}^{-1}\boldsymbol{W}\boldsymbol{V}^{-1}$ with $\boldsymbol{W}$ being the

2×2 *matrix whose* (i,j)*-th elements are respectively given by*

$$W_{ij}=E\left(u_t^2(\alpha)\frac{\partial g(\alpha;X_{t-1})}{\partial \alpha_i}\frac{\partial g(\alpha;X_{t-1})}{\partial \alpha_j}\right),\qquad i,j=1,2,$$

and $u_t(\alpha)=X_t-g(\alpha;X_{t-1})$.

The CP-INGARCH(1) process considered is an ergodic and stationary (Theorem 3.11) Markov chain, since it depends upon its past history only through X_{t-1}, with all moments finite (Theorem 3.4) and the regularity conditions (i)-(iii) are satisfied. In fact, as $g(\alpha;X_{t-1})=\alpha_0+\alpha_1X_{t-1}$ one obtains

$$\frac{\partial g(\alpha;X_{t-1})}{\partial \alpha_0}=1,\qquad \frac{\partial g(\alpha;X_{t-1})}{\partial \alpha_1}=X_{t-1},$$

$$\frac{\partial^2 g(\alpha;X_{t-1})}{\partial \alpha_i\partial\alpha_j}=\frac{\partial^3 g(\alpha;X_{t-1})}{\partial \alpha_i\partial\alpha_j\partial\alpha_k}=0,\qquad i,j,k=0,1,$$

which are continuous for all α, and from where we see that the regularity conditions (i)-(iii) are satisfied. Thus, from Lemma 4.1, the CLS estimators (4.14) are consistent and asymptotically normal.

Let us obtain the covariance matrix of the asymptotic distribution. We have

$$\mathbf{V}=\begin{bmatrix} E\left(\frac{\partial g}{\partial\alpha_0}\frac{\partial g}{\partial\alpha_0}\right) & E\left(\frac{\partial g}{\partial\alpha_0}\frac{\partial g}{\partial\alpha_1}\right)\\ E\left(\frac{\partial g}{\partial\alpha_1}\frac{\partial g}{\partial\alpha_0}\right) & E\left(\frac{\partial g}{\partial\alpha_1}\frac{\partial g}{\partial\alpha_1}\right)\end{bmatrix}=\begin{bmatrix} E(1) & E(X_{t-1})\\ E(X_{t-1}) & E\left(X_{t-1}^2\right)\end{bmatrix}=\begin{bmatrix} 1 & \frac{\alpha_0}{1-\alpha_1}\\ \frac{\alpha_0}{1-\alpha_1} & \frac{\alpha_0(v_0+\alpha_0(1+\alpha_1))}{(1-\alpha_1)(1-\alpha_1^2)}\end{bmatrix},$$

taking into account the expressions stated in Theorem 4.2. $\mathbf{V}$ is invertible and its inverse is equal to

$$\mathbf{V}^{-1}=\frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\begin{bmatrix} \frac{\alpha_0(v_0+\alpha_0(1+\alpha_1))}{(1-\alpha_1)(1-\alpha_1^2)} & -\frac{\alpha_0}{1-\alpha_1}\\ -\frac{\alpha_0}{1-\alpha_1} & 1\end{bmatrix}=\begin{bmatrix} 1+\frac{\alpha_0}{v_0}(1+\alpha_1) & -\frac{1}{v_0}(1-\alpha_1^2)\\ -\frac{1}{v_0}(1-\alpha_1^2) & \frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\end{bmatrix}.$$

Furthermore,

$$\begin{aligned} E\left[f(X_{t-1})\cdot u_t^2(\alpha)\right] &= E\left[f(X_{t-1})\cdot E\left[(X_t-\alpha_0-\alpha_1X_{t-1})^2|X_{t-1}\right]\right]\\ &= E\left[f(X_{t-1})\cdot V\left[X_t-\alpha_0-\alpha_1X_{t-1}|X_{t-1}\right]+0\right]\\ &= E\left[f(X_{t-1})\cdot V\left[X_t|X_{t-1}\right]\right]=E\left[f(X_{t-1})\cdot v_0(\alpha_0+\alpha_1X_{t-1})\right],\end{aligned}$$

because of the conditional compound Poisson distribution, and then

$$\begin{aligned}\mathbf{W}&=\begin{bmatrix} E\left(u_t^2\frac{\partial g}{\partial\alpha_0}\frac{\partial g}{\partial\alpha_0}\right) & E\left(u_t^2\frac{\partial g}{\partial\alpha_0}\frac{\partial g}{\partial\alpha_1}\right)\\ E\left(u_t^2\frac{\partial g}{\partial\alpha_1}\frac{\partial g}{\partial\alpha_0}\right) & E\left(u_t^2\frac{\partial g}{\partial\alpha_1}\frac{\partial g}{\partial\alpha_1}\right)\end{bmatrix}\\ &=\begin{bmatrix} E[1\cdot v_0(\alpha_0+\alpha_1X_{t-1})] & E[X_{t-1}\cdot v_0(\alpha_0+\alpha_1X_{t-1})]\\ E[X_{t-1}\cdot v_0(\alpha_0+\alpha_1X_{t-1})] & E[X_{t-1}^2\cdot v_0(\alpha_0+\alpha_1X_{t-1})]\end{bmatrix}\\ &=\frac{v_0\alpha_0}{1-\alpha_1}\begin{bmatrix} 1 & \frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2}\\ \frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2} & \frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)}+\frac{\alpha_0^2}{(1-\alpha_1)^2}+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{(1-\alpha_1^2)(1-\alpha_1^3)}\end{bmatrix},\end{aligned}$$

since

$$E[v_0\,(\alpha_0+\alpha_1 X_{t-1})] = v_0\left[\alpha_0+\alpha_1\frac{\alpha_0}{1-\alpha_1}\right] = \frac{v_0\alpha_0}{1-\alpha_1},$$

$$E[X_{t-1}\cdot v_0\,(\alpha_0+\alpha_1 X_{t-1})] = v_0\left[\frac{\alpha_0^2}{1-\alpha_1}+\frac{\alpha_1\alpha_0(v_0+\alpha_0(1+\alpha_1))}{(1-\alpha_1)(1-\alpha_1^2)}\right] = \frac{v_0\alpha_0}{1-\alpha_1}\cdot\frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2},$$

$$\begin{aligned}E[X_{t-1}^2\cdot v_0\,(\alpha_0+\alpha_1 X_{t-1})] &= v_0\left[\frac{\alpha_0^2(v_0+\alpha_0(1+\alpha_1))}{(1-\alpha_1)(1-\alpha_1^2)}\right.\\ &\quad\left.+\frac{\alpha_1\alpha_0}{(1-\alpha_1)^3}\left(\frac{d_0+(3v_0^2-d_0)\alpha_1^2}{(1+\alpha_1)(1+\alpha_1+\alpha_1^2)}+\frac{3v_0\alpha_0}{1+\alpha_1}+\alpha_0^2\right)\right]\\ &= \frac{v_0\alpha_0}{1-\alpha_1}\left[\frac{v_0\alpha_0(1-\alpha_1)+3v_0\alpha_0\alpha_1}{(1-\alpha_1)^2(1+\alpha_1)}+\frac{\alpha_0^2(1-\alpha_1)+\alpha_0^2\alpha_1}{(1-\alpha_1)^2}+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{(1-\alpha_1^2)(1-\alpha_1^3)}\right]\\ &= \frac{v_0\alpha_0}{1-\alpha_1}\left[\frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)}+\frac{\alpha_0^2}{(1-\alpha_1)^2}+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{(1-\alpha_1^2)(1-\alpha_1^3)}\right],\end{aligned}$$

using again the expressions stated in Theorem 4.2.

Now, we provide some calculations to obtain the entries of the matrix $\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1}$:
The product of $\mathbf{V}^{-1}\mathbf{W}$ is given by

$$\begin{bmatrix} 1+\frac{\alpha_0}{v_0}(1+\alpha_1) & -\frac{1}{v_0}(1-\alpha_1^2) \\ -\frac{1}{v_0}(1-\alpha_1^2) & \frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\end{bmatrix}\begin{bmatrix} 1 & \frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2} \\ \frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2} & \frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)}+\frac{\alpha_0^2}{(1-\alpha_1)^2}+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{(1-\alpha_1^2)(1-\alpha_1^3)}\end{bmatrix}$$

$$=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} = \begin{bmatrix} 1-\alpha_1 & \frac{v_0\alpha_1}{1-\alpha_1^2}-\frac{\alpha_0\alpha_1}{1-\alpha_1}-\frac{\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)}{v_0(1-\alpha_1^3)} \\ \frac{\alpha_1(1-\alpha_1)}{\alpha_0} & 1+\alpha_1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\end{bmatrix},$$

since

$$\begin{aligned}a_{11} &= 1+\frac{\alpha_0(1+\alpha_1)}{v_0}-\frac{1-\alpha_1^2}{v_0}\,\frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2} = 1-\alpha_1,\\ a_{12} &= \left(1+\frac{\alpha_0}{v_0}(1+\alpha_1)\right)\frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2}\\ &\quad-\frac{1-\alpha_1^2}{v_0}\left[\frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)}+\frac{\alpha_0^2}{(1-\alpha_1)^2}+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{(1-\alpha_1^2)(1-\alpha_1^3)}\right]\\ &= \frac{v_0\alpha_1}{1-\alpha_1^2}+\frac{\alpha_0}{1-\alpha_1}+\frac{\alpha_0\alpha_1}{1-\alpha_1}+\frac{\alpha_0^2(1+\alpha_1)}{v_0(1-\alpha_1)}-\frac{\alpha_0(1+2\alpha_1)}{1-\alpha_1}\\ &\quad-\frac{\alpha_0^2(1+\alpha_1)}{v_0(1-\alpha_1)}-\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0(1-\alpha_1^3)}\\ &= \frac{v_0\alpha_1}{1-\alpha_1^2}-\frac{\alpha_0\alpha_1}{1-\alpha_1}-\frac{\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)}{v_0(1-\alpha_1^3)},\\ a_{21} &= -\frac{(1-\alpha_1^2)}{v_0}+\frac{(1-\alpha_1)(1-\alpha_1^2)(v_0\alpha_1+\alpha_0(1+\alpha_1))}{v_0\alpha_0(1-\alpha_1^2)}\\ &= -\frac{(1-\alpha_1^2)}{v_0}+\frac{\alpha_1(1-\alpha_1)}{\alpha_0}+\frac{(1-\alpha_1^2)}{v_0} = \frac{\alpha_1(1-\alpha_1)}{\alpha_0},\end{aligned}$$

$$
\begin{aligned}
a_{22} &= -\frac{(1-\alpha_1^2)(v_0\alpha_1+\alpha_0(1+\alpha_1))}{v_0(1-\alpha_1^2)} + \frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\left[\frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)}\right.\\
&\quad \left. + \frac{\alpha_0^2}{(1-\alpha_1)^2} + \frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{(1-\alpha_1^2)(1-\alpha_1^3)}\right]\\
&= -\alpha_1 - \frac{\alpha_0(1+\alpha_1)}{v_0} + 1 + 2\alpha_1 + \frac{\alpha_0(1+\alpha_1)}{v_0} + \frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\\
&= 1+\alpha_1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}.
\end{aligned}
$$

So, the asymptotic covariance matrix is given by

$$
\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}
$$

$$
= \frac{v_0\alpha_0}{1-\alpha_1}\begin{bmatrix} 1-\alpha_1 & \frac{v_0\alpha_1}{1-\alpha_1^2} - \frac{\alpha_0\alpha_1}{1-\alpha_1} - \frac{\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)}{v_0(1-\alpha_1^3)} \\ \frac{\alpha_1(1-\alpha_1)}{\alpha_0} & 1+\alpha_1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)} \end{bmatrix}\begin{bmatrix} 1+\frac{\alpha_0}{v_0}(1+\alpha_1) & -\frac{1}{v_0}(1-\alpha_1^2) \\ -\frac{1}{v_0}(1-\alpha_1^2) & \frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0} \end{bmatrix}
$$

where

$$
\begin{aligned}
b_{11} &= \frac{\alpha_0}{1-\alpha_1}\left(\alpha_0(1+\alpha_1)+\frac{v_0^2+(d_0-v_0^2)\alpha_1(1+\alpha_1-\alpha_1^2)+(3v_0^2-d_0)\alpha_1^4}{v_0(1+\alpha_1+\alpha_1^2)}\right),\\
b_{12} &= b_{21} = v_0\alpha_1-\alpha_0(1+\alpha_1)-\frac{\alpha_1(1+\alpha_1)(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0(1+\alpha_1+\alpha_1^2)},\\
b_{22} &= (1-\alpha_1^2)\left(1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right).
\end{aligned}
$$

In fact, we have

$$
\begin{aligned}
b_{11} &= \frac{v_0\alpha_0}{1-\alpha_1}\left[(1-\alpha_1)\left(1+\frac{\alpha_0}{v_0}(1+\alpha_1)\right)\right.\\
&\quad \left. -\frac{1}{v_0}(1-\alpha_1^2)\left(\frac{v_0\alpha_1}{1-\alpha_1^2}-\frac{\alpha_0\alpha_1}{1-\alpha_1}-\frac{\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)}{v_0(1-\alpha_1^3)}\right)\right]\\
&= \frac{\alpha_0}{1-\alpha_1}\left[v_0(1-\alpha_1)+\alpha_0(1-\alpha_1^2)-v_0\alpha_1+\alpha_0\alpha_1(1+\alpha_1)\right.\\
&\quad \left. +\frac{\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)(1+\alpha_1)}{v_0(1+\alpha_1+\alpha_1^2)}\right]\\
&= \frac{\alpha_0}{1-\alpha_1}\left[\alpha_0(1+\alpha_1)+\frac{v_0^2(1-2\alpha_1)(1+\alpha_1+\alpha_1^2)+\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)(1+\alpha_1)}{v_0(1+\alpha_1+\alpha_1^2)}\right]\\
&= \frac{\alpha_0}{1-\alpha_1}\left(\alpha_0(1+\alpha_1)+\frac{v_0^2+(d_0-v_0^2)\alpha_1(1+\alpha_1-\alpha_1^2)+(3v_0^2-d_0)\alpha_1^4}{v_0(1+\alpha_1+\alpha_1^2)}\right),\\
b_{12} &= \frac{v_0\alpha_0}{1-\alpha_1}\left[-\frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0}\right.\\
&\quad \left. +\frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\left(\frac{v_0\alpha_1}{1-\alpha_1^2}-\frac{\alpha_0\alpha_1}{1-\alpha_1}-\frac{\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)}{v_0(1-\alpha_1^3)}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\alpha_0(1-\alpha_1^2)+v_0\alpha_1-\alpha_0\alpha_1(1+\alpha_1)-\frac{\alpha_1(1+\alpha_1)\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)}{v_0(1+\alpha_1+\alpha_1^2)}\\
&= v_0\alpha_1-\alpha_0(1+\alpha_1)-\frac{\alpha_1(1+\alpha_1)(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0(1+\alpha_1+\alpha_1^2)},\\
b_{21} &= \frac{v_0\alpha_0}{1-\alpha_1}\left[\frac{\alpha_1(1-\alpha_1)}{\alpha_0}\left(1+\frac{\alpha_0(1+\alpha_1)}{v_0}\right)-\frac{1-\alpha_1^2}{v_0}\left(1+\alpha_1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right)\right]\\
&= v_0\alpha_1+\alpha_0\alpha_1(1+\alpha_1)-\alpha_0(1+\alpha_1)-\alpha_0\alpha_1(1+\alpha_1)-\frac{\alpha_1(1+\alpha_1)(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0(1+\alpha_1+\alpha_1^2)}\\
&= v_0\alpha_1-\alpha_0(1+\alpha_1)-\frac{\alpha_1(1+\alpha_1)(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0(1+\alpha_1+\alpha_1^2)},\\
b_{22} &= \frac{v_0\alpha_0}{1-\alpha_1}\left[-\frac{\alpha_1(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}+\frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\left(1+\alpha_1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right)\right]\\
&= -\alpha_1(1-\alpha_1^2)+\alpha_1(1-\alpha_1^2)+(1-\alpha_1^2)\left(1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right)\\
&= (1-\alpha_1^2)\left(1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right).
\end{aligned}
$$

We have proved the following theorem.

Theorem 4.3 *Let $\widehat{\alpha}_n$ be the CLS estimate of α given in* (4.14). *We have*

$$\sqrt{n}(\widehat{\alpha}_n-\alpha)\xrightarrow{d}N(\boldsymbol{0}_{2\times 1},\boldsymbol{V}^{-1}\boldsymbol{W}\boldsymbol{V}^{-1}),$$

as $n\to\infty$, where the entries of the matrix $\boldsymbol{V}^{-1}\boldsymbol{W}\boldsymbol{V}^{-1}=(b_{ij})$, $i,j=1,2$, are those given above.

We carried out simulation studies to illustrate the CLS method for computing the estimates given in (4.14) and also to examine their performance. Table 4.1 presents the computation of the sample mean values, and the associated standard errors, for the CLS estimates of α_0 and α_1 for the INARCH(1) model. These estimates are obtained considering different sample sizes n, namely $n=100$, 500, 1000, and different combinations of parameters $\alpha_0=0.5$, 1, 2 and $\alpha_1=0.2$, 0.6, 0.9. We generated a sample of size n for a specified value of the parameters α_0 and α_1. Then, for this sample size we obtain its CLS estimates. We repeated this procedure 1000 times for these fixed n, α_0 and α_1, and the mean values, along with standard errors in parenthesis, are presented in Table 4.1 ([4]).

These simulations show that, as expected, the estimates of (α_0,α_1) perform well as the sample size increases. For instance, if $\alpha_0=2$ and $\alpha_1=0.2$, the average of $\widehat{\alpha}_0$ is 2.0434 and the average of $\widehat{\alpha}_1$ is 0.1815, for $n=100$. When the sample size is increased to 1000, the average of the estimates of $\widehat{\alpha}_0$ and $\widehat{\alpha}_1$ are respectively 2.0012 and 0.1989. Further, the standard errors of the estimates decrease when the sample size increases. For the other parameter combinations of α_0 and α_1 we have a similar behavior. It is also observed from the Table 4.1 that as α_1 decreases, the standard error of the $\widehat{\alpha}_1$ increases and that of $\widehat{\alpha}_0$ decreases. For example, for $\alpha_0=2$, we have standard error of $\widehat{\alpha}_1$ equal to 0.0741, 0.0949 and 0.1053 (increases) for $\alpha_1=0.9$, 0.6, 0.2, respectively. The standard errors of $\widehat{\alpha}_0$,

[4]The detailed algorithm (in MATLAB code) is given in Appendix *D*.3.

when $\alpha_0 = 2$, are equal to 1.3762, 0.4676 and 0.2944 (decreases) for $\alpha_1 = 0.9, 0.6, 0.2$, respectively. The standard errors of $\widehat{\alpha}_0$ and $\widehat{\alpha}_1$ behave in a analogous way when α_0 decreases.

Table 4.1 CLS estimators performance for the INARCH(1) model-tentativa2.

Parameters		$n=100$		$n=500$		$n=1000$	
α_0	α_1	$E_{est}(\widehat{\alpha}_0)$	$E_{est}(\widehat{\alpha}_1)$	$E_{est}(\widehat{\alpha}_0)$	$E_{est}(\widehat{\alpha}_1)$	$E_{est}(\widehat{\alpha}_0)$	$E_{est}(\widehat{\alpha}_1)$
	0.2	0.5071	0.1871	0.5033	0.1951	0.5034	0.1954
		(0.0902)	(0.1089)	(0.0418)	(0.0504)	(0.0295)	(0.0366)
0.5	0.6	0.5442	0.5527	0.5107	0.5892	0.5071	0.5946
		(0.1307)	(0.1157)	(0.0594)	(0.0497)	(0.0432)	(0.0365)
	0.9	0.7309	0.8352	0.5611	0.8832	0.5301	0.8921
		(0.3838)	(0.0979)	(0.1373)	(0.0362)	(0.0904)	(0.0228)
	0.2	1.0134	0.1840	1.0035	0.1967	1.0008	0.1987
		(0.1582)	(0.1072)	(0.0711)	(0.0477)	(0.0496)	(0.0333)
1	0.6	1.0809	0.5634	1.0151	0.5926	1.0091	0.5966
		(0.2465)	(0.1027)	(0.1099)	(0.0447)	(0.0757)	(0.0308)
	0.9	1.4260	0.8524	1.1033	0.8876	1.0483	0.8940
		(0.7271)	(0.0792)	(0.2412)	(0.0278)	(0.1648)	(0.0189)
	0.2	2.0434	0.1815	2.0068	0.1954	2.0012	0.1989
		(0.2944)	(0.1053)	(0.1250)	(0.0456)	(0.0899)	(0.0316)
2	0.6	2.1559	0.5662	2.0177	0.5952	2.0088	0.5982
		(0.4676)	(0.0949)	(0.1975)	(0.0403)	(0.1377)	(0.0282)
	0.9	2.8086	0.8553	2.1746	0.8903	2.0885	0.8949
		(1.3762)	(0.0741)	(0.4791)	(0.0259)	(0.3153)	(0.0177)

Now, we focus on the INARCH(1) model with true parameters $\alpha_0 = 2$ and $\alpha_1 = 0.2$ and, for different sample sizes $n = 100$, 250, 500, 750, 1000, we present in Table 4.2 the means, variances and covariance of $\widehat{\alpha}_0$ and $\widehat{\alpha}_1$. In the last column of this table we represent the entries of the asymptotic matrix $\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1}$, respectively b_{11}, b_{22} and b_{12}. It is clear from Table 4.2 (and Table 4.1) that $\widehat{\alpha}_1$ is biased down and $\widehat{\alpha}_0$ is biased up. The negative covariance between $\widehat{\alpha}_0$ and $\widehat{\alpha}_1$ is a consequence of this inverse relationship. The asymptotic and the sample values are quite similar for larger values of n.

Table 4.2 Expected values, variances and covariances for the CLS estimates of the INARCH(1) model for different sample sizes n and with coefficients $\alpha_0 = 2$ and $\alpha_1 = 0.2$.

	$n=$ 100	250	500	750	1000	
$E_{est}(\widehat{\alpha}_0)$	2.0434	2.0117	2.0068	2.0037	2.0012	
$E_{est}(\widehat{\alpha}_1)$	0.1815	0.1941	0.1954	0.1975	0.1989	
$n \cdot V_{est}(\widehat{\alpha}_0)$	8.3919	8.0550	7.7741	8.0491	8.0647	8.0226
$n \cdot V_{est}(\widehat{\alpha}_1)$	1.0638	1.0581	1.0267	1.0349	0.9974	1.0436
$n \cdot Cov_{est}(\widehat{\alpha}_0, \widehat{\alpha}_1)$	−2.4616	−2.4310	−2.3139	−2.4114	−2.3783	−2.4090

Until now we have shown how to obtain the CLS estimator for α. To estimate the parameter b we use a two-step estimation inspired in the moments estimation method. Taking into consideration the expression of the variance of the CP-INARCH(1) model stated in Remark 3.5, namely

$$V(X_t) = \frac{\alpha_0 v_0}{(1-\alpha_1)(1-\alpha_1^2)},$$

we estimate v_0 from the estimated equality

$$\frac{\widehat{\alpha}_0 v_0}{(1-\widehat{\alpha}_1)(1-\widehat{\alpha}_1^2)} = \frac{1}{n}\sum_{t=1}^{n}\left(X_t - \overline{X}_n\right)^2,$$

where $\overline{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t$ denotes the empirical mean of $(X_1,...,X_n)$.

Thus, we deduce the following estimator for the additional parameter present in the

- NTA-INARCH(1) model (since $v_0 = 1+\phi$):

$$\widehat{\phi} = -1 + \frac{(1-\widehat{\alpha}_1)(1-\widehat{\alpha}_1^2)}{n\widehat{\alpha}_0}\sum_{t=1}^{n}\left(X_t - \overline{X}_n\right)^2.$$

- GEOMP2-INARCH(1) model (since $v_0 = \frac{2-p^*}{p^*}$):

$$\widehat{p^*} = 2\left[1 + \frac{(1-\widehat{\alpha}_1)(1-\widehat{\alpha}_1^2)}{n\widehat{\alpha}_0}\sum_{t=1}^{n}\left(X_t - \overline{X}_n\right)^2\right]^{-1}.$$

The strong consistence of these estimators is a consequence from the strict stationarity and ergodicity of process X. In fact, from the Ergodic Theorem, the empirical variance is a strongly consistent estimator for the variance of X which together with the compatibility of the almost sure convergence with the algebraic operations confirms the statement.

Table 4.3 Expected values, variances and covariances for the CLS estimates of the NTA-INARCH(1) model for different sample sizes n and with coefficients $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $\phi = 2$.

	$n=$ 100	250	500	750	1000	
$E_{est}(\widehat{\alpha}_0)$	2.0502	2.0078	2.0114	2.0086	2.0058	2
$E_{est}(\widehat{\alpha}_1)$	0.1816	0.1931	0.1939	0.1958	0.1969	0.2
$E_{est}(\widehat{\phi})$	1.9017	1.9864	1.9861	1.9947	2.0004	2
$n\cdot V_{est}(\widehat{\alpha}_0)$	12.1944	12.7152	12.4894	12.3982	12.3714	12.3774
$n\cdot V_{est}(\widehat{\alpha}_1)$	1.2926	1.2501	1.3499	1.2655	1.2836	1.2604
$n\cdot V_{est}(\widehat{\phi})$	18.6919	21.9114	23.3308	22.5843	21.3290	
$n\cdot Cov_{est}(\widehat{\alpha}_0,\widehat{\alpha}_1)$	−2.4064	−2.5024	−2.7620	−2.5567	−2.4962	−2.5510

Table 4.4 Expected values, variances and covariances for the CLS estimates of the GEOMP2-INARCH(1) model for different sample sizes n and with coefficients $\alpha_0 = 2$, $\alpha_1 = 0.4$ and $p^* = 0.1$.

	$n=$ 100	250	500	750	1000	
$E_{est}(\widehat{\alpha}_0)$	2.0527	2.0758	2.0400	2.0177	2.0182	2
$E_{est}(\widehat{\alpha}_1)$	0.3367	0.3637	0.3817	0.3890	0.3892	0.4
$E_{est}(\widehat{p^*})$	0.1170	0.1071	0.1040	0.1026	0.1012	0.1
$n\cdot V_{est}(\widehat{\alpha}_0)$	50.6433	53.6845	61.9295	56.6796	59.6529	61.5325
$n\cdot V_{est}(\widehat{\alpha}_1)$	2.9292	3.3704	3.5536	4.2723	3.8424	4.3979
$n\cdot V_{est}(\widehat{p^*})$	0.0961	0.0881	0.0841	0.0856	0.0800	
$n\cdot Cov_{est}(\widehat{\alpha}_0,\widehat{\alpha}_1)$	−1.5730	−3.2463	−5.9278	−5.5029	−5.7433	−7.0598

In Table 4.3 and Table 4.4, we present the means, variances and covariance of $\widehat{\alpha}_0$, $\widehat{\alpha}_1$ and $\widehat{b}$ for the CP-INARCH(1) model considering different sample sizes n and with Neyman type-A and geometric Poisson conditional distributions, respectively. In the last column of these tables we represent the true values of α_0, α_1 and b as well as the entries b_{11}, b_{22} and b_{12} of the asymptotic matrix $\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1}$. As before, we verify that the asymptotic and the sample values are quite similar for larger values of n.

Table 4.5 Empirical correlations for the CLS estimates of the NTA-INARCH(1) model for different sample sizes n and with coefficients $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $\phi = 2$.

	$n=$ 250	750	1000	5000	10000
$\rho_{est}(\widehat{\alpha}_0,\widehat{\alpha}_1)$	-0.6277	-0.6455	-0.6264	-0.6379	-0.6273
$\rho_{est}(\widehat{\alpha}_0,\widehat{\phi})$	0.1242	0.1163	0.0828	0.0683	0.0636
$\rho_{est}(\widehat{\alpha}_1,\widehat{\phi})$	0.0061	0.0265	0.0043	0.0703	0.0462

From the empirical results presented in the two last lines of Table 4.5, we can presume that the estimators of α and b are possibly asymptotically uncorrelated. In fact, for the NTA-INARCH(1) model in study, the empirical correlations $\rho_{est}(\widehat{\alpha}_0,\widehat{\phi})$ and $\rho_{est}(\widehat{\alpha}_1,\widehat{\phi})$ are significantly low. So, under our conjecture, we estimate α and b separately without lost of efficiency. To support this statement we use the Monte Carlo method to determine confidence intervals for the mean of $\rho_{est}(\widehat{\alpha}_0,\widehat{\phi})$ and for the mean of $\rho_{est}(\widehat{\alpha}_1,\widehat{\phi})$ which we denote by m_0 and m_1, respectively. The confidence intervals are obtained considering $\widetilde{n} = 35$ and $\widetilde{n} = 50$ replications of n-dimensional samples ($n = 500$ and $n = 1000$) of a Neyman type-A INARCH(1) model. Such intervals with confidence level 0.99 are presented in Table 4.6 ([5]), where we stress the lower values when n or $\widetilde{n}$ increase.

Table 4.6 Confidence intervals for the mean of $\rho_{est}(\widehat{\alpha}_0,\widehat{\phi})$ and for the mean of $\rho_{est}(\widehat{\alpha}_1,\widehat{\phi})$, with confidence level $\gamma = 0.99$ and for different sample sizes n and $\widetilde{n}$.

	$\widetilde{n}=35$		$\widetilde{n}=50$	
	$n=500$	$n=1000$	$n=500$	$n=1000$
m_0	$[0.0917, 0.1180]$	$[0.0883, 0.1162]$	$[0.0940, 0.1160]$	$[0.0814, 0.1064]$
m_1	$[0.0113, 0.0412]$	$[0.0165, 0.0412]$	$[0.0137, 0.0354]$	$[0.0132, 0.0397]$

As the standard error is an estimative of the variability in a parameter estimate, we would generally like standard errors to be small because that indicates better precision in our coefficient estimates. In what follows, we measure the performance of the CLS standard errors through the evaluation of a confidence interval and its coverage probabilities ([79]). A coverage probability is the proportion of simulated samples for which the estimated confidence interval includes the true parameter. This way, computing a coverage probability is similar to assessing if the method for computing confidence intervals (and, thus, standard errors) is living up to its definition, that is, if the same formula is used to compute a $\gamma \times 100\%$ confidence interval in repeated simulated samples, the confidence interval should enclose the true parameter in $\gamma \times 100\%$ of the samples, on average.

Let us consider the asymptotically exact simultaneous confidence region for α on level γ given by:

$$I_\gamma = \left\{ \alpha : (\widehat{\alpha}_{0,n} - \alpha_0, \widehat{\alpha}_{1,n} - \alpha_1) V W^{-1} V (\widehat{\alpha}_{0,n} - \alpha_0, \widehat{\alpha}_{1,n} - \alpha_1)^\top < \frac{z}{n-1} \right\},$$

[5] The method (in MATLAB code) is given in Appendix D.3.

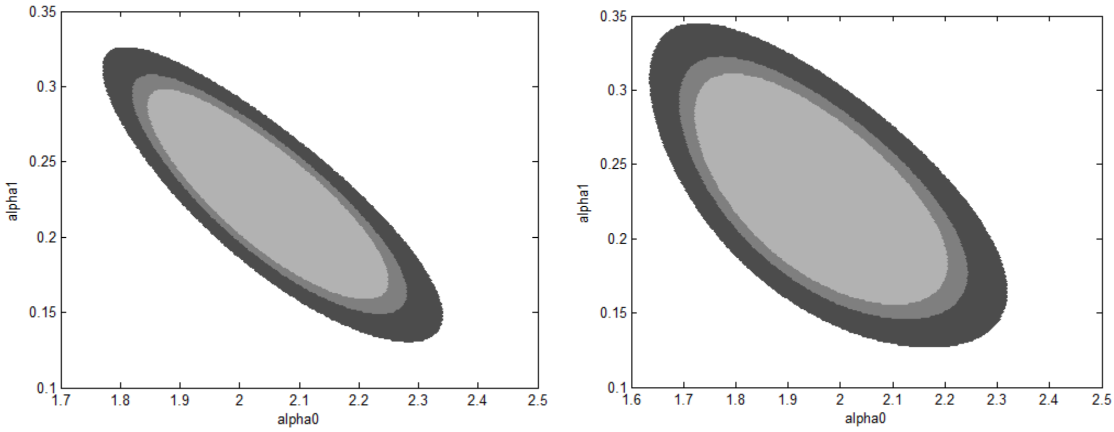

Fig. 4.1 Simultaneous confidence region based on CLS estimates for the INARCH(1) (on the left) and the NTA-INARCH(1) (on the right, with $\phi = 2$) models with $\alpha_0 = 2$ and $\alpha_1 = 0.2$. Confidence level $\gamma = 0.9$ (lightest gray), 0.95 (lightest and medium gray) and 0.99 (dark to lightest gray).

where z denotes the γ-quantile of the χ_2^2 distribution ([6]).

Examples of the (non-rectangular) shape of this confidence region for different confidence levels $\gamma = 0.9$, 0.95 and 0.99 are represented in Figure 4.1 with the same true values $\alpha_0 = 2$ and $\alpha_1 = 0.2$.

To estimate the true coverage probability of the confidence region I_γ a simulation study was done. For each combination of true parameters (α_0, α_1), for each confidence level γ and sample sizes n, 10 000 time series were generated. From each time series, the respective estimates and confidence region were computed. Then it was checked if this region contained the true parameter tuple. The number of "successes" divided by 10 000 is an estimate of the true coverage probability ([7]). The results are presented in Tables 4.7 to 4.9.

Table 4.7 Estimated coverage probabilities of the confidence region CLS $I_{0.90}$

		$\gamma = 0.90$,					
α_0	α_1	$n = 100$	250	500	750	1000	5000
2	0.2	0.8965	0.8945	0.9007	0.8970	0.9021	0.8977
	0.4	0.8816	0.8978	0.8931	0.9021	0.9027	0.9023
	0.6	0.8632	0.8837	0.8880	0.8963	0.8918	0.9005
	0.8	0.8133	0.8609	0.8841	0.8822	0.8889	0.8979
4	0.2	0.8975	0.8997	0.8980	0.8995	0.8961	0.9041
	0.4	0.8859	0.8944	0.8958	0.8983	0.8989	0.8934
	0.6	0.8600	0.8824	0.8917	0.8924	0.8990	0.8942
	0.8	0.8025	0.8543	0.8777	0.8865	0.8881	0.8952
6	0.2	0.8964	0.8971	0.8945	0.8940	0.8981	0.9055
	0.4	0.8860	0.8910	0.8957	0.8939	0.8956	0.8954
	0.6	0.8611	0.8785	0.8913	0.9004	0.8961	0.8966
	0.8	0.8124	0.8631	0.8734	0.8870	0.8882	0.8968

[6] Let us observe that this confidence region is based on the fact that $(Y-\mu)^\top \Sigma^{-1}(Y-\mu)$ is χ_p^2 distributed for a p-dimensional normal vector $Y \sim N(\mu, \Sigma)$.

[7] The detailed algorithm (in MATLAB code) is given in Appendix D.3.

Table 4.8 Estimated coverage probabilities of the confidence region CLS $I_{0.95}$

		$\gamma = 0.95$,					
α_0	α_1	$n = 100$	250	500	750	1000	5000
2	0.2	0.9427	0.9451	0.9485	0.9473	0.9522	0.9484
	0.4	0.9295	0.9473	0.9486	0.9513	0.9475	0.9528
	0.6	0.9131	0.9376	0.9401	0.9436	0.9446	0.9496
	0.8	0.8648	0.9127	0.9305	0.9316	0.9383	0.9479
4	0.2	0.9462	0.9479	0.9487	0.9486	0.9462	0.9501
	0.4	0.9374	0.9446	0.9477	0.9481	0.9494	0.9446
	0.6	0.9117	0.9361	0.9462	0.9450	0.9469	0.9434
	0.8	0.8573	0.9046	0.9290	0.9350	0.9360	0.9491
6	0.2	0.9434	0.9466	0.9480	0.9465	0.9506	0.9514
	0.4	0.9364	0.9387	0.9493	0.9422	0.9484	0.9492
	0.6	0.9120	0.9314	0.9436	0.9455	0.9479	0.9477
	0.8	0.8648	0.9133	0.9273	0.9381	0.9417	0.9481

Table 4.9 Estimated coverage probabilities of the confidence region CLS $I_{0.99}$

		$\gamma = 0.99$,					
α_0	α_1	$n = 100$	250	500	750	1000	5000
2	0.2	0.9841	0.9880	0.9892	0.9899	0.9901	0.9895
	0.4	0.9779	0.9851	0.9893	0.9872	0.9871	0.9902
	0.6	0.9677	0.9826	0.9859	0.9884	0.9851	0.9883
	0.8	0.9350	0.9658	0.9783	0.9797	0.9828	0.9880
4	0.2	0.9863	0.9873	0.9888	0.9894	0.9904	0.9899
	0.4	0.9797	0.9836	0.9890	0.9882	0.9881	0.9897
	0.6	0.9676	0.9806	0.9871	0.9857	0.9886	0.9896
	0.8	0.9285	0.9616	0.9772	0.9804	0.9830	0.9886
6	0.2	0.9850	0.9888	0.9888	0.9889	0.9911	0.9906
	0.4	0.9779	0.9835	0.9889	0.9881	0.9890	0.9894
	0.6	0.9651	0.9793	0.9844	0.9858	0.9880	0.9902
	0.8	0.9331	0.9674	0.9765	0.9801	0.9848	0.9899

The results of these tables show that the performance of the asymptotically exact CLS region depends heavily on the length n of the available time series and on the parameter α_1. In general, the difference between γ and the estimated coverage probability, which is mainly less than γ, decreases for increasing n. For example, when $\gamma = 0.9$, $\alpha_0 = 2$ and $\alpha_1 = 0.8$ we have the differences 0.0867, 0.0391, 0.0159, 0.0178, 0.0111 and 0.0021 for n = 100, 250, 500, 750, 1000 and 5000, respectively. It is also observed from the Tables 4.7-4.9 that when we increase α_1 the difference between the estimated coverage probabilities and γ also increases. For instance, when $n = 100$, $\alpha_0 = 6$ and $\gamma = 0.95$ we have the estimated coverage probabilities 0.9434, 0.9364, 0.9120, 0.8648 (decreases, so the difference increases) for $\alpha_1 = 0.2$, 0.4, 0.6 and 0.8, respectively.

It seams correct to say that we can trust in the region I_γ for a small value of the parameter α_1, say $\alpha_1 \leq 0.2$, or if the sample size satisfies $n \geq 500$ $(0.4 \leq \alpha_1 \leq 0.6)$ or even $n \geq 1000$ $(\alpha_1 \approx 0.8)$.

Chapter 5

The zero-inflated CP-INGARCH model

In recent years there has been considerable and growing interest in modeling zero-inflated count data since there is a lot of phenomenon where such data may occur and zero counts frequently have special status. With the aim of responding to this kind of data within conditionally heteroscedastic integer-valued time series we propose a generalization of the CP-INGARCH(p,q) model that may capture, in addition to the conditional volatility, overdispersion and zero inflation.

The chapter is organized as follows. In Section 5.1 we recall the definition of zero-inflated distributions and some of their properties. Using this class of distributions we present in Section 5.2 the zero-inflated CP-INGARCH model and we specify some particular cases. In parallel to what was done in Chapter 3 for the CP-INGARCH model, we analyse first and second-order stationarity conditions in Section 5.3. Moreover, we show how to construct a strictly stationary solution for the model proposed. We conclude presenting additional probabilistic developments of these processes, particularly, its autocorrelation function and expressions for cumulants.

5.1 Zero-inflated distributions

In some applications the count distribution considered might be inappropriate due to the excess of zeros in the data compared with what is expected from the distribution. This phenomenon can be a consequence of the existence of the so called structural zeros. For example, in the counting of the number of cigarettes smoked in a day, if the sample includes both smokers and non-smokers, there will be some of them non-smokers who report 0 cigarettes smoked in a day (meaning they are structural zeros) and others that did not smoke that day (meaning they are expected or sampling zeros instead). Count data collected from biology is another typical example that contains a lot of zeros. For instance, in counting disease lesions on plants, a plant may have no lesions either because it is resistant to the disease (structural zeros), or simply because no disease spores have landed on it [67].

Ignoring the excess of zeros, often referred to as zero inflation, we can have at least two consequences: first, the estimated parameters and standard errors may be biased, and second, the excessive number of zeros can cause overdispersion [87]. To overcome this issue, zero-inflated distributions should be specified. These distributions are a two-component mixture where one of the components is a point mass at zero and the other component is a count distribution. This way, such distributions distinguish between structural zeros, for units where zero is the only observable value, and sample zeros, for units on which we observe a zero, but other values might also have been recorded.

Definition 5.1 (Zero-inflated distribution) *The probability mass function of a zero-inflated (ZI for brevity) random variable X can be expressed as*

$$P(X=x)=\begin{cases}\omega+(1-\omega)P(Y=0), & \text{for } x=0,\\ (1-\omega)P(Y=x), & \text{for } x=1,2,\ldots\end{cases} \tag{5.1}$$

where the random variable Y follows a standard count distribution and $0<\omega<1$.

So, the distribution of Y is modified by an additional excess zero function and ω is the proportion of zeros added. Let us observe that these distributions are also referred as zero-modified or with added zeros [46, p. 351]. It is also possible to take ω less than zero, which corresponds to zero deflation, provided that $\omega+(1-\omega)P(Y=0)\geq 0$. In the limiting case, that is $\omega\to 0$, the zero-inflated distribution corresponds to the distribution of the random variable Y.

The characteristic function and moments of a zero-inflated distribution can be derived from those of the distribution of Y. In fact, if φ_Y represents the characteristic function of Y, then

$$\begin{aligned}\varphi_X(u)&=E\left[e^{iuX}\right]=P(X=0)+\sum_{x=1}^{\infty}e^{iux}P(X=x)\\&=\omega+(1-\omega)P(Y=0)+\sum_{x=1}^{\infty}e^{iux}(1-\omega)P(Y=x)=\omega+(1-\omega)\varphi_Y(u),\quad u\in\mathbb{R},\end{aligned}$$

which imply $E(X^r)=(1-\omega)E(Y^r)$, $r\in\mathbb{N}$. Thus, the mean and the variance of the zero-inflated distribution, if they exist, are given respectively by

$$E(X)=(1-\omega)E(Y)\quad\text{and}\quad V(X)=(1-\omega)[V(Y)+\omega(E(Y))^2].$$

In the same way, the probability generating function of X is given by $g_X(z)=\omega+(1-\omega)g_Y(z)$ and its r-th descending factorial moment ([1]), which can be obtained from $g_X^{(r)}(1)$, equals $(1-\omega)g_Y^{(r)}(1)$.

In addition to being viewed as a mixture, a zero-inflated distribution belongs to the family of the compound laws with Bernoulli counting distribution. This means that X can be written as $X\stackrel{d}{=}Y_1+...+Y_N$ where N is a Bernoulli random variable with parameter $1-\omega$ ($0<\omega<1$) and Y_j, $j=1,2,...$, are i.i.d. random variables which are also independent of N and following the same law as Y. In fact, from the proof of Theorem 2.4 the statement holds, i.e., $\varphi_X(u)=\omega+(1-\omega)\varphi_Y(u)$, $u\in\mathbb{R}$.

Regarding its importance within this study, we define in the following the zero-inflated Poisson (ZIP for brevity), the zero-inflated negative binomial (ZINB) and the zero-inflated generalized Poisson (ZIGP) laws. Some other examples of zero-inflated distributions can be found in [46, Section 8.2.4].

Example 5.1 (Zero-inflated Poisson distribution) *The random variable X follows a ZIP distribution with parameters* (λ,ω), $\lambda>0$ *and* $0<\omega<1$, *if its probability mass function is given by*

[1] Recall Remark 2.3.

$$P(X=x)=\begin{cases}\omega+(1-\omega)e^{-\lambda} & \textit{for } x=0,\\ (1-\omega)\dfrac{e^{-\lambda}\lambda^x}{x!}, & \textit{for } x=1,2,\ldots\end{cases}$$

Since $E(Y)=V(Y)=\lambda$, the mean and variance of X are given, respectively, by $E(X)=(1-\omega)\lambda$ and $V(X)=(1-\omega)\lambda(1+\omega\lambda)$. In Figure 5.1*, we represent the probability mass function of the ZIP distribution considering different values for the parameters (λ,ω) from which the effect of increasing the parameter ω becomes clear.*

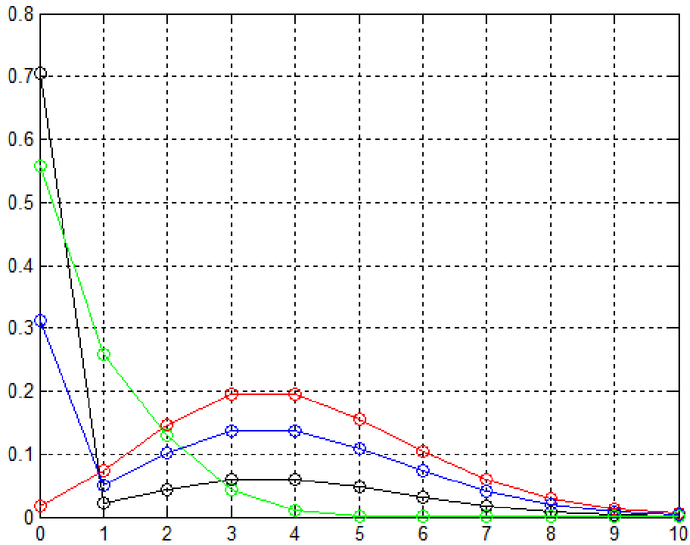

Fig. 5.1 Probability mass function of $X\sim ZIP(\lambda,\omega)$. From the top to the bottom in abscissa $x=4$, $(\lambda,\omega)=(4,0)$ (simple Poisson case with parameter 4), $(4,0.3)$, $(4,0.7)$, $(1,0.3)$.

Example 5.2 (Zero-inflated negative binomial distribution) *The random variable X follows a ZINB distribution with parameters (r,p,ω), $r\in\mathbb{N}$, $p\in]0,1[$ and $0<\omega<1$, if its probability mass function can be written in the form*

$$P(X=x)=\omega\delta_{x,0}+(1-\omega)\begin{pmatrix}x+r-1\\ r-1\end{pmatrix}p^r(1-p)^x,\quad x=0,1,2,\ldots,$$

where $\delta_{x,0}$ is the Kronecker delta, i.e., $\delta_{x,0}$ is 1 when $x=0$ and is zero when $x\neq 0$. Since the mean of the negative binomial distribution is $r(1-p)/p$ and the variance is $r(1-p)/p^2$, we have $E(X)=(1-\omega)\lambda$ and $V(X)=(1-\omega)r(1-p)[1+\omega r(1-p)]/p^2$.

Example 5.3 (Zero-inflated generalized Poisson distribution) *The random variable X follows a ZIGP distribution with parameters (λ,κ,ω) if its probability mass function can be written in the form*

$$P(X=x)=\begin{cases}\omega+(1-\omega)e^{-\lambda}, & \textit{for } x=0,\\ \frac{(1-\omega)\lambda(\lambda+\kappa x)^{x-1}e^{-(\lambda+\kappa x)}}{x!}, & \textit{for } x=1,2,\ldots\\ 0, & \textit{for } x>m \textit{ if } \kappa<0,\end{cases}$$

where $\lambda > 0$*,* $0 < \omega < 1$*,* $\max(-1, -\lambda/m) < \kappa < 1$ *and* $m(\geq 4)$ *is the largest positive integer for which* $\lambda + \kappa m > 0$*. Since the mean of the GP law is* $\lambda/(1-\kappa)$ *and the variance is* $\lambda/(1-\kappa)^3$*, we get*

$$E(X) = \frac{1-\omega}{1-\kappa}\,\lambda \quad \textit{and} \quad V(X) = (1-\omega)\left\{\frac{\omega\lambda^2}{(1-\kappa)^2} + \frac{\lambda}{(1-\kappa)^3}\right\}.$$

5.2 The definition of the ZICP-INGARCH model

Since its introduction [25] the INGARCH model has been generalized and extended in various directions in order to increase its flexibility. In Chapter 2 we presented the CP-INGARCH model as a solution to capture different kind of overdispersion in count data. As reported in literature with examples from manufacturing defects, road safety, medical consultations or species abundance [67], many count time series also display the zero-inflation characteristic. So, in this Section we will define the zero-inflated CP-INGARCH model and present some particular cases; a general procedure to generate this kind of processes will conclude the Section.

Definition 5.2 (ZICP-INGARCH(p, q) model) *The process X is said to follow a zero-inflated compound Poisson integer-valued GARCH model with orders p and q, (where* $p, q \in \mathbb{N}$*), briefly a ZICP-INGARCH*(p,q)*, if,* $\forall t \in \mathbb{Z}$*, the characteristic function of* X_t *conditioned on* $\underline{X}_{t-1}$ *is given by*

$$\Phi_{X_t|\underline{X}_{t-1}}(u) = \omega + (1-\omega)\exp\left\{i\frac{\lambda_t}{\varphi_t'(0)}\left[\varphi_t(u)-1\right]\right\}, \quad u \in \mathbb{R}, \tag{5.2}$$

with

$$\frac{E(X_t|\underline{X}_{t-1})}{1-\omega} = \lambda_t = \alpha_0 + \sum_{j=1}^{p}\alpha_j X_{t-j} + \sum_{k=1}^{q}\beta_k\lambda_{t-k}, \tag{5.3}$$

for $0 \leq \omega < 1$*,* $\alpha_0 > 0$*,* $\alpha_j \geq 0$ *(*$j = 1,...,p$*),* $\beta_k \geq 0$ *(*$k = 1,...,q$*), and where* $(\varphi_t, t \in \mathbb{Z})$ *is a family of characteristic functions on* $\mathbb{R}$*,* $\underline{X}_{t-1}$*-measurables, associated to a family of discrete laws with support in* $\mathbb{N}_0$ *and finite mean. If* $q = 1$ *and* $\beta_1 = 0$*, the model is simply denoted by ZICP-INARCH*(p)*.*

As a consequence of this definition and assuring the existence of variance by considering that the characteristic functions $(\varphi_t, t \in \mathbb{Z})$ are twice differentiable at zero, the conditional mean and the conditional variance of X_t are, respectively, given by $E(X_t|\underline{X}_{t-1}) = (1-\omega)\lambda_t$ and

$$V(X_t|\underline{X}_{t-1}) = -\Phi''_{X_t|\underline{X}_{t-1}}(0) - (1-\omega)^2\lambda_t^2 = (1-\omega)\lambda_t\left(-i\frac{\varphi_t''(0)}{\varphi_t'(0)} + \omega\lambda_t\right). \ (^2)$$

Clearly, the CP-INGARCH model is recovered when $\omega = 0$. Besides this, a wide class of processes is included in the family of ZICP-INGARCH models. To show that, we use the fact that the conditional distribution of X_t belongs to the class of the compound laws. In fact, let M_t be a Bernoulli random variable with parameter $1-\omega$, with $0 \leq \omega < 1$, and let us define the process $X = (X_t, t \in \mathbb{Z})$ as

[2] Recall the expressions stated in page 18.

$$X_t \stackrel{d}{=} \begin{cases} 0, & M_t = 0, \\ \\ \sum_{j=1}^{N_t} X_{t,j}, & M_t = 1, \end{cases} \tag{5.4}$$

where, conditionally on $\underline{X}_{t-1}$, N_t is a random variable following a Poisson distribution with parameter $\lambda_t^* = \lambda_t / E(X_{t,j})$ and $X_{t,1}$, $X_{t,2}$,... are discrete random variables with support in $\mathbb{N}_0$ and common characteristic function φ_t with finite mean. In addition, all the random variables involved in the equality (5.4) are, for each t, independent. Then the process X satisfies the ZICP-INGARCH model since its characteristic function conditioned on $\underline{X}_{t-1}$ is given by

$$\Phi_{X_t|\underline{X}_{t-1}}(u) = \omega + (1-\omega) \sum_{n=0}^{\infty} E\left[e^{iu\left(X_{t,1}+...+X_{t,N_t}\right)} \mid N_t = n\right] \cdot P(N_t = n)$$

$$= \omega + (1-\omega)e^{-\lambda_t^*} \sum_{n=0}^{\infty} \varphi_t^n(u) \frac{(\lambda_t^*)^n}{n!} = \omega + (1-\omega)\exp\left\{\lambda_t^* \left[\varphi_t(u) - 1\right]\right\}, \quad u \in \mathbb{R}.$$

It should be noted that the ZICP-INGARCH model can also be obtained using another different representation. Indeed, let us define the process X as

$$X_t \stackrel{d}{=} X_{t,1} + \ldots + X_{t,U_t}, \tag{5.5}$$

where U_t is a random variable that, conditionally on $\underline{X}_{t-1}$, follows a zero-inflated Poisson law and $X_{t,1}$, $X_{t,2}$, ... are discrete random variables with support in $\mathbb{N}_0$ that, conditionally on $\underline{X}_{t-1}$, are independent, independent of U_t and with characteristic function φ_t with finite mean. If the parameters of the probability mass function of U_t are (λ_t^*, ω), with $\lambda_t^* = \lambda_t / E(X_{t,j})$ and $0 \leq \omega < 1$, then the process X still satisfies the ZICP-INGARCH model as we have

$$\Phi_{X_t|\underline{X}_{t-1}}(u) = \sum_{n=0}^{\infty} E[\exp\{iu(X_{t,1} + ... + X_{t,U_t})\}|U_t = n] \cdot P(U_t = n)$$

$$= \omega \sum_{n=0}^{\infty} \varphi_t^n(u)\delta_{n,0} + (1-\omega)e^{-\lambda_t^*} \sum_{n=0}^{\infty} \varphi_t^n(u) \frac{(\lambda_t^*)^n}{n!}$$

$$= \omega + (1-\omega)\exp\left\{\lambda_t^* \left[\varphi_t(u) - 1\right]\right\}, \quad u \in \mathbb{R}.$$

This representation shows that the conditional distribution of the ZICP-INGARCH process can be represented as a compound distribution with a zero-inflated Poisson law as counting distribution.

We can use any of these representations in law of the model. We remark that representation (5.4) is particularly important in the construction of a strictly stationary solution whereas (5.5) will be useful to state a condition for the existence of all moments of a ZICP-INGARCH$(1,1)$ model.

Many particular models can be deduced as we illustrate next:

1. To model overdispersion and zero inflation in the same framework, Zhu [84] proposed the zero-inflated Poisson INGARCH(p,q) model (ZIP-INGARCH for brevity) defined as

$$X_t \mid \underline{X}_{t-1} \sim \mathbf{ZIP}\left(\lambda_t, \omega\right), \qquad \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k},$$

where $0 < \omega < 1$, $\alpha_0 > 0$, $\alpha_j \geq 0$, $\beta_k \geq 0$, $j = 1, ..., p$, $k = 1, ..., q$.

We observe that $E(X_t|\underline{X}_{t-1}) = (1-\omega)\lambda_t$ and $V(X_t|\underline{X}_{t-1}) = (1-\omega)\lambda_t(1+\omega\lambda_t)$ (recall Example 5.1). This corresponds to a ZICP-INGARCH(p,q) model considering in representation (5.4) $\lambda_t^* = \lambda_t$ and the characteristic function $\varphi_t(u) = e^{iu}$, $u \in \mathbb{R}$.

2. In the same paper, Zhu [84] proposed also the zero-inflated negative binomial INGARCH(p,q) process (ZINB-INGARCH for brevity), defined as

$$X_t \mid \underline{X}_{t-1} \sim \mathbf{ZINB}\left(\frac{\lambda_t^{1-c}}{a}, \frac{1}{1+a\lambda_t^c}, \omega\right), \qquad \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k},$$

where $0 < \omega < 1$, $a > 0$, $c \in \{0,1\}$, $\alpha_0 > 0$, $\alpha_j \geq 0$, $\beta_k \geq 0$, $j = 1, ..., p$, $k = 1, ..., q$. We note that $V(X_t|\underline{X}_{t-1}) = (1-\omega)\lambda_t(1+\omega\lambda_t + a\lambda_t^c)$ (recall Example 5.2). Considering in the representation (5.4), the random variables $X_{t,j}$, $j = 1, 2, ...$, having a logarithmic distribution with parameter $\frac{a\lambda_t^c}{1+a\lambda_t^c}$ and $\lambda_t^* = \frac{\lambda_t^{1-c}}{a}\ln(1+a\lambda_t^c)$ we recover the ZINB-INGARCH(p,q) model.

3. Recently, Lee et al. [52] proposed a zero-inflated generalized Poisson autoregressive model (ZIGP AR) by considering

$$X_t \mid \underline{X}_{t-1} \sim \mathbf{ZIGP}\left((1-\kappa)\lambda_t, \kappa, \omega\right), \qquad \lambda_t = f(\lambda_{t-1}, X_{t-1}),$$

where $0 \leq \omega < 1$, $\max\{-1, -(1-\kappa)\lambda_t/m\} < \kappa < 1$ and f is a positive function on $[0,\infty[\times\mathbb{N}_0$ irrespective of ω and κ. We have $V(X_t|\underline{X}_{t-1}) = (1-\omega)\lambda_t(1/(1-\kappa)^2 + \omega\lambda_t)$ (recall Example 5.3) and $E(X_t|\underline{X}_{t-1}) = (1-\omega)\lambda_t$. For $0 < \kappa < 1$ and $f(\lambda_{t-1}, X_{t-1}) = \alpha_0 + \alpha_1 X_{t-1} + \beta_1\lambda_{t-1}$, we recover a particular case of the ZIGP AR model, which we will called ZIGP-INGARCH, considering that in the representation (5.4) the common distribution of the random variables $X_{t,j}$, $j = 1, 2, ...$, is the Borel law with parameter κ and $\lambda_t^* = (1-\kappa)\lambda_t$.

4. Let us consider independent random variables $X_{t,j}$, $j = 1, 2, ...$, following a Poisson distribution with parameter $\phi > 0$ and let N_t follow a Poisson distribution with parameter $\lambda_t^* = \frac{\lambda_t}{\phi}$, independent of $X_{t,j}$. So, the process X defined as in (5.4) satisfies a ZICP-INGARCH model with $V(X_t|\underline{X}_{t-1}) = (1-\omega)\lambda_t(1+\phi+\omega\lambda_t)$ (recall Example 2.4). It is denoted by ZINTA-INGARCH, since the associated conditional law is a zero-inflated Neyman type-A law, i.e.,

$$X_t \mid \underline{X}_{t-1} \sim \mathbf{ZINTA}\left(\frac{\lambda_t}{\phi}, \phi, \omega\right), \quad \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k},$$

with $\phi > 0$, $0 \leq \omega < 1$, $\alpha_0 > 0$, $\alpha_j \geq 0$, $\beta_k \geq 0$, $j = 1, ..., p$, $k = 1, ..., q$.

Following the same idea, all the processes defined in Section 2.3 can be extended in order to obtain the ZIGEOMP-INGARCH model, the ZIGEOMP2-INGARCH model, and so on.

Figure 5.2 presents trajectories and the basic descriptives of the ZIP-INGARCH$(1,1)$ model with $\alpha_0 = 10$, $\alpha_1 = 0.4$, $\beta_1 = 0.5$, considering two different values for ω, namely $\omega = 0.2$ and 0.6, illustrating clearly the zero-inflated characteristic of these models (compare with Figure 2.3, where $\omega = 0$). The descriptive resumes presented in Figure 5.2 and Figure 5.3, suggest overdispersion in the class of the ZICP-INGARCH processes. In fact, for the model specified by (5.2) and (5.3) we have

$$\begin{aligned}\frac{V(X_t|\underline{X}_{t-1})}{E(X_t|\underline{X}_{t-1})} &= -i\frac{\varphi_t''(0)}{\varphi_t'(0)} + \omega\lambda_t = \frac{E(X_{t,1}^2)}{E(X_{t,1})} + \omega\lambda_t \\ &= 1 + \frac{E(X_{t,1}(X_{t,1}-1))}{E(X_{t,1})} + \omega\lambda_t \geq 1 + \omega\lambda_t > 1,\end{aligned}$$

whenever $\omega > 0$, and so

$$\frac{V(X_t)}{E(X_t)} > 1,$$

that is, the ZICP-INGARCH process is always overdispersed.

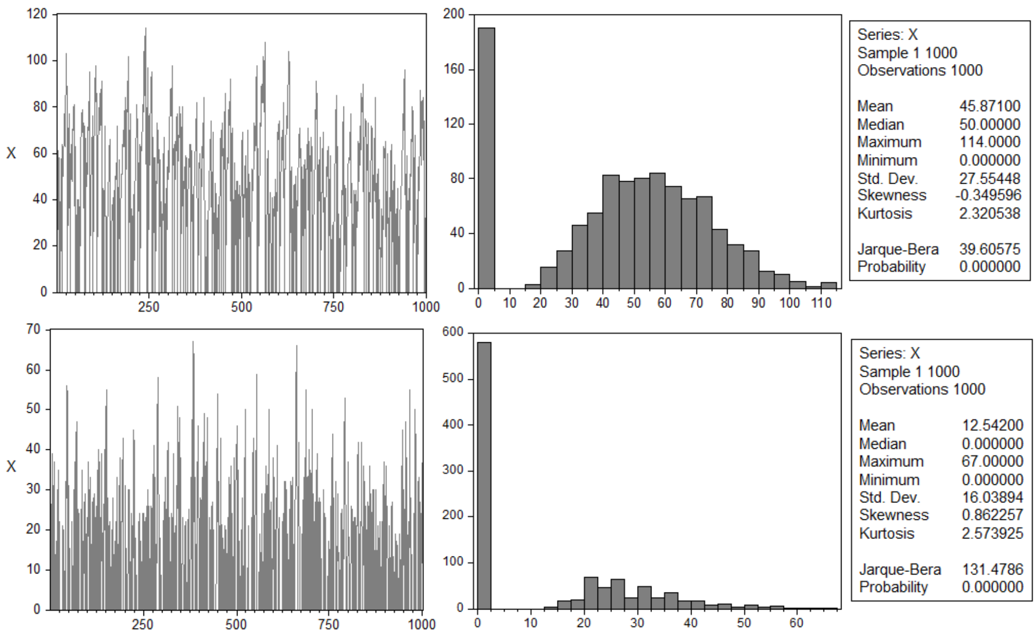

Fig. 5.2 Trajectories and descriptives of ZIP-INGARCH$(1,1)$ models with $\alpha_0 = 10$, $\alpha_1 = 0.4$ and $\beta_1 = 0.5$: $\omega = 0.2$ (on top) and $\omega = 0.6$ (below).

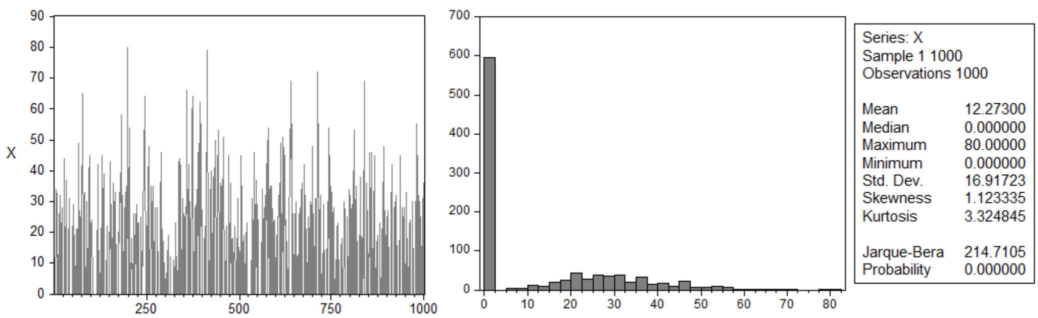

Fig. 5.3 Trajectory and descriptives of a ZINTA-INGARCH$(1,1)$ model with $\alpha_0 = 10$, $\alpha_1 = 0.4$, $\beta_1 = 0.5$, $\phi = 2$ and $\omega = 0.6$.

5.3 Weak and strict stationarity

The study of first-order and weak stationarity of these processes follows the approach developed for the CP-INGARCH processes in Sections 3.1 and 3.2. In the following we resume the main conclusions of this study for the ZICP-INGARCH process, which naturally, are affected by the parameter ω.

In what concerns first-order stationarity considering $\mu_t = E(X_t)$, we deduce from the difference equation $\mu_t = (1-\omega)\alpha_0 + \sum_{j=1}^{p}(1-\omega)\alpha_j\mu_{t-j} + \sum_{k=1}^{q}\beta_k\mu_{t-k}$, that X is first-order stationary if and only if $(1-\omega)\sum_{j=1}^{p}\alpha_j + \sum_{k=1}^{q}\beta_k < 1$. Moreover, under this condition, the processes (X_t) and (λ_t) are both first-order stationary and we have

$$\mu = E(X_t) = (1-\omega)E(\lambda_t) = \frac{(1-\omega)\alpha_0}{1-(1-\omega)\sum_{j=1}^{p}\alpha_j - \sum_{k=1}^{q}\beta_k}.$$

In Figure 5.4 we plot first-order stationarity regions of a ZICP-INGARCH$(2,1)$ model considering different values for ω, namely, $\omega = 0$, 0.4 and 0.6. We conclude that when the value of ω decrease the first-order stationarity region becomes smaller. To better view, we present in Figure 5.5 the three planes that define the mentioned regions where it is now clear that the first-order stationarity region with $\omega = 0$ (lightest gray) is contained in the region with $\omega = 0.4$ (light to medium gray) and this one is contained in the region with $\omega = 0.6$ (light to darkest gray).

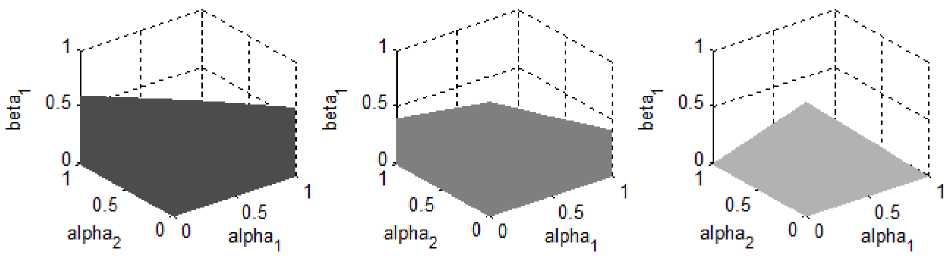

Fig. 5.4 First-order stationarity regions of a ZICP-INGARCH$(2,1)$ model considering $\omega = 0.6$ (darkest gray), 0.4 (medium gray) and 0 (lightest gray).

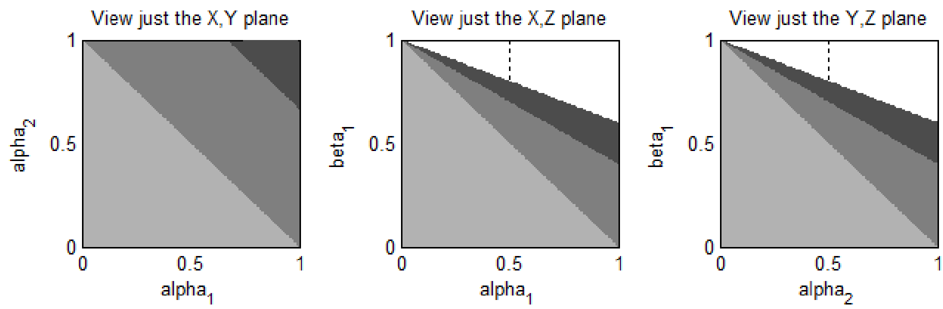

Fig. 5.5 The three planes that define the first-order stationarity regions of Figure 5.4.

The weak stationarity is developed for processes X satisfying the hypothesis **H2**. This hypothesis in the context of the ZICP-INGARCH model can be interpreted in the following way:

$$-i\frac{\varphi_t''(0)}{\varphi_t'(0)} = \frac{V(X_t|\underline{X}_{t-1})}{E(X_t|\underline{X}_{t-1})} - \omega\lambda_t.$$

We stress that, in addition to the examples presented in Section 3.2, the ZIP-INGARCH, the ZINB-INGARCH, the ZIGP-INGARCH and the ZINTA-INGARCH processes also satisfy **H2**. Namely,

- ZIP-INGARCH(p,q) with $v_0 = 1$;
- ZIGP-INGARCH(p,q) with $v_0 = \frac{1}{(1-\kappa)^2}$;
- ZINTA-INGARCH(p,q) with $v_0 = 1+\phi$;

all of them with $v_1 = 0$, and also the

- ZINB-INGARCH(p,q) with $\begin{cases} v_0 = 1+a \ \text{ and } \ v_1 = 0, & \text{when } c = 0, \\ v_0 = 1 \ \text{ and } \ v_1 = a, & \text{when } c = 1. \end{cases}$

A necessary and sufficient condition of weak stationarity of X is easily deduced from the vectorial state space representation presented below. To establish it, let us take into account the following facts:

$$E(X_{t-j}\lambda_{t-k}) = E\left[E(X_{t-j}|\underline{X}_{t-j-1})\lambda_{t-k}\right] = (1-\omega)E(\lambda_{t-j}\lambda_{t-k}), \quad \text{if } k \geq j, \tag{5.6}$$

$$E(X_{t-j}\lambda_{t-k}) = \frac{1}{1-\omega}E\left[X_{t-j}E(X_{t-k}|\underline{X}_{t-k-1})\right] = \frac{E(X_{t-j}X_{t-k})}{1-\omega}, \quad \text{if } k < j, \tag{5.7}$$

from where we can deduce the expressions

$$\begin{aligned} E(X_tX_{t-h}) &= E\left[E(X_t|\underline{X}_{t-1})X_{t-h}\right] = (1-\omega)E(\lambda_t X_{t-h}) \\ &= (1-\omega)E\left(\left[\alpha_0 + \sum_{j=1}^{p}\alpha_j X_{t-j} + \sum_{k=1}^{q}\beta_k\lambda_{t-k}\right]X_{t-h}\right) \\ &= (1-\omega)\left\{\alpha_0 E\left(X_{t-h}\right) + \sum_{j=1}^{p}\alpha_j E\left(X_{t-j}X_{t-h}\right) + \sum_{k=1}^{q}\beta_k E\left(\lambda_{t-k}X_{t-h}\right)\right\} \\ &= (1-\omega)\left\{\alpha_0 E\left(X_{t-h}\right) + \sum_{j=1}^{p}\alpha_j E\left(X_{t-j}X_{t-h}\right)\right. \\ &\quad \left. + \sum_{k=1}^{h-1}\frac{\beta_k}{1-\omega}E(X_{t-k}X_{t-h}) + \sum_{k=h}^{q}(1-\omega)\beta_k E(\lambda_{t-k}\lambda_{t-h})\right\}, \qquad h \geq 1, \end{aligned} \tag{5.8}$$

and in a similar way

$$\begin{aligned} E(\lambda_t\lambda_{t-h}) &= E\left(\left[\alpha_0 + \sum_{j=1}^{p}\alpha_j X_{t-j} + \sum_{k=1}^{q}\beta_k\lambda_{t-k}\right]\lambda_{t-h}\right) \\ &= \alpha_0 E\left(\lambda_{t-h}\right) + \sum_{j=1}^{h}(1-\omega)\alpha_j E\left(\lambda_{t-j}\lambda_{t-h}\right) \\ &\quad + \sum_{j=h+1}^{p}\frac{\alpha_j}{1-\omega}E(X_{t-j}X_{t-h}) + \sum_{k=1}^{q}\beta_k E\left(\lambda_{t-k}\lambda_{t-h}\right), \qquad h \geq 0. \end{aligned} \tag{5.9}$$

Proposition 5.1 *Suppose that the process X following the ZICP-INGARCH(p,q) model is first-order stationary and satisfies the hypothesis* **H2**. *The vector W_t, $t \in \mathbb{Z}$, of dimension $p+q-1$ given by*

$$W_t = \begin{bmatrix} E(X_t^2) \\ E(X_t X_{t-1}) \\ \cdots \\ E(X_t X_{t-(p-1)}) \\ E(\lambda_t \lambda_{t-1}) \\ \cdots \\ E(\lambda_t \lambda_{t-(q-1)}) \end{bmatrix}$$

satisfies an autoregressive equation of order $\max(p,q)$*:*

$$W_t = A_0 + \sum_{k=1}^{\max(p,q)} A_k W_{t-k}, \tag{5.10}$$

where A_0 is a real vector of dimension $p+q-1$ and A_k ($k=1,\ldots,\max(p,q)$) are real squared matrices of order $p+q-1$.

Proof: The proof is similar to that of Proposition 3.1. Once again, let us focus on the case $p=q$. From the hypothesis of first-order stationarity, the expressions (5.6) and (5.7), and since

$$E(X_t^2) = (1-\omega)\{v_0 E(\lambda_t) + (1+v_1)E(\lambda_t^2)\} \;\Leftrightarrow\; E(\lambda_t^2) = \frac{E(X_t^2) - v_0\mu}{(1-\omega)(1+v_1)},$$

we deduce the following expressions:

$$\begin{aligned}
E(X_t^2) &= D + (1+v_1)\left[\sum_{j=1}^{p}\left((1-\omega)\alpha_j^2 + \frac{2(1-\omega)\alpha_j\beta_j + \beta_j^2}{1+v_1}\right)E(X_{t-j}^2)\right. \\
&\quad +2\sum_{j=1}^{p-1}\sum_{k=j+1}^{p}\alpha_k((1-\omega)\alpha_j+\beta_j)E(X_{t-j}X_{t-k}) \\
&\quad \left. +2(1-\omega)\sum_{j=1}^{p-1}\sum_{k=j+1}^{p}\beta_k((1-\omega)\alpha_j+\beta_j)E(\lambda_{t-j}\lambda_{t-k})\right], \\
E(X_tX_{t-h}) &= \left[\alpha_0 - \frac{v_0\beta_h}{1+v_1}\right](1-\omega)\mu + (1-\omega)\left[\alpha_h + \frac{\beta_h}{1+v_1}\right]E(X_{t-h}^2) \\
&\quad + \sum_{j=h+1}^{p}(1-\omega)^2\beta_j E(\lambda_{t-j}\lambda_{t-h}) + \sum_{j=h+1}^{p}(1-\omega)\alpha_j E(X_{t-j}X_{t-h}) \\
&\quad + \sum_{j=1}^{h-1}((1-\omega)\alpha_j+\beta_j)E(X_{t-j}X_{t-h}), \qquad h \geq 1, \\
E(\lambda_t\lambda_{t-h}) &= \left[\alpha_0 - \frac{v_0(\alpha_h+\beta_h)}{1+v_1}\right]\frac{\mu}{1-\omega} + \frac{\alpha_h+\beta_h}{(1-\omega)(1+v_1)}E(X_{t-h}^2) \\
&\quad + \sum_{j=h+1}^{p}\beta_j E(\lambda_{t-j}\lambda_{t-h}) + \sum_{j=h+1}^{p}\frac{\alpha_j}{1-\omega}E(X_{t-j}X_{t-h})
\end{aligned}$$

$$+\sum_{j=1}^{h-1}((1-\omega)\alpha_j+\beta_j)E(\lambda_{t-j}\lambda_{t-h}), \qquad h\geq 1,$$

where $D = v_0\mu[1-\sum_{j=1}^{p}(2(1-\omega)\alpha_j\beta_j+\beta_j^2)]+(1+v_1)\left[2\alpha_0\mu-\alpha_0^2\right]$ is a positive constant independent of t. Using these expressions it is clear now that the vector W_t satisfies the autoregressive equation of order p, $W_t = A_0+\sum_{k=1}^{p}A_kW_{t-k}$, with $A_0=(a_j)$ the vector such that

$$a_j = \begin{cases} D, & j=1,\\ (1-\omega)\mu\left[\alpha_0-\dfrac{v_0\beta_{j-1}}{1+v_1}\right], & j=2,...,p,\\ \dfrac{\mu}{1-\omega}\left[\alpha_0-\dfrac{v_0(\alpha_{j-p}+\beta_{j-p})}{1+v_1}\right], & j=p+1,...,2p-1,\end{cases}$$

and A_k ($k=1,...,p$) the squared matrices having generic element $a_{ij}^{(k)}$ given by

- row $i=1$:

$$a_{1j}^{(k)} = \begin{cases} (1-\omega)(1+v_1)\alpha_k^2+2(1-\omega)\alpha_k\beta_k+\beta_k^2, & j=1,\\ 2(1+v_1)[(1-\omega)\alpha_k+\beta_k]\alpha_{j+k-1}, & j=2,...,p,\\ 2(1-\omega)(1+v_1)((1-\omega)\alpha_k+\beta_k)\beta_{j+k-p}, & j=p+1,...,2p-1,\end{cases}$$

- row $i=k+1$, ($k\neq p$):

$$a_{k+1,j}^{(k)} = \begin{cases} (1-\omega)\left[\alpha_k+\dfrac{\beta_k}{1+v_1}\right], & j=1,\\ (1-\omega)\alpha_{j+k-1}, & j=2,...,p,\\ (1-\omega)^2\beta_{j+k-p}, & j=p+1,...,2p-1,\end{cases}$$

- row $i=k+p$:

$$a_{k+p,j}^{(k)} = \begin{cases} \dfrac{\alpha_k+\beta_k}{(1-\omega)(1+v_1)}, & j=1,\\ \dfrac{\alpha_{j+k-1}}{1-\omega}, & j=2,...,p,\\ \beta_{j+k-p}, & j=p+1,...,2p-1,\end{cases}$$

- row $i=k+j$:

$$a_{k+j,j}^{(k)} = \begin{cases} (1-\omega)\alpha_k+\beta_k, & j=2,...,p-k,p+1,...,2p-1-k,\\ 0 & j=p-k+1,...,p,\end{cases}$$

and for any other case $a_{ij}^{(k)}=0$, where we consider $\alpha_j=\beta_j=0$, for $j>p$. The general form of these matrices A_k can be found in Appendix $A.2$. ■

In the following theorem we present the referred necessary and sufficient condition for weak stationarity of the ZICP-INGARCH process. The proof is analogous of that of Theorem 3.2.

Theorem 5.1 *Let X be a first-order stationary process following a ZICP-INGARCH(p,q) model such that* **H2** *is satisfied. This process is weakly stationary if and only if*

$$Q(L) = I_{p+q-1} - \sum_{k=1}^{\max(p,q)} A_k L^k$$

is a polynomial matrix such that $\det Q(z)$ *has all its roots outside the unit circle, with* A_k $(k = 1,...,\max(p,q))$ *the squared matrices of the autoregressive equation* (5.10). *Moreover, we have under the weak stationarity of X, the covariance function of X and* λ*, respectively* Γ *and* $\widetilde{\Gamma}$*, given by*

$$\Gamma(j) = e_{j+1}[Q(1)]^{-1}A_0 - \mu^2, \qquad j = 0,...,p-1,$$

$$\widetilde{\Gamma}(j) = e_{p+j}[Q(1)]^{-1}A_0 - \frac{\mu^2}{(1-\omega)^2}, \qquad j = 1,...,q-1,$$

with e_j *denoting the* j*-th row of the identity matrix.*

For a first-order stationary ZICP-INGARCH(p,p) process where only the order p coefficients are nonzero we can deduce, following the same steps than in Example 3.1, the weak stationarity characterization presented below.

Corollary 5.1 *Let X be a first-order stationary process following a ZICP-INGARCH(p,p) model such that* $\alpha_1 = ... = \alpha_{p-1} = \beta_1 = ... = \beta_{p-1} = 0$ *and hypothesis* **H2** *is satisfied. A necessary and sufficient condition for weak stationarity is* $(1-\omega)(1+v_1)\alpha_p^2 + 2(1-\omega)\alpha_p\beta_p + \beta_p^2 < 1$.

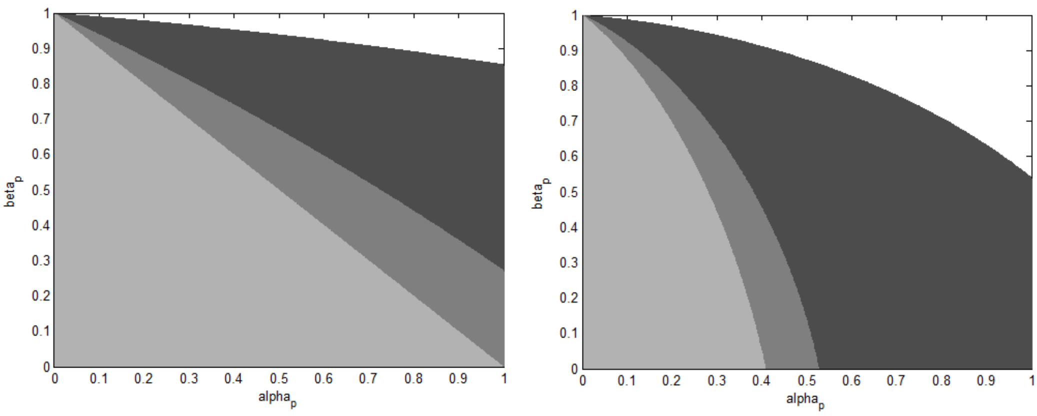

Fig. 5.6 Weak stationarity regions of a ZICP-INGARCH(p,p) model under $(1-\omega)\alpha_p + \beta_p < 1$, considering $\alpha_1 = ... = \alpha_{p-1} = \beta_1 = ... = \beta_{p-1} = 0$ and $\omega = 0$ (lightest gray), 0.4 (lightest and medium gray) and 0.9 (light to darkest gray). We have $v_1 = 0$ (on the left) and $v_1 = 5$ (on the right).

In Figure 5.6, on the left, we present weak stationarity regions of a ZICP-INGARCH(p,p) model, in the conditions of Corollary 5.1, with φ_t deterministic and independent of t characteristic function (recall in this case $v_1 = 0$) and considering different values of ω, namely $\omega = 0$, 0.4 and 0.9. We see that when we increase the parameter ω the weak stationarity region also increases. The same happens when we change the parameter v_1 to be equal to 5 (see Figure 5.6, on the right) corresponding to random characteristics functions φ_t in the ZICP-INGARCH(p,p) model. For instance, these regions corresponds to a ZINB-INGARCH(1,1) model where $a=5$ and $c=1$.

In order to derive the autocovariance function of the ZICP-INGARCH(p,q) model, we extend Theorem 3.3 omitting its proof since it is obtained using the same arguments. This general result includes those of Zhu [84] on ZIP-INGARCH and ZINB-INGARCH models.

Theorem 5.2 *Let X be a weakly stationary ZICP-INGARCH(p,q) process. The autocovariances of the processes X and λ satisfy the linear equations*

$$\Gamma(h)=(1-\omega)\sum_{j=1}^{p}\alpha_j\cdot\Gamma(h-j)+\sum_{k=1}^{\min(h-1,q)}\beta_k\cdot\Gamma(h-k)+(1-\omega)^2\sum_{k=h}^{q}\beta_k\cdot\widetilde{\Gamma}(k-h),\quad h\geq 1,$$

$$\widetilde{\Gamma}(h)=(1-\omega)\sum_{j=1}^{\min(h,p)}\alpha_j\cdot\widetilde{\Gamma}(h-j)+\frac{1}{1-\omega}\sum_{j=h+1}^{p}\alpha_j\cdot\Gamma(j-h)+\sum_{k=1}^{q}\beta_k\cdot\widetilde{\Gamma}(h-k),\qquad h\geq 0,$$

assuming that $\sum_{k=h}^{q}\beta_k\cdot\widetilde{\Gamma}(k-h)=0$ *if* $h>q$ *and* $\sum_{j=h+1}^{p}\alpha_j\cdot\Gamma(j-h)=0$ *if* $h+1>p$.

The following example illustrate the expression stated for $p=q=1$.

Example 5.4 *Let X be a ZICP-INGARCH$(1,1)$ model. From Theorem 5.2 we have, for $h\geq 2$,*

$$\begin{aligned}\Gamma(h) &= (1-\omega)\alpha_1\cdot\Gamma(h-1)+\beta_1\cdot\Gamma(h-1)=\ \dots\\ &= (1-\omega)\alpha_1+\beta_1]^{h-1}[(1-\omega)\alpha_1\cdot\Gamma(0)+(1-\omega)^2\beta_1\cdot\widetilde{\Gamma}(0)].\end{aligned}$$

To determine an expression for $V(\lambda_t)=\widetilde{\Gamma}(0)$, we note first that

$$\begin{aligned}\widetilde{\Gamma}(h) &= (1-\omega)\alpha_1\cdot\widetilde{\Gamma}(h-1)+\beta_1\cdot\widetilde{\Gamma}(h-1)\ =\ \dots\\ &= [(1-\omega)\alpha_1+\beta_1]^h\cdot\widetilde{\Gamma}(0), \qquad\qquad h\geq 1,\end{aligned}$$

$$\begin{aligned}\widetilde{\Gamma}(0) &= \frac{\alpha_1}{1-\omega}\cdot\Gamma(1)+\beta_1\cdot\widetilde{\Gamma}(1)\\ &= \alpha_1^2\cdot\Gamma(0)+(1-\omega)\alpha_1\beta_1\cdot\widetilde{\Gamma}(0)+\beta_1[(1-\omega)\alpha_1+\beta_1]\cdot\widetilde{\Gamma}(0)\\ &= \alpha_1^2\cdot\Gamma(0)+[2(1-\omega)\alpha_1\beta_1+\beta_1^2]\cdot\widetilde{\Gamma}(0)\end{aligned}$$

$$\Leftrightarrow\ \widetilde{\Gamma}(0)\ =\ \frac{\alpha_1^2\cdot\Gamma(0)}{1-2(1-\omega)\alpha_1\beta_1+\beta_1^2}.$$

Replacing this in the expression of $\Gamma(h)$ *above, we obtain*

$$\begin{aligned}\Gamma(h) &= [(1-\omega)\alpha_1+\beta_1]^{h-1}\cdot\left[(1-\omega)\alpha_1\cdot\Gamma(0)+\frac{(1-\omega)^2\alpha_1^2\beta_1\cdot\Gamma(0)}{1-2(1-\omega)\alpha_1\beta_1+\beta_1^2}\right]\\ &= [(1-\omega)\alpha 1+\beta_1]^{h-1}\frac{(1-\omega)\alpha_1[1-(1-\omega)\alpha_1\beta_1-\beta_1^2]}{1-2(1-\omega)\alpha_1\beta_1-\beta_1^2}\cdot\Gamma(0),\end{aligned}$$

and so the autocorrelations of X and λ are given by

$$\begin{aligned}\rho(h) &= \frac{\Gamma(h)}{\Gamma(0)} = [(1-\omega)\alpha_1+\beta_1]^{h-1}\frac{(1-\omega)\alpha_1[1-(1-\omega)\alpha_1\beta_1-\beta_1^2]}{1-2(1-\omega)\alpha_1\beta_1-\beta_1^2}, \qquad h\geq 1,\\ \widetilde{\rho}(h) &= [(1-\omega)\alpha_1+\beta_1]^h, \qquad h\geq 0.\end{aligned}$$

Under the hypothesis **H2**, *the value of* $\Gamma(0)$ *can be deduced, for instance, using the expression derived in Theorem* 5.1. *Indeed,* $\Gamma(0)=V(X_t)=[Q(1)]^{-1}A_0-\mu^2$, *where*

$$\begin{aligned}Q(1) &= 1-A_1 = 1-(1-\omega)(1+v_1)\alpha_1^2-2(1-\omega)\alpha_1\beta_1-\beta_1^2,\\ A_0 &= v_0\mu+(1+v_1)[2\alpha_0\mu-(1-\omega)\alpha_0^2]-v_0\mu[2(1-\omega)\alpha_1\beta_1+\beta_1^2],\\ \mu &= \frac{(1-\omega)\alpha_0}{1-(1-\omega)\alpha_1-\beta_1}.\end{aligned}$$

Thus, we obtain

$$\Gamma(0) = \frac{1-2(1-\omega)\alpha_1\beta_1-\beta_1^2}{1-(1-\omega)(1+v_1)\alpha_1^2-2(1-\omega)\alpha_1\beta_1-\beta_1^2}\left(v_0\mu+\frac{(v_1+\omega)\mu^2}{1-\omega}\right).$$

Using this result when $\omega=0$ *we recover the expression stated in Remark* 3.5. *When* $\omega\neq 0$, *for example, the variance of the ZINB-INGARCH*$(1,1)$ *process is given by*

$$\Gamma(0)=\begin{cases}\frac{1-2(1-\omega)\alpha_1\beta_1-\beta_1^2}{1-(1-\omega)\alpha_1^2-2(1-\omega)\alpha_1\beta_1-\beta_1^2}\left((1+a)\mu+\frac{\omega\mu^2}{1-\omega}\right), & c=0,\\[2ex] \frac{1-2(1-\omega)\alpha_1\beta_1-\beta_1^2}{1-(1-\omega)(1+a)\alpha_1^2-2(1-\omega)\alpha_1\beta_1-\beta_1^2}\left(\mu+\frac{(a+\omega)\mu^2}{1-\omega}\right), & c=1.\end{cases}$$

Going back to the study of the stationarity properties, we now present the construction of a strictly stationary solution of the ZICP-INGARCH(p,q) model. The study developed in Section 3.4 will be of extreme importance here as well as the representation (5.4) of this process stated in Section 5.2.

With this aim, let us consider a sequence $M=(M_t,t\in\mathbb{Z})$ of i.i.d. Bernoulli random variables with parameter $1-\omega$, $0<\omega<1$, and let us define a process $X^*=(X_t^*,t\in\mathbb{Z})$ such that

$$X_t^*=\begin{cases}0, & M_t=0,\\ Y_t, & M_t=1,\end{cases} \tag{5.11}$$

where $Y=(Y_t,t\in\mathbb{Z})$ is a CP-INGARCH(p,q) process, independent of M, for which the conditional distribution of Y_t given $\underline{Y}_{t-1}$ satisfies

$$\begin{aligned}\Phi_{Y_t|\underline{Y}_{t-1}}(u) &= \exp\left\{i\frac{\lambda_t}{\varphi_t'(0)}[\varphi_t(u)-1]\right\}, \qquad u\in\mathbb{R},\\ E(Y_t|\underline{Y}_{t-1}) &= \lambda_t = \alpha_0+\sum_{j=1}^p\alpha_jY_{t-j}+\sum_{k=1}^q\beta_k\lambda_{t-k}.\end{aligned}$$

We remember that in order to have this process it is sufficient to consider $Y_t = \sum_{j=1}^{N_t} Y_{t,j}$ where, conditionally to the past, N_t follows a Poisson law with parameter $\lambda_t/E(Y_{t,j})$ and $Y_{t,1}$, $Y_{t,2}$,... are discrete independent random variables, independent of N_t and with characteristic function φ_t.

If $X=(X_t, t\in\mathbb{Z})$ is the process following the model ZICP-INGARCH specified by (5.2) and (5.3), we show in the following that $\Phi_{X_t^*|(\underline{Y}_{t-1},\underline{M}_{t-1})}(u)=\Phi_{X_t|\underline{X}_{t-1}}(u),\ u\in\mathbb{R}$. In fact, we have

$$\begin{aligned}\Phi_{X_t^*|(\underline{Y}_{t-1},\underline{M}_{t-1})}(u) &= E\left(\exp\left(iu\mathbb{I}_{\{M_t=1\}}Y_t\right)|\left(\underline{Y}_{t-1},\underline{M}_{t-1}\right)\right)\\ &= E\left(\mathbb{I}_{\{M_t=0\}}+\exp\left(iuY_t\right)\mathbb{I}_{\{M_t=1\}}|\left(\underline{Y}_{t-1},\underline{M}_{t-1}\right)\right)\\ &= \omega+E\left(\exp\left(iuY_t\right)\mathbb{I}_{\{M_t=1\}}|\left(\underline{Y}_{t-1},\underline{M}_{t-1}\right)\right)\\ &= \omega+(1-\omega)E\left(\exp\left(iuY_t\right)|\left(\underline{Y}_{t-1},\underline{M}_{t-1}\right)\right),\quad u\in\mathbb{R},\end{aligned}$$

since $\{Y_t\}$ is independent of $\{M_t\}$ and $\{M_t\}$ are independent variables. So, from the independence between the processes Y and M, we obtain

$$\begin{aligned}\Phi_{X_t^*|(\underline{Y}_{t-1},\underline{M}_{t-1})}(u) &= \omega+(1-\omega)E\left(\exp\left(iuY_t\right)|\underline{Y}_{t-1}\right)\\ &= \omega+(1-\omega)\exp\left\{i\frac{\lambda_t}{\varphi_t'(0)}\left[\varphi_t(u)-1\right]\right\} = \Phi_{X_t|\underline{X}_{t-1}}(u),\quad u\in\mathbb{R}.\end{aligned}$$

So, a solution of our ZICP-INGARCH model given by (5.2) and (5.3) may be obtained by (5.11).

We are now in conditions to state the strict stationarity of this solution of the model, when φ_t is a characteristic function deterministic and independent of t.

Theorem 5.3 *Let us consider the ZICP-INGARCH(p,q) model as specified in* (5.2) *and* (5.3) *and such that hypothesis* **H4** *is satisfied. There is a strictly stationary process in L^1 that is a solution of this model if and only if $\sum_{j=1}^{p}\alpha_j+\sum_{k=1}^{q}\beta_k<1$. Moreover, this process is also weakly stationary.*

Proof: Let us consider the ZICP-INGARCH(p,q) model associated to a deterministic and independent of t characteristic function φ. From Remark 3.8 **(b)**, there is a strictly stationary solution in L^1 of a CP-INGARCH(p,q) model associated to the referred characteristic function φ if and only if $\sum_{j=1}^{p}\alpha_j+\sum_{k=1}^{q}\beta_k<1$. Let us denote this solution by $Y^*=(Y_t^*, t\in\mathbb{Z})$.

Then the process $X^*=(X_t^*, t\in\mathbb{Z})$ defined as

$$X_t^*=Y_t^*\mathbb{I}_{\{M_t=1\}}$$

with $M=(M_t, t\in\mathbb{Z})$ a sequence of i.i.d. Bernoulli random variables with parameter $1-\omega$, $0<\omega<1$, and independent of Y, is a solution of the ZICP-INGARCH(p,q) model since

$$\Phi_{X_t^*|\left(\underline{Y}_{t-1}^*,\underline{M}_{t-1}\right)}(u)=\omega+(1-\omega)\exp\left\{i\frac{\lambda_t}{\varphi'(0)}\left[\varphi(u)-1\right]\right\},\quad u\in\mathbb{R}.$$

The process X^* is a strictly stationary process as it is a measurable function of the process $\{(Y_t^*, M_t), t\in\mathbb{Z}\}$ which is strictly stationary as Y^* and M are independent and strictly stationary processes. As Y^* is a first and a second-order process (Theorem 3.7 and 3.8), the same happens to X^*, and so X^* is also a weakly stationary process in L^1 if $\sum_{j=1}^{p}\alpha_j+\sum_{k=1}^{q}\beta_k<1$. ■

We study now the moments and the cumulants of the ZICP-INGARCH$(1,1)$ process. To this end, let us assume in the following that the characteristic functions φ_t are differentiable as many times as necessary. We start by enunciating a necessary and sufficient condition for the existence of all its moments, a result where the representation (5.5) stated in Section 5.2 will be useful.

Theorem 5.4 *The moments of a ZICP-INGARCH*$(1,1)$ *model satisfying the hypothesis* **H3** *are all finite if and only if* $(1-\omega)\alpha_1+\beta_1<1$.

Proof: To prove the statement, let us recall that the conditional distribution of X_t can be viewed as a compound law with a zero-inflated Poisson counting distribution with parameters (λ_t^*,ω) provided by the representation (5.5). Since the expression (2.1) presented in Remark 2.3 is valid for any compound law, and the r-th descending factorial moment of U_t is $(1-\omega)(\lambda_t^*)^r$, $r\geq 1$, we deduce

$$
\begin{aligned}
E[X_t^m|\underline{X}_{t-1}] &= (1-\omega)\sum_{r=0}^{m}\frac{1}{r!}\frac{\lambda_t^r}{(\varphi_t'(0))^r}\sum_{k=0}^{r}\begin{pmatrix} r \\ k\end{pmatrix}\frac{(-1)^{r-k}}{i^{m-r}}\left(\varphi_t^k\right)^{(m)}(0),\\
E[X_t^m] &= (1-\omega)\sum_{r=0}^{m}\sum_{j=0}^{r}\frac{1}{r!}\begin{pmatrix} r \\ j\end{pmatrix}\frac{(-1)^{r-j}\left(\varphi_t^j\right)^{(m)}(0)}{i^{m-r}(\varphi_t'(0))^r}\,E[\lambda_t^r], \qquad m\geq 1,
\end{aligned}
$$

$$
\begin{aligned}
E[\lambda_t^r|\underline{X}_{t-2}] &= \alpha_0^r+\sum_{n=1}^{r}\begin{pmatrix} r \\ n\end{pmatrix}\alpha_0^{r-n}\beta_1^n\lambda_{t-1}^n+(1-\omega)\sum_{n=1}^{r}\begin{pmatrix} r \\ n\end{pmatrix}\alpha_0^{r-n}\sum_{l=1}^{n}\begin{pmatrix} n \\ l\end{pmatrix}\times\\
&\quad\times\sum_{v=0}^{l}\frac{\alpha_1^l\beta_1^{n-l}}{v!(\varphi_{t-1}'(0))^v}\sum_{x=0}^{v}\begin{pmatrix} v \\ x\end{pmatrix}\frac{(-1)^{v-x}}{i^{l-v}}\left(\varphi_{t-1}^x\right)^{(l)}(0)\lambda_{t-1}^{v+n-l},
\end{aligned}
$$

with $\left(\varphi_t^k\right)^{(m)}$ given in Lemma 3.1 and following the same steps as in Theorem 3.4. Then we obtain the equation $E[\Lambda_t|\underline{X}_{t-2}]=\mathbf{d}+\mathbf{D}\Lambda_{t-1}$, with the constant vector $\mathbf{d}=(\alpha_0^m,\cdots,\alpha_0^2,\alpha_0)^T$ and $\mathbf{D}=(d_{ij})$, $i,j=1,...,m$, the upper triangular matrix given by

$$
\mathbf{D}=\begin{bmatrix}
(1-\omega)(\alpha_1+\beta_1)^m+\omega\beta_1^m & \cdots & d_{1,m-1} & d_{1m}\\
\vdots & \ddots & \vdots & \vdots\\
0 & \cdots & (1-\omega)(\alpha_1+\beta_1)^2+\omega\beta_1^2 & d_{m-1,m}\\
0 & \cdots & 0 & (1-\omega)\alpha_1+\beta_1
\end{bmatrix},
$$

from here the required condition holds. ■

We conclude this section presenting expressions for the first three cumulants and moments up to order 3 of the ZICP-INARCH(1) model. As in Chapter 4, this study can have interest in statistical developments of these models. Let us consider a first-order stationary ZICP-INARCH(1) model satisfying the hypothesis **H4**. In the following we illustrate the derivation of its first three cumulants.

Using the characteristic function of the conditional distribution and $\lambda_t=\alpha_0+\alpha_1X_{t-1}$ we obtain

$$
\Phi_{X_t}(z) = E\left(e^{izX_t}\right)=E\left[E\left(e^{izX_t}|\underline{X}_{t-1}\right)\right]
$$

$$
\begin{aligned}
&= E\left[\omega + (1-\omega)\exp\left\{i\frac{\lambda_t}{\varphi'(0)}[\varphi(z)-1]\right\}\right] \\
&= \omega + (1-\omega)\exp\left\{\frac{i\alpha_0}{\varphi'(0)}[\varphi(z)-1]\right\}\cdot\Phi_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}[\varphi(z)-1]\right) \\
&= \omega + (1-\omega)\cdot A(z), \qquad\qquad z\in\mathbb{R},
\end{aligned}
$$

hence, the cumulant generating function of X_t is given by

$$\kappa_{X_t}(z) = \ln(\Phi_{X_t}(z)) = \ln(\omega + (1-\omega)\cdot A(z)), \qquad z\in\mathbb{R}.$$

Taking derivatives on both sides, it follows that

$$
\begin{aligned}
\kappa'_{X_t}(z) &= \frac{\Phi'_{X_t}(z)}{\Phi_{X_t}(z)} = \frac{(1-\omega)A'(z)}{\omega+(1-\omega)A(z)} = \frac{(1-\omega)\frac{A'(z)}{A(z)}}{(1-\omega)+\frac{\omega}{A(z)}}, \\
\kappa''_{X_t}(z) &= \frac{\Phi''_{X_t}(z)}{\Phi_{X_t}(z)} - \left[\frac{\Phi'_{X_t}(z)}{\Phi_{X_t}(z)}\right]^2 = \frac{(1-\omega)\frac{A''(z)}{A(z)}}{(1-\omega)+\frac{\omega}{A(z)}} - \left(\kappa'_{X_t}(z)\right)^2, \\
\kappa'''_{X_t}(z) &= \frac{\Phi'''_{X_t}(z)}{\Phi_{X_t}(z)} - 3\frac{\Phi'_{X_t}(z)\Phi''_{X_t}(z)}{(\Phi_{X_t}(z))^2} + 2\left[\frac{\Phi'_{X_t}(z)}{\Phi_{X_t}(z)}\right]^3 \\
&= \frac{(1-\omega)\frac{A'''(z)}{A(z)}}{(1-\omega)+\frac{\omega}{A(z)}} - 3\kappa'_{X_t}(z)\kappa''_{X_t}(z) - \left(\kappa'_{X_t}(z)\right)^3,
\end{aligned}
$$

where, taking as $h(z) = \frac{\varphi(z)-1}{\varphi'(0)}$,

$$
\begin{aligned}
A'(z) &= \frac{i\alpha_0\varphi'(z)}{\varphi'(0)}\cdot\exp\{i\alpha_0\cdot h(z)\}\cdot\Phi_{X_{t-1}}(\alpha_1\cdot h(z)) \\
&+ \frac{\alpha_1\varphi'(z)}{\varphi'(0)}\cdot\exp\{i\alpha_0\cdot h(z)\}\cdot\Phi'_{X_{t-1}}(\alpha_1\cdot h(z)),
\end{aligned}
$$

$$
\begin{aligned}
A''(z) &= \left[\frac{i\alpha_0\varphi''(z)}{\varphi'(0)} + \left(\frac{i\alpha_0\varphi'(z)}{\varphi'(0)}\right)^2\right]\cdot\exp\{i\alpha_0\cdot h(z)\}\cdot\Phi_{X_{t-1}}(\alpha_1\cdot h(z)) \\
&+ \left[\frac{\alpha_1\varphi''(z)}{\varphi'(0)} + 2i\alpha_0\alpha_1\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^2\right]\cdot\exp\{i\alpha_0\cdot h(z)\}\cdot\Phi'_{X_{t-1}}(\alpha_1\cdot h(z)) \\
&+ \left(\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\right)^2\cdot\exp\{i\alpha_0\cdot h(z)\}\cdot\Phi''_{X_{t-1}}(\alpha_1\cdot h(z)),
\end{aligned}
$$

and

$$
\begin{aligned}
A'''(z) &= \left[\frac{i\alpha_0\varphi'''(z)}{\varphi'(0)} - 3\alpha_0^2\frac{\varphi'(z)\varphi''(z)}{(\varphi'(0))^2} - i\alpha_0^3\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^3\right]\cdot\exp\{i\alpha_0\cdot h(z)\}\cdot\Phi_{X_{t-1}}(\alpha_1\cdot h(z)) \\
&+ \left[6i\alpha_0\alpha_1\frac{\varphi'(z)\varphi''(z)}{(\varphi'(0))^2} - 3\alpha_0^2\alpha_1\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^3 + \alpha_1\frac{\varphi'''(z)}{\varphi'(0)}\right]\cdot\exp\{i\alpha_0\cdot h(z)\}\cdot\Phi'_{X_{t-1}}(\alpha_1\cdot h(z))
\end{aligned}
$$

$$+\left[3\alpha_1^2\frac{\varphi'(z)\varphi''(z)}{(\varphi'(0))^2}+3i\alpha_0\alpha_1^2\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^3\right]\cdot\exp\{i\alpha_0\cdot h(z)\}\cdot\Phi''_{X_{t-1}}(\alpha_1\cdot h(z))$$

$$+\left(\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\right)^3\cdot\exp\{i\alpha_0\cdot h(z)\}\cdot\Phi'''_{X_{t-1}}(\alpha_1\cdot h(z)).$$

So, we have

$$\frac{A'(z)}{A(z)}=\frac{i\alpha_0\varphi'(z)}{\varphi'(0)}+\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\cdot\kappa'_{X_{t-1}}(\alpha_1\cdot h(z)),$$

$$\frac{A''(z)}{A(z)}=\frac{i\alpha_0\varphi''(z)}{\varphi'(0)}+\left(\frac{i\alpha_0\varphi'(z)}{\varphi'(0)}\right)^2+\left[\frac{\alpha_1\varphi''(z)}{\varphi'(0)}+2i\alpha_0\alpha_1\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^2\right]\cdot\kappa'_{X_{t-1}}(\alpha_1\cdot h(z))$$
$$+\left(\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\right)^2\cdot\left(\kappa''_{X_{t-1}}(\alpha_1\cdot h(z))+\left(\kappa'_{X_{t-1}}(\alpha_1\cdot h(z))\right)^2\right),$$

$$\frac{A'''(z)}{A(z)}=\frac{i\alpha_0\varphi'''(z)}{\varphi'(0)}-\frac{3\alpha_0^2\varphi'(z)\varphi''(z)}{(\varphi'(0))^2}-i\alpha_0^3\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^3$$
$$+\left[6i\alpha_0\alpha_1\frac{\varphi'(z)\varphi''(z)}{(\varphi'(0))^2}-3\alpha_0^2\alpha_1\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^3+\alpha_1\frac{\varphi'''(z)}{\varphi'(0)}\right]\cdot\kappa'_{X_{t-1}}(\alpha_1\cdot h(z))$$
$$+\left[3\alpha_1^2\frac{\varphi'(z)\varphi''(z)}{(\varphi'(0))^2}+3i\alpha_0\alpha_1^2\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^3\right]\cdot\left(\kappa''_{X_{t-1}}(\alpha_1\cdot h(z))+\left(\kappa'_{X_{t-1}}(\alpha_1\cdot h(z))\right)^2\right)$$
$$+\left(\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\right)^3\cdot\left[\kappa'''_{X_{t-1}}(\alpha_1\cdot h(z))+3\kappa'_{X_{t-1}}(\alpha_1\cdot h(z))\,\kappa''_{X_{t-1}}(\alpha_1\cdot h(z))+\left(\kappa'_{X_{t-1}}(\alpha_1\cdot h(z))\right)^3\right].$$

Let us recall the notations stated in (4.6) and the hypothesis of first-order stationarity. Inserting $z=0$ into the previous equations and noting that $A(0)=1$ and $h(0)=0$, we obtain

$$\begin{aligned}
\kappa'_{X_t}(0) &= (1-\omega)\left[i\alpha_0+\alpha_1\cdot\kappa'_{X_{t-1}}(0)\right]\\
\Rightarrow\ \kappa_1(X_t) &= \mu=\frac{(1-\omega)\alpha_0}{1-(1-\omega)\alpha_1},\\
\kappa''_{X_t}(0) &= (1-\omega)\left\{\frac{i\alpha_0\varphi''(0)}{\varphi'(0)}-\alpha_0^2+\left[\frac{\alpha_1\varphi''(0)}{\varphi'(0)}+2i\alpha_0\alpha_1\right]\cdot\kappa'_{X_{t-1}}(0)\right\}\\
&\quad+(1-\omega)\alpha_1^2\cdot\left[\kappa''_{X_{t-1}}(0)+\left(\kappa'_{X_{t-1}}(0)\right)^2\right]-\left(\kappa'_{X_t}(0)\right)^2\\
&\quad-(1-\omega)\alpha_0^2+iv_0(1-\omega)[i\alpha_0+\alpha_1\cdot\kappa'_{X_{t-1}}(0)]+2i(1-\omega)\alpha_0\alpha_1\cdot\kappa'_{X_{t-1}}(0)\\
&\quad+(1-\omega)\alpha_1^2\cdot\left[\kappa''_{X_{t-1}}(0)+\left(\kappa'_{X_{t-1}}(0)\right)^2\right]+\mu^2\\
&= -(1-\omega)\alpha_0^2+iv_0\cdot\kappa'_{X_t}(0)+2i(1-\omega)\alpha_0\alpha_1\cdot\kappa'_{X_t}(0)\\
&\quad+(1-\omega)\alpha_1^2\cdot\kappa''_{X_{t-1}}(0)+(1-\omega)\alpha_1^2\cdot\left(\kappa'_{X_t}(0)\right)^2+\mu^2
\end{aligned}$$

$$
\begin{aligned}
&= (1-\omega)\alpha_1^2 \cdot \kappa''_{X_{t-1}}(0) - v_0\mu \\
&\quad -[(1-\omega)\alpha_0^2 + 2(1-\omega)\alpha_0\alpha_1\mu + (1-\omega)\alpha_1^2\mu^2 - \mu^2] \\
&= (1-\omega)\alpha_1^2 \cdot \kappa''_{X_{t-1}}(0) - v_0\mu - \frac{\omega\mu^2}{1-\omega},
\end{aligned}
$$

since $(1-\omega)\alpha_0 = \mu(1-(1-\omega)\alpha_1)$, and then, as in Example 5.4,

$$
\kappa_2(X_t) = \Gamma(0) = \frac{v_0\mu + \frac{\omega\mu^2}{1-\omega}}{1-(1-\omega)\alpha_1^2}.
$$

Moreover,

$$
\begin{aligned}
\kappa'''_{X_t}(0) &= (1-\omega)\Bigg\{ \frac{i\alpha_0\varphi'''(0)}{\varphi'(0)} - \frac{3\alpha_0^2\varphi''(0)}{\varphi'(0)} - i\alpha_0^3 \\
&\quad + \left[6i\alpha_0\alpha_1\frac{\varphi''(0)}{\varphi'(0)} - 3\alpha_0^2\alpha_1 + \alpha_1\frac{\varphi'''(0)}{\varphi'(0)}\right] \cdot \kappa'_{X_{t-1}}(0) \\
&\quad + \left[3\alpha_1^2\frac{\varphi''(0)}{\varphi'(0)} + 3i\alpha_0\alpha_1^2\right]\left[\kappa''_{X_{t-1}}(0) + \left[\kappa'_{X_{t-1}}(0)\right]^2\right] \\
&\quad + \alpha_1^3 \cdot \left[\kappa'''_{X_{t-1}}(0) + 3\kappa'_{X_{t-1}}(0)\,\kappa''_{X_{t-1}}(0) + \left[\kappa'_{X_{t-1}}(0)\right]^3\right]\Bigg\} \\
&\quad -3\kappa'_{X_t}(0)\kappa''_{X_t}(0) - \left[\kappa'_{X_t}(0)\right]^3 \\
&= -d_0(1-\omega)[i\alpha_0 + \alpha_1 \cdot \kappa'_{X_{t-1}}(0)] - i(1-\omega)\alpha_0[3\alpha_0 v_0 + \alpha_0^2] \\
&\quad -(1-\omega)\alpha_1[6\alpha_0 v_0 + 3\alpha_0^2] \cdot \kappa'_{X_t}(0) \\
&\quad +3i(1-\omega)\alpha_1^2(v_0+\alpha_0) \cdot [\kappa''_{X_t}(0) + (\kappa'_{X_t}(0))^2] \\
&\quad +(1-\omega)\alpha_1^3 \cdot \kappa'''_{X_{t-1}}(0) + 3(1-\omega)\alpha_1^3 \cdot \kappa'_{X_t}(0) \cdot \kappa''_{X_t}(0) \\
&\quad +(1-\omega)\alpha_1^3 \cdot (\kappa'_{X_t}(0))^3 + 3i\mu\Gamma(0) + i\mu^3 \\
&= (1-\omega)\alpha_1^3 \cdot \kappa'''_{X_{t-1}}(0) - id_0\mu - i(1-\omega)\alpha_0[3\alpha_0 v_0 + \alpha_0^2] \\
&\quad -(1-\omega)i\alpha_1[6\alpha_0 v_0 + 3\alpha_0^2]\mu - 3i(1-\omega)\alpha_1^2(v_0+\alpha_0)[\Gamma(0)+\mu^2] \\
&\quad -3(1-\omega)i\alpha_1^3\mu\Gamma(0) - (1-\omega)i\alpha_1^3\mu^3 + 3i\mu\Gamma(0) + i\mu^3.
\end{aligned}
$$

Using the equalities

$$
\begin{aligned}
(1-\omega)\alpha_1\mu &= \mu - (1-\omega)\alpha_0, \\
(1-\omega)\alpha_1^2\mu^2 &= (1-\omega)\left(\frac{\mu}{1-\omega} - \alpha_0\right)^2 = \frac{\mu^2}{1-\omega} - 2\alpha_0\mu + (1-\omega)\alpha_0^2,
\end{aligned}
$$

we deduce

$$
\begin{aligned}
\kappa'''_{X_t}(0) &= (1-\omega)\alpha_1^3 \cdot \kappa'''_{X_{t-1}}(0) - id_0\mu - i(1-\omega)\alpha_0[3\alpha_0 v_0 + \alpha_0^2] \\
&\quad -i[6\alpha_0 v_0 + 3\alpha_0^2]\mu + i(1-\omega)\alpha_0[6\alpha_0 v_0 + 3\alpha_0^2] \\
&\quad -3i(1-\omega)\alpha_1^2(v_0+\alpha_0)\Gamma(0) \\
&\quad + \left\{-\frac{3i\mu^2}{1-\omega} + 6i\alpha_0\mu - 3i(1-\omega)\alpha_0^2\right\}(v_0+\alpha_0) - 3i\alpha_1^2\Gamma(0)\mu
\end{aligned}
$$

$$
\begin{aligned}
&+3i(1-\omega)\alpha_0\alpha_1^2\Gamma(0)-\frac{i\alpha_1\mu^3}{1-\omega}+2i\alpha_0\alpha_1\mu^2\\
&-i\alpha_0^2\mu+i(1-\omega)\alpha_0^3+3i\mu\Gamma(0)+i\mu^3\\
= \;& (1-\omega)\alpha_1^3\cdot\kappa'''_{X_{t-1}}(0)-id_0\mu-3iv_0(1-\omega)\alpha_1^2\Gamma(0)-\frac{3i\mu^2}{1-\omega}(v_0+\alpha_0)\\
&+2i\alpha_0^2\mu-3i\alpha_1^2\mu\Gamma(0)-\frac{i\alpha_1\mu^3}{1-\omega}+2i\alpha_0\alpha_1\mu^2+3i\mu\Gamma(0)+i\mu^3\\
= \;& (1-\omega)\alpha_1^3\cdot\kappa'''_{X_{t-1}}(0)-id_0\mu+-3i(1-\omega)\alpha_1^2v_0\Gamma(0)\\
&-\frac{i\omega\mu^2}{1-\omega}[\alpha_0+\mu(1+\alpha_1)]-3i\mu\left[\frac{v_0\mu}{1-\omega}-\Gamma(0)(1-\alpha_1^2)\right]\\
= \;& (1-\omega)\alpha_1^3\cdot\kappa'''_{X_{t-1}}(0)-id_0\mu-3iv_0(1-\omega)\alpha_1^2\Gamma(0)\\
&-\frac{i\omega\mu^2}{1-\omega}[\alpha_0+\mu(1+\alpha_1)]-3i\omega\mu\left[\frac{\mu}{1-\omega}(v_0-\mu)+\alpha_1^2\Gamma(0)\right]\\
= \;& (1-\omega)\alpha_1^3\cdot\kappa'''_{X_{t-1}}(0)-id_0\mu-3i\alpha_1^2\Gamma(0)[(1-\omega)v_0+\omega\mu]\\
&-\frac{i\omega\mu^2}{1-\omega}[3v_0+\alpha_0+\mu(\alpha_1-2)]
\end{aligned}
$$

and from that we get

$$
\kappa_3(X_t)=\frac{d_0\mu+3\alpha_1^2\Gamma(0)[(1-\omega)v_0-\omega\mu],+\frac{\omega\mu^2}{1-\omega}[3v_0+\alpha_0+\mu(\alpha_1-2)]}{1-(1-\omega)\alpha_1^3}.
$$

Now we derive an explicit expression for $\mu(k,l)$, with $0\leq k\leq l$. In order to do this, we remember the following conditional moments:

$$
\begin{aligned}
E(X_t|\underline{X}_{t-1}) &= (1-\omega)\lambda_t=(1-\omega)[\alpha_0+\alpha_1X_{t-1}],\\
E(X_t^2|\underline{X}_{t-1}) &= (1-\omega)\left[v_0\lambda_t+\lambda_t^2\right]\\
&= (1-\omega)\left[v_0(\alpha_0+\alpha_1X_{t-1})+(\alpha_0+\alpha_1X_{t-1})^2\right]\\
&= (1-\omega)\left[\alpha_1^2X_{t-1}^2+\alpha_1(v_0+2\alpha_0)X_{t-1}+\alpha_0(v_0+\alpha_0)\right],
\end{aligned}
$$

and we distinguish the following three cases:

Case 1: $l>k$.

$$
\begin{aligned}
\mu(k,l) &= E(X_tX_{t+k}X_{t+l})=E[X_tX_{t+k}E(X_{t+l}|\underline{X}_{t+l-1})]\\
&= (1-\omega)\left[\alpha_0E(X_tX_{t+k})+\alpha_1E(X_tX_{t+k}X_{t+l-1})\right]\\
&= (1-\omega)\alpha_0\mu(k)+(1-\omega)\alpha_1\mu(k,l-1)\\
&= (1-\omega)\alpha_0\mu(k)+(1-\omega)^2\alpha_1[\alpha_0\mu(k)+\alpha_1\mu(k,l-2)]\\
&= (1-\omega)\alpha_0\mu(k)(1+(1-\omega)\alpha_1)+(1-\omega)^2\alpha_1^2\mu(k,l-2)\\
&= \ldots=(1-\omega)\alpha_0\mu(k)\sum_{j=0}^{l-k-1}(1-\omega)^j\alpha_1^j+(1-\omega)^{l-k}\alpha_1^{l-k}\mu(k,k)\\
&= \mu\cdot\mu(k)(1-(1-\omega)^{l-k}\alpha_1^{l-k})+(1-\omega)^{l-k}\alpha_1^{l-k}\mu(k,k)
\end{aligned}
$$

$$= \quad (1-\omega)^{l-k}\alpha_1^{l-k}[\mu(k,k)-\mu\cdot\mu(k)]+\mu\cdot\mu(k).$$

Case 2: $l=k>0$.

$$\begin{aligned}
\mu(k,k) &= E[X_t E(X_{t+k}^2|\underline{X}_{t+k-1})]\\
&= (1-\omega)\left[\alpha_1^2 E(X_t X_{t+k-1}^2)+\alpha_1(v_0+2\alpha_0)E(X_t X_{t+k-1})+\alpha_0(v_0+\alpha_0)E(X_t)\right]\\
&= (1-\omega)\left[\alpha_1^2\mu(k-1,k-1)+\alpha_1(v_0+2\alpha_0)\mu(k-1)+\alpha_0(v_0+\alpha_0)\mu\right]\\
&= (1-\omega)\alpha_1^2\mu(k-1,k-1)+(1-\omega)^k\alpha_1^k(v_0+2\alpha_0)\Gamma(0)\\
&\quad +(1-\omega)\alpha_1(v_0+2\alpha_0)\mu^2+(1-(1-\omega)\alpha_1)(v_0+\alpha_0)\mu^2\\
&= (1-\omega)\alpha_1^2\mu(k-1,k-1)+(1-\omega)^k\alpha_1^k(v_0+2\alpha_0)\Gamma(0)\\
&\quad +[\alpha_0[1+(1-\omega)\alpha_1]+v_0]\,\mu^2\\
&= (1-\omega)\alpha_1^2\mu(k-1,k-1)+(1-\omega)^k\alpha_1^k(v_0+2\alpha_0)\Gamma(0)\\
&\quad +\left[1-(1-\omega)\alpha_1^2\right]\mu\cdot\mu(0)\\
&= \ldots=(1-\omega)^k\alpha_1^{2k}\mu(0,0)+(1-\omega)^k(v_0+2\alpha_0)\Gamma(0)\sum_{j=0}^{k-1}\alpha_1^{k+j}\\
&\quad +\left[1-(1-\omega)\alpha_1^2\right]\mu\cdot\mu(0)\sum_{j=0}^{k-1}\alpha_1^{2j}\\
&= \alpha_1^{2k}\left[(1-\omega)^k\mu(0,0)-\frac{v_0+2\alpha_0}{1-\alpha_1}(1-\omega)^k\Gamma(0)-(1-\omega)\mu\cdot\mu(0)\right]\\
&\quad +\frac{v_0+2\alpha_0}{1-\alpha_1}(1-\omega)^k\Gamma(0)\alpha_1^k+\left[1+\omega\sum_{j=0}^{k-1}\alpha_1^{2j}\right]\mu\cdot\mu(0).
\end{aligned}$$

Case 3: $l=k=0$.

$$\begin{aligned}
\mu(0,0) &= E(X_t^3)=\kappa_3+3\kappa_2\mu+\mu^3\\
&= \frac{d_0\mu+3\alpha_1^2\kappa_2[(1-\omega)v_0-\omega\mu]+\frac{\omega\mu^2}{1-\omega}[3v_0+\alpha_0+\mu(\alpha_1-2)]}{1-(1-\omega)\alpha_1^3}+3\kappa_2\mu+\mu^3\\
&= \frac{d_0\mu+\kappa_2[3(1-\omega)v_0\alpha_1^2-3\omega\mu\alpha_1^2+2\mu-2(1-\omega)\mu\alpha_1^3]}{1-(1-\omega)\alpha_1^3}\\
&\quad +\frac{\omega\mu^2[3v_0+\alpha_0+\mu(\alpha_1-2)]}{(1-\omega)(1-(1-\omega)\alpha_1^3)}+\mu\cdot\mu(0).
\end{aligned}$$

Chapter 6

Conclusions and future developments

"*Science never solves a problem without creating ten more.*" - George Bernard Shaw

This thesis focuses on establishing theoretical results in order to develop applications in which the responses are nonnegative integer-valued values, developing a unified study in the class of the nonnegative integer-valued conditional heteroscedastic models. In the last years diverse models have been proposed contributing to the existence of a vast literature for analysing count time series. In our opinion, the work developed under this thesis strongly improves the existing results on this area.

A new class of models which includes the main INGARCH processes present in literature is proposed and developed enlarging and unifying the analysis of those processes, and accomplishing the practical goal of modeling simultaneously different stylized facts that have been recorded in real count data: different kinds of conditional heteroscedasticity, overdispersion and zero inflation.

We first proposed a broad class of nonnegative integer-valued conditional heteroscedastic models that is based upon the family of discrete infinitely divisible distributions which incorporates various well-known and important distributions. The establishment of the strict stationarity and ergodicity of these processes is addressed by presenting the construction of a solution. A necessary and sufficient condition on the model coefficients for weak stationarity is also stated with some particular cases displayed. This general class of processes allows the easy and straightforward introduction of new models as well as the inclusion of recent contributions. In fact, a general procedure to obtain this kind of models is developed showing the main nature of the processes that are solution of the model equations, namely the fact that they may be expressed, conditioned by the past, as a random sum of random variables. This study resulted in the publication of the paper "*Infinitely divisible distributions in integer-valued GARCH models*" [34]. Moreover, an overall estimation procedure based on the conditional least squares and on the moments estimation method is analysed in this thesis.

Among the new models that can be presented and because of its practical potential, we underline in the paper "*A new approach to integer-valued times series modeling: The Neyman Type-A INGARCH model*" [33] the study of the particular Neyman Type-A INGARCH model.

Thereafter, it seemed to be natural to consider zero-inflated conditional distributions in the previous model in order to take also into account in the same framework the characteristic of zero inflation. This generalization of the CP-INGARCH model was analysed recently in "*Zero-inflated compound Poisson distributions in integer-valued GARCH models*" [35]. The previous study plays an important role mainly to state strict stationarity. In addition, the autocorrelation function, expressions for moments and cumulants and a condition ensuring the finiteness of the moments of the process are treated.

Although some of the particular models of the ZICP-INGARCH process have already probabilistic developments, our study, in many cases, enlarges or completes them. For instance, in Example 4.2 we computed the first four cumulants, the skewness and the kurtosis of the GP-INARCH(1) model. As we mentioned, Zhu [82] only presented the mean and the variance claiming the difficulty for deriving higher-order cumulants with the techniques adopted. For the ZIP-INGARCH model defined by Zhu [84] we provide an important contribution on the stationarity. In fact, from Theorem 5.3 the strict stationarity is analysed displaying a solution whereas in the literature, to the best of our knowledge, only first and second-order stationarity conditions were established ([84]) and the strict stationarity for $p = q = 1$ ([52]), by using different techniques than ours. Another example is the NB-DINARCH(p) model. Xu et al. [80] presented a sufficient condition for the strictly stationarity and ergodicity of the process but a solution was not shown. We answered it in Theorem 3.11. New results for this model, for $p = 1$, are presented as the existence of all moments (Theorem 3.4) and expressions for moments and cumulants (Theorem 4.2 and Corollary 4.2). For the NB-INGARCH model, Zhu [81] proposed a necessary and sufficient condition for the weak stationarity analogous to the necessary condition for the weak stationarity of our model presented in Appendix B. Nevertheless, we stress that the study developed to discuss the weak stationarity of our general model follows a new vectorial approach.

Other probabilistic studies may be considered in future as, for instance, those related to the Taylor property ([32]), the behavior of the tails or the analysis of the strict stationarity of this model when φ_t is a random function which is still an open question that deserves further development. The estimation of the distribution of a ZICP-INGARCH process is also a subject of future interest. To evaluate it by bounding as for real heteroscedastic processes, ([61]), could be a first approach. In Section 4.2, we presented a preliminary study on the estimation of the model parameters. In the simulation study presented, the empirical results lead us to think that the proposed estimators of α and b are asymptotically uncorrelated, an assertion which requires theoretical developments as well as the properties of the estimator b. It seems also interesting to develop alternative parameters estimate methods, for example, using a Poisson quasi-maximum likelihood estimator ([3]).

This work can be viewed as a starting point for new extensions of the INGARCH model. One of them will be to consider instead of a conditional compound Poisson distribution other particular compound distributions such as the compound negative binomial or the compound binomial. Using the same methodology and slightly heavier calculations, the study presented can also be made when a discrete set of points present probability inflation. Different specifications for the conditional mean can also be considered. More specifically we can assume, for example, a two-regime structure of the conditional mean process according to the magnitude of the lagged observations (similar to Wang et al. [75] who proposed the self-excited threshold integer-valued Poisson autoregressive model); the coefficients α's and β's of the evolution of the conditional mean being periodic in t, with a certain period S (similar to Bentarzi and Bentarzi [9] who investigated the periodic INGARCH model); or the conditional mean of the form $\lambda_t = f(\lambda_{t-1}, X_{t-1})$, with the function f defined on $[0,\infty[\times\mathbb{N}_0$ (as in [52]).

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Appendix A

A.1 Matrices B_k of the autoregressive equation (3.5)

Let us consider $v = 1 + v_1$. The general form of these matrices is, for $k = 1, \ldots, p$,

$$B_k = \begin{bmatrix} B_{1,1}^{(k)} & B_{1,2}^{(k)} \\ B_{2,1}^{(k)} & B_{2,2}^{(k)} \end{bmatrix},$$

where $B_{1,1}^{(k)}$ is the squared matrix of order p,

$$B_{1,1}^{(1)} = \begin{bmatrix} v\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 & 2v\alpha_2(\alpha_1+\beta_1) & \cdots & 2v\alpha_{p-1}(\alpha_1+\beta_1) & 2v\alpha_p(\alpha_1+\beta_1) \\ \alpha_1 + \frac{\beta_1}{v} & \alpha_2 & \cdots & \alpha_{p-1} & \alpha_p \\ 0 & \alpha_1+\beta_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_1+\beta_1 & 0 \end{bmatrix},$$

$$B_{1,1}^{(2)} = \begin{bmatrix} v\alpha_2^2 + 2\alpha_2\beta_2 + \beta_2^2 & 2v\alpha_3(\alpha_2+\beta_2) & \cdots & 2v\alpha_{p-1}(\alpha_2+\beta_2) & 2v\alpha_p(\alpha_2+\beta_2) & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha_2 + \frac{\beta_2}{v} & \alpha_3 & \cdots & \alpha_{p-1} & \alpha_p & 0 \\ 0 & \alpha_2+\beta_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_2+\beta_2 & 0 & 0 \end{bmatrix},$$

...

$$B_{1,1}^{(p)} = \begin{bmatrix} v\alpha_p^2 + 2\alpha_p\beta_p + \beta_p^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$B_{1,2}^{(k)}$ is the $p \times (p-1)$ matrix,

$$B_{1,2}^{(1)} = \begin{bmatrix} 2v\beta_2(\alpha_1+\beta_1) & \cdots & 2v\beta_{p-1}(\alpha_1+\beta_1) & 2v\beta_p(\alpha_1+\beta_1) \\ \beta_2 & \cdots & \beta_{p-1} & \beta_p \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$B_{1,2}^{(2)} = \begin{bmatrix} 2v\beta_3(\alpha_2+\beta_2) & \cdots & 2v\beta_{p-1}(\alpha_2+\beta_2) & 2v\beta_p(\alpha_2+\beta_2) & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ \beta_3 & \cdots & \beta_{p-1} & \beta_p & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \cdots, \; B_{1,2}^{(p)} = \mathbf{0},$$

$B_{2,1}^{(k)}$ is the $(p-1)\times p$ matrix,

$$B_{2,1}^{(1)} = \begin{bmatrix} \frac{\alpha_1+\beta_1}{v} & \alpha_2 & \cdots & \alpha_{p-1} & \alpha_p \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \; B_{2,1}^{(2)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{\alpha_2+\beta_2}{v} & \alpha_3 & \cdots & \alpha_{p-1} & \alpha_p & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \cdots, \; B_{2,1}^{(p)} = \mathbf{0},$$

and $B_{2,2}^{(k)}$ is the squared matrix of order $p-1$,

$$B_{2,2}^{(1)} = \begin{bmatrix} \beta_2 & \cdots & \beta_{p-1} & \beta_p \\ \alpha_1+\beta_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \alpha_1+\beta_1 & 0 \end{bmatrix}, \; B_{2,2}^{(2)} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ \beta_3 & \cdots & \beta_{p-1} & \beta_p & 0 \\ \alpha_2+\beta_2 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \alpha_2+\beta_2 & 0 & 0 \end{bmatrix}, \cdots, \; B_{2,2}^{(p)} = \mathbf{0}.$$

A.2 Matrices A_k of the autoregressive equation (5.10)

Let us consider $v = 1+v_1$ and $\widetilde{\omega} = 1-\omega$. The general form of these matrices is, for $k = 1,\ldots,p$,

$$A_k = \begin{bmatrix} ,A_{1,1}^{(k)} & A_{1,2}^{(k)} \\ A_{2,1}^{(k)} & A_{2,2}^{(k)} \end{bmatrix},$$

where $A_{1,1}^{(k)}$ is the squared matrix of order p,

$$A_{1,1}^{(1)} = \begin{bmatrix} \widetilde{\omega}v\alpha_1^2+2\widetilde{\omega}\alpha_1\beta_1+\beta_1^2 & 2v\alpha_2(\widetilde{\omega}\alpha_1+\beta_1) & \cdots & 2v\alpha_{p-1}(\widetilde{\omega}\alpha_1+\beta_1) & 2v\alpha_p(\widetilde{\omega}\alpha_1+\beta_1) \\ \widetilde{\omega}\left(\alpha_1+\frac{\beta_1}{v}\right) & \widetilde{\omega}\alpha_2 & \cdots & \widetilde{\omega}\alpha_{p-1} & \widetilde{\omega}\alpha_p \\ 0 & \widetilde{\omega}\alpha_1+\beta_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \widetilde{\omega}\alpha_1+\beta_1 & 0 \end{bmatrix},$$

$$A_{1,1}^{(2)} = \begin{bmatrix} \widetilde{\omega}v\alpha_2^2+2\widetilde{\omega}\alpha_2\beta_2+\beta_2^2 & 2v\alpha_3(\widetilde{\omega}\alpha_2+\beta_2) & \cdots & 2v\alpha_{p-1}(\widetilde{\omega}\alpha_2+\beta_2) & 2v\alpha_p(\widetilde{\omega}\alpha_2+\beta_2) & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \widetilde{\omega}\left(\alpha_2+\frac{\beta_2}{v}\right) & \widetilde{\omega}\alpha_3 & \cdots & \widetilde{\omega}\alpha_{p-1} & \widetilde{\omega}\alpha_p & 0 \\ 0 & \widetilde{\omega}\alpha_2+\beta_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \widetilde{\omega}\alpha_2+\beta_2 & 0 & 0 \end{bmatrix},$$

...

$$A_{1,1}^{(p)} = \begin{bmatrix} \widetilde{\omega}v\alpha_p^2 + 2\widetilde{\omega}\alpha_p\beta_p + \beta_p^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$A_{1,2}^{(k)}$ is the $p \times (p-1)$ matrix,

$$A_{1,2}^{(1)} = \begin{bmatrix} 2\widetilde{\omega}v\beta_2(\widetilde{\omega}\alpha_1 + \beta_1) & \cdots & 2\widetilde{\omega}v\beta_{p-1}(\widetilde{\omega}\alpha_1 + \beta_1) & 2\widetilde{\omega}v\beta_p(\widetilde{\omega}\alpha_1 + \beta_1) \\ \widetilde{\omega}^2\beta_2 & \cdots & \widetilde{\omega}^2\beta_{p-1} & \widetilde{\omega}^2\beta_p \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$A_{1,2}^{(2)} = \begin{bmatrix} 2\widetilde{\omega}v\beta_3(\widetilde{\omega}\alpha_2 + \beta_2) & \cdots & 2\widetilde{\omega}v\beta_{p-1}(\widetilde{\omega}\alpha_2 + \beta_2) & 2\widetilde{\omega}v\beta_p(\widetilde{\omega}\alpha_2 + \beta_2) & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ \widetilde{\omega}^2\beta_3 & \cdots & \widetilde{\omega}^2\beta_{p-1} & \widetilde{\omega}^2\beta_p & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \cdots, \quad A_{1,2}^{(p)} = \mathbf{0},$$

$A_{2,1}^{(k)}$ is the $(p-1) \times p$ matrix,

$$A_{2,1}^{(1)} = \begin{bmatrix} \frac{\alpha_1 + \beta_1}{\widetilde{\omega}v} & \frac{\alpha_2}{\widetilde{\omega}} & \cdots & \frac{\alpha_{p-1}}{\widetilde{\omega}} & \frac{\alpha_p}{\widetilde{\omega}} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad A_{2,1}^{(2)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{\alpha_2 + \beta_2}{\widetilde{\omega}v} & \frac{\alpha_3}{\widetilde{\omega}} & \cdots & \frac{\alpha_{p-1}}{\widetilde{\omega}} & \frac{\alpha_p}{\widetilde{\omega}} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$\cdots, \quad A_{2,1}^{(p)} = \mathbf{0},$$

and $A_{2,2}^{(k)}$ is the squared matrix of order $p-1$,

$$A_{2,2}^{(1)} = \begin{bmatrix} \beta_2 & \cdots & \beta_{p-1} & \beta_p \\ \widetilde{\omega}\alpha_1 + \beta_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \widetilde{\omega}\alpha_1 + \beta_1 & 0 \end{bmatrix}, \quad A_{2,2}^{(2)} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ \beta_3 & \cdots & \beta_{p-1} & \beta_p & 0 \\ \widetilde{\omega}\alpha_2 + \beta_2 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \widetilde{\omega}\alpha_2 + \beta_2 & 0 & 0 \end{bmatrix},$$

$$\cdots, \quad A_{2,2}^{(p)} = \mathbf{0}.$$

Appendix B

Another necessary condition for weak stationarity of CP-INGARCH processes

B.1 Some matrix results

We now turn our attention to the class of strictly diagonally dominant by rows matrices that will help us to obtain a second-order stationarity necessary condition of a CP-INGARCH(p,q) process. For more details concerning the results presented in this section see, e.g., [41] and [64].

Definition B.1 (Strictly diagonally dominant by rows matrix) *The real $n \times n$ matrix $A = (a_{ij})$ is said to be strictly diagonally dominant by rows when*

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad \textit{holds for every row index } \ i = 1, ..., n.$$

The relevance of diagonal dominance in our study is related to the problem of verifying the nonsingularity of a such matrix which is a consequence of the Levy-Desplanques Theorem.

Theorem B.1 (Levy-Desplanques) *A strictly diagonally dominant by rows matrix A is nonsingular.*

Proof: Suppose $\det(A) = 0$. Then the system $Ax = 0$ has a nontrivial solution, say $x = (x_1, ..., x_n)$.

Let r be the integer for which $|x_r| \geq |x_i|$, $i = 1, ..., n$. Then

$$|a_{rr}||x_r| = \left| -\sum_{j=1, j\neq r}^{n} a_{rj}x_j \right| \leq \sum_{j=1, j\neq r}^{n} |a_{rj}||x_j| \leq |x_r| \sum_{j=1, j\neq i}^{n} |a_{ij}|,$$

which contradicts the hypothesis of strictly diagonally dominance by rows. ■

Another fundamental question to establish the required necessary condition is to prove the positivity of a certain constant. To that end we need the following notion of M-matrix.

Definition B.2 (M-matrix) *A nonsingular real $n \times n$ matrix $A = (a_{ij})$ is said to be an M-matrix if $a_{ij} \leq 0$ for $i \neq j$ and if all the entries of its inverse are nonnegative.*

M-matrices are related to strictly diagonally dominant matrices by the following result.

Theorem B.2 *A $n\times n$ matrix $A=(a_{ij})$ that is strictly diagonally dominant by rows and whose entries satisfy the relations $a_{ij}\leq 0$ for $i\neq j$ and $a_{ii}>0$, is an M-matrix.*

Proof: We only need to prove that all the entries of A^{-1} are nonnegative.

Let $B=(b_{ij})$ be defined by $B=I_n-D^{-1}A$ where $D=diag(a_{11},...,a_{nn})$ and I_n is the identity matrix of order n. Note that $b_{ii}=0$ for each $i=1,...,n$, and $b_{ij}=-a_{ij}/a_{ii}\geq 0$ for $i\neq j$. Also, the fact that A is strictly diagonally dominant by rows implies that

$$|a_{ii}|>\sum_{j=1,j\neq i}^{n}\left|a_{ij}\right| \Leftrightarrow 1>\sum_{j=1}^{n}|b_{ij}|,\quad i=1,...,n.$$

It follows immediately from the Gershgorin Theorem ([64, p. 184]) that all the eigenvalues of B are less than 1. So, it implies that I_n-B is nonsingular and $(I_n-B)^{-1}=\sum_{k=0}^{\infty}B^k$. Since all the entries of B are nonnegative it is clear that the same holds for $(I_n-B)^{-1}$. As the elements of I_n-B of the main diagonal are nonpositive then $I_n-B=D^{-1}A$ is an M-matrix. Consequently, A is also an M-matrix. ■

B.2 The necessary condition

The necessary condition presented in this section has resulted from an initial study on the weak stationarity of the CP-INGARCH processes. In fact, this condition intended to generalize an existing result in the literature established by Zhu [81] for a NB-INARCH(p) model. We note that this result is not only an extension for the class of conditional distributions, but it is also a generalization in terms of orders since Zhu stated the condition only for $q=1$, $\beta_1=0$ and any $p\in\mathbb{N}$.

Let us now develop a necessary condition of weak stationarity for a general CP-INGARCH(p,q) model, using arguments similar to that of Zhu [81]. In that sense, we consider $B=(b_{ij})$ the squared matrix of order $p+q-2$ whose terms are, for $i=1,...,p-1$, given by

$$b_{ij}=\begin{cases}\sum\limits_{|k-i|=j}\alpha_k+\beta_{i-j}, & 1\leq j\leq i-1\\ \alpha_{2i}-1, & j=i\\ \sum\limits_{|k-i|=j}\alpha_k, & i+1\leq j\leq p-1\\ \beta_{i+j}, & p\leq j\leq p+q-i-1\\ 0, & \text{otherwise,}\end{cases}$$

and for $i=p,...,p+q-2$, given by

$$b_{ij} = \begin{cases} \alpha_{j+i-p+1}, & 1 \le j \le p-1 \\ \sum\limits_{|k-i|=j} \beta_k + \alpha_{i-j}, & p \le j \le i-1 \\ \beta_{2(i-p+1)} - 1, & j = i \\ \sum\limits_{|k-i|=j} \beta_k, & i+1 \le j \le p+q-2 \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha_i = 0$ for $i > p$ and $\beta_j = 0$ for $j > q$. If B has an inverse, B^{-1}, we denote its elements by d_{ij}. Consider also the vector $b = (b_{i0})$ with components

$$b_{i0} = \begin{cases} \alpha_i + \dfrac{\beta_i}{1+v_1}, & i = 1, ..., p-1, \\ \dfrac{\alpha_{i-p+1} + \beta_{i-p+1}}{1+v_1}, & i = p, ..., p+q-2. \end{cases}$$

Theorem B.3 *Let X be a process following a CP-INGARCH(p,q) model satisfying* **H2** *and such that $\alpha_0(1+v_1) > v_0$. If the process is weakly stationary then $C_1 + ... + C_r < 1$, with $r = \max(p,q)$ and where for $v = 1, ..., r-1$, the coefficients are given by*

$$C_v = (1+v_1)\left[\alpha_v^2 + \frac{2\alpha_v\beta_v + \beta_v^2}{1+v_1} - 2\sum_{\substack{(i,j)\in\{1,...,p\}\times\{1,...,q\}:\\ j-i=v}} (\alpha_i+\beta_i)\sum_{u=1}^{p+q-2}(\alpha_j d_{vu} + \beta_j d_{v+r-1,u})\, b_{u0}\right],$$

$$C_r = (1+v_1)\alpha_r^2 + 2\alpha_r\beta_r + \beta_r^2.$$

Proof: Let us start by recalling the existence of the CP-INARCH(∞) representation which results from the assumption of first-order stationarity. From this representation and using the hypothesis that X is a second-order stationary process we conclude the second-order stationarity of λ.

In what follows we use the notation $\gamma_k = E(X_t X_{t-k})$ and $\widetilde{\gamma}_k = E(\lambda_t \lambda_{t-k})$, with $k \in \mathbb{Z}$, and once again we restrict ourselves to the case $p = q$ since the other cases can be obtained from this one setting additional parameters to 0. Let us take into account the expressions obtained for γ_0, γ_k and $\widetilde{\gamma}_k$ in Proposition 3.1. From (3.8) we have

$$\begin{aligned} \gamma_0 &= C + (1+v_1)\left[\sum_{j=1}^{p}\left(\alpha_j^2 + \frac{2\alpha_j\beta_j + \beta_j^2}{1+v_1}\right)\gamma_0 + 2\sum_{j=1}^{p-1}\sum_{k=j+1}^{p}\alpha_k(\alpha_j+\beta_j)\gamma_{k-j}\right. \\ &\left. \quad +2\sum_{j=1}^{p-1}\sum_{k=j+1}^{p}\beta_k(\alpha_j+\beta_j)\widetilde{\gamma}_{k-j}\right] \end{aligned}$$

$$= \quad C+(1+v_1)\left[\sum_{j=1}^{p}\left(\alpha_j^2+\frac{2\alpha_j\beta_j+\beta_j^2}{1+v_1}\right)\gamma_0+2\sum_{v=1}^{p-1}\sum_{k-j=v}(\alpha_j+\beta_j)(\alpha_k\gamma_v+\beta_k\widetilde{\gamma}_v)\right] \qquad \text{(B.1)}$$

with $C=v_0\mu\left[1-\sum_{j=1}^{p}(2\alpha_j\beta_j+\beta_j^2)\right]+(1+v_1)\left[2\alpha_0\mu-\alpha_0^2\right]>0$ and independent of t.

From (3.9) it follows that, for $k=1,...,p-1$,

$$\gamma_k=\left(\alpha_0-\frac{v_0\beta_k}{1+v_1}\right)\mu+\left(\alpha_k+\frac{\beta_k}{1+v_1}\right)\gamma_0+\sum_{j=k+1}^{p}\beta_j\widetilde{\gamma}_{j-k}+\sum_{j=1}^{k-1}(\alpha_j+\beta_j)\gamma_{k-j}+\sum_{j=k+1}^{p}\alpha_j\gamma_{j-k}$$

$$\Leftrightarrow\ \gamma_k-\sum_{|j-k|=1}\alpha_j\gamma_1-...-\sum_{|j-k|=k}\alpha_j\gamma_k-...-\sum_{|j-k|=p-1}\alpha_j\gamma_{p-1}-\sum_{k-j=1}\beta_j\gamma_1-...-\sum_{k-j=k-1}\beta_j\gamma_{k-1}$$

$$-\sum_{j-k=1}\beta_j\widetilde{\gamma}_1-...-\sum_{j-k=p-k}\beta_j\widetilde{\gamma}_{p-k}=\left(\alpha_0-\frac{v_0\beta_k}{1+v_1}\right)\mu+\left(\alpha_k+\frac{\beta_k}{1+v_1}\right)\gamma_0,$$

or equivalently,

$$\sum_{u=1}^{p-1}b_{ku}\gamma_u+\sum_{u=1}^{p-k}b_{k,u+p-1}\widetilde{\gamma}_u=-\left[\left(\alpha_0-\frac{v_0\beta_k}{1+v_1}\right)\mu+\left(\alpha_k+\frac{\beta_k}{1+v_1}\right)\gamma_0\right] \qquad \text{(B.2)}$$

with

$$b_{ku}=\begin{cases}\sum_{|j-k|=u}\alpha_j+\beta_{k-u}, & 1\le u\le k-1\\ \alpha_{2k}-1, & u=k\\ \sum_{|j-k|=u}\alpha_j, & k+1\le u\le p-1,\end{cases}$$

and $b_{k,u+p-1}=\beta_{u+k}$, for $u=1,...,p-k$, where we consider $\alpha_j=\beta_j=0$, $j>p$.

Similarly we get from (3.10), for $k=1,...,p-1$,

$$\widetilde{\gamma}_k=\left(\alpha_0-\frac{v_0(\alpha_k+\beta_k)}{1+v_1}\right)\mu+\frac{\alpha_k+\beta_k}{1+v_1}\gamma_0+\sum_{j=k+1}^{p}\alpha_j\gamma_{j-k}+\sum_{j=1}^{k-1}(\alpha_j+\beta_j)\widetilde{\gamma}_{k-j}+\sum_{j=k+1}^{p}\beta_j\widetilde{\gamma}_{j-k}$$

$$\Leftrightarrow\ (1-\beta_{2k})\widetilde{\gamma}_k-\sum_{|j-k|=1}\beta_j\widetilde{\gamma}_1-...-\sum_{|j-k|=p-1}\beta_j\widetilde{\gamma}_{p-1}-\sum_{k-j=1}\alpha_j\widetilde{\gamma}_1-...-\sum_{k-j=k-1}\alpha_j\widetilde{\gamma}_{k-1}$$

$$-\sum_{j-k=1}\alpha_j\gamma_1-...-\sum_{j-k=p-k}\alpha_j\gamma_{p-k}=\left(\alpha_0-\frac{v_0(\alpha_k+\beta_k)}{1+v_1}\right)\mu+\frac{\alpha_k+\beta_k}{1+v_1}\gamma_0,$$

or equivalently,

$$\sum_{u=1}^{p-k}b_{k+p-1,u}\gamma_u+\sum_{u=1}^{p-1}b_{k+p-1,u+p-1}\widetilde{\gamma}_u=-\left[\left(\alpha_0-\frac{v_0(\alpha_k+\beta_k)}{1+v_1}\right)\mu+\frac{\alpha_k+\beta_k}{1+v_1}\gamma_0\right] \qquad \text{(B.3)}$$

with

$$b_{k+p-1,u+p-1} = \begin{cases} \sum_{|j-k|=u} \beta_j + \alpha_{k-u}, & 1 \leq u \leq k-1 \\ \\ \beta_{2k} - 1, & u = k \\ \\ \sum_{|j-k|=u} \beta_j, & k+1 \leq u \leq p-1, \end{cases}$$

and $b_{k+p-1,u} = \alpha_{u+k}$, for $u = 1,...,p-k$.

Let $B = (b_{ij})_{i,j=1}^{2p-2}$ and $B^{-1} = (d_{ij})_{i,j=1}^{2p-2}$ its inverse, whose existence is assured by the second-order stationarity. Indeed, as X is second-order stationary it is also first-order stationary and then, from Theorem 3.1, we have

$$\sum_{l=1}^{p} (\alpha_l + \beta_l) < 1 \quad \Leftrightarrow \quad \alpha_{2i} + \beta_{2i} + \sum_{|l-i| \neq i} (\alpha_l + \beta_l) < 1$$

$$\Rightarrow \begin{cases} |\alpha_{2i} - 1| > \beta_{2i} + \sum_{|l-i| \neq i} (\alpha_l + \beta_l) - (\alpha_i + \beta_i), & \text{if } i = 1,...,p-1 \\ \\ |\beta_{2i} - 1| > \alpha_{2i} + \sum_{|l-i| \neq i} (\alpha_l + \beta_l) - (\alpha_i + \beta_i), & \text{if } i = p,...,2p-2 \end{cases}$$

which proves that B is a strictly diagonally dominant by rows matrix and therefore invertible. Thus, from expressions $(B.2)$ and $(B.3)$ and using the invertibility of B we obtain

$$\widehat{\gamma} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_{p-1} \\ \widetilde{\gamma}_1 \\ \vdots \\ \widetilde{\gamma}_{p-1} \end{bmatrix} = -B^{-1} \left(\begin{bmatrix} \alpha_0\mu \\ \vdots \\ \alpha_0\mu \\ \alpha_0\mu \\ \vdots \\ \alpha_0\mu \end{bmatrix} + v_0\mu \begin{bmatrix} \alpha_1 - b_{10} \\ \vdots \\ \alpha_{p-1} - b_{p-1,0} \\ -b_{p,0} \\ \vdots \\ -b_{2p-2,0} \end{bmatrix} + \gamma_0 b \right),$$

where b is the vector previously introduced. So, for $l = 1,...,2p-2$,

$$\widehat{\gamma}_l = -\alpha_0\mu \sum_{u=1}^{2p-2} d_{lu} + v_0\mu \left[\sum_{u=1}^{p-1} (b_{u0} - \alpha_u) d_{lu} + \sum_{u=p}^{2p-2} b_{u0} d_{lu} \right] - \sum_{u=1}^{2p-2} d_{lu} b_{u0} \gamma_0.$$

Taking the last part of $(B.1)$ and using the previous expression, we get

$$\begin{aligned} 2\sum_{v=1}^{p-1} \sum_{j-i=v} (\alpha_i + \beta_i)(\alpha_j \gamma_v + \beta_j \widetilde{\gamma}_v) &= \widehat{C} - 2\sum_{v=1}^{p-1} \sum_{j-i=v} (\alpha_i + \beta_i) \left[\alpha_j \sum_{u=1}^{2p-2} d_{vu} b_{u0} \right. \\ &\quad \left. + \beta_j \sum_{u=1}^{2p-2} d_{v+p-1,u} b_{u0} \right] \gamma_0, \end{aligned}$$

where the constant $\widehat{C}$ given by

$$\begin{aligned}
\widehat{C} &= -2\alpha_0\mu\sum_{v=1}^{p-1}\sum_{j-i=v}(\alpha_i+\beta_i)\left[\alpha_j\sum_{u=1}^{2p-2}d_{vu}+\beta_j\sum_{u=1}^{2p-2}d_{v+p-1,u}\right]\\
&\quad +2\,\frac{v_0\mu}{1+v_1}\sum_{v=1}^{p-1}\sum_{j-i=v}(\alpha_i+\beta_i)\left[\alpha_j\sum_{u=1}^{p-1}\beta_u d_{vu}+\alpha_j\sum_{u=p}^{2p-2}(\alpha_{u-p+1}+\beta_{u-p+1})d_{vu}\right.\\
&\quad \left.+\beta_j\sum_{u=1}^{p-1}\beta_u d_{v+p-1,u}+\beta_j\sum_{u=p}^{2p-2}(\alpha_{u-p+1}+\beta_{u-p+1})d_{v+p-1,u}\right]
\end{aligned}$$

is positive and independent of t. We notice that the positivity of $\widehat{C}$ follows from the assumption that $\alpha_0(1+v_1)>v_0$, and once again, from the strictly diagonally dominance by rows of B. In fact, since B is strictly diagonally dominant by rows the same happens to $-B$. As $-b_{lu}\leq 0$, for $u\neq l$, and $-b_{ll}>0$ we conclude that $-B$ is an M-matrix so, all the entries of $(-B)^{-1}$ are nonnegative and thus $d_{ij}\leq 0$.

Then replacing this expression in $(B.1)$, we finally get

$$\begin{aligned}
\gamma_0 &= C_0+(1+v_1)\left[\sum_{i=1}^{p}\left(\alpha_i^2+\frac{2\alpha_i\beta_i+\beta_i^2}{1+a_1}\right)\gamma_0\right.\\
&\quad \left.-2\sum_{v=1}^{p-1}\sum_{j-i=v}(\alpha_i+\beta_i)\left(\alpha_j\sum_{u=1}^{2p-2}d_{vu}b_{u0}+\beta_j\sum_{u=1}^{2p-2}d_{v+p-1,u}b_{u0}\right)\gamma_0\right]\\
&= C_0+(1+v_1)\left[\left(\alpha_p^2+\frac{2\alpha_p\beta_p+\beta_p^2}{1+v_1}\right)\gamma_0+\sum_{v=1}^{p-1}\left\{\left(\alpha_v^2+\frac{2\alpha_v\beta_v+\beta_v^2}{1+v_1}\right)\right.\right.\\
&\quad \left.\left.-2\sum_{j-i=v}(\alpha_i+\beta_i)\sum_{u=1}^{2p-2}(\alpha_j d_{vu}+\beta_j d_{v+p-1,u})\,b_{u0}\right\}\gamma_0\right],
\end{aligned}$$

or equivalently

$$\gamma_0=C_0+\sum_{v=1}^{p}C_v\gamma_0 \;\Leftrightarrow\; \left(1-\sum_{v=1}^{p}C_v\right)\gamma_0=C_0, \tag{B.4}$$

where $C_0=C+(1+v_1)\widehat{C}>0$ and C_v are the coefficients defined in the statement of the theorem.

Hence, the equality $(B.4)$ implies $1-\sum_{v=1}^{p}C_v>0$. ■

Let us point out that when X follows a CP-INARCH(p) model, we easily obtain, in the proof of Theorem $B.3$, the constant $\widehat{C}=-2\alpha_0\mu\sum_{v=1}^{p-1}\sum_{j-i=v}\alpha_i\alpha_j\sum_{u=1}^{p-1}d_{vu}$. Therefore, in this case, we do not need to ensure that $\alpha_0(1+v_1)>v_0$ to assure the positivity of this constant since the fact that $d_{ij}\leq 0$ is sufficient. Accordingly, the previous theorem assumes the following form:

Corollary B.1 *Let X be a first-order stationary process following a CP-INARCH(p) model that satisfies* **H2**. *If the process is weakly stationary, then $C_1+...+C_p<1$, where for $u,l=1,...,p-1$,*

$$C_u=(1+v_1)\left[\alpha_u^2-\sum_{v=1}^{p-1}\sum_{|i-j|=v}\alpha_i\alpha_j d_{vu}b_{u0}\right],\;\; C_p=(1+v_1)\,\alpha_p^2,$$

$$b_{l0}=\alpha_l,\quad b_{ll}=\sum_{|i-l|=l}\alpha_i-1\quad \textit{and for}\;\; u\neq l,\; b_{lu}=\sum_{|i-l|=u}\alpha_i,$$

with $B = (b_{ij})$ and $B^{-1} = (d_{ij})$ squared matrices of order $p-1$.

Remark B.1 *We note that since all the entries of the matrix B^{-1} are nonpositive, from the previous proof, then the coefficients $C_v > 0$ for all $v = 1,...,r$, and the necessary condition is equivalent to say that the roots of the equation $1 - C_1 z - ... - C_r z^r = 0$ are outside the unit circle.*

The following examples illustrate the necessary condition stated in Theorem $B.3$.

Example B.1 *Let us consider a CP-INGARCH$(2,2)$ model satisfying the hypothesis **H2** and such that $\alpha_1+\alpha_2+\beta_1+\beta_2 < 1$. In order to obtain the necessary condition of Theorem $B.3$, let us consider $\alpha_0(1+v_1) > v_0$ and the coefficients*

$$\begin{aligned} C_1 &= (\alpha_1+\beta_1)^2 + v_1\alpha_1^2 - 2(1+v_1)\left[(\alpha_1+\beta_1)\sum_{u=1}^{2}(\alpha_2 d_{1u}+\beta_2 d_{2u})b_{u0}\right] \\ &= (\alpha_1+\beta_1)^2 + v_1\alpha_1^2 + 2(1+v_1)(\alpha_1+\beta_1)\frac{\alpha_2 b_{10}+\beta_2 b_{20}}{1-\alpha_2-\beta_2} \\ &= (\alpha_1+\beta_1)^2\left[\frac{1+\alpha_2+\beta_2}{1-\alpha_2-\beta_2}\right] + v_1\alpha_1\left[\frac{\alpha_1(1+\alpha_2-\beta_2)+2\alpha_2\beta_1}{1-\alpha_2-\beta_2}\right], \\ C_2 &= (\alpha_2+\beta_2)^2 + v_1\alpha_2^2, \end{aligned}$$

since the matrices B and B^{-1} and the vector b are respectively given by

$$B = \begin{bmatrix} \alpha_2-1 & \beta_2 \\ \alpha_2 & \beta_2-1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} \frac{\beta_2-1}{1-\alpha_2-\beta_2} & \frac{-\beta_2}{1-\alpha_2-\beta_2} \\ \frac{-\alpha_2}{1-\alpha_2-\beta_2} & \frac{\alpha_2-1}{1-\alpha_2-\beta_2} \end{bmatrix}, \quad b = \begin{bmatrix} \alpha_1+\frac{\beta_1}{1+v_1} \\ \frac{\alpha_1+\beta_1}{1+v_1} \end{bmatrix}.$$

Hence, the necessary condition of weak stationarity is

$$\begin{aligned} C_1+C_2<1 \quad\Leftrightarrow\quad & (\alpha_1+\beta_1)^2(1+\alpha_2+\beta_2) + (\alpha_2+\beta_2)\left[1+\alpha_2+\beta_2-(\alpha_2+\beta_2)^2\right] \\ & + v_1\left(\alpha_1^2[1+\alpha_2-\beta_2]+\alpha_2^2[1-\alpha_2-\beta_2]+2\alpha_1\alpha_2\beta_1\right) < 1. \end{aligned} \tag{B.5}$$

Taking $\alpha_2 = \beta_2 = 0$, the condition $(B.5)$ reduces to $(\alpha_1+\alpha_2)^2 + v_1\alpha_1^2 < 1$. So, this necessary condition of weak stationarity of the CP-INGARCH$(1,1)$ model equals the condition stated in Example 3.1. The same happens in the CP-INGARCH$(2,2)$ where $\alpha_1 = \beta_1 = 0$. In fact, from $(B.5)$ we get

$$(\alpha_2+\beta_2)\left[1+\alpha_2+\beta_2-(\alpha_2+\beta_2)^2\right] + v_1\alpha_2^2(1-\alpha_2-\beta_2) < 1 \quad\Leftrightarrow\quad (\alpha_2+\beta_2)^2 + v_1\alpha_2^2 < 1.$$

Example B.2 *Let us now consider a CP-INARCH(3) model satisfying the hypothesis **H2** and such that $\alpha_1+\alpha_2+\alpha_3 < 1$. To obtain the necessary condition we analyze the coefficients*

$$\begin{aligned} C_1 &= (1+v_1)\left[\alpha_1^2 - \sum_{|i-j|=1}\alpha_i\alpha_j d_{11}b_{10} - \sum_{|i-j|=2}\alpha_i\alpha_j d_{21}b_{10}\right] \\ &= (1+v_1)[\alpha_1^2 - 2(\alpha_1\alpha_2 d_{11}+\alpha_2\alpha_3 d_{11}+\alpha_1\alpha_3 d_{21})b_{10}] \\ &= (1+v_1)\left[\alpha_1^2 + 2\,\frac{\alpha_1^2\alpha_2+\alpha_1\alpha_2\alpha_3+\alpha_1^2\alpha_3^2+\alpha_1^3\alpha_3}{1-\alpha_2-\alpha_1\alpha_3-\alpha_3^2}\right], \end{aligned}$$

$$\begin{aligned}
C_2 &= (1+v_1)\left[\alpha_2^2 - \sum_{|i-j|=1}\alpha_i\alpha_j d_{12}b_{20} - \sum_{|i-j|=2}\alpha_i\alpha_j d_{22}b_{20}\right] \\
&= (1+v_1)[\alpha_2^2 - 2(\alpha_1\alpha_2 d_{12} + \alpha_2\alpha_3 d_{12} + \alpha_1\alpha_3 d_{22})b_{20}] \\
&= (1+v_1)\left[\alpha_2^2 + 2\,\frac{\alpha_2^2\alpha_3^2 + \alpha_1\alpha_2\alpha_3}{1-\alpha_2-\alpha_1\alpha_3-\alpha_3^2}\right], \\
C_3 &= (1+v_1)\alpha_3^2,
\end{aligned}$$

as the matrices B and B^{-1} and the vector b are respectively given by

$$B = \begin{bmatrix} \alpha_2 - 1 & \alpha_3 \\ \alpha_1 + \alpha_3 & -1 \end{bmatrix},\ B^{-1} = \begin{bmatrix} \frac{-1}{1-\alpha_2-\alpha_1\alpha_3-\alpha_3^2} & \frac{-\alpha_3}{1-\alpha_2-\alpha_1\alpha_3-\alpha_3^2} \\ \frac{-(\alpha_1+\alpha_3)}{1-\alpha_2-\alpha_1\alpha_3-\alpha_3^2} & \frac{\alpha_2-1}{1-\alpha_2-\alpha_1\alpha_3-\alpha_3^2} \end{bmatrix},\ b = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

Finally, the necessary condition of weak stationarity is

$$\begin{aligned}
C_1+C_2+C_3<1 \quad\Leftrightarrow\quad & (1+v_1)\left[(\alpha_1^2+\alpha_2^2+\alpha_3^2)(1-\alpha_2-\alpha_1\alpha_3-\alpha_3^2)+2\alpha_1^2\alpha_2+2\alpha_1\alpha_2\alpha_3\right. \\
&\left.+2\alpha_1^2\alpha_3^2+2\alpha_1^3\alpha_3+2\alpha_2^2\alpha_3^2+2\alpha_1\alpha_2\alpha_3\right] < 1-\alpha_2-\alpha_1\alpha_3-\alpha_3^2,
\end{aligned}$$

We finish this Appendix saying that we are strongly convinced that the necessary condition of weakly stationarity stated in Theorem $B.3$ is also a sufficient one and consequently it is equivalent to the condition of Theorem 3.2, as we observe in the cases developed in example $B.1$.

Appendix C

Auxiliary Results

C.1 Lemma 3.1

In Lemma 3.1 we state that for $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$, the m-th derivative of the function $\varphi_t^k = \prod_{j=1}^k \varphi_t$ is given by

$$\left(\varphi_t^k\right)^{(m)}(u) = \sum_{n=\max\{0,m-k\}}^{m-1} \frac{k!}{(k-m+n)!}\, \varphi_t^{k-m+n}(u) \times$$

$$\sum_{\substack{k_1+\ldots+k_m=m-n \\ k_1+2k_2+\ldots+mk_m=m \\ k_r \in \mathbb{N}_0}} (m;k_1,\ldots,k_m) \left[\varphi_t'(u)\right]^{k_1} \ldots [\varphi_t^{(m)}(u)]^{k_m}, \qquad u \in \mathbb{R},$$

where

$$(m;k_1,\ldots,k_m) = \frac{m!}{(1!)^{k_1} k_1!(2!)^{k_2} k_2! \ldots (m!)^{k_m} k_m!}.$$

Let us prove this result by induction with respect to m. Without loss of generality, let us consider $m \geq j$. For $m = 1$ the result is valid since $\left(\varphi_t^k\right)'(u) = k\varphi_t^{k-1}(u)\varphi_t'(u)$, $u \in \mathbb{R}$. Now, let us assume that the formula is also valid for an arbitrarily fixed value of m and let us prove that it holds for $m+1$.

We have

$$\left(\varphi_t^k\right)^{(m+1)}(u) = \frac{d}{du}\left(\sum_{n=m-k}^{m-1} \frac{k!\; \varphi_t^{k-m+n}(u)}{(k-m+n)!} \sum_{\substack{k_1+\ldots+k_m=m-n \\ k_1+\ldots+mk_m=m}} (m;k_1,\ldots,k_m) \left[\varphi_t'(u)\right]^{k_1} \ldots [\varphi_t^{(m)}(u)]^{k_m} \right)$$

$$= \sum_{n=m-k+1}^{m-1} \frac{k!(j-m+n)}{(k-m+n)!}\, \varphi_t^{k-m+n-1}(u) \sum_{\substack{k_1+\ldots+k_m=m-n \\ k_1+\ldots+mk_m=m}} (m;k_1,\ldots,k_m) \left[\varphi_t'(u)\right]^{k_1+1} \left[\varphi_t''(u)\right]^{k_2} \ldots [\varphi_t^{(m)}(u)]^{k_m}$$

$$+ \sum_{n=m-k}^{m-1} \frac{k!\; \varphi_t^{k-m+n}(u)}{(k-m+n)!} \left(\sum_{\substack{k_1+\ldots+k_m=m-n \\ k_1+\ldots+mk_m=m}} k_1(m;k_1,\ldots,k_m) \left[\varphi_t'(u)\right]^{k_1-1} \left[\varphi_t''(u)\right]^{k_2+1} \ldots [\varphi_t^{(m)}(u)]^{k_m} \right.$$

$$+ \sum_{\substack{k_1+\ldots+k_m=m-n \\ k_1+\ldots+mk_m=m}} k_2(m;k_1,\ldots,k_m) \left[\varphi_t'(u)\right]^{k_1} \left[\varphi_t''(u)\right]^{k_2-1} \left[\varphi_t'''(u)\right]^{k_3+1} [\varphi_t^{(iv)}(u)]^{k_4} \ldots [\varphi_t^{(m)}(u)]^{k_m}$$

$$+ \dots + \sum_{\substack{k_1+\dots+k_m=m-n \\ k_1+\dots+mk_m=m}} k_m(m;k_1,\dots,k_m)\left[\varphi_t'(u)\right]^{k_1}\dots[\varphi_t^{(m)}(u)]^{k_m-1}\varphi_t^{(m+1)}(u)\Bigg)$$

$$= \sum_{n=m-k+1}^{m-1} \frac{k!\ \varphi_t^{k-m+n-1}(u)}{(k-m+n-1)!}\Bigg[\sum_{\substack{k_1+\dots+k_m=m-n \\ k_1+\dots+mk_m=m}} (m;k_1,\dots,k_m)\left[\varphi_t'(u)\right]^{k_1+1}\left[\varphi_t''(u)\right]^{k_2}\dots[\varphi_t^{(m)}(u)]^{k_m}$$

$$+ \sum_{\substack{k_1+\dots+k_m=m-n+1 \\ k_1+\dots+mk_m=m}} k_1(m;k_1,\dots,k_m)\left[\varphi_t'(u)\right]^{k_1-1}\left[\varphi_t''(u)\right]^{k_2+1}\left[\varphi_t'''(u)\right]^{k_3}\dots[\varphi_t^{(m)}(u)]^{k_m}$$

$$+ \sum_{\substack{k_1+\dots+k_m=m-n+1 \\ k_1+\dots+lk_m=m}} k_2(m;k_1,\dots,k_m)\left[\varphi_t'(u)\right]^{k_1}\left[\varphi_t''(u)\right]^{k_2-1}\left[\varphi_t'''(u)\right]^{k_3+1}[\varphi_t^{(iv)}(u)]^{k_4}\dots[\varphi_t^{(m)}(u)]^{k_m}$$

$$+ \dots + \sum_{\substack{k_1+\dots+k_m=m-n+1 \\ k_1+\dots+mk_m=m}} k_m(m;k_1,\dots,k_m)\left[\varphi_t'(u)\right]^{k_1}\dots[\varphi_t^{(m)}(u)]^{k_m-1}\varphi_t^{(m+1)}(u)\Bigg]$$

$$+ k\ \varphi_t^{k-1}(u)\varphi_t^{(m+1)}(u),$$

where this last term results from taking $n = m-1$ in the summation $\sum_{n=m-k}^{m-1}$, since in this case we have $k_1+\dots+k_m = 1$ and $k_1+2k_2+\dots+mk_m = m$ which just occurs if $k_1 = \dots = k_{m-1} = 0$ and $k_m = 1$.

Therefore,

$$\left(\varphi_t^k\right)^{(m+1)}(u) = \sum_{n=m-k+1}^{m-1} \frac{k!\ \varphi_t^{k-m+n-1}(u)}{(k-m+n-1)!}\Bigg[\sum_{\substack{c_1+\dots+c_m=m+1-n \\ c_1+\dots+mc_m=m+1}} (m;c_1-1,c_2,\dots,c_m)\left[\varphi_t'(u)\right]^{c_1}\dots[\varphi_t^{(m)}(u)]^{c_m}$$

$$+ \sum_{\substack{c_1+\dots+c_m=m+1-n \\ c_1+\dots+mc_m=m+1}} (c_1+1)\ (m;c_1+1,c_2-1,c_3,\dots,c_m)\left[\varphi_t'(u)\right]^{c_1}\dots[\varphi_t^{(m)}(u)]^{c_m}$$

$$+ \sum_{\substack{c_1+\dots+c_m=m+1-n \\ c_1+\dots+mc_m=m+1}} (c_2+1)\ (m;c_1,c_2+1,c_3-1,c_4,\dots,c_m)\left[\varphi_t'(u)\right]^{c_1}\dots[\varphi_t^{(m)}(u)]^{c_m}$$

$$+ \dots + \sum_{\substack{c_1+\dots+c_{m+1}=m+1-n \\ c_1+\dots+(m+1)c_{m+1}=m+1}} (c_m+1)\ (m;c_1,\dots,c_{m-1},c_m+1)\left[\varphi_t'(u)\right]^{c_1}\dots[\varphi_t^{(m)}(u)]^{c_m}\varphi_t^{(m+1)}(u)\Bigg]$$

$$+ k\ \varphi_t^{k-1}(u)\varphi_t^{(m+1)}(u)$$

$$= \sum_{n=m-k+1}^{m-1} \frac{k!\ \varphi_t^{k-m+n-1}(u)}{(k-m+n-1)!}\Bigg[\sum_{\substack{c_1+\dots+c_m=m+1-n \\ c_1+\dots+mc_m=m+1}} (m+1;c_1,\dots,c_m,0)\left[\varphi_t'(u)\right]^{c_1}\dots[\varphi_t^{(m)}(u)]^{c_m}$$

$$+ \sum_{\substack{c_1+\dots+c_m=m+1-n \\ c_1+\dots+mc_m=m+1}} (m+1;c_1,\dots,c_m,1)\left[\varphi_t'(u)\right]^{c_1}\dots[\varphi_t^{(m)}(u)]^{c_m}\varphi_t^{(m+1)}(u)\Bigg]$$

$$+\ k\ \varphi_t^{k-1}(u)\ (m+1;0,...,0,1)\varphi_t^{(m+1)}(u)$$

$$=\sum_{n=m-k+1}^{m}\frac{k!\ \varphi_t^{k-m+n-1}(u)}{(k-m+n-1)!}\sum_{\substack{c_1+...+c_{m+1}=m+1-n\\ c_1+...+(m+1)c_{m+1}=m+1}}(m+1;c_1,...,c_{m+1})\left[\varphi_t'(u)\right]^{c_1}\dots[\varphi_t^{(m+1)}(u)]^{c_{m+1}},$$

where we used the following equalities:

$$(m;c_1-1,c_2,...,c_m)=\frac{c_1}{m+1}(m+1;c_1,c_2,...,c_m,0), \tag{C.1}$$

$$(c_j+1)\ (m;c_1,...,c_{j-1},c_j+1,c_{j+1}-1,...,c_m)$$

$$=\frac{(j+1)c_{j+1}}{m+1}(m+1;c_1,...,c_m,0), \qquad \text{for } j=1,...,m-1, \tag{C.2}$$

$$(c_m+1)\ (m;c_1,...,c_{m-1},c_m+1)=(m+1;c_1,...,c_{m-1},c_m,1), \tag{C.3}$$

and

$$(m;c_1-1,c_2,...,c_m)+\sum_{j=1}^{m-1}(c_j+1)\ (m;c_1,...,c_{j-1},c_j+1,c_{j+1}-1,...,c_m)$$

$$=(m+1;c_1,...,c_m,0)\left[\frac{c_1+2c_2+...+mc_m}{m+1}\right]=(m+1;c_1,...,c_m,0).\quad \blacksquare \tag{C.4}$$

C.2 Proof of formula (4.3)

Let us suppose φ_t derivable as many times as necessary and that X admits moments of all orders.

Let us prove by induction that we have (4.3):

$$\kappa_{X_t}^{(n)}(z)=\frac{i\alpha_0\varphi_t^{(n)}(z)}{\varphi_t'(0)}+\sum_{j=1}^{n-1}a_{n-1,j}(z)\cdot\kappa_{X_{t-1}}^{(j)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)$$

$$+\left[\frac{\alpha_1\varphi_t'(z)}{\varphi_t'(0)}\right]^n\cdot\kappa_{X_{t-1}}^{(n)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right), \qquad n\geq 2,$$

where the coefficients $a_{n-1,j}$ are given by

$$a_{n-1,j}(z)=\left[\frac{\alpha_1}{\varphi_t'(0)}\right]^j\sum_{\substack{k_1+...+k_n=j\\ k_1+2k_2+...+nk_n=n\\ k_r\in\mathbb{N}_0}}(n;k_1,...,k_n)\left[\varphi_t'(z)\right]^{k_1}...[\varphi_t^{(n)}(z)]^{k_n}, \quad j\geq 1.$$

Taking derivatives on both sides of the expression (4.2), one obtains

$$\kappa_{X_t}''(z)=\frac{i\alpha_0\varphi_t''(z)}{\varphi_t'(0)}+\frac{\alpha_1\varphi_t''(z)}{\varphi_t'(0)}\cdot\kappa_{X_{t-1}}'\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)$$

$$+\left[\frac{\alpha_1\varphi_t'(z)}{\varphi_t'(0)}\right]^2\cdot\kappa''_{X_{t-1}}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right),$$

so, (4.3) is valid for $n=2$ since $a_{11}(z)=\frac{\alpha_1\varphi_t''(z)}{\varphi_t'(0)}$. Now, we assume that formula (4.3) is true for an arbitrarily fixed integer n, $n\geq 2$. Considering that $a_{n-1,0}(z)=0$, it then follows that

$$\begin{aligned}
\kappa_{X_t}^{(n+1)}(z) &= \frac{d}{dz}\left(\frac{i\alpha_0\varphi_t^{(n)}(z)}{\varphi_t'(0)}+\sum_{j=1}^{n-1}a_{n-1,j}(z)\cdot\kappa_{X_{t-1}}^{(j)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)\right.\\
&\quad\left.+\left[\frac{\alpha_1\varphi_t'(z)}{\varphi_t'(0)}\right]^n\cdot\kappa_{X_{t-1}}^{(n)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)\right)\\
&= \frac{i\alpha_0\varphi_t^{(n+1)}(z)}{\varphi_t'(0)}+\sum_{j=1}^{n}a_{n-1,j-1}(z)\frac{\alpha_1\varphi_t'(z)}{\varphi_t'(0)}\cdot\kappa_{X_{t-1}}^{(j)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)\\
&\quad+\sum_{j=1}^{n-1}\left(\frac{\alpha_1}{\varphi_t'(0)}\right)^j\left[\sum_{\substack{k_1+\ldots+k_n=j\\k_1+\ldots+nk_n=n}}k_1(n;k_1,\ldots,k_n)\left[\varphi_t'(z)\right]^{k_1-1}\left[\varphi_t''(z)\right]^{k_2+1}\ldots\left[\varphi_t^{(n)}(z)\right]^{k_n}\right.\\
&\quad+\sum_{\substack{k_1+\ldots+k_n=j\\k_1+\ldots+nk_n=n}}k_2(n;k_1,\ldots,k_n)\left[\varphi_t'(z)\right]^{k_1}\left[\varphi_t''(z)\right]^{k_2-1}\left[\varphi_t'''(z)\right]^{k_3+1}\ldots\left[\varphi_t^{(n)}(z)\right]^{k_n}+\ldots\\
&\quad\left.+\sum_{\substack{k_1+\ldots+k_n=j\\k_1+\ldots+nk_n=n}}k_n(n;k_1,\ldots,k_n)\left[\varphi_t'(z)\right]^{k_1}\ldots\left[\varphi_t^{(n)}(z)\right]^{k_n-1}\varphi_t^{(n+1)}(z)\right]\cdot\kappa_{X_{t-1}}^{(j)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)\\
&\quad+n\left(\frac{\alpha_1}{\varphi_t'(0)}\right)^n\left[\varphi_t'(z)\right]^{n-1}\varphi_t''(z)\cdot\kappa_{X_{t-1}}^{(n)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)\\
&\quad+\left[\frac{\alpha_1\varphi_t'(z)}{\varphi_t'(0)}\right]^{n+1}\cdot\kappa_{X_{t-1}}^{(n+1)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)\\
&= \frac{i\alpha_0\varphi_t^{(n+1)}(z)}{\varphi_t'(0)}+\sum_{j=1}^{n-1}\left(\frac{\alpha_1}{\varphi_t'(0)}\right)^j\left[\sum_{\substack{c_1+\ldots+c_n=j\\c_1+\ldots+nc_n=n+1}}\frac{c_1}{n+1}(n+1;c_1,\ldots,c_n,0)\left[\varphi_t'(z)\right]^{c_1}\ldots\left[\varphi_t^{(n)}(z)\right]^{c_n}\right.\\
&\quad+\sum_{\substack{c_1+\ldots+c_n=j\\c_1+\ldots+nc_n=n+1}}\frac{2c_2}{n+1}(n+1;c_1,\ldots,c_n,0)\left[\varphi_t'(z)\right]^{c_1}\ldots\left[\varphi_t^{(n)}(z)\right]^{c_n}+\ldots\\
&\quad+\sum_{\substack{c_1+\ldots+c_n=j\\c_1+\ldots+nc_n=n+1}}\frac{nc_n}{n+1}(n+1;c_1,\ldots,c_n,0)\left[\varphi_t'(z)\right]^{c_1}\ldots\left[\varphi_t^{(n)}(z)\right]^{c_n}\\
&\quad\left.+\sum_{\substack{c_1+\ldots+c_{n+1}=j\\c_1+\ldots+nc_n=n}}(n+1;c_1,\ldots,c_n,1)\left[\varphi_t'(z)\right]^{c_1}\ldots\left[\varphi_t^{(n)}(z)\right]^{c_n}\varphi_t^{(n+1)}(z)\right]\cdot\kappa_{X_{t-1}}^{(j)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)\\
&\quad+\left(\frac{\alpha_1}{\varphi_t'(0)}\right)^n(n+1;n-1,1,0,\ldots,0)\left[\varphi_t'(z)\right]^{n-1}\varphi_t''(z)\cdot\kappa_{X_{t-1}}^{(n)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)\\
&\quad+\left[\frac{\alpha_1\varphi_t'(z)}{\varphi_t'(0)}\right]^{n+1}\cdot\kappa_{X_{t-1}}^{(n+1)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right),
\end{aligned}\tag{C.5}$$

since we have

$$a_{n-1,j-1}\frac{\alpha_1\varphi_t'(z)}{\varphi_t'(0)} = \left(\frac{\alpha_1}{\varphi_t'(0)}\right)^j \sum_{\substack{k_1+\ldots+k_n=j-1\\ k_1+\ldots+nk_n=n}} (n;k_1,\ldots,k_n)\left[\varphi_t'(z)\right]^{k_1+1}\ldots[\varphi_t^{(n)}(z)]^{k_n}$$

$$= \left(\frac{\alpha_1}{\varphi_t'(0)}\right)^j \sum_{\substack{c_1+\ldots+c_n=j\\ c_1+\ldots+nc_n=n+1}} (n;c_1-1,c_2,\ldots,c_n)\left[\varphi_t'(z)\right]^{c_1}\ldots[\varphi_t^{(n)}(z)]^{c_n}$$

$$= \left(\frac{\alpha_1}{\varphi_t'(0)}\right)^j \sum_{\substack{c_1+\ldots+c_n=j\\ c_1+\ldots+nc_n=n+1}} \frac{c_1}{n+1}(n+1;c_1,\ldots,c_n,0)\left[\varphi_t'(z)\right]^{c_1}\ldots[\varphi_t^{(n)}(z)]^{c_n}, \qquad \text{from } (C.1),$$

$$a_{n-1,n-1}\frac{\alpha_1\varphi_t'(z)}{\varphi_t'(0)} + n\left(\frac{\alpha_1}{\varphi_t'(0)}\right)^n [\varphi_t'(z)]^{n-1}\varphi_t''(z) = \left[\frac{n(n-1)}{2}+n\right][\varphi_t'(z)]^{n-1}\varphi_t''(z)$$

$$= (n+1;n-1,1,0,\ldots,0),$$

$$\sum_{\substack{k_1+\ldots+k_n=j\\ k_1+\ldots+nk_n=n}} k_1(n;k_1,\ldots,k_n)\left[\varphi_t'(z)\right]^{k_1-1}\left[\varphi_t''(z)\right]^{k_2+1}\ldots[\varphi_t^{(n)}(z)]^{k_n}$$

$$= \sum_{\substack{c_1+\ldots+c_n=j\\ c_1+\ldots+nc_n=n+1}} (c_1+1)(n;c_1+1,c_2-1,c_3,\ldots,c_n)\left[\varphi_t'(z)\right]^{c_1}\left[\varphi_t''(z)\right]^{c_2}\ldots[\varphi_t^{(n)}(z)]^{c_n}$$

$$= \sum_{\substack{c_1+\ldots+c_n=j\\ c_1+\ldots+nc_n=n+1}} \frac{2c_2}{n+1}(n+1;c_1,\ldots,c_n,0)\left[\varphi_t'(z)\right]^{c_1}\left[\varphi_t''(z)\right]^{c_2}\ldots[\varphi_t^{(n)}(z)]^{c_n}, \qquad \text{from } (C.2),$$

and

$$\sum_{\substack{k_1+\ldots+k_n=j\\ k_1+\ldots+nk_n=n}} k_n(n;k_1,\ldots,k_n)\left[\varphi_t'(z)\right]^{k_1}\ldots[\varphi_t^{(n)}(z)]^{k_n-1}\varphi_t^{(n+1)}(z)$$

$$= \sum_{\substack{c_1+\ldots+c_n+1=j\\ c_1+\ldots+nc_n+(n+1)=n+1}} (c_n+1)(n;c_1,\ldots,c_{n-1},c_n+1)\left[\varphi_t'(z)\right]^{c_1}\ldots[\varphi_t^{(n)}(z)]^{c_n}\varphi_t^{(n+1)}(z)$$

$$= \sum_{\substack{c_1+\ldots+c_n+1=j\\ c_1+\ldots+nc_n+(n+1)=n+1}} (n+1;c_1,\ldots,c_n,1)\left[\varphi_t'(z)\right]^{c_1}\ldots[\varphi_t^{(n)}(z)]^{c_n}\varphi_t^{(n+1)}(z), \qquad \text{from } (C.3).$$

Thus, from $(C.4)$, we can finally conclude that equality $(C.5)$ is equivalent to

$$\kappa_{X_t}^{(n+1)}(z) = \frac{i\alpha_0\varphi_t^{(n+1)}(z)}{\varphi_t'(0)} + \left[\frac{\alpha_1\varphi_t'(z)}{\varphi_t'(0)}\right]^{n+1}\cdot\kappa_{X_{t-1}}^{(n+1)}\left(\frac{\alpha_1}{\varphi_t'(0)}[\varphi_t(z)-1]\right)$$

$$+\sum_{j=1}^{n}\left(\frac{\alpha_1}{\varphi_t'(0)}\right)^j \sum_{\substack{c_1+\ldots+c_{n+1}=j\\ c_1+\ldots+(n+1)c_{n+1}=n+1}} (n+1;c_1,...,c_{n+1})\left[\varphi_t'(z)\right]^{c_1}\ldots[\varphi_t^{(n+1)}(z)]^{c_{n+1}}$$

$$\times\ \kappa_{X_{t-1}}^{(j)}\left(\frac{\alpha_1}{\varphi_t'(0)}[\varphi_t(z)-1]\right). \ \blacksquare$$

C.3 Proof of Theorem 4.2

Let us recall that we have to show that if X is a first-order stationary process following a CP-INARCH(1) model such that **H4** is satisfied, then

(a) For any $k\geq 0$, we have

$$\mu(k) \ = \ f_2(v_0\alpha_1^k+\alpha_0(1+\alpha_1)).$$

(b) For any $l\geq k\geq 0$, we have

$$\begin{aligned}\mu(k,l) \ = \ & [d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{l+k}\\ &+\frac{v_0(\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^l+v_0f_1f_2\alpha_1^{l-k}+f_1\mu(k).\end{aligned}$$

(c) For any $m\geq l\geq k\geq 0$, we have

$$\begin{aligned}\mu(k,l,m) \ = \ & \alpha_1^{m-l}\left[\left\{(c_0-4v_0d_0+3v_0^3)+3v_0(v_0^2-d_0)\alpha_1+(3v_0d_0-c_0)\alpha_1^2\right.\right.\\ &\left.+(7v_0d_0-6v_0^3-c_0)\alpha_1^3+3v_0(d_0-2v_0^2)\alpha_1^4+(6v_0^3-6v_0d_0+c_0)\alpha_1^5\right\}f_4\alpha_1^{2l+k}\\ &+\frac{2v_0+\alpha_0}{1-\alpha_1}f_3\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]\alpha_1^{2l}\\ &+\frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)}f_2\left[2v_0\alpha_0+d_0(1-\alpha_1)+v_0^2(2\alpha_1-1)\right]\alpha_1^{2l-k}\\ &+\frac{\alpha_0f_3}{1-\alpha_1}\left\{d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right\}\alpha_1^{2(l-k)}+\frac{v_0+\alpha_0}{1-\alpha_1}\mu(k,l)\\ &\left.-f_2\mu(k)[\alpha_0+(v_0+\alpha_0)\alpha_1]\right]+f_1\mu(k,l).\end{aligned}$$

Proof: Let us recall the notations presented in formula (4.6) and (4.7) and take into account the following conditional moments:

$$E(X_t|\underline{X}_{t-1}) \ = \ \lambda_t=\alpha_0+\alpha_1X_{t-1},$$

$$\begin{aligned}E(X_t^2|\underline{X}_{t-1}) \ &= \ v_0\lambda_t+\lambda_t^2=v_0(\alpha_0+\alpha_1X_{t-1})+(\alpha_0+\alpha_1X_{t-1})^2\\ &= \ \alpha_1^2X_{t-1}^2+\alpha_1(2\alpha_0+v_0)X_{t-1}+\alpha_0(\alpha_0+v_0),\end{aligned} \qquad \text{(C.6)}$$

$$\begin{aligned}
E(X_t^3|\underline{X}_{t-1}) &= i\,\Phi'''_{X_t|\underline{X}_{t-1}}(0) = d_0\lambda_t + 3v_0\lambda_t^2 + \lambda_t^3 \quad (^1)\\
&= d_0(\alpha_0+\alpha_1 X_{t-1}) + 3v_0(\alpha_0^2 + 2\alpha_0\alpha_1 X_{t-1} + \alpha_1^2 X_{t-1}^2) + \alpha_0^3\\
&\quad +3\alpha_0^2\alpha_1 X_{t-1} + 3\alpha_0\alpha_1^2 X_{t-1}^2 + \alpha_1^3 X_{t-1}^3\\
&= \alpha_1^3 X_{t-1}^3 + 3\alpha_1^2(v_0+\alpha_0)X_{t-1}^2 + \alpha_1(3\alpha_0^2 + 6v_0\alpha_0 + d_0)X_{t-1}\\
&\quad +\alpha_0(d_0 + 3v_0\alpha_0 + \alpha_0^2). \qquad (C.7)
\end{aligned}$$

(a) Using the fact that $V(X_t) = f_2$ and that the autocovariance function, according to Corollary 3.2, equals $\Gamma(k) = \alpha_1^k f_2$ we get

$$\begin{aligned}
\mu(k) &= E(X_t X_{t+k}) = Cov(X_t, X_{t+k}) + E(X_t)^2 = v_0\alpha_1^k f_2 + \mu^2\\
&= f_2(v_0\alpha_1^k + \alpha_0(1+\alpha_1)). \qquad (C.8)
\end{aligned}$$

(b) To derive an explicit expression for $\mu(k,l)$, $0 \le k \le l$, we distinguish the following three cases:

Case 1: $l > k$**.** We have

$$\begin{aligned}
\mu(k,l) &= E(X_t X_{t+k} X_{t+l}) \ = \ E[X_t X_{t+k} E(X_{t+l}|\underline{X}_{t+l-1})]\\
&= \alpha_0 E(X_t X_{t+k}) + \alpha_1 E(X_t X_{t+k} X_{t+l-1})\\
&= \alpha_0\mu(k) + \alpha_1\mu(k,l-1)\\
&= \alpha_0\mu(k) + \alpha_1[\alpha_0\mu(k) + \alpha_1\mu(k,l-2)]\\
&= \alpha_0\mu(k)(1+\alpha_1) + \alpha_1^2\mu(k,l-2)\\
&= \ldots \ = \ \alpha_0\mu(k)\sum_{j=0}^{l-k-1}\alpha_1^j + \alpha_1^{l-k}\mu(k,k)\\
&= \alpha_0\mu(k)\frac{1-\alpha_1^{l-k}}{1-\alpha_1} + \alpha_1^{l-k}\mu(k,k)\\
&= \alpha_1^{l-k}[\mu(k,k) - f_1\mu(k)] + f_1\mu(k).
\end{aligned}$$

Case 2: $l = k > 0$**.** We have

$$\begin{aligned}
\mu(k,k) &= E[X_t E(X_{t+k}^2|\underline{X}_{t+k-1})]\\
&= \alpha_1^2 E(X_t X_{t+k-1}^2) + \alpha_1(2\alpha_0+v_0)E(X_t X_{t+k-1}) + \alpha_0(\alpha_0+v_0)E(X_t)\\
&= \alpha_1^2\mu(k-1,k-1) + \alpha_1(2\alpha_0+v_0)\mu(k-1) + \alpha_0(\alpha_0+v_0)f_1\\
&= \alpha_1^2\mu(k-1,k-1) + v_0(2\alpha_0+v_0)f_2\alpha_1^k + f_1[\alpha_1(2\alpha_0+v_0)f_1 + \alpha_0(\alpha_0+v_0)]\\
&= \alpha_1^2\mu(k-1,k-1) + v_0(2\alpha_0+v_0)f_2\alpha_1^k\\
&\quad + f_1[\alpha_1(2\alpha_0+v_0)f_1 + f_1(1-\alpha_1)(\alpha_0+v_0)]\\
&= \alpha_1^2\mu(k-1,k-1) + v_0(2\alpha_0+v_0)f_2\alpha_1^k + f_1\mu(0)(1-\alpha_1^2)
\end{aligned}$$

[1] We note that

$$\Phi'''_{X_t|\underline{X}_{t-1}}(u) = \left[i\frac{\varphi_t'''(u)\lambda_t}{\varphi_t'(0)} - 3\left(\frac{\lambda_t}{\varphi_t'(0)}\right)^2\varphi_t'(u)\varphi_t''(u) - i\left(\frac{\varphi_t'(u)\lambda_t}{\varphi_t'(0)}\right)^3\right]\exp\left\{i\frac{\lambda_t}{\varphi_t'(0)}[\varphi_t(u)-1]\right\}, \quad u\in\mathbb{R}.$$

$$
\begin{aligned}
&= \quad \ldots = \alpha_1^{2k}\mu(0,0) + v_0(2\alpha_0+v_0)f_2\sum_{j=0}^{k-1}\alpha_1^{k+j} + f_1\mu(0)(1-\alpha_1^2)\sum_{j=0}^{k-1}\alpha_1^{2j} \\
&= \alpha_1^{2k}\mu(0,0) + v_0(2\alpha_0+v_0)f_2\alpha_1^k\,\frac{1-\alpha_1^k}{1-\alpha_1} + f_1\mu(0) \\
&= \alpha_1^{2k}\left[\mu(0,0) - \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1}f_2 - f_1\mu(0)\right] + \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^k + f_1\mu(0).
\end{aligned}
$$

Case 3: $l = k = 0$. According to the relations between the moments and the cumulants (e.g., formula (15.10.4) in [15, p. 186]) and Theorem 4.1, we have

$$
\begin{aligned}
\mu(0,0) &= E(X_t^3) = \kappa_3 + 3\kappa_2\mu + \mu^3 = f_3[d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2] + 3v_0f_2f_1 + f_1^3 \\
&= [d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2]f_3 + \frac{3\alpha_0v_0}{1-\alpha_1}f_2 + \alpha_0^2\,\frac{1+\alpha_1}{1-\alpha_1}f_2 \\
&= [d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2]f_3 + \frac{2\alpha_0v_0}{1-\alpha_1}f_2 + \frac{\alpha_0(v_0+\alpha_0(1+\alpha_1))}{1-\alpha_1}f_2 \\
&= [d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2]f_3 + \frac{2\alpha_0v_0}{1-\alpha_1}f_2 + f_1\mu(0),
\end{aligned}
$$

since $f_1 = (1-\alpha_1^2)f_2$.

So the above formula for $\mu(k,k)$ simplifies to

$$
\begin{aligned}
\mu(k,k) &= \alpha_1^{2k}\left[[d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2]f_3 - \frac{v_0^2}{1-\alpha_1}f_2\right] + \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^k + f_1\mu(0) \\
&= \alpha_1^{2k}\left[[d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2]f_3 - \frac{v_0^2}{1-\alpha_1}f_3(1-\alpha_1^3)\right] \\
&\quad + \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^k + f_1\mu(0) \\
&= \alpha_1^{2k}f_3\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\right] + \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^k + f_1\mu(0),
\end{aligned}
$$

which also holds for $k = 0$. Replacing this expression in $\mu(k,l)$ above, it follows that

$$
\begin{aligned}
\mu(k,l) &= \alpha_1^{l-k}[\mu(k,k) - f_1\mu(k)] + f_1\mu(k) \\
&= \alpha_1^{l-k}\left[[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{2k} + \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^k\right. \\
&\quad \left. + f_1\mu(0) - f_1\mu(k)\right] + f_1\mu(k).
\end{aligned}
$$

As

$$
\begin{aligned}
f_1\mu(0) - f_1\mu(k) &= f_1f_2(v_0+\alpha_0(1+\alpha_1)) - f_1[v_0\alpha_1^k + \alpha_0(1+\alpha_1)]f_2 \\
&= v_0f_1f_2 - \frac{v_0\alpha_0}{1-\alpha_1}f_2\alpha_1^k,
\end{aligned}
$$

we finally obtain, for any $0 \le k \le l$,

$$
\begin{aligned}
\mu(k,l) &= \alpha_1^{l-k}\left[[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{2k} + \frac{v_0(\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^k + v_0f_1f_2\right] + f_1\mu(k) \\
&= [d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{l+k} + \frac{v_0(\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^l + v_0f_1f_2\alpha_1^{l-k} + f_1\mu(k).
\end{aligned}
$$

(c) In what concerns the fourth-order moments $\mu(k,l,m)$ with $0 \leq k \leq l \leq m$, we proceed in a similar way as above and distinguish the following four cases:

Case 1: $m > l$**.** As above we have

$$\begin{aligned}
\mu(k,l,m) &= E(X_t X_{t+k} X_{t+l} X_{t+m}) = E[X_t X_{t+k} X_{t+l} E(X_{t+m}|\underline{X}_{t+m-1})] \\
&= \alpha_0 E(X_t X_{t+k} X_{t+l}) + \alpha_1 E(X_t X_{t+k} X_{t+l} X_{t+m-1}) \\
&= \alpha_0 \mu(k,l) + \alpha_1 \mu(k,l,m-1) \\
&= \ldots = \alpha_1^{m-l}[\mu(k,l,l) - f_1\mu(k,l)] + f_1\mu(k,l).
\end{aligned}$$

Case 2: $m = l > k$**.** For this case, using formula $(C.6)$, we obtain

$$\begin{aligned}
\mu(k,l,l) &= E[X_t X_{t+k} E(X_{t+l}^2|\underline{X}_{t+l-1})] \\
&= \alpha_1^2 \mu(k,l-1,l-1) + \alpha_1(v_0 + 2\alpha_0)\mu(k,l-1) + \alpha_0(v_0+\alpha_0)\mu(k) \\
&= \alpha_1^2 \mu(k,l-1,l-1) + \alpha_0(v_0+\alpha_0)\mu(k) \\
&\quad + \alpha_1(v_0+2\alpha_0)\Big\{[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{l-1+k} \\
&\quad + \frac{v_0(\alpha_0+v_0)}{1-\alpha_1} f_2 \alpha_1^{l-1} + v_0 f_1 f_2 \alpha_1^{l-1-k} + f_1\mu(k)\Big\},
\end{aligned}$$

by replacing $\mu(k,l-1)$. So,

$$\begin{aligned}
\mu(k,l,l) &= \alpha_1^2 \mu(k,l-1,l-1) + f_1\mu(k)[\alpha_1(v_0+2\alpha_0) + (1-\alpha_1)(v_0+\alpha_0)] \\
&\quad + \alpha_1^l(v_0+2\alpha_0)\Big\{[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^k \\
&\quad + \frac{v_0(\alpha_0+v_0)}{1-\alpha_1} f_2 + v_0 f_1 f_2 \alpha_1^{-k}\Big\} \\
&= \ldots = \alpha_1^{2(l-k)}\mu(k,k,k) + f_1\mu(k)(v_0+\alpha_0(1+\alpha_1)) \sum_{j=0}^{l-k-1} \alpha_1^{2j} \\
&\quad + (v_0+2\alpha_0)\Big\{[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^k \\
&\quad + \frac{v_0(\alpha_0+v_0)}{1-\alpha_1} f_2 + v_0 f_1 f_2 \alpha_1^{-k}\Big\} \sum_{j=0}^{l-k-1} \alpha_1^{l+j} \\
&= \alpha_1^{2(l-k)}\mu(k,k,k) + \mu(k)(v_0+\alpha_0(1+\alpha_1)) f_2 (1-\alpha_1^{2(l-k)}) \\
&\quad + \frac{v_0+2\alpha_0}{1-\alpha_1}\Big\{[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^k \\
&\quad + \frac{v_0(\alpha_0+v_0)}{1-\alpha_1} f_2 + v_0 f_1 f_2 \alpha_1^{-k}\Big\}(\alpha_1^l - \alpha_1^{2l-k}).
\end{aligned}$$

Since $\mu(0) = (v_0+\alpha_0(1+\alpha_1))f_2$ and replacing $\mu(k)$, we get

$$\begin{aligned}
\mu(k,l,l) &= \alpha_1^{2(l-k)}\mu(k,k,k) + \mu(k)\mu(0) - f_2(v_0\alpha_1^k + \alpha_0(1+\alpha_1))\mu(0)\alpha_1^{2(l-k)} \\
&\quad + \frac{v_0+2\alpha_0}{1-\alpha_1}\Big\{[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{l+k} \\
&\quad + \frac{v_0(\alpha_0+v_0)}{1-\alpha_1} f_2 \alpha_1^l + v_0 f_1 f_2 \alpha_1^{l-k}\Big\}
\end{aligned}$$

$$
\begin{aligned}
& -\frac{v_0+2\alpha_0}{1-\alpha_1}\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]f_3\alpha_1^{2l} \\
& -\frac{(v_0+2\alpha_0)v_0(v_0+\alpha_0)}{(1-\alpha_1)^2}f_2\alpha_1^{2l-k}-\frac{v_0(v_0+2\alpha_0)}{1-\alpha_1}f_1f_2\alpha_1^{2(l-k)} \\
= \; & \alpha_1^{2(l-k)}\mu(k,k,k)+\mu(k)\mu(0) \\
& -f_2v_0\left[f_2(v_0+\alpha_0(1+\alpha_1))+\frac{(v_0+2\alpha_0)(v_0+\alpha_0)}{(1-\alpha_1)^2}\right]\alpha_1^{2l-k} \\
& -f_1\left[f_1\mu(0)+\frac{v_0(v_0+2\alpha_0)}{1-\alpha_1}f_2\right]\alpha_1^{2(l-k)}+\frac{v_0+2\alpha_0}{1-\alpha_1}[\mu(k,l)-f_1\mu(k)] \\
& -\frac{v_0+2\alpha_0}{1-\alpha_1}\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]f_3\alpha_1^{2l}.
\end{aligned}
$$

So, replacing $\mu(0)$, recalling $\mu(0,0)$ and taking into account that $\frac{f_1}{1-\alpha_1}=(1+\alpha_1)f_2$, we get

$$
\begin{aligned}
\mu(k,l,l) = \; & \alpha_1^{2(l-k)}\mu(k,k,k)+\mu(k)f_2[v_0+\alpha_0(1+\alpha_1)] \\
& -\frac{f_2v_0}{(1-\alpha_1)(1-\alpha_1^2)}\left[v_0^2(1+\alpha_1)+v_0\alpha_0(4+3\alpha_1)+3\alpha_0^2(1+\alpha_1)\right]\alpha_1^{2l-k} \\
& -f_1\left\{\mu(0,0)-[d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2]f_3+\frac{v_0^2f_2}{1-\alpha_1}\right\}\alpha_1^{2(l-k)} \\
& +\frac{v_0+2\alpha_0}{1-\alpha_1}\mu(k,l)-(v_0+2\alpha_0)(1+\alpha_1)f_2\mu(k) \\
& -\frac{v_0+2\alpha_0}{1-\alpha_1}\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]f_3\alpha_1^{2l} \\
= \; & \alpha_1^{2(l-k)}\mu(k,k,k)-\mu(k)f_2[\alpha_0+(v_0+\alpha_0)\alpha_1] \\
& -\frac{f_2v_0}{(1-\alpha_1)(1-\alpha_1^2)}\left[v_0^2(1+\alpha_1)+v_0\alpha_0(4+3\alpha_1)+3\alpha_0^2(1+\alpha_1)\right]\alpha_1^{2l-k} \\
& -f_1\left\{\mu(0,0)-[d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2]f_3+\frac{v_0^2f_2}{1-\alpha_1}\right\}\alpha_1^{2(l-k)} \\
& +\frac{v_0+2\alpha_0}{1-\alpha_1}\mu(k,l)-\frac{v_0+2\alpha_0}{1-\alpha_1}\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]f_3\alpha_1^{2l}. \qquad \text{(C.9)}
\end{aligned}
$$

Case 3: $m=l=k>0$. From formula $(C.7)$ we have

$$
\begin{aligned}
\mu(k,k,k) = \; & E[X_tE(X_{t+k}^3|\underline{X}_{t+k-1})] \\
= \; & \alpha_1^3\mu(k-1,k-1,k-1)+3\alpha_1^2(v_0+\alpha_0)\mu(k-1,k-1) \\
& +\alpha_1(d_0+6v_0\alpha_0+3\alpha_0^2)\mu(k-1)+\alpha_0(d_0+3v_0\alpha_0+\alpha_0^2)\mu.
\end{aligned}
$$

Replacing $\mu(k-1,k-1)$ and thereafter $\mu(k-1)$, we deduce

$$
\begin{aligned}
\mu(k,k,k) = \; & \alpha_1^3\mu(k-1,k-1,k-1) \\
& +3\alpha_1^2(v_0+\alpha_0)\Big\{[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{2k-2} \\
& +\frac{v_0(\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^{k-1}+v_0f_1f_2+f_1\mu(k-1)\Big\} \\
& +\alpha_1(d_0+6v_0\alpha_0+3\alpha_0^2)\mu(k-1)+\alpha_0(d_0+3v_0\alpha_0+\alpha_0^2)\mu \\
= \; & \alpha_1^3\mu(k-1,k-1,k-1)
\end{aligned}
$$

$$
\begin{aligned}
&+3(v_0+\alpha_0)\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]f_3\alpha_1^{2k}+3\alpha_1\frac{v_0(v_0+\alpha_0)^2}{1-\alpha_1}f_2\alpha_1^k\\
&+3\alpha_1^2(v_0+\alpha_0)\left\{v_0f_1f_2+f_1f_2(v_0\alpha_1^{k-1}+\alpha_0(1+\alpha_1))\right\}\\
&+(d_0+6v_0\alpha_0+3\alpha_0^2)f_2\left[v_0\alpha_1^k+\alpha_0\alpha_1(1+\alpha_1)\right]+\alpha_0(d_0+3v_0\alpha_0+\alpha_0^2)f_1\\
= \;& \alpha_1^3\mu(k-1,k-1,k-1)\\
&+3(v_0+\alpha_0)\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]f_3\alpha_1^{2k}\\
&+\frac{v_0f_2}{1-\alpha_1}\left[3\alpha_1(v_0+\alpha_0)^2+3\alpha_1(v_0+\alpha_0)\alpha_0+(d_0+6v_0\alpha_0+3\alpha_0^2)(1-\alpha_1)\right]\alpha_1^k\\
&+f_1f_2\left\{3\alpha_1^2(v_0+\alpha_0)(v_0+\alpha_0(1+\alpha_1))+(d_0+6v_0\alpha_0+3\alpha_0^2)\alpha_1(1-\alpha_1)(1+\alpha_1)\right.\\
&\left.+(d_0+3v_0\alpha_0+\alpha_0^2)(1-\alpha_1)(1-\alpha_1^2)\right\}.
\end{aligned}
$$

Making some calculations and then recalling the expression of $\mu(0,0)$, we obtain

$$
\begin{aligned}
\mu(k,k,k) \;=\;& \alpha_1^3\mu(k-1,k-1,k-1)\\
&+3(v_0+\alpha_0)\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]f_3\alpha_1^{2k}\\
&+\frac{v_0f_2}{1-\alpha_1}\left[3\alpha_0^2(1+\alpha_1)+3v_0\alpha_0(2+\alpha_1)+d_0(1-\alpha_1)+3v_0^2\alpha_1\right]\alpha_1^k\\
&+f_1f_2\left\{d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2+\frac{3v_0\alpha_0}{1-\alpha_1}(1-\alpha_1^3)+\frac{\alpha_0^2(1+\alpha_1)}{1-\alpha_1}(1-\alpha_1^3)\right\}\\
=\;& \alpha_1^3\mu(k-1,k-1,k-1)\\
&+3(v_0+\alpha_0)\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]f_3\alpha_1^{2k}\\
&+\frac{v_0f_2}{1-\alpha_1}\left[3\alpha_0^2(1+\alpha_1)+3v_0\alpha_0(2+\alpha_1)+d_0(1-\alpha_1)+3v_0^2\alpha_1\right]\alpha_1^k\\
&+f_1(1-\alpha_1^3)\mu(0,0).
\end{aligned}
$$

Replacing successively the expression of $\mu(k-j,k-j,k-j)$, $j=1,...,k-1$, it remains

$$
\begin{aligned}
\mu(k,k,k) \;=\;& \alpha_1^{3k}\mu(0,0,0)+3(v_0+\alpha_0)\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]f_3\sum_{j=0}^{k-1}\alpha_1^{2k+j}\\
&+\frac{v_0f_2}{1-\alpha_1}\left[3\alpha_0^2(1+\alpha_1)+3v_0\alpha_0(2+\alpha_1)+d_0(1-\alpha_1)+3v_0^2\alpha_1\right]\sum_{j=0}^{k-1}\alpha_1^{k+2j}\\
&+f_1(1-\alpha_1^3)\mu(0,0)\sum_{j=0}^{k-1}\alpha_1^{3j}\\
=\;& \alpha_1^{3k}\left\{\mu(0,0,0)-3(v_0+\alpha_0)\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]\frac{f_3}{1-\alpha_1}\right.\\
&-\frac{v_0f_2}{(1-\alpha_1)(1-\alpha_1^2)}\left[3\alpha_0^2(1+\alpha_1)+3v_0\alpha_0(2+\alpha_1)+d_0(1-\alpha_1)+3v_0^2\alpha_1\right]\\
&\left.-f_1\mu(0,0)\right\}+\frac{3(v_0+\alpha_0)f_3\alpha_1^{2k}}{1-\alpha_1}\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]\\
&+\frac{v_0f_2\alpha_1^k}{(1-\alpha_1)(1-\alpha_1^2)}\left[3\alpha_0^2(1+\alpha_1)+3v_0\alpha_0(2+\alpha_1)+d_0(1-\alpha_1)+3v_0^2\alpha_1\right]\\
&+f_1\mu(0,0).
\end{aligned}
\tag{C.10}
$$

Let us now take into account that replacing $\mu(0,0)$, we obtain

$$\begin{aligned}
& 3(v_0+\alpha_0)\frac{f_3}{1-\alpha_1}\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right] \\
& +\frac{v_0 f_2}{(1-\alpha_1)(1-\alpha_1^2)}\left[3\alpha_0^2(1+\alpha_1)+3v_0\alpha_0(2+\alpha_1)+d_0(1-\alpha_1)+3v_0^2\alpha_1\right] \\
& +f_1\mu(0,0) \\
= \; & 3(v_0+\alpha_0)\frac{f_3}{1-\alpha_1}\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right] \\
& +\frac{v_0 f_2}{(1-\alpha_1)(1-\alpha_1^2)}\left[3\alpha_0^2(1+\alpha_1)+3v_0\alpha_0(2+\alpha_1)+d_0(1-\alpha_1)+3v_0^2\alpha_1\right] \\
& +f_1 f_3[d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2]+f_1 f_2\frac{2v_0\alpha_0}{1-\alpha_1}+f_1^2 f_2(v_0+\alpha_0(1+\alpha_1))
\end{aligned}$$

which, highlighting $\frac{f_3}{1-\alpha_1^2}$ and noting that $f_2=(1-\alpha_1^3)f_3$, equals

$$\begin{aligned}
& \frac{f_3}{1-\alpha_1^2}\Big\{3(v_0+\alpha_0)(1+\alpha_1)[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)] \\
& +\frac{1-\alpha_1^3}{1-\alpha_1}[3v_0\alpha_0^2(1+\alpha_1)+3v_0^2\alpha_0(2+\alpha_1)+v_0d_0(1-\alpha_1)+3v_0^3\alpha_1] \\
& +\alpha_0(1+\alpha_1)[d_0(1-\alpha_1^2)+3v_0^2\alpha_1^2]+(1+\alpha_1)\frac{1-\alpha_1^3}{1-\alpha_1}[2v_0\alpha_0^2+v_0\alpha_0^2+\alpha_0^3(1+\alpha_1)]\Big\} \\
= \; & \frac{f_3}{1-\alpha_1^2}\Big\{4v_0d_0-3v_0^3+3v_0(d_0-v_0^2)\alpha_1+3v_0(2v_0^2-d_0)\alpha_1^2+v_0(9v_0^2-4d_0)\alpha_1^3 \\
& +\left(3v_0^2+4d_0+(3v_0^2+4d_0)\alpha_1+(15v_0^2-4d_0)\alpha_1^2+(12v_0^2-4d_0)\alpha_1^3\right)\alpha_0 \\
& +6v_0\alpha_0^2(1+\alpha_1)(1+\alpha_1+\alpha_1^2)+\alpha_0^3(1+\alpha_1)^2(1+\alpha_1+\alpha_1^2)\Big\} \\
= \; & f_4\Big\{4v_0d_0-3v_0^3+3v_0(d_0-v_0^2)\alpha_1+3v_0(v_0^2+d_0)\alpha_1^2 \\
& +v_0(6v_0^2-d_0)\alpha_1^3+3v_0(2v_0^2-d_0)\alpha_1^4+v_0(9v_0^2-4d_0)\alpha_1^5 \\
& +(1+\alpha_1^2)\left(3v_0^2+4d_0+(3v_0^2+4d_0)\alpha_1+(15v_0^2-4d_0)\alpha_1^2+(12v_0^2-4d_0)\alpha_1^3\right)\alpha_0 \\
& +6v_0\alpha_0^2(1+\alpha_1^2)(1+\alpha_1)(1+\alpha_1+\alpha_1^2)+\alpha_0^3(1+\alpha_1^2)(1+\alpha_1)^2(1+\alpha_1+\alpha_1^2)\Big\},
\end{aligned}$$

since $\frac{f_3}{1-\alpha_1^2}=f_4(1+\alpha_1^2)$ and developing the calculations.

So, replacing the expression above in the coefficient of α_1^{3k} in formula $(C.10)$, we finally get

$$\begin{aligned}
\mu(k,k,k) \;=\; & \Big\{\mu(0,0,0)-f_4\left[4v_0d_0-3v_0^3+3v_0(d_0-v_0^2)\alpha_1+v_0(3v_0^2+d_0)\alpha_1^2\right. \\
& +v_0(6v_0^2-d_0)\alpha_1^3+3v_0(2v_0^2-d_0)\alpha_1^4+v_0(9v_0^2-4d_0)\alpha_1^5 \\
& +\alpha_0(1+\alpha_1^2)\left[3v_0^2+4d_0+(3v_0^2+4d_0)\alpha_1+(15v_0^2-4d_0)\alpha_1^2+(12v_0^2-4d_0)\alpha_1^3\right] \\
& +6v_0\alpha_0^2(1+\alpha_1^2)(1+\alpha_1)(1+\alpha_1+\alpha_1^2) \\
& \left.+\alpha_0^3(1+\alpha_1^2)(1+\alpha_1)^2(1+\alpha_1+\alpha_1^2)\right]\Big\}\,\alpha_1^{3k} \\
& +3\frac{v_0+\alpha_0}{1-\alpha_1}f_3\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]\alpha_1^{2k}+f_1\mu(0,0) \\
& +\frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)}f_2\left[3\alpha_0^2(1+\alpha_1)+3v_0\alpha_0(2+\alpha_1)\right. \\
& \left.+d_0(1-\alpha_1)+3v_0^2\alpha_1\right]\alpha_1^k. \qquad \text{(C.11)}
\end{aligned}$$

Case 4: $m = l = k = 0$**.** Once again, according to the relations between the moments and the cumulants (e.g., formula (15.10.4) in [15, p. 186]) and Theorem 4.1, we obtain

$$\begin{aligned}
\mu(0,0,0) &= E(X_t^4) = \kappa_4 + 3\kappa_2^2 + 6\kappa_2\mu^2 + 4\kappa_3\mu + \mu^4 \\
&= f_4\left\{c_0(1-\alpha_1^2)(1-\alpha_1^3) + v_0^3(3\alpha_1^2 + 15\alpha_1^5) + v_0d_0(4\alpha_1^2 + 6\alpha_1^3 - 10\alpha_1^5)\right\} \\
&\quad +3v_0^2f_2^2 + 6v_0f_2f_1^2 + 4f_3[d_0(1-\alpha_1^2) + 3v_0^2\alpha_1^2]f_1 + f_1^4 \\
&= f_4\left\{c_0(1-\alpha_1^2)(1-\alpha_1^3) + v_0^3(3\alpha_1^2 + 15\alpha_1^5) + v_0d_0(4\alpha_1^2 + 6\alpha_1^3 - 10\alpha_1^5)\right. \\
&\quad +3v_0^2\alpha_0\frac{(1-\alpha_1^4)(1-\alpha_1^3)}{(1-\alpha_1)(1-\alpha_1^2)} + 6v_0\alpha_0^2\frac{(1-\alpha_1^4)(1-\alpha_1^3)}{(1-\alpha_1)^2} \\
&\quad \left. +4\alpha_0\frac{(1-\alpha_1^4)}{1-\alpha_1}[d_0(1-\alpha_1^2) + 3v_0^2\alpha_1^2] + \alpha_0^3\frac{(1-\alpha_1^4)(1-\alpha_1^3)(1-\alpha_1^2)}{(1-\alpha_1)^3}\right\} \\
&= f_4\left\{c_0(1-\alpha_1^2)(1-\alpha_1^3) + v_0^3(3\alpha_1^2 + 15\alpha_1^5) + v_0d_0(4\alpha_1^2 + 6\alpha_1^3 - 10\alpha_1^5)\right. \\
&\quad +3v_0^2\alpha_0(1+\alpha_1^2)(1+\alpha_1+\alpha_1^2) + 6v_0\alpha_0^2(1+\alpha_1)(1+\alpha_1^2)(1+\alpha_1+\alpha_1^2) \\
&\quad \left. +4\alpha_0(1+\alpha_1)(1+\alpha_1^2)[d_0(1-\alpha_1^2) + 3v_0^2\alpha_1^2] + \alpha_0^3(1+\alpha_1)^2(1+\alpha_1^2)(1+\alpha_1+\alpha_1^2)\right\} \\
&= f_4\left\{c_0 + (3v_0^3 + 4v_0d_0 - c_0)\alpha_1^2 + (6v_0d_0 - c_0)\alpha_1^3 + (15v_0^3 - 10v_0d_0 + c_0)\alpha_1^5\right. \\
&\quad +\alpha_0(1+\alpha_1^2)\left[3v_0^2 + 4d_0 + (3v_0^2 + 4d_0)\alpha_1 + (15v_0^2 - 4d_0)\alpha_1^2 + (12v_0^2 - 4d_0)\alpha_1^3\right] \\
&\quad \left. +6v_0\alpha_0^2(1+\alpha_1)(1+\alpha_1^2)(1+\alpha_1+\alpha_1^2) + \alpha_0^3(1+\alpha_1)^2(1+\alpha_1^2)(1+\alpha_1+\alpha_1^2)\right\}.
\end{aligned}$$

So the formula $(C.11)$ for $\mu(k,k,k)$ studied in case 3 simplifies to

$$\begin{aligned}
\mu(k,k,k) &= f_4\left\{c_0 - 4v_0d_0 + 3v_0^3 + 3v_0(v_0^2 - d_0)\alpha_1 + (3v_0d_0 - c_0)\alpha_1^2\right. \\
&\quad \left. +(7v_0d_0 - 6v_0^3 - c_0)\alpha_1^3 + 3v_0(d_0 - 2v_0^2)\alpha_1^4 + (6v_0^3 - 6v_0d_0 + c_0)\alpha_1^5\right\}\alpha_1^{3k} \\
&\quad +3\frac{v_0+\alpha_0}{1-\alpha_1}f_3\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\right]\alpha_1^{2k} + f_1\mu(0,0) \\
&\quad +\frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)}f_2\left[3\alpha_0^2(1+\alpha_1) + 3v_0\alpha_0(2+\alpha_1) + d_0(1-\alpha_1) + 3v_0^2\alpha_1\right]\alpha_1^k.
\end{aligned}$$

Inserting into the formula $(C.9)$ for $\mu(k,l,l)$ stated in case 2, we obtain

$$\begin{aligned}
\mu(k,l,l) &= f_4\left\{c_0 - 4v_0d_0 + 3v_0^3 + 3v_0(v_0^2 - d_0)\alpha_1 + (3v_0d_0 - c_0)\alpha_1^2\right. \\
&\quad \left. +(7v_0d_0 - 6v_0^3 - c_0)\alpha_1^3 + 3v_0(d_0 - 2v_0^2)\alpha_1^4 + (6v_0^3 - 6v_0d_0 + c_0)\alpha_1^5\right\}\alpha_1^{2l+k} \\
&\quad +\frac{3v_0 + 3\alpha_0 - v_0 - 2\alpha_0}{1-\alpha_1}f_3\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\right]\alpha_1^{2l} \\
&\quad +f_1\left\{\mu(0,0) - \mu(0,0) - \frac{v_0^2f_2}{1-\alpha_1} + [d_0(1-\alpha_1^2) + 3v_0^2\alpha_1^2]f_3\right\}\alpha_1^{2(l-k)} \\
&\quad +\frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)}f_2\left[3\alpha_0^2(1+\alpha_1) + 3v_0\alpha_0(2+\alpha_1) + d_0(1-\alpha_1) + 3v_0^2\alpha_1\right. \\
&\quad \left. -v_0^2(1+\alpha_1) - v_0\alpha_0(4+3\alpha_1) - 3\alpha_0^2(1+\alpha_1)\right]\alpha_1^{2l-k} \\
&\quad +\frac{v_0+2\alpha_0}{1-\alpha_1}\mu(k,l) - f_2\mu(k)[\alpha_0 + (v_0+\alpha_0)\alpha_1] \\
&= f_4\left\{c_0 - 4v_0d_0 + 3v_0^3 + 3v_0(v_0^2 - d_0)\alpha_1 + (3v_0d_0 - c_0)\alpha_1^2\right. \\
&\quad \left. +(7v_0d_0 - 6v_0^3 - c_0)\alpha_1^3 + 3v_0(d_0 - 2v_0^2)\alpha_1^4 + (6v_0^3 - 6v_0d_0 + c_0)\alpha_1^5\right\}\alpha_1^{2l+k}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2v_0+\alpha_0}{1-\alpha_1} f_3\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]\alpha_1^{2l} \\
&+\left\{\frac{\alpha_0 f_3}{1-\alpha_1}[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]\right\}\alpha_1^{2(l-k)} \\
&+\frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)} f_2\left[2v_0\alpha_0+d_0(1-\alpha_1)+v_0^2(2\alpha_1-1)\right]\alpha_1^{2l-k} \\
&+\frac{v_0+2\alpha_0}{1-\alpha_1}\mu(k,l)-f_2\mu(k)[\alpha_0+(v_0+\alpha_0)\alpha_1].
\end{aligned}
$$

So it follows that we have

$$
\begin{aligned}
\mu(k,l,m) &= \alpha_1^{m-l}[\mu(k,l,l)-f_1\mu(k,l)]+f_1\mu(k,l) \\
&= \alpha_1^{m-l}\left[f_4\left\{c_0-4v_0d_0+3v_0^3+3v_0(v_0^2-d_0)\alpha_1+(3v_0d_0-c_0)\alpha_1^2\right.\right. \\
&\quad \left.+(7v_0d_0-6v_0^3-c_0)\alpha_1^3+3v_0(d_0-2v_0^2)\alpha_1^4+(6v_0^3-6v_0d_0+c_0)\alpha_1^5\right\}\alpha_1^{2l+k} \\
&\quad +\frac{2v_0+\alpha_0}{1-\alpha_1} f_3\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]\alpha_1^{2l} \\
&\quad +\left\{\frac{\alpha_0 f_3}{1-\alpha_1}[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]\right\}\alpha_1^{2(l-k)} \\
&\quad +\frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)} f_2\left[2v_0\alpha_0+d_0(1-\alpha_1)+v_0^2(2\alpha_1-1)\right]\alpha_1^{2l-k} \\
&\quad \left.+\frac{v_0+\alpha_0}{1-\alpha_1}\mu(k,l)-f_2\mu(k)[\alpha_0+(v_0+\alpha_0)\alpha_1]\right]+f_1\mu(k,l),
\end{aligned}
$$

which holds for all $0 \leq k \leq l \leq m$. ■

C.4 Proof of Corollary 4.2

We recall that we have to prove that if X is a first-order stationary process following a CP-INARCH(1) model such that **H4** is satisfied, then

(a) For any $s \geq 0$, we have

$$
\widetilde{\mu}(s) = \kappa(s) = v_0\alpha_1^s f_2.
$$

(b) For any $l \geq s \geq 0$, we have

$$
\begin{aligned}
\widetilde{\mu}(s,l) &= \kappa(s,l) \\
&= f_3\alpha_1^l[v_0^2(1+\alpha_1+\alpha_1^2)-\{v_0^2(1+\alpha_1-2\alpha_1^2)-d_0(1-\alpha_1^2)\}\alpha_1^s].
\end{aligned}
$$

(c) For any $m \geq l \geq s \geq 0$, we have

$$
\begin{aligned}
\kappa(s,l,m) &= \alpha_1^m f_4\left[\left\{c_0+3v_0^3-4v_0d_0+3v_0(v_0^2-d_0)\alpha_1+(3\alpha_0d_0-c_0)\alpha_1^2\right.\right. \\
&\quad \left.+(7v_0d_0-6v_0^3-c_0)\alpha_1^3+3v_0(d_0-2v_0^2)\alpha_1^4+(6v_0^3-6v_0d_0+c_0)\alpha_1^5\right\}\alpha_1^{l+s} \\
&\quad +v_0(1+\alpha_1+\alpha_1^2+\alpha_1^3)[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)](2\alpha_1^l+\alpha_1^s)
\end{aligned}
$$

$$+v_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)[(1+\alpha_1)v_0^2+(d_0(1-\alpha_1)+v_0^2(2\alpha_1-1))\alpha_1^{l-s}]\Big],$$

$$\widetilde{\mu}(s,l,m) \;=\; \kappa(s,l,m)+v_0^2 f_2^2\alpha_1^{m-l+s}+2v_0^2 f_2^2\alpha_1^{m+l-s}.$$

Proof:

(a) From (4.11) and (4.8) we have, for any $s \geq 0$,

$$\widetilde{\mu}(s) \;=\; \kappa(s) \;=\; Cov(X_t, X_{t+s}) \;=\; v_0\alpha_1^s f_2.$$

(b) From (4.12), (4.9) and using the expressions stated in Theorem 4.2, the third-order central moment and cumulant of X are given by

$$\begin{aligned}
\widetilde{\mu}(s,l) &= \kappa(s,l)\\
&= [d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{l+s}+\frac{v_0(\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^l\\
&\quad +v_0 f_1 f_2\alpha_1^{l-s}+f_1\mu(s)-f_1\mu(s)\\
&\quad -f_1[f_2(v_0\alpha_1^{l-s}+\alpha_0(1+\alpha_1))+f_2(v_0\alpha_1^l+\alpha_0(1+\alpha_1))-2f_1^2]\\
&= [d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{l+s}+\frac{v_0(\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^l\\
&\quad +v_0 f_1 f_2\alpha_1^{l-s}-f_1 f_2 v_0(\alpha_1^{l-s}+\alpha_1^l)\\
&= [d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{l+s}+\frac{v_0^2 f_2}{1-\alpha_1}\alpha_1^l\\
&= f_3\alpha_1^l[v_0^2(1+\alpha_1+\alpha_1^2)-\{v_0^2(1+\alpha_1-2\alpha_1^2)-d_0(1-\alpha_1^2)\}\alpha_1^s],
\end{aligned}$$

for $l \geq s \geq 0$.

(c) In what concerns the fourth-order cumulant we have from (4.10) and the expressions stated in Theorem 4.2, for $m \geq l \geq s \geq 0$,

$$\begin{aligned}
\kappa(s,l,m) &= \boldsymbol{\alpha_1^{m-l}}\Big[\;\alpha_1^{2l+s}f_4\{c_0-4v_0d_0+3v_0^3+3v_0(v_0^2-d_0)\alpha_1+(3v_0d_0-c_0)\alpha_1^2\\
&\quad +(7v_0d_0-6v_0^3-c_0)\alpha_1^3+3v_0(d_0-2v_0^2)\alpha_1^4+(6v_0^3-6v_0d_0+c_0)\alpha_1^5\}\\
&\quad +\frac{2v_0+\alpha_0}{1-\alpha_1}f_3\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]\alpha_1^{2l}\\
&\quad \boldsymbol{+\left\{\frac{\alpha_0 f_3}{1-\alpha_1}[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]\right\}\alpha_1^{2(l-s)}}\\
&\quad +\frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)}f_2\left[2v_0\alpha_0+d_0(1-\alpha_1)+v_0^2(2\alpha_1-1)\right]\alpha_1^{2l-s}\\
&\quad +\frac{v_0+\alpha_0}{1-\alpha_1}\mu(s,l)-f_2\mu(s)[\alpha_0+(v_0+\alpha_0)\alpha_1]\;\Big]\boldsymbol{+f_1\mu(s,l)-f_1\mu(s,l)}\\
&\quad \boldsymbol{-f_1}\Big(\;\boldsymbol{[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{m+l-2s}}+\frac{v_0(v_0+\alpha_0)}{1-\alpha_1}f_2\alpha_1^{m-s}
\end{aligned}$$

$$
\begin{aligned}
&+v_0 f_1 f_2 \alpha_1^{m-l} \boldsymbol{+f_1\mu(l-s)-f_1 f_2(v_0\alpha_1^{l-s}+\alpha_0(1+\alpha_1))}\\
&+[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{m+l}\\
&+\frac{v_0(v_0+\alpha_0)}{1-\alpha_1}f_2\alpha_1^m+f_1f_2v_0\alpha_1^{m-l}\boldsymbol{+f_1\mu(l)-f_1\mu(l)}+\frac{v_0(v_0+\alpha_0)}{1-\alpha_1}f_2\alpha_1^m\\
&+[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{m+s}+v_0f_1f_2\alpha_1^{m-s}\boldsymbol{+f_1\mu(s)-f_1\mu(s)}\quad)\\
&-(f_2[v_0\alpha_1^s+\alpha_0(1+\alpha_1)]-f_1^2)(f_2[v_0\alpha_1^{m-l}+\alpha_0(1+\alpha_1)]-f_1^2)\\
&-(f_2[v_0\alpha_1^l+\alpha_0(1+\alpha_1)]-f_1^2)(f_2[v_0\alpha_1^{m-s}+\alpha_0(1+\alpha_1)]-f_1^2)\\
&-(f_2[v_0\alpha_1^m+\alpha_0(1+\alpha_1)]-f_1^2)(f_2[v_0\alpha_1^{l-s}+\alpha_0(1+\alpha_1)]-f_1^2)\\
&+f_1^2\Big(\ f_2[v_0\alpha_1^m+\alpha_0(1+\alpha_1)]+f_2[v_0\alpha_1^{m-s}+\alpha_0(1+\alpha_1)]\\
&+f_2[v_0\alpha_1^{m-l}+\alpha_0(1+\alpha_1)]-3f_1^2\ \Big),
\end{aligned}
$$

where we highlight, using bold, expressions whose sum equals zero.

So, taking into account that

$$
-f_2\mu(s)[\alpha_0+(v_0+\alpha_0)\alpha_1]\alpha_1^{m-l}=\left[-f_1\frac{\alpha_0+v_0}{1-\alpha_1}\mu(s)+v_0f_2\mu(s)\right]\alpha_1^{m-l}
$$

and

$$
\begin{aligned}
&-(f_2[v_0\alpha_1^s+\alpha_0(1+\alpha_1)]-f_1^2)(f_2[v_0\alpha_1^{m-l}+\alpha_0(1+\alpha_1)]-f_1^2)\\
&-(f_2[v_0\alpha_1^l+\alpha_0(1+\alpha_1)]-f_1^2)(f_2[v_0\alpha_1^{m-s}+\alpha_0(1+\alpha_1)]-f_1^2)\\
&-(f_2[v_0\alpha_1^m+\alpha_0(1+\alpha_1)]-f_1^2)(f_2[v_0\alpha_1^{l-s}+\alpha_0(1+\alpha_1)]-f_1^2)\\
&+f_1^2\left(f_2[v_0\alpha_1^m+\alpha_0(1+\alpha_1)]+f_2[v_0\alpha_1^{m-s}+\alpha_0(1+\alpha_1)]\right.\\
&\left.+f_2[v_0\alpha_1^{m-l}+\alpha_0(1+\alpha_1)]-3f_1^2\right)\\
=\ &-f_2^2(v_0\alpha_1^s+\alpha_0(1+\alpha_1))(v_0\alpha_1^{m-l}+\alpha_0(1+\alpha_1))\\
&+f_1^2f_2(v_0\alpha_1^s+\alpha_0(1+\alpha_1))+f_1^2f_2(v_0\alpha_1^{m-l}+\alpha_0(1+\alpha_1))-f_1^4\\
&-f_2^2(v_0\alpha_1^l+\alpha_0(1+\alpha_1))(v_0\alpha_1^{m-s}+\alpha_0(1+\alpha_1))\\
&+f_1^2f_2(v_0\alpha_1^l+\alpha_0(1+\alpha_1))+f_1^2f_2(v_0\alpha_1^{m-s}+\alpha_0(1+\alpha_1))-f_1^4\\
&-f_2^2(v_0\alpha_1^m+\alpha_0(1+\alpha_1))(v_0\alpha_1^{l-s}+\alpha_0(1+\alpha_1))\\
&+f_1^2f_2(v_0\alpha_1^m+\alpha_0(1+\alpha_1))+f_1^2f_2(v_0\alpha_1^{l-s}+\alpha_0(1+\alpha_1))-f_1^4\\
&+f_1^2f_2(v_0\alpha_1^m+\alpha_0(1+\alpha_1))+f_1^2f_2(v_0\alpha_1^{m-s}+\alpha_0(1+\alpha_1))\\
&+f_1^2f_2[v_0\alpha_1^{m-l}+\alpha_0(1+\alpha_1)]-3f_1^4\\
=\ &-f_2^2\left[v_0^2\alpha_1^{m-l+s}+\alpha_0(1+\alpha_1)v_0\alpha_1^s+\alpha_0(1+\alpha_1)v_0\alpha_1^{m-l}+\alpha_0^2(1+\alpha_1)^2\right]\\
&+f_1^2f_2(v_0\alpha_1^s+\alpha_0(1+\alpha_1))+2f_1^2f_2(v_0\alpha_1^{m-l}+\alpha_0(1+\alpha_1))-6f_1^4\\
&-f_2^2\left[v_0^2\alpha_1^{m+l-s}+\alpha_0(1+\alpha_1)v_0\alpha_1^l+\alpha_0(1+\alpha_1)v_0\alpha_1^{m-s}+\alpha_0^2(1+\alpha_1)^2\right]\\
&+f_1^2f_2(v_0\alpha_1^l+\alpha_0(1+\alpha_1))+2f_1^2f_2(v_0\alpha_1^{m-s}+\alpha_0(1+\alpha_1))\\
&-f_2^2\left[v_0^2\alpha_1^{m+l-s}+\alpha_0(1+\alpha_1)v_0\alpha_1^{l-s}+\alpha_0(1+\alpha_1)v_0\alpha_1^m+\alpha_0^2(1+\alpha_1)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
& +f_1^2 f_2(v_0\alpha_1^{l-s}+\alpha_0(1+\alpha_1))+2f_1^2 f_2(v_0\alpha_1^m+\alpha_0(1+\alpha_1)) \\
= & -v_0^2 f_2^2\alpha_1^{m-l+s}+[f_1^2 f_2 v_0 - f_2^2 v_0\alpha_0(1+\alpha_1)][\alpha_1^s+\alpha_1^l+\alpha_1^{l-s}] \\
& +[2f_1^2 f_2 v_0 - f_2^2 v_0\alpha_0(1+\alpha_1)][\alpha_1^{m-s}+\alpha_1^{m-l}+\alpha_1^m]-2f_2^2 v_0^2\alpha_1^{m+l-s} \\
& -3f_2^2\alpha_0^2(1+\alpha_1)^2+9f_1^2 f_2\alpha_0(1+\alpha_1)-6f_1^4 \\
= & -v_0^2 f_2^2[\alpha_1^{m-l+s}+2\alpha_1^{m+l-s}]+v_0 f_1^2 f_2[\alpha_1^{m-l}+\alpha_1^{m-s}+\alpha_1^m]
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\kappa(s,l,m) = & \alpha_1^{m+l+s} f_4\{c_0-4v_0d_0+3v_0^3+3v_0(v_0^2-d_0)\alpha_1+(3v_0d_0-c_0)\alpha_1^2 \\
& +(7v_0d_0-6v_0^3-c_0)\alpha_1^3+3v_0(d_0-2v_0^2)\alpha_1^4+(6v_0^3-6v_0d_0+c_0)\alpha_1^5\} \\
& +\frac{2v_0+\boldsymbol{\alpha_0}}{\mathbf{1-\alpha_1}}\boldsymbol{f_3\alpha_1^{m+l}\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right]} \\
& \boldsymbol{+\frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)}f_2\alpha_1^{m+l-s}}\left[\boldsymbol{2v_0\alpha_0}+d_0(1-\alpha_1)+v_0^2(2\alpha_1-1)\right] \\
& +\frac{v_0+\alpha_0}{1-\alpha_1}\alpha_1^{m-l}\{\mu(s,l)-f_1\mu(s)\}+v_0f_2\mu(s)\alpha_1^{m-l} \\
& \boldsymbol{-f_1}\left(\frac{v_0(v_0+2\alpha_0)}{1-\alpha_1}f_2\alpha_1^{m-s}+\frac{2v_0(v_0+\alpha_0)}{1-\alpha_1}f_2\alpha_1^m+2v_0f_1f_2\alpha_1^{m-l}\right. \\
& \left.\boldsymbol{+[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]f_3[\alpha_1^{m+l}}+\alpha_1^{m+s}]\right) \\
& \boldsymbol{-v_0^2f_2^2}(\alpha_1^{m-l+s}+\boldsymbol{2\alpha_1^{m+l-s}})+v_0f_1^2f_2(\alpha_1^m+\alpha_1^{m-s}+\alpha_1^{m-l}).
\end{aligned}
$$

As the sum of the expressions in bold equals 0 and replacing $\mu(s,l)$, we obtain

$$
\begin{aligned}
\kappa(s,l,m) = & \alpha_1^{m+l+s} f_4\{c_0-4v_0d_0+3v_0^3+3v_0(v_0^2-d_0)\alpha_1+(3v_0d_0-c_0)\alpha_1^2 \\
& +(7v_0d_0-6v_0^3-c_0)\alpha_1^3+3v_0(d_0-2v_0^2)\alpha_1^4+(6v_0^3-6v_0d_0+c_0)\alpha_1^5\} \\
& +\frac{2v_0}{1-\alpha_1}f_3\alpha_1^{m+l}\left[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)\right] \\
& +\frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)}f_2\alpha_1^{m+l-s}\left[d_0(1-\alpha_1)+v_0^2(2\alpha_1-1)\right] \\
& +\frac{v_0+\boldsymbol{\alpha_0}}{\mathbf{1-\alpha_1}}\boldsymbol{\alpha_1^{m-l}}\left\{\boldsymbol{[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{l+s}}\right. \\
& \left.+\frac{v_0(v_0+\alpha_0)}{1-\alpha_1}f_2\alpha_1^l+v_0f_1f_2\alpha_1^{l-s}\right\} \\
& \boldsymbol{-f_1}\left(\frac{v_0(v_0+2\alpha_0)}{1-\alpha_1}f_2\alpha_1^{m-s}+\boldsymbol{[d_0(1-\alpha_1^2)-v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{m+s}}\right. \\
& \left.-v_0f_1f_2\alpha_1^m+\frac{2v_0(v_0+\alpha_0)}{1-\alpha_1}f_2\alpha_1^m-v_0f_1f_2\alpha_1^{m-s}\right) \\
& \boldsymbol{+v_0f_2\alpha_1^{m-l}[\mu(s)-f_1^2]-v_0^2f_2^2\alpha_1^{m-l+s}}.
\end{aligned}
$$

The sum of the expressions in bold equals 0 and then we conclude

$$\begin{aligned}
\kappa(s,l,m) &= \alpha_1^{m+l+s} f_4 \left\{c_0 - 4v_0 d_0 + 3v_0^3 + 3v_0(v_0^2 - d_0)\alpha_1 + (3v_0 d_0 - c_0)\alpha_1^2 \right. \\
&\quad \left. +(7v_0 d_0 - 6v_0^3 - c_0)\alpha_1^3 + 3v_0(d_0 - 2v_0^2)\alpha_1^4 + (6v_0^3 - 6v_0 d_0 + c_0)\alpha_1^5 \right\} \\
&\quad + \frac{2v_0}{1-\alpha_1} f_3 \alpha_1^{m+l} \left\{d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1 - 2\alpha_1^2)\right\} \\
&\quad + \frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)} f_2 \alpha_1^{m+l-s} \left\{d_0(1-\alpha_1) + v_0^2(2\alpha_1 - 1)\right\} \\
&\quad + \frac{v_0+\alpha_0}{1-\alpha_1}\alpha_1^{m-l} \left\{\frac{v_0(v_0+\alpha_0)}{1-\alpha_1} f_2 \alpha_1^l + v_0 f_1 f_2 \alpha_1^{l-s}\right\} \\
&\quad + \frac{v_0}{1-\alpha_1} f_3 \alpha_1^{m+s}[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1 - 2\alpha_1^2)] \\
&\quad - f_1 \left(\frac{v_0(v_0+\alpha_0)}{1-\alpha_1} f_2 \alpha_1^{m-s} + \frac{v_0(2v_0+\alpha_0)}{1-\alpha_1} f_2 \alpha_1^m\right) \\
&= \alpha_1^m f_4 \left[\left\{c_0 - 4v_0 d_0 + 3v_0^3 + 3v_0(v_0^2 - d_0)\alpha_1 + (3\alpha_0 d_0 - c_0)\alpha_1^2 \right.\right. \\
&\quad \left. +(7v_0 d_0 - 6v_0^3 - c_0)\alpha_1^3 + 3v_0(d_0 - 2v_0^2)\alpha_1^4 + (6v_0^3 - 6v_0 d_0 + c_0)\alpha_1^5\right\} \alpha_1^{l+s} \\
&\quad + v_0(1+\alpha_1+\alpha_1^2+\alpha_1^3)[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1 - 2\alpha_1^2)](2\alpha_1^l + \alpha_1^s) \\
&\quad \left. + v_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)[(1+\alpha_1)v_0^2 + (d_0(1-\alpha_1) + v_0^2(2\alpha_1 - 1))\alpha_1^{l-s}]\right],
\end{aligned}$$

for any $m \geq l \geq s \geq 0$.

Finally, the fourth-order central moment of X, using the expression (4.13), is given by

$$\begin{aligned}
\widetilde{\mu}(s,l,m) &= \kappa(s,l,m) + v_0\alpha_1^s f_2 v_0 \alpha_1^{m-l} f_2 + v_0 \alpha_1^l f_2 v_0 \alpha_1^{m-s} f_2 + v_0 \alpha_1^{l-s} f_2 v_0 \alpha_1^m f_2 \\
&= \kappa(s,l,m) + v_0^2 f_2^2 \alpha_1^{m-l+s} + 2v_0^2 f_2^2 \alpha_1^{m+l-s}. \quad \blacksquare
\end{aligned}$$

Appendix D

Programs

In this appendix we summarize the EViews and the MATLAB programs developed to obtain the trajectories of the particular ZICP-INGARCH(p,q) models of the Sections 2.3 and 5.2, the stationarity regions of the Chapters 2 and 5, and the tables and confidence regions of the Section 4.2.

D.1 Trajectories of Sections 2.3 and 5.2

The programs of this section are developed in EViews code. Let us note that after generating each trajectory with 1100 elements, the first 100 observations were discarded to eliminate the effect of choosing the values of the initial observations.

To simulate a particular realization from the CP-INGARCH model we use the representation (2.5) stated in Section 2.3. Thus, the main idea of these programs is, firstly, to generate a scalar N that follows a Poisson law and then generate N scalars following the conditional compounding distribution.

To generate the INGARCH$(1,1)$ model of Figure 2.3:

```
smpl @first @last
series x=100.0
smpl @first @last
series lambda=100.0
for !i = 2 to 1100
    lambda(!i)=10.0+0.4*x(!i-1)+0.5*lambda(!i-1)
    x(!i)=@rpoisson(lambda(!i))
next
```

To generate the NTA-INGARCH$(1,1)$ model of Figure 2.4:

```
scalar y
scalar parametro
scalar N
smpl @first @last
series x=100.0
```

```
smpl @first @last
series lambda=100.0
scalar phi=2
for !i = 2 to 1100
    lambda(!i)=10.0+0.4*x(!i-1)+0.5*lambda(!i-1)
    parametro=lambda(!i)/phi
    N=@rpoisson(parametro)
    y=0.0
    for !j = 1 to N
        y=y+@rpoisson(phi)
    next
    x(!i)=y
next
```

To generate the GEOMP-INGARCH$(1,1)$ model of Figure 2.5:

```
scalar y
scalar conta
scalar fcum
scalar v
scalar u
scalar p
scalar N
smpl @first @last
series x=100.0
smpl @first @last
series lambda=100.0
scalar r=2
for !i = 2 to 1100
    lambda(!i)=10.0+0.4*x(!i-1)+0.5*lambda(!i-1)
    p=r/(r+lambda(!i))
    N=@rpoisson(r)
    y=0.0
    for !j=1 to N
        u=@runif(0,1)
        conta=0
        v=p
        fcum=v
        while u >= fcum
        v=v*(1-p)
        fcum=fcum+v
        conta=conta+1
```

```
        wend
        y=y+conta
    next
    x(!i)=y
next
```

<u>To generate the CP-INGARCH$(1,1)$ model of Figure 2.6</u>:

```
scalar q
scalar s
series y
series lambda
series p
series n
scalar x
y=0.0
lambda=10.0
scalar r=5000
for !t=2 to 1100
    lambda(!t)=10.0+0.4*y(!t-1)+0.5*lambda(!t-1)
    q=1/((!t)^2+1)
    p(!t)=lambda(!t)/(r*q)
    n(!t)=@rpoisson(p(!t))
    s=0
    for !j=1 to n(!t)
        x=@rbinom(r,q)
        s=s+x
    next
    y(!t)=s
next
```

<u>To generate the GEOMP2-INARCH(1) model of Figure 2.7</u>:

```
scalar parametro
scalar u
scalar fcum
scalar conta
scalar y
scalar N
smpl @first @last
series x=100.0
smpl @first @last
```

```
series lambda=100.0
scalar p=0.3
for !i = 2 to 1100
    lambda(!i)=10.0+0.4*x(!i-1)
    parametro=(p*lambda(!i))/(1-p)
    N=@rpoisson(parametro)
    y=0.0
    for !j = 1 to N
        u=@runif(0,1)
        fcum=p
        conta=0
        while u>= fcum
        conta=conta +1
        fcum=fcum +p*(1-p)^(conta)
        wend
        y=y+conta
    next
    x(!i)=y
next
```

To generate the GP-INARCH(1) model of Figure 2.8:

```
scalar parametro
scalar y
scalar u
scalar fcum
scalar conta
scalar N
smpl @first @last
series x=100.0
smpl @first @last
series lambda=100.0
scalar kappa=0.5
for !i = 2 to 1100
    lambda(!i)=10.0+0.4*x(!i-1)
    parametro=(1-kappa)*lambda(!i)
    N=@rpoisson(parametro)
    y=0.0
    for !j=1 to N
        u=@runif(0,1)
        fcum=@exp(-kappa)
        conta=1
```

```
            while u>= fcum
            conta=conta +1
            fcum=fcum +((conta*kappa)∧(conta-1)*@exp(-kappa*conta))/@fact(conta)
            wend
            y=y+conta
      next
      x(!i)=y
next
```

To simulate a particular realization from the ZICP-INGARCH model we use the representation (5.4) stated in Section 5.2. The main idea of the next programs is, firstly, to generate a scalar M that follows a Bernoulli law with parameter $1-\omega$ and then, if M$\neq 0$, we generate M scalars following the conditional compound Poisson law which have the additional proportion of zeros.

To generate the ZIP-INGARCH$(1,1)$ model of Figure 5.2:

```
scalar M
scalar N
smpl @first @last
series x=44.0
smpl @first @last
series lambda=44.0
scalar omega=0.2
for !i = 2 to 1100
      lambda(!i)=10.0+0.4*x(!i-1)+0.5*lambda(!i-1)
      M=@rbinom(1,1-omega)
      if M=0 then
      x(!i)=0
      else
      x(!i)=@rpoisson(lambda(!i))
      endif
next
```

To generate the ZINTA-INGARCH$(1,1)$ model of Figure 5.3:

```
scalar M
scalar N
scalar parametro
series x=12.0
series lambda=12.0
scalar phi=2
scalar omega=0.6
for !i = 2 to 1100
```

```
    lambda(!i)=10.0+0.4*x(!i-1)+0.5*lambda(!i-1)
    M=@rbinom(1,1-omega)
    if M=0 then
    x(!i)=0
    else
    parametro=lambda(!i)/phi
    N=@rpoisson(parametro)
    scalar y=0.0
    for !j = 1 to N
        y=y+@rpoisson(phi)
    next
    x(!i)=y
    endif
next
```

D.2 Stationarity regions of Sections 3.2 and 5.3

The programs of this section were developed in MATLAB code.

To get the weak stationarity regions $\{(\alpha_p,\beta_p)\in(\mathbb{R}_0^+)^2:(1-\omega)(1+v_1)\alpha_p^2+2(1-\omega)\alpha_p\beta_p+\beta_p^2<1\}$ of a ZICP-INGARCH(p,p) model with the coefficients $\alpha_1=...=\alpha_{p-1}=\beta_1=...=\beta_{p-1}=0$, under the condition $(1-\omega)\alpha_p+\beta_p<1$ and considering different values for v_1, we run the program:

a) **function** [] = regioes_est_fraca_ZICP_INGARCH_p_p_graf()

```
grey1 = [0.75,0.75,0.75];
grey2 = [0.55,0.55,0.55];
grey3 = [0.4,0.4,0.4];
alphap = [0.001:0.001:1];
betap = [0.001:0.001:1];

B = regiao_est_fraca_zero_iflated_p_p(alphap, betap, 5, 0);
C = regiao_est_fraca_zero_iflated_p_p(alphap, betap, 0.5, 0);
D = regiao_est_fraca_zero_iflated_p_p(alphap, betap, 0, 0);

t1 = size(B);   t2 = size(C);   t3 = size(D);
i = 1:1:t1(1);   j = 1:1:t2(1);   k = 1:1:t3(1);
f1(i) = B(i,1);   g1(i) = B(i,2);
f2(j) = C(j,1);   g2(j) = C(j,2);
f3(k) = D(k,1);   g3(k) = D(k,2);

plot(f3(k),g3(k), 'Color', grey1)
hold on
plot(f2(j),g2(j), 'Color', grey2)
hold on
```

```
    plot(f1(i),g1(i), 'Color', grey3)
    xlabel('alpha_p')
    ylabel('beta_p')
```

where

b) **function** B = regiao_est_fraca_zero_iflated_p_p(alphap, betap, v1, omega)

```
    b1 = size(alphap,2);
    b2 = size(betap,2);
    b = b1*b2;
    A = zeros(b,2);
    conta = 0;

    for i=1:size(alphap,2)
        for j=1:size(betap,2)
            if ((1-omega)*alphap(i) + betap(j))<1
                aux1 = (1-omega)*(1+v1)*alphap(i)^2;
                aux2 = 2*(1-omega)*alphap(i)*betap(j)+betap(j)^2;
                if (aux1 + aux2)<1
                    conta = conta + 1;
                    A(conta,1) = alphap(i);
                    A(conta,2) = betap(j);
                end
            end
        end
    end

    B = zeros(conta,2);
    for i=1:conta
        B(i,:)=A(i,:);
    end
```

Let us observe that when we run the program *regioes_est_fraca_ZICP_INGARCH_p_p_graf* we obtain the regions presented in Figure 3.1. To get the regions on the left in Figure 5.6 we consider

```
B = regiao_est_fraca_zero_iflated_p_p(alphap, betap, 0, 0);
C = regiao_est_fraca_zero_iflated_p_p(alphap, betap, 0, 0.4);
D = regiao_est_fraca_zero_iflated_p_p(alphap, betap, 0, 0.9);
```

and for the regions on the right in Figure 5.6 we need to consider

```
B = regiao_est_fraca_zero_iflated_p_p(alphap, betap, 5, 0);
C = regiao_est_fraca_zero_iflated_p_p(alphap, betap, 5, 0.4);
D = regiao_est_fraca_zero_iflated_p_p(alphap, betap, 5, 0.9);
```

instead of the lines 7, 8 and 9 of the program.

To represent the weak stationarity regions of a CP-INGARCH$(2,1)$ process considering different values for the parameter v_1 (namely, $v_1 = 0, 0.5, 5$) and under the condition $\alpha_1 + \alpha_2 + \beta_1 < 1$, which we presented in figures 3.2 and 3.3 we used the following program:

c)
```
function [] = regiao_est_fraca_CP_INGARCH_2_1()

grey1 = [0.75,0.75,0.75];
grey2 = [0.55,0.55,0.55];
grey3 = [0.4,0.4,0.4];

B = regiao(0);
C = regiao(0.5);
D = regiao(5);

t = size(B);   t1 = size(C);   t2 = size(D);
i = 1:1:t(1);   j = 1:1:t1(1);   k = 1:1:t2(1);
f(i) = B(i,1);   f1(j) = C(j,1);   f2(k) = D(k,1);
g(i) = B(i,2);   g1(j) = C(j,2);   g2(k) = D(k,2);
h(i) = B(i,3);   h1(j) = C(j,3);   h2(k) = D(k,3);

subplot(2,3,1)
plot3(f(i),g(i),h(i), 'Color', grey1)
grid on
xlabel('alpha_1')
ylabel('alpha_2')
zlabel('beta_1')

subplot(2,3,2)
plot3(f1(j),g1(j),h1(j), 'Color', grey2)
grid on
xlabel('alpha_1')
ylabel('alpha_2')
zlabel('beta_1')

subplot(2,3,3)
plot3(f2(k),g2(k),h2(k), 'Color', grey3)
grid on
xlabel('alpha_1')
ylabel('alpha_2')
zlabel('beta_1')

subplot(2,3,4)
plot(f(i),g(i), 'Color', grey1)
hold on
plot(f1(j),g1(j), 'Color', grey2)
hold on
```

```
plot(f2(k),g2(k), 'Color', grey3)
grid
xlabel('alpha_1')
ylabel('alpha_2')
title('View just the X,Y plane')

subplot(2,3,5)
plot(f(i),h(i), 'Color', grey1)
hold on
plot(f1(j),h1(j), 'Color', grey2)
hold on
plot(f2(k),h2(k), 'Color', grey3)
grid
xlabel('alpha_1')
ylabel('beta_1')
title('View just the X,Z plane')

subplot(2,3,6)
plot(g(i),h(i), 'Color', grey1)
hold on
plot(g1(j),h1(j), 'Color', grey2)
hold on
plot(g2(k),h2(k), 'Color', grey3)
grid
xlabel('alpha_2')
ylabel('beta_1')
title('View just the Y,Z plane')
```

where we have

d) **function** B = regiao(v1)

% INPUT:
% parameter v_1
% OUTPUT:
% matrix B containing the points $(\alpha_1, \alpha_2, \beta_1)$ **that belong to the weak**
% stationarity region for this particular v_1

```
syms alpha1 alpha2 beta1
g = obter_zeros(v1);
syms f1(alpha1,alpha2,beta1,v1)
syms f2(alpha1,alpha2,beta1,v1)
syms f3(alpha1,alpha2,beta1,v1)
f1(alpha1,alpha2,beta1,v1) = g(1);
```

```
f2(alpha1,alpha2,beta1,v1) = g(2);
f3(alpha1,alpha2,beta1,v1) = g(3);
x = [0.001:0.02:1];
y = [0.001:0.02:1];
z = [0.001:0.02:1];
b1 = size(x,2);   b2 = size(y,2);   b3 = size(z,2);
b = b1*b2*b3;
A = zeros(b,3);
conta = 0;

for i=1:size(x,2)
    for j=1:size(y,2)
        for k=1:size(z,2)
            if (x(i) + y(j) + z(k))<1
                aux1 = double(f1(x(i),y(j),z(k),v1));
                aux2 = double(f2(x(i),y(j),z(k),v1));
                aux3 = double(f3(x(i),y(j),z(k),v1));
                if (abs(aux1) > 1) && (abs(aux2) > 1) && (abs(aux3) > 1)
                    conta = conta + 1;
                    A(conta,1) = x(i);
                    A(conta,2) = y(j);
                    A(conta,3) = z(k);
                end
            end
        end
    end
end

B = zeros(conta,3);
for i=1:conta
    B(i,:)=A(i,:);
end
```

and also

e)
```
function g = obter_zeros(v1)
% INPUT:
% parameter v1
% OUTPUT:
% the roots of the equation det P(z) = 0 for this particular v1

syms alpha1 alpha2 beta1
aux1 = (alpha1+beta1)^2*alpha2+alpha2^2;
aux2 = v1*(alpha2^2+alpha1^2*alpha2+2*alpha1*alpha2*beta1);
```

```
p1 = alpha2^3+v1*alpha2^3;
p2 = aux1+aux2;
p3 = (alpha1+beta1)^2+alpha2+v1*alpha1^2;
p = [p1 -p2 -p3 1];
g = roots(p);
```

D.3 Simulation Study - Section 4.2

The following programs are developed to use in MATLAB.

In what concerns the Table 4.1 and Table 4.2 we developed the following algorithm:

a)

```
function [g] = valores(alpha0, alpha1, n)

% INPUT:
% real parameters (α0, α1) of the INARCH(1) model
% length n of the trajectory
% OUTPUT:
% means, variances and covariances of the estimates

A = zeros(1000,2);
b1 = ones(1000,1);
b2 = ones(1000,1);
u1 = ones(1000,1);
u2 = ones(1000,1);
d = zeros(1000,1);

for i=1:1000
    x = valores_trajectoria_Poisson(alpha0, alpha1, n);
    est = estimativaCLS_Poisson(x);
    A(i,1) = est(1);
    A(i,2) = est(2);
end

% A is the matrix that in the column j has the estimates for the parameter αj,
% j = 0,1, that we obtain from the 1000 trajectories that were generated

S = sum(A);
% S is the 1x2 matrix that in the column j has the sum of the estimates of αj, j = 0,1

media(1) = S(1)/1000;
media(2) = S(2)/1000;
b1 = media(1)*b1;
b2 = media(2)*b2;
B = [b1  b2];
% B is the 1000x2 matrix that in the column j has the mean of the 1000
% estimates of αj, j = 0,1
```

```
D = A - B;
C = D.^2;
V = sum(C);
variancia(1) = n*(V(1)/1000);
variancia(2) = n*(V(2)/1000);
u1 = alpha0*u1;
u2 = alpha1*u2;
U = [u1  u2];
R = (U - A).^2;
Q = sum(R);
standarderror(1) = sqrt(Q(1)/1000);
standarderror(2) = sqrt(Q(2)/1000);

for j=1:1000
    d(j) = D(j,1)*D(j,2);
end

covariancia = n*(sum(d)/1000);
g = [media(1)  media(2)  variancia(1)  standarderror(1)  variancia(2)  standarderror(2)
    covariancia];
```

where we have

b)
```
function [y] = valores_trajectoria_Poisson(alpha0, alpha1, n)

% INPUT:
% parameters α0, α1
% length n of the trajectory
% OUTPUT:
% one trajectory with n elements of an INARCH(1) process with parameters α0 and α1

x = zeros(1,n+100);
y = zeros(1,n);
lambda = zeros(1,n+100);
lambda(1) = alpha0/(1-alpha1);

for i=2:n+100
    lambda(i)= alpha0 + alpha1*x(i-1);
    x(i) = poissrnd(lambda(i));
end

for k=1:n
    y(k)=x(k+100);
end
```

and to calculate the CLS estimates we use

c) **function** [estimativas] = estimativaCLS_Poisson(y)

```
% INPUT:
% trajectory y
% OUTPUT:
% CLS estimates of the parameters (α0,α1) of the process INARCH(1) from trajectory y

n=length(y);
soma = 0;
soma0 = 0;

for i=2:n
    soma = soma + y(i)*y(i-1);
    soma0 = soma0 + y(i-1)*y(i-1);
end

somar=0;
mediaemp=sum(y)/n;

for i=1:n
    somar= somar + (y(i)- mediaemp)^2;
end

soma1=sum(y)-y(1);
soma2=sum(y)-y(n);
soma3=1/(n-1)*soma1*soma2;
soma4=1/(n-1)*soma2*soma2;

alpha1=(soma -soma3)/(soma0-soma4);
alpha0=(soma1-alpha1*soma2)/(n-1);
estimativas(1)=alpha0;
estimativas(2)=alpha1;
```

Concerning the tables 4.3 and 4.5 we used the following algorithm:

d) **function** [g] = valores_tabelas(alpha0, alpha1, b, n)

```
% INPUT:
% real parameters (α0, α1, ϕ) of the NTA-INARCH(1) model
% (b corresponds to parameter ϕ)
% OUTPUT:
% means, variances and covariances of the estimates

A = zeros(1000,3);
b1 = ones(1000,1);
b2 = ones(1000,1);
b3 = ones(1000,1);
```

```
d = zeros(1000,1);
e = zeros(1000,1);
f = zeros(1000,1);
for i=1:1000
    x = valores_trajectoria_NTA(alpha0, alpha1, b, n);
    est = estimativaCLS_NTA(x);
    A(i,1) = est(1);
    A(i,2) = est(2);
    A(i,3) = est(3);
end
S = sum(A);
media(1) = S(1)/1000;
media(2) = S(2)/1000;
media(3) = S(3)/1000;
b1 = media(1)*b1;
b2 = media(2)*b2;
b3 = media(3)*b3;
B = [b1  b2  b3];
D = A - B;
C = D.^2;
V = sum(C);
variancia(1) = n*(V(1)/1000);
variancia(2) = n*(V(2)/1000);
variancia(3) = V(3)/1000;
varb = n*variancia(3);
for j=1:1000
    d(j) = D(j,1)*D(j,2);
    e(j) = D(j,1)*D(j,3);
    f(j) = D(j,2)*D(j,3);
end
covariancia(1) = n*(sum(d)/1000);
covariancia(2) = sum(e)/1000;
covariancia(3) = sum(f)/1000;
corr(1) = covariancia(1)/(sqrt(variancia(1))*sqrt(variancia(2)));
corr(2) = covariancia(2)/(sqrt(variancia(1)/n)*sqrt(variancia(3)));
corr(3) = covariancia(3)/(sqrt(variancia(2)/n)*sqrt(variancia(3)));
g = [media(1)  media(2)  media(3)  variancia(1)  variancia(2)  varb  covariancia(1)  corr(1)
    corr(2)   corr(3)];
```

where to generate the trajectory of the NTA-INARCH(1) process we use the program:

e) **function** [y] = valores_trajectoria_NTA(alpha0, alpha1, phi, n)

```
% INPUT:
% parameters α0, α1 and φ
% length n of the trajectory
% OUTPUT:
% trajectory with n elements of an NTA-INARCH(1) process with parameters α0, α1, φ

x = zeros(1,n+100);
y = zeros(1,n);
lambda = zeros(1,n+100);
lambda(1) = alpha0/(1-alpha1);

for i=2:n+100
    lambda(i)= alpha0 + alpha1*x(i-1);
    parametro=lambda(i)/phi;
    N=poissrnd(parametro);
    z=0;
    for j=1:N
        z=z+poissrnd(phi);
    end
    x(i)=z;
end

for k=1:n
    y(k)=x(k+100);
end
```

and to calculate the estimates of the three parameters we use

f) **function** [estimativas] = estimativaCLS_NTA(y)

```
% INPUT:
% trajectory y
% OUTPUT:
% CLS estimates of the parameters (α0,α1) and estimate the parameter φ based on
% the moments estimation method of the process NTA-INARCH(1) from trajectory y

n=length(y);
soma = 0;
soma0 = 0;

for i=2:n
    soma = soma + y(i)*y(i-1);
    soma0 = soma0 + y(i-1)*y(i-1);
end
```

```
somar=0;
mediaemp=sum(y)/n;
for i=1:n
    somar= somar + (y(i)- mediaemp)^2;
end

VarEmp=somar/n;
soma1=sum(y)-y(1);
soma2=sum(y)-y(n);
soma3=1/(n-1)*soma1*soma2;
soma4=1/(n-1)*soma2*soma2;

alpha1=(soma -soma3)/(soma0-soma4);
alpha0=(soma1-alpha1*soma2)/(n-1);
phi= -1 + ((1-alpha1)*(1-alpha1^2)*VarEmp)/alpha0;
estimativas(1)=alpha0;
estimativas(2)=alpha1;
estimativas(3)=phi;
```

The program to get the values presented in Table 4.4 is quite similar to the above. In fact, in the program *valores_tabelas(alpha0, alpha1, b, n)* we only need to change

"x = valores_trajectoria_NTA(alpha0, alpha1, b, n);" to

"x = valores_trajectoria_GEOMP2(alpha0, alpha1, b, n);"

and

"est = estimativaCLS_NTA(x);" to

"est = estimativaCLS_GEOMP2(x);"

where now b corresponds to the parameter p^*, and then consider

g)

```
function [y] = valores_trajectoria_GEOMP2(alpha0, alpha1, p, n)

% INPUT:
% parameters α0, α1 and p
% length n of the trajectory
% OUTPUT:
% trajectory with n elements of a GEOMP2-INARCH(1) process
% with parameters α0, α1, p
x = zeros(1,n+100);
y = zeros(1,n);
lambda = zeros(1,n+100);
lambda(1) = alpha0/(1-alpha1);

for i=2:n+100
    lambda(i)= alpha0 + alpha1*x(i-1);
    parametro=(p*lambda(i))/(1-p);
```

```
        N=poissrnd(parametro);
        z=0;
        for j=1:N
            z=z+ geornd(p);
        end
        x(i)=z;
    end
    for k=1:n
        y(k)=x(k+100);
    end
```

and

h)
```
function [estimativas] = estimativaCLS_GEOMP2(y)
% INPUT:
% trajectory y
% OUTPUT:
% CLS estimates of the parameters (α0,α1) and estimate the parameter p based on the
% moments estimation method of the process GEOMP2-INARCH(1) from trajectory y
n=length(y);
soma=0;
soma0=0;
for i=2:n
    soma = soma + y(i)*y(i-1);
    soma0 = soma0 + y(i-1)*y(i-1);
end
somar=0;
mediaemp=sum(y)/n;
for i=1:n
    somar=somar + (y(i)- mediaemp)^2;
end
VarEmp=somar/n;
soma1=sum(y)-y(1);
soma2=sum(y)-y(n);
soma3=1/(n-1)*soma1*soma2;
soma4=1/(n-1)*soma2*soma2;
alpha1=(soma -soma3)/(soma0-soma4);
alpha0=(soma1-alpha1*soma2)/(n-1);
outro= (1-alpha1)/alpha0;
```

```
p= 2/(1+VarEmp*(1-alpha1^2)*outro);
estimativas(1)=alpha0;
estimativas(2)=alpha1;
estimativas(3)=p;
```

To get the confidence intervals presented in Table 4.6 we developed the following program:

i) **function** [CI] = intervalo_confianca(alpha0, alpha1, phi, gamma, n)

% INPUT:
% true parameters α_0, α_1, ϕ of the NTA-INARCH(1) model
% the confidence level γ
% length n of the trajectory
% OUTPUT:
% Confidence interval for m_0 and m_1

```
ntil = 35;
a = zeros(ntil,1);   b = zeros(ntil,1);
alpha = 1-gamma;
aux = alpha/2;
z = norminv([aux 1-aux],0,1);

for j=1:ntil
    g = valores_tabelas(alpha0, alpha1, phi, n);
    a(j) = g(9);
    b(j) = g(10);
end

media1 = sum(a)/ntil;
media2 = sum(b)/ntil;
c = (a-media1).^2;
d = (b-media2).^2;
e = sum(c)/(ntil-1);
f = sum(d)/(ntil-1);

limiteInf1 = media1+z(1)*sqrt(e/ntil);
limiteSup1 = media1+z(2)*sqrt(e/ntil);
limiteInf2 = media2+z(1)*sqrt(f/ntil);
limiteSup2 = media2+z(2)*sqrt(f/ntil);
CI =[limiteInf1 limiteSup1; limiteInf2 limiteSup2];
```

where the function *valores_tabelas(alpha0, alpha1, phi, n)* is given above in page 153.

To obtain the values presented in tables 4.7, 4.8 and 4.9 we use the following program:

j) **function** [probCobertura] = Prob_Cobertura_varias(alpha0, alpha1, n)

```
% INPUT:
% true parameters α0, α1 of the INARCH(1) model
% length n of the trajectory
% OUTPUT:
% Estimated coverage probabilities of the confidence region for three different
% confidence levels: 0.9, 0.95 and 0.99

v0=1; d0=1;
B = matriz(alpha0, alpha1, v0, d0);
% gamma is the confidence level
gamma = [0.9, 0.95, 0.99];
conta1 = 0; conta2 = 0; conta3 = 0;
% determine the gamma-quantile of the chi-squared distribution
z1 = chi2inv(gamma(1),2)/(n-1);
z2 = chi2inv(gamma(2),2)/(n-1);
z3 = chi2inv(gamma(3),2)/(n-1);

for i=1:10000
    x = valores_trajectoria_Poisson(alpha0, alpha1, n);
    est = estimativaCLS_Poisson(x);
    y = [est(1)-alpha0; est(2)-alpha1];
    CR = y'*B*y;
    if (CR < z1)
        conta1 = conta1 + 1;
    end
    if (CR < z2)
        conta2 = conta2 + 1;
    end
    if (CR < z3)
        conta3 = conta3 + 1;
    end
end

probCobertura(1) = conta1/10000;
probCobertura(2) = conta2/10000;
probCobertura(3) = conta3/10000;
```

where

k)

```
function [B] = matriz(alpha0, alpha1, v0, d0)

% OUTPUT:
% asymptotic matrix V^-1 W V^-1

A = zeros(2,2);
```

```
num1 = v0^2+(d0-v0^2)*alpha1*(1+alpha1-alpha1^2)+(3*v0^2-d0)*alpha1^4;
num2 = alpha1*(d0+(3*v0^2-d0)*alpha1^2);
num3 = (1+alpha1)*num2;
den1 = v0*(1+alpha1+alpha1^2);
den2 = den1*alpha0;
A(1,1) = (alpha0/(1-alpha1))*(alpha0*(1+alpha1)+num1/den1);
A(2,2) = (1-alpha1^2)*(1+num2/den2);
A(1,2) = v0*alpha1-alpha0*(1+alpha1)-num3/den1;
A(2,1) = A(1,2);
B = inv(A);
```

Let us note that when we consider the parameters $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $n = 100$, we obtain the first element of tables 4.7, 4.8 and 4.9. Then we need to run again the program to get the other values.