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ASYMPTOTIC RESULTS FOR CERTAIN WEAK DEPENDENT VARIABLES

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Abstract. We consider a special class of weak dependent random variables with control on covariances of Lipschitz transformations. This class includes, but is not limited to, positively, negatively associated variables and a few other classes of weakly dependent structures. We prove the Strong Law of Large Numbers with the characterization of convergence rates which is almost optimal, in the sense that it is arbitrarily close to the optimal rate for independent variables. Moreover, we prove an inequality comparing the joint distributions with the product distributions of the margins, similar to the well known Newman's inequality for characteristic functions of associated variables. As a consequence, we prove the Central Limit Theorem together with its functional counterpart, and also the convergence of the empirical process for this class of weak dependent variables.

Key words and phrases. Central Limit Theorem, Convergence rate, L-weak dependence, Strong law of large numbers.

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1. Introduction

Limit theorems, either with respect to almost sure convergence or convergence in distribution are a central subject in statistics. In more recent years, many authors have expressed interested in the asymptotics of sequences of dependent variables. Several forms of controlling the dependence have been proposed, many of them describing a control on covariances of transformations of variables. Mostly, this control may be thought of as measuring the degree of dependence between the past and a sufficiently separated future. These dependence structures are commonly named weak dependence, and are described using specific families of transformations of the random variables. We refer the reader to Doukhan and Louhichi [8] or Dedecker et al. [6] for some examples and relations between such dependence notions. Many of these notions stemmed from the positive dependence and the association introduced by Lehmann [15] and Esary, Proschan and Walkup [10], respectively. The association was the first of these two dependence notions to attract the interest of researchers, and as expected, Strong Laws of Large Numbers and Central Limit Theorems were eventually proved. We refer the reader to the monographs by Bulinski and Shashkin [4], Oliveira [21] or Prakasa Rao [23] for an account of relevant literature. Inevitably, several variations and extensions of these dependence notions were introduced and limit theorems were established. Among these, the negative association defined by Joag-Dev and Proschan [13] was one of the most popular, with various different extensions introduced in more recent years: extended negative dependent (END) introduced by Liu [16], widely orthant dependent (WOD) introduced by Wang, Wang and Gao [27] among other variations.

In this paper, we will be interested in a particular version of weak dependence defined in the same spirit as in Doukhan and Louhichi [8], assuming an appropriate control on covariances after transformation through Lipschitz functions, instead of a direct variation

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on the inequalities that express the positive or the negative dependence. We note that using covariances of Lipschtiz transformations to prove limit theorems in probability seems to have appeared for the first time in Bulinski and Shabanovich [3]. The dependence we will be considering is somewhat similar to the quasi-association as introduced in Bulinski and Suquet [5], that includes the positive, negative dependence notions referred above and the quasi-association. We will also provide nontrivial examples showing that the inclusions between these classes of dependent variables are strict. For the weak dependence notion we are defining, we will prove the Strong Law of Large Numbers, with the characterization of rates for both bounded and unbounded random variables, the Central Limit Theorem and the invariance principle. We should compare the results proved here with the ones already available in the literature for the various dependence structures. As to the convergence in distribution, our results are of similar strength, essentially only providing a unified approach to the different frameworks. For the almost sure convergence, the assumptions and the derived rates are again similar to most of the known results for negatively or positively associated variables. However, for weak dependent families of variables, the only inequality controlling tail probabilities (see Corollary 1 in [8]) is a Bernstein type inequality, that has a relatively weak form. Later, Corollary 4.1 and Theorem 4.5 in [6] and Kallabis and Neumann [14] also prove exponential inequalities that are analogous to the Bernstein inequalities, but again with weaker exponents in their upper bounds. This means that although the Strong Laws of Large Numbers may be derived, not only the assumptions will become stronger, but the convergence rates that follow will not be almost optimal, in the sense that these rates may be arbitrarily close to the well known rates for independent variables. In the present paper, the version of weak dependence we will be studying allows for the adaptation of techniques used for associated variables (see, for example, Ioannides and Roussas [12], Oliveira [20], Sung [26]) providing stronger forms of the Bernstein-type inequality, meaning that we will obtain almost optimal convergence rates.

The paper is organized as follows: Section 2 defines the framework, Section 3 proves some basic inequalities needed for the control of the almost sure convergence, which is the object of Section 4, where the Strong Laws of Large Numbers for bounded and unbounded random variables, with characterization of rates, are proved. Finally, in Section 5, we extend the Newman inequality for characteristic functions to the present dependence structure, from which the Central Limit Theorem, the invariance principle and the convergence of the empirical process follow.

2. Definitions and framework

Let X_n, $n\geq 1$, be centered random variables and define $S_n=X_1+\cdots+X_n$. As mentioned before, we will be interested in a particular form of weak dependence, according to the following definition.

Definition 2.1. *The random variables X_i, $i=1,\ldots,n$, are said to be L-weakly dependent if there exist nonnegative coefficients γ_k, $k\geq 1$, such that for every disjoint subsets $I,J\subset\{1,\ldots,n\}$ and real valued Lipschitz functions f and g, defined on the appropriate Euclidean spaces, the following inequality is satisfied:*

$$\left|\mathrm{Cov}\left(f\left(X_i,i\in I\right),g\left(X_j,j\in J\right)\right)\right|\leq \|f\|_L\|g\|_L\sum_{i\in I}\sum_{j\in J}\gamma_{|j-i|},$$

where $\|f\|_L$ represents the Lipschitz norm of f:

$$\|f\|_L=\sup_{x\neq y}\frac{|f(x)-f(y)|}{\|x-y\|}.$$

An infinite family of random variables is said to be L-weakly dependent if every finite subfamily is L-weakly dependent and the coefficients define a convergent series.

This is a form of weak dependence in the same spirit as in Doukhan and Louhichi [8] or Dedecker et al. [6]. With respect to the discussion in [6], the L-weak dependence implies what these authors called the κ or the ζ coefficients. In fact, the κ and the ζ coefficients are defined in a quite similar way as our L-weak dependence by multiplying the Lipschtiz norms by the number of arguments, or by the minimum number of arguments, used to define the transformations f and g. This means the examples of L-weakly dependent sequences include positively associated, negatively associated, Gaussian sequences or models for interacting particles systems (see Section 3.5.3 in [6] for details for this last example). Moreover, the notion of quasi-association, introduced by Bulinski and Suquet [5], is also included in the L-weak dependence structure by choosing $\gamma_k = \mathrm{Cov}(X_1, X_{k+1})$ assuming, of course, the stationarity of the random variables. The inclusion between these families of dependent variables is strict, as we will show by presenting a few examples. In particular, the examples of the construction of L-weak dependent sequences that are not quasi-associated are of interest because they show the advantage of replacing covariances by some family of coefficients in the definition.

Example 2.2. *Let ξ_n, $n \in \mathbb{Z}$, be a sequence of independent random variables with variances σ_n^2. Given $p \geq 1$ and $\alpha_1, \ldots, \alpha_p \in \mathbb{R}$, define, for each $n \geq 1$, $X_n = \sum_{j=1}^{p} \alpha_j \xi_{n-j}$. It is well known that the sequence X_n is positively associated if and only if all α_i have the same sign, as follows from property (P4) in Esary, Proschan and Walkup* [10]*. Now, if we choose the coefficients α_1 and α_p positive, and α_2, and α_{p-1} negative, it follows that*

$$\mathrm{Cov}(X_n, X_{n+p-1}) = \alpha_1 \alpha_p \sigma_{n-1}^2 > 0,$$
$$\mathrm{Cov}(X_n, X_{n+p-2}) = \alpha_1 \alpha_{p-1} \sigma_{n-1}^2 + \alpha_2 \alpha_p \sigma_{n-2}^2 < 0.$$

Hence, the sequence X_n, $n \geq 1$, is neither negatively associated nor positively associated. However, it is easily verified that it is quasi-associated.

Remark 2.3. *As composition of Lipschitz functions is still Lipschitzian, and quasi-associated variables are L-weak dependent, it follows that Lipschitz transformations of quasi-associated variables are L-weak dependent. However, as shown by the following example, the transformed variables are not necessarily quasi-associated.*

Example 2.4. *Let ξ_n, $n \geq 1$, be a sequence of independent and identically distributed random variables, α_n, $n \geq 1$, a sequence of nonnegative real numbers, and define, for each $n \geq 1$, $X_n = \sum_{i=1}^{n} \alpha_i \xi_i$. Therefore, the variables X_n, $n \geq 1$, are positively associated, hence, also quasi-associated. Consider the Lipschitz function $g(x) = e^{ax}$, for some $a \in \mathbb{R}$. Remark that g^{-1} is also a Lipschitzian function in every domain bounded away from 0. Finally, define $Y_n = g(X_n)$. It is now easily verifiable that $\mathrm{Cov}(X_1, X_2) = \alpha_1^2 \mathrm{Var}(\xi_1)$, while $\mathrm{Cov}(Y_1, Y_2) = \mathrm{Var}(g(\alpha_1 \xi_1)) \mathbb{E} g(\alpha_2 \xi_2)$. If we choose the common distribution of ξ_n and the function g such that $\lim_{\alpha_1 \to +\infty} \mathrm{Var}(g(\alpha_1 \xi_1)) = 0$, it follows that the inequality*

$$\mathrm{Cov}(X_1, X_2) = \mathrm{Cov}(g^{-1}(g(X_1)), g^{-1}(g(X_2))) \leq \|g^{-1}\|_L^2 \mathrm{Cov}(g(X_1), g(X_2)) \qquad (1)$$

can not be fulfilled, at least for α_1 large enough. Therefore, the random variables $Y_n = g(X_n)$, $n \geq 1$, can not be quasi-associated.

Example 2.5. *A more concrete example may be obtained taking $g(x) = e^{-x}$ and the ξ_n uniform on some closed interval. As before, note that, although $g(x)$ and $g^{-1}(x) = -\log x$ are not Lipschtizian in all their entire domain, they are Lipschitz in the support of the variables to which we will be applying these transformations and, as we will be computing expectations, this is enough to characterize the L-weak dependence. The uniform distribution is just an easily verifiable example. Other distributions may be considered. In* *fact, representing by M_ξ the moment generating function of the initial random variables ξ_n, we have that* $\mathrm{Var}(g(\alpha_1\xi_1)) = M_\xi(-2\alpha_1) - M_\xi^2(-\alpha_1)$ *and this converges to 0 under the assumption* $\lim_{\alpha\to+\infty} M_\xi(-\alpha) = 0$*. Besides, to have the Lipschitzianity of the transformations considered, ξ_n should have a compact support.*

Example 2.6. *A family of examples showing that L-weak dependence is broader than quasi-association may be constructed by adding perturbations to the random variables that are more significant when these variables are close to the origin and become negligible when the variables are large. To be specific, let us start with a family of independent and non-negative random variables ξ_n, $n \geq 1$, and define $X_n = \sum_{i=1}^n \alpha_i\xi_i$, where $\alpha_i > 0$. This implies that X_n, $n \geq 1$, are associated and that* $\mathrm{Cov}(X_1, X_2) = \alpha_1^2\mathrm{Var}(\xi_1)$*. Now, choose some decreasing Lipschitz function h such that $h'(0) = 0$ and $\beta > 0$, and define $g(x) = h(\beta x) + x$. The function h should be chosen such that g is strictly increasing. Consider now*

$$\begin{aligned}\mathrm{Cov}(g(X_1), g(X_2)) &= \alpha_1^2\mathrm{Var}(\xi_1) + \mathrm{Cov}\Big(\alpha_1\xi_1, h(\beta(\alpha_1\xi_1 + \alpha_2\xi_2))\Big) \\ &\quad +\mathrm{Cov}\Big(h(\beta\alpha_1\xi_1), g(\alpha_1\xi_1 + \alpha_2\xi_2)\Big).\end{aligned}$$

Both covariances in the right hand side above consider an increasing transformation of the (ξ_1, ξ_2) and a decreasing transformation of the same random vector. So the association of the vector implies that these covariances are negative. Let us denote $N(\beta) = \|g^{-1}\|_L = \sup_x \left|\frac{1}{\beta h'(\beta g^{-1}(x))+1}\right|$, and choose $\alpha_1 = \alpha_2 = \frac{c}{\beta}$, where $c > 0$ is such that $\mathrm{Cov}(\xi_1, h(c(\xi_1+\xi_2))) < 0$*. Note that, h being Lipschitzian implies that h' is bounded, so for $\beta > 0$ small enough, we have $N(\beta) = \frac{1}{1+\inf_x \beta h'(\beta g^{-1}(x))}$. If the random variables $g(X_1)$ and $g(X_2)$ are to be quasi-associated, then (1) must be fulfilled. For the present framework, this is equivalent to*

$$\mathrm{Var}(\xi_1) \leq N^2(\beta)\left(\mathrm{Var}(\xi_1) + \beta\mathrm{Cov}\Big(\xi_1, h(\xi_1+\xi_2)\Big) + \beta^2\mathrm{Cov}\Big(h(\xi_1), g(\alpha_1\xi_1 + \alpha_2\xi_2)\Big)\right).$$

To prove the above inequality can not hold, assume the weaker condition (remember the last covariance is negative): $\mathrm{Var}(\xi_1) \leq N^2(\beta)\left(\mathrm{Var}(\xi_1) + \beta\mathrm{Cov}\Big(\xi_1, h(\xi_1+\xi_2)\Big)\right)$*. Dividing by* $\mathrm{Var}(\xi_1)$*, this is equivalent to $1 \leq N^2(\beta)(1-d\beta)$, where $d = \frac{-\mathrm{Cov}(\xi_1, h(\xi_1+\xi_2))}{\mathrm{Var}(\xi_1)} > 0$, which may be rewritten as $\beta\left(\beta(h'(\beta g^{-1}(x)))^2 + 2h'(\beta g^{-1}(x)) + d\right) \leq 0$, and this, for small enough β, can not be fulfilled. Hence, for such choice of the parameters, the random variables $g(X_1)$ and $g(X_2)$ can not be quasi-associated. However, being Lipschitz transformations of quasi-associated random variables, they are L-weak dependent. A concrete example is obtained choosing $h(x) = e^{-x^2}$, for which we have $\|g^{-1}\|_L = \frac{1}{1-\sqrt{2\beta}\,e^{-1/2}}$.*

We will be assuming throughout this paper that

$$\frac{1}{n}\mathbb{E}S_n^2 \longrightarrow \sigma^2 \in (0, \infty). \tag{2}$$

Remark 2.7. *This condition follows immediately from the convergence of the series of L-weak dependence coefficients γ_k (see Lemma 1.1 in Rio* [24]*).*

This, obviously, implies that for n large enough, we have $\mathbb{E}S_n^2 \leq 2\sigma^2 n$. Besides, we will need to decompose S_n into an appropriate sum of blocks in order to apply the classical Bernstein block decomposition method, here with equal sized blocks. For this purpose, consider an increasing sequence of integers $p_n \leq \frac{n}{2}$ such that $p_n \longrightarrow +\infty$, put $r_n = \lfloor \frac{n}{2p_n} \rfloor$, where $\lfloor x \rfloor$ represents the integer part of x, and define the blocks:

$$Y_{j,n} = \sum_{k=(j-1)p_n+1}^{jp_n} X_k, \quad j = 1, \ldots, 2r_n. \tag{3}$$

Notice that, if the random variables are bounded by $c>0$, then $|Y_{j,n}| \leq cp_n$. Moreover, define the alternate sums:

$$Z_{n,od} = \sum_{j=1}^{r_n} Y_{2j-1,n} \qquad \text{and} \qquad Z_{n,ev} = \sum_{j=1}^{r_n} Y_{2j,n}.$$

Note that $S_n = Z_{n,od} + Z_{n,ev} + R_n$, where

$$R_n = \sum_{j=2r_np_n+1}^{n} Y_j.$$

Finally, we introduce the generalized Cox-Grimmett coefficients adapted to the L-weak dependence structure,

$$v(n) = \sum_{k=n}^{\infty} \gamma_k. \tag{4}$$

3. Inequalities for bounded variables

This section establishes a few inequalities that are the basic tools for proving the almost sure convergence results. The inequalities below are extensions of analogous results for associated random variables. We start by establishing a bound for the Laplace transform of the blocks $Y_{j,n}$.

Lemma 3.1. *Assume that the sequence X_n, $n \geq 1$, is stationary, there exists some $c>0$ such that for every $n \geq 1$, $|X_n| \leq c$ almost surely, and that (2) holds. Let $d_n > 1$, $n \geq 1$, be a sequence of real numbers. Then, for every $t \leq \frac{d_n-1}{d_n}\frac{1}{cp_n}$ and n large enough,*

$$\mathbb{E}e^{tY_{j,n}} \leq \exp\left(2t^2\sigma^2 p_n d_n\right).$$

Proof. Using the Taylor expansion and taking into account the boundedness of the random variables, we have

$$\mathbb{E}e^{tY_{j,n}} = 1 + \sum_{k=2}^{\infty} \frac{t^k \mathbb{E}Y_{j,n}^k}{k!} \leq 1 + \sum_{k=2}^{\infty} \frac{t^k c^{k-2} p_n^{k-2} \mathbb{E}Y_{j,n}^2}{k!} \leq 1 + t^2 \mathbb{E}Y_{j,n}^2 \sum_{k=2}^{\infty} (tcp_n)^{k-2}.$$

It follows from the assumption on t that $tcp_n \leq \frac{d_n-1}{d_n} < 1$, thus, as the sequence X_n, $n \geq 1$, is stationary, we may write

$$\mathbb{E}e^{tY_{j,n}} \leq 1 + \frac{t^2 \mathbb{E}S_{p_n}^2}{1 - tcp_n}.$$

Now, as $p_n \longrightarrow +\infty$, we have that for n large enough, $\mathbb{E}S_{p_n}^2 \leq 2\sigma^2 p_n$. Moreover, we have $\frac{1}{1-tcp_n} \leq d_n$, so $\mathbb{E}e^{tY_{j,n}} \leq 1 + 2t^2\sigma^2 p_n d_n \leq \exp\left(2t^2\sigma^2 p_n d_n\right)$. ■

Considering now L-weakly dependent variables, we establish an upper bound for $\mathbb{E}e^{tZ_{n,od}}$.

Lemma 3.2. *Assume the conditions of Lemma 3.1 are satisfied and the sequence of random variables X_n, $n \geq 1$, is L-weakly dependent. Then, for every $t \leq \frac{d_n-1}{d_n}\frac{1}{cp_n}$ and n large enough, we have*

$$\mathbb{E}e^{tZ_{n,od}} \leq t^2 e^{\frac{tcn}{2}} p_n v(p_n) \sum_{j=0}^{r_n-2} \exp\left(jtp_n(2t\sigma^2 d_n - c)\right) + \exp\left(t^2\sigma^2 n d_n\right). \tag{5}$$

Proof. First, remark that $\mathbb{E}e^{tZ_{n,od}} = \mathbb{E}\left(\prod_{j=1}^{r_n} e^{tY_{2j-1,n}}\right)$. Now, by adding and subtracting appropriate terms, we find that

$$\begin{aligned}\mathbb{E}\left(\prod_{j=1}^{r_n} e^{tY_{2j-1,n}}\right) &= \operatorname{Cov}\left(\prod_{j=1}^{r_n-1} e^{tY_{2j-1,n}}, e^{tY_{2r_n-1,n}}\right) + \mathbb{E}\left(\prod_{j=1}^{r_n-1} e^{tY_{2j-1,n}}\right)\mathbb{E}e^{tY_{2r_n-1,n}} \\ &= \operatorname{Cov}\left(\prod_{j=1}^{r_n-1} e^{tY_{2j-1,n}}, e^{tY_{2r_n-1,n}}\right) \\ &\quad +\operatorname{Cov}\left(\prod_{j=1}^{r_n-2} e^{tY_{2j-1,n}}, e^{tY_{2r_n-3,n}}\right)\mathbb{E}e^{tY_{2r_n-1,n}} \\ &\quad +\mathbb{E}\left(\prod_{j=1}^{r_n-3} e^{tY_{2j-1,n}}\right)\mathbb{E}e^{tY_{2r_n-3,n}}\mathbb{E}e^{tY_{2r_n-1,n}}.\end{aligned}$$

Before iterating this procedure note that due to the stationarity of the sequence of random variables X_i, $\mathbb{E}e^{tY_{2r_n-3,n}} = \mathbb{E}e^{tY_{2r_n-1,n}} = \mathbb{E}e^{tY_{1,n}}$, so the previous expression may be rewritten as

$$\begin{aligned}&\mathbb{E}\left(\prod_{j=1}^{r_n} e^{tY_{2j-1,n}}\right) \\ &\quad = \operatorname{Cov}\left(\prod_{j=1}^{r_n-1} e^{tY_{2j-1,n}}, e^{tY_{2r_n-1,n}}\right) + \operatorname{Cov}\left(\prod_{j=1}^{r_n-2} e^{tY_{2j-1,n}}, e^{tY_{2r_n-3,n}}\right)\mathbb{E}e^{tY_{1,n}} \\ &\quad\quad +\mathbb{E}\left(\prod_{j=1}^{r_n-3} e^{tY_{2j-1,n}}\right)\left(\mathbb{E}e^{tY_{1,n}}\right)^2.\end{aligned}$$

Now, we iterate the procedure above to decompose the mathematical expectation of the product to find

$$\mathbb{E}\left(\prod_{j=1}^{r_n} e^{tY_{2j-1,n}}\right) = \sum_{j=1}^{r_n-1}\left(\mathbb{E}e^{tY_{1,n}}\right)^{j-1}\operatorname{Cov}\left(\prod_{k=1}^{r_n-j} e^{tY_{2k-1,n}}, e^{tY_{2(r_n-j)+1,n}}\right) + \left(\mathbb{E}e^{tY_{1,n}}\right)^{r_n}.$$

The L-weak dependence of the variables implies that

$$\begin{aligned}&\left|\operatorname{Cov}\left(\prod_{k=1}^{r_n-j} e^{tY_{2k-1,n}}, e^{tY_{2(r_n-j)+1,n}}\right)\right| \\ &\qquad \le t^2 e^{tcp_n(r_n-j+1)}\sum_{k=1}^{r_n-j}\sum_{\ell=2(k-2)p_n+1}^{(2k-1)p_n}\sum_{\ell'=2(r_n-j)p_n+1}^{(2(r_n-j)+1)p_n}\gamma_{\ell'-\ell}.\end{aligned} \tag{6}$$

The summation above is similar to the one treated in the course of proof of Lemma 3.1 in [12]. Adapting their arguments, one easily finds that

$$\begin{aligned}\sum_{\ell=2(k-2)p_n+1}^{(2k-1)p_n}\sum_{\ell'=2(r_n-j)p_n+1}^{(2(r_n-j)+1)p_n}\gamma_{\ell'-\ell} &= \sum_{\ell=0}^{p_n-1}(p_n-\ell)\gamma_{2kp_n+\ell} + \sum_{\ell=1}^{p_n-1}(p_n-\ell)\gamma_{2kp_n-\ell} \\ &\le p_n\sum_{\ell=(2k-1)p_n+1}^{(2k+1)p_n-1}\gamma_\ell,\end{aligned}$$

thus,

$$\sum_{k=1}^{r_n-j}\sum_{\ell=2(k-2)p_n+1}^{(2k-1)p_n}\sum_{\ell'=2(r_n-j)p_n+1}^{(2(r_n-j)+1)p_n}\gamma_{\ell'-\ell}\leq\sum_{k=1}^{r_n-j}p_n\sum_{\ell=(2k-1)p_n+1}^{(2k+1)p_n-1}\gamma_\ell\leq p_n v(p_n).$$

Insert this into (6) and use the inequality proved in Lemma 3.1 to obtain the upper bounds for $\left(\mathbb{E}e^{tY_{1,n}}\right)^{j-1}$ and $\left(\mathbb{E}e^{tY_{1,n}}\right)^{r_n}$. Finally, recall that $2p_nr_n\leq n$ to conclude the proof. ■

We may now find an upper bound for the tail probabilities of $Z_{n,od}$.

Lemma 3.3. *Assume that the sequence X_n, $n\geq 1$, is stationary, there exists some $c>0$ such that for every $n\geq 1$, $|X_n|\leq c$ almost surely, that (2) holds, and that the sequence of random variables is L-weakly dependent. Then, for each fixed x and n large enough,*

$$\mathbb{P}\left(Z_{n,od}>x\right)\leq\left(\frac{k_1(x,n)x^2}{4\sigma^4n^2d_n^2}e^{\frac{cx}{4\sigma^2d_n}}p_nv(p_n)+1\right)\exp\left(-\frac{x^2}{4\sigma^2nd_n}\right),\tag{7}$$

where $k_1(x,n)=\left(1-\exp(\frac{xp_n}{2\sigma^2nd_n}(\frac{x}{n}-c))\right)^{-1}$.

Proof. Using Markov's inequality and taking into account (5), we get that

$$\begin{aligned}\mathbb{P}\left(Z_{n,od}>x\right)\quad\leq\quad & t^2e^{\frac{tcn}{2}}p_nv(p_n)e^{-tx}\sum_{j=0}^{r_n-2}\exp\left(jtp_n(2t\sigma^2d_n-c)\right)\\ & +\exp\left(t^2\sigma^2nd_n-tx\right).\end{aligned}\tag{8}$$

The minimization of the exponent in the second term above leads to the choice $t=\frac{x}{2\sigma^2nd_n}$, which implies that

$$t^2\sigma^2nd_n-tx=-\frac{x^2}{4\sigma^2nd_n}.$$

We still have to control the summation in the first term. For this purpose, remark that for the choice of t as above, $2t\sigma^2d_n-c=\frac{x}{n}-c$. Thus, as x is fixed, for n large enough $2t\sigma^2d_n-c<0$, so the series corresponding to the summation appearing in (7) is convergent, its sum being equal to $k_1(x,n)$. Finally, remark that, again for the choice made for t, we have $tx=\frac{x^2}{2\sigma^2nd_n}$, so $e^{-tx}\leq\exp\left(-\frac{x^2}{4\sigma^2nd_n}\right)$, and the proof is concluded. ■

4. Strong laws and convergence rates

With the tools proved in the previous section, we may now find conditions for the Strong Law of Large Numbers and characterize its convergence rate. The first subsection will deal with bounded random variables, using directly the inequalities of Section 3, while in the second subsection we will extend these results to arbitrary (unbounded) L-weakly dependent variables by using a truncation technique.

4.1. **The case of bounded variables.** The aim is, of course, to prove that $\frac{1}{n}S_n$ converges almost surely to 0, and we will conclude this by verifying that it is possible to decompose S_n into sums with appropriate block sizes.

Lemma 4.1. *Assume that the sequence X_n, $n\geq 1$, is stationary and L-weakly dependent, there exists some $c>0$ such that for every $n\geq 1$, $|X_n|\leq c$ almost surely and that (2) holds. Assume that the generalized Cox-Grimmett coefficients (4) satisfy $v(n)=O(\rho^n)$, for some $\rho\in(0,1)$. Then for a suitable choice of the sequence p_n we can achieve $\frac{1}{n}Z_{n,od}\longrightarrow 0$ almost surely.*

Proof. We will bound $\mathbb{P}(Z_{n,od} > n\varepsilon)$ where, without loss of generality, we choose $\varepsilon > 0$ sufficiently small. Applying (7) with $x = n\varepsilon$, we find the upper bound

$$\mathbb{P}(Z_{n,od} > n\varepsilon) \leq \left(\frac{k_1(n\varepsilon, n)\varepsilon^2}{4\sigma^4 d_n^2} e^{\frac{cn\varepsilon}{4\sigma^2 d_n}} p_n v(p_n) + 1 \right) \exp\left(-\frac{n\varepsilon^2}{4\sigma^2 d_n} \right). \tag{9}$$

Choosing $d_n = \frac{n\varepsilon^2}{4\sigma^2 \alpha \log n}$, for some $\alpha > 1$, the final term above becomes $n^{-\alpha}$, thus defining a convergent series. So, to prove the convergence, it is enough to control the term in the large parentheses in (9) and verify that the assumptions of Lemma 3.1 are satisfied. We start by looking at

$$k_1(n\varepsilon, n) = \frac{1}{1 - \exp(\frac{\varepsilon(\varepsilon - c)p_n}{2\sigma^2 d_n})}.$$

To prevent $k_1(n\varepsilon, n)$ from exploding, we choose the sequences such that $p_n = c_0 d_n$, for some strictly positive c_0, making $k_1(n\varepsilon, n)$ independent of n. On the other hand, taking into account the assumption on the Cox-Grimmett coefficients and the choice for p_n,

$$\begin{aligned} \frac{\varepsilon^2}{d_n^2} e^{\frac{cn\varepsilon}{4\sigma^2 d_n}} p_n v(p_n) &= \frac{\varepsilon^2 c_0}{d_n} \exp\left(\frac{c\varepsilon n}{4\sigma^2 d_n} + p_n \log\rho \right) \\ &= 4c_0\sigma^2\alpha \frac{\log n}{n} \exp\left(\frac{c\alpha}{\varepsilon} \log n + \frac{c_0\varepsilon^2 \log\rho}{4\sigma^2\alpha} \frac{n}{\log n} \right), \end{aligned}$$

which is bounded as $\rho \in (0,1)$. Concerning the assumptions of Lemma 3.1, we need to verify that $t = \frac{\varepsilon}{2\sigma^2 d_n} \leq \frac{d_n - 1}{d_n} \frac{1}{cp_n}$, that is equivalent to

$$\varepsilon \leq 2\frac{\sigma^2}{c} \frac{(d_n - 1)}{p_n} = \frac{2\sigma^2}{c} \left(c_0 - \frac{1}{p_n} \right),$$

which is fulfilled for n large enough as, for the choice we made, $p_n \longrightarrow +\infty$. ■

It is obvious that the result just proved also holds if we replace $Z_{n,od}$ by $Z_{n,ev}$, thus we have the almost sure convergence of $\frac{1}{n}S_n$. For the sake of completeness, we state this result.

Theorem 4.2. *Assume that the conditions of Lemma 4.1 are satisfied. Then,* $\frac{1}{n}S_n \longrightarrow 0$ *almost surely.*

Remark 4.3. *We did not mention the remaining term* R_n. *In fact, in our setting, this term is negligible, taking into account that* $|R_n| \leq 2cp_n$ *and, for the choice made for the sequence* p_n, *we have that* $\frac{2cp_n}{n} \longrightarrow 0$, *so*

$$\begin{aligned} \mathbb{P}\left(\left| \frac{S_n}{n} \right| \geq \varepsilon \right) &\leq \mathbb{P}\left(\left| \frac{Z_{n,od}}{n} \right| + \left| \frac{Z_{n,ev}}{n} \right| + \left| \frac{R_n}{n} \right| \geq \varepsilon \right) \\ &\leq \mathbb{P}\left(\left| \frac{Z_{n,od}}{n} \right| + \left| \frac{Z_{n,ev}}{n} \right| + \frac{2cp_n}{n} \geq \varepsilon \right) \leq \mathbb{P}\left(\left| \frac{Z_{n,od}}{n} \right| + \left| \frac{Z_{n,ev}}{n} \right| \geq \frac{\varepsilon}{2} \right). \end{aligned}$$

We may further identify the convergence rate for the almost sure convergence above.

Theorem 4.4. *Assume that the conditions of Lemma 4.1 are satisfied. Then, for a suitable choice of the sequence* p_n, *we can achieve that* $\frac{1}{n}Z_{n,od} \longrightarrow 0$ *almost surely with convergence rate* $\frac{\log n}{n^{1/2-\delta}}$, *where* $\delta > 0$ *is arbitrarily small.*

Proof. We follow the proof of Lemma 4.1, allowing now ε to depend on n, that is, considering ε_n such that

$$\varepsilon_n^2 = \frac{4\sigma^2\alpha d_n \log n}{n}, \qquad \alpha > 1. \tag{10}$$

We need to verify that the condition on t in Lemma 3.1 is satisfied for an appropriate choice of the sequences p_n and d_n, that is, $t = \frac{\varepsilon_n}{2\sigma^2 d_n} \leq \frac{d_n-1}{d_n}\frac{1}{cp_n}$. Hence, we require that $\frac{\varepsilon_n p_n}{d_n}$ is bounded. On the other hand, looking at

$$k_1(n\varepsilon, n) = \left(1 - \exp\left(\frac{\varepsilon_n p_n}{2\sigma^2 d_n}(\varepsilon_n - c)\right)\right)^{-1},$$

we need to require that $\frac{\varepsilon_n p_n}{d_n}$ is bounded away from 0, so that we can still have $\varepsilon_n \longrightarrow 0$. Therefore, we choose the sequences such that $\frac{\varepsilon_n p_n}{d_n} = O(1)$. Taking into account (10), this leads to $d_n = O\left(\frac{p_n^2 \log n}{n}\right)$, which implies that

$$\varepsilon_n = O\left(\frac{p_n \log n}{n}\right).$$

If we now choose $p_n = O(n^\theta)$, for some $\theta > \frac{1}{2}$, it follows that $\varepsilon_n = O(n^{\theta-1}\log n)$, identifying the convergence rate stated after representing $\theta = \frac{1}{2} + \delta$. However, we still need to control the upper bound for the tail probability that follows from (7), which is now written as

$$\mathbb{P}\left(Z_{n,od} > n\varepsilon_n\right) \leq \left(\frac{k_1(n\varepsilon, n)\varepsilon_n^2}{4\sigma^2 d_n^2} e^{\frac{cn\varepsilon_n}{4\sigma^2 d_n}} p_n \rho^{p_n} + 1\right) n^{-\alpha}.$$

Replacing the choices for the sequences made so far, we find that the term inside the parentheses above behaves like

$$\frac{1}{n^\theta}\exp\left(\frac{c\alpha^{1/2}}{2\sigma}\left(\frac{n\log n}{d_n}\right)^{1/2}\right)\rho^{n^\theta} + 1 = \frac{1}{n^\theta}\exp\left(n^{1-\theta} + n^\theta \log\rho\right) + 1,$$

which, as $\theta > \frac{1}{2}$, is bounded. ■

Remark 4.5. *For convenience in the next section, we write explicitly the exponential inequality that follows from the arguments used in course of proof of the previous result: there exists a constant* $C > 0$ *such that*

$$\mathbb{P}\left(Z_{n,od} > n\varepsilon_n\right) \leq \left[C\frac{1}{n^\theta}\exp\left(n^{1-\theta} + n^\theta \log\rho\right) + 1\right] n^{-\alpha}, \tag{11}$$

where $\theta > \frac{1}{2}$ *and* $\alpha > 1$.

Theorem 4.4 was proved for $\frac{1}{n}Z_{n,od}$ for the convenience of the exposition. An analogous version obviously holds for $\frac{1}{n}Z_{n,ev}$, thus implying the same result for $\frac{1}{n}S_n$ by using the same argument as stated in Remark 4.3. Again, for the sake of completeness, we state the final result.

Theorem 4.6. *Assume that the conditions of Lemma 4.1 are satisfied. Then,* $\frac{1}{n}S_n \longrightarrow 0$ *almost surely with convergence rate* $\frac{\log n}{n^{1/2-\delta}}$, *where* $\delta > 0$ *is arbitrarily small.*

4.2. **General random variables.** We now want to drop the boundedness assumption. To extend the results just proved, we will use a truncation technique together with a control on the tails of the distributions. Define, for a given fixed $c > 0$, the nondecreasing function $g_c(x) = \max(\min(x, c), -c)$, performing a truncation at level c. Remark that, for every $c > 0$, g_c is Lipschtizian with $\|g_c\|_L = 1$. Choose some sequence $c_n \longrightarrow +\infty$, to be made precise later, and define, for $j, n \geq 1$, the random variables

$$X_{1,j,n} = g_{c_n}(X_j), \qquad X_{2,j,n} = X_j - X_{1,j,n},$$

and the partial summations

$$S_{1,n} = \sum_{j=1}^{n} (X_{1,j,n} - \mathbb{E}X_{1,j,n}), \qquad S_{2,n} = \sum_{j=1}^{n} (X_{2,j,n} - \mathbb{E}X_{2,j,n}).$$

Theorem 4.7. *Assume that the L-weakly dependent sequence X_n, $n \geq 1$, is stationary, (2) holds, and the generalized Cox-Grimmett coefficients (4) satisfy $v(n) = O(\rho^n)$, for some $\rho \in (0,1)$. Assume further that,*

$$\exists\, \tau > 0, U > 0 : \sup_{0 \leq r \leq \tau} \mathbb{E}e^{r|X_1|} \leq U. \tag{12}$$

Then, $\frac{1}{n} S_n \longrightarrow 0$ almost surely with convergence rate $\frac{\log^2 n}{n^{1/2-\delta}}$, where $\delta > 0$ is arbitrarily small.

Proof. It is obvious that $\mathbb{P}(|S_n| > 2n\varepsilon) \leq \mathbb{P}(|S_{1,n}| > n\varepsilon) + \mathbb{P}(|S_{2,n}| > n\varepsilon)$. The random variables $X_{1,j,n}$ are bounded by c_n, hence, to follow the proof of Theorem 4.4, we need to verify that the construction of the auxiliary sequences satisfies the suitable assumptions. In order to obtain an analogous control on the probabilities, we will again choose ε_n according to (10). As in Theorem 4.4, we need to start by verifying that the assumptions of Lemma 3.1 are satisfied. Now, let us rewrite these assumptions as $t = \frac{\varepsilon_n}{2\sigma^2 d_n} \leq \frac{d_n - 1}{d_n} \frac{1}{c_n p_n}$. Thus, $\frac{c_n \varepsilon_n p_n}{d_n}$ needs to be bounded. Now, we need to prevent

$$k_1(n\varepsilon_n, n) = \left(1 - \exp\left(\frac{\varepsilon_n p_n}{2\sigma^2 d_n}(\varepsilon_n - c_n)\right)\right)^{-1}$$

from exploding to $+\infty$. As $\varepsilon_n \longrightarrow 0$, the argument of the exponential behaves like $\frac{-c_n \varepsilon_n p_n}{d_n}$, so we require that $\frac{c_n \varepsilon_n p_n}{d_n}$ is bounded away from 0. These considerations lead to $d_n = O(c_n \varepsilon_n p_n)$. Squaring this representation and taking into account the behaviour of ε_n described in (10), it follows that $d_n = O\left(c_n^2 p_n^2 \frac{\log n}{n}\right)$. We take the remaining auxiliary sequences as $p_n = O(n^\theta)$, for some $\theta > \frac{1}{2}$ and $c_n = \beta \log n$, for some $\beta > 0$. Applying now (11), replacing the bounding constant there by c_n, and using Remark 4.3, which requires $\frac{c_n p_n}{n} \longrightarrow 0$, that is satisfied, it follows that,

$$\mathbb{P}(|S_{1,n}| > n\varepsilon_n) \leq 2\left[C \frac{1}{n^\theta} \exp\left(n^{1-\theta} \log n + n^\theta \log \rho\right) + 1\right] n^{-\alpha},$$

and this upper bound defines a convergent series if $\theta = \frac{1}{2} + \delta$, for some arbitrary $\delta > 0$. To complete the proof, we need now to control $\mathbb{P}(|S_{2,n}| > n\varepsilon_n)$. Note first that, taking into account the stationarity,

$$\mathbb{P}(|S_{2,n}| > n\varepsilon_n) \leq n\mathbb{P}(|X_{2,1,n} - \mathbb{E}X_{2,1,n}| > \varepsilon_n) \leq \frac{n}{\varepsilon_n^2} \mathbb{E}X_{2,1,n}^2.$$

Denoting $\bar{F}(x) = \mathbb{P}(|X_1| > x)$, we have that

$$\mathbb{E}X_{2,1,n}^2 = -\int_{(c_n, +\infty)} (x - c_n)^2\, \bar{F}(dx) = \int_{c_n}^{+\infty} 2(x - c_n)\bar{F}(x)\, dx.$$

Now, using Markov's inequality, it follows that $\bar{F}(x) \leq e^{-rx}\mathbb{E}e^{r|X_1|} \leq Ue^{-rx}$, if $r \in (0, \tau)$. Thus, for $r \in (0, \tau)$, integrating the above expression above, we get that

$$\mathbb{E}X_{2,1,n}^2 \leq \frac{2U}{r^2} e^{-rc_n},$$

so finally, taking into account the choices made for the several sequences,

$$\mathbb{P}(|S_{2,n}| > n\varepsilon_n) \leq \frac{2nU}{r^2 \varepsilon_n^2} e^{-rc_n} = O\left(\frac{n^{3-2\theta-\beta r}}{\log^4 n}\right),$$

and take $r = \frac{3}{\beta}$, so this upper bound defines a convergent series, as $\theta > \frac{1}{2}$. Finally, insert these choices into the expression for ε_n to explicitly identify the convergence rate, finding

$$\varepsilon_n = O\left(\frac{\log^2 n}{n^{1-\theta}}\right),$$

and remember that $\theta = \frac{1}{2} + \delta$. ■

Remark 4.8. *The convergence rate proved in Theorem 4.7 is, up to logarithmic factors, close to the rates derived, using an approach based on the same methodologies, for positively associated variables, where a denominator equal to* $n^{1/2}$ *has been attained (see Corollary 2.4 in Henriques and Oliveira* [11]*, or the examples discussed after Corollary 5.4 in Xing, Yang and Liu* [30]*). Moreover, note that the convergence rate proved here is, up to logarithmic factors, arbitrarily close to the optimal convergence rate for the Strong Law of Large Numbers for associated random variables which is of order* $\frac{(\log n)^{1/2}(\log\log n)^{\eta/2}}{n^{1/2}}$ *for arbitrarily small* $\eta > 0$*, as proved by Yang, Su and Yu* [28].

5. Asymptotic normality

We now look at the convergence in distribution of sums of L-weakly dependent variables, extending the Central Limit Theorem (CLT) for associated random variables by Newman [18, 19] to the L-weak dependence structure. The proof of Newman's result (see Theorem 2 in [18] or Theorem 12 in [19]) relies on the inequality for characteristic functions, the Newman inequality for characteristic functions (Theorem 1 in Newman [18] or Theorem 10 in Newman [19]) that controls the approximation between the joint distribution and the product of the marginal distributions. So, we start by proving a version of that inequality for the present dependence structure.

Theorem 5.1. *(Newman's inequality for L-weakly dependent random variables) Let* X_1*,* $X_2, \ldots, X_n$ *be L-weakly dependent random variables. Then, for every* $t \in \mathbb{R}$*, we have*

$$\left|\mathbb{E}\left(\prod_{j=1}^{n} e^{itX_j}\right) - \prod_{j=1}^{n} \mathbb{E}\left(e^{itX_j}\right)\right| \le 4t^2 \sum_{j=1}^{n-1}(n-j)\gamma_j. \tag{13}$$

Proof. First, in the left side of (13), we add and subtract the appropriate terms to find,

$$\begin{aligned}
&\left|\mathbb{E}\left(\prod_{j=1}^{n} e^{itX_j}\right) - \prod_{j=1}^{n} \mathbb{E}\left(e^{itX_j}\right)\right| \\
&\quad\le \left|\mathbb{E}\left(\prod_{j=1}^{n} e^{itX_j}\right) - \mathbb{E}\left(e^{itX_n}\right)\mathbb{E}\left(\prod_{j=1}^{n-1} e^{itX_j}\right)\right| + \left|\mathbb{E}\left(e^{itX_n}\right)\mathbb{E}\left(\prod_{j=1}^{n-1} e^{itX_j}\right) - \prod_{j=1}^{n} \mathbb{E}\left(e^{itX_j}\right)\right| \\
&\quad\le \left|\mathrm{Cov}\left(\prod_{j=1}^{n-1} e^{itX_j}, e^{itX_n}\right)\right| + \left|\mathbb{E}\left(\prod_{j=1}^{n-1} e^{itX_j}\right) - \prod_{j=1}^{n-1} \mathbb{E}\left(e^{itX_j}\right)\right|.
\end{aligned}$$

Iterating now this procedure, we find that

$$\left|\mathbb{E}\left(\prod_{j=1}^{n} e^{itX_j}\right) - \prod_{j=1}^{n} \mathbb{E}\left(e^{itX_j}\right)\right| \le \sum_{m=2}^{n}\left|\mathrm{Cov}\left(\prod_{j=1}^{m-1} e^{itX_j}, e^{itX_m}\right)\right|.$$

To bound the covariance terms above, expand this covariance using the trigonometric representation of the complex exponential to find four terms involving cosine or sinus functions. Now, for example,

$$\left|\mathrm{Cov}\left(\cos\left(t\sum_{j=1}^{m-1}X_j\right),\cos\left(tX_m\right)\right)\right|\leq t^2\sum_{j=1}^{m-1}\gamma_{m-j},$$

taking into account that $\|\cos(tx)\|_L=t$ and using the L-weak dependence of the sequence X_i of random variables. Obviously, the same upper bound applies to the remaining terms, so we finally have

$$\left|\mathbb{E}\left(\prod_{j=1}^{n}e^{itX_j}\right)-\prod_{j=1}^{n}\mathbb{E}\left(e^{itX_j}\right)\right|\leq 4t^2\sum_{m=2}^{n}\sum_{j=1}^{m-1}\gamma_{m-j}=4t^2\sum_{j=1}^{n-1}(n-j)\gamma_j.$$

■

Newman's inequality is the main tool for proving the Central Limit Theorem for associated random variables (see, for example, Theorem 4.1 in Oliveira [21]). So, having extended Newman's inequality to L-weakly dependent variables, we immediately may state the corresponding CLT. The arguments for the proof are similar to those of Theorem 5 in Newman [18], except for what regards the control of the approximation to independence.

Theorem 5.2. *Let the sequence X_n, $n\geq 1$, of random variables be centered, square integrable, L-weakly dependent, strictly stationary, and satisfying (2). Then, $\frac{1}{\sqrt{n}}S_n$ converges in distribution to a centered normal random variable with variance σ^2.*

Proof. The proof is based on the decomposition of S_n similar to (3), into the sum of blocks of size $p\in\mathbb{N}$, now independent from n, and using (13). So, given $p\in\mathbb{N}$, put $m=\lfloor\frac{n}{p}\rfloor$, and redefine the blocks

$$Y_{j,p}=\sum_{k=(j-1)p+1}^{jp}X_k,\ j=1,\dots,m,\qquad\text{and}\qquad Y_{m+1,p}=\sum_{k=mp+1}^{n}X_k.$$

Represent by $\varphi_n(t)$ the characteristic function of $\frac{1}{\sqrt{n}}S_n$. We will start by establishing that $\left|\varphi_n(t)-e^{-t^2\sigma^2/2}\right|\longrightarrow 0$. Let us start by writing

$$\begin{aligned}\left|\varphi_n(t)-e^{-\frac{t^2\sigma^2}{2}}\right| &\leq |\varphi_n(t)-\varphi_{mp}(t)|+\left|\varphi_{mp}(t)-\varphi_p^m(t)\right|\\ &\quad+\left|\varphi_p^m(t)-e^{-\frac{t^2\sigma_p^2}{2}}\right|+\left|e^{-\frac{t^2\sigma_p^2}{2}}-e^{-\frac{t^2\sigma^2}{2}}\right|,\end{aligned}\tag{14}$$

where $\sigma_p^2=\frac{1}{\sqrt{p}}\mathrm{Var}(S_p)$, and prove that each term of the right hand side goes to zero. Let p be fixed for the time being. As to the first term of the upper bound in (14), we have, using Cauchy's inequality,

$$\begin{aligned}|\varphi_n(t)-\varphi_{mp}(t)| &\leq \mathbb{E}\left|\exp\left(\frac{it}{\sqrt{n}}S_n\right)-\exp\left(\frac{it}{\sqrt{mp}}S_{mp}\right)\right|\\ &\leq |t|\,\mathbb{E}\left|\frac{S_n}{\sqrt{n}}-\frac{S_{mp}}{\sqrt{mp}}\right|\leq|t|\left(\mathbb{E}\left(\frac{S_n}{\sqrt{n}}-\frac{S_{mp}}{\sqrt{mp}}\right)^2\right)^{1/2}\\ &\leq |t|\left(\frac{1}{\sqrt{mp}}-\frac{1}{\sqrt{n}}\right)\left(\mathbb{E}S_{mp}^2\right)^{1/2}+\frac{|t|}{\sqrt{n}}\left(\mathbb{E}Y_{m+1,p}^2\right)^{1/2}.\end{aligned}$$

It follows from the stationarity of the sequence of random variables X_i that, for m large enough, $\mathbb{E}S_{mp}^2 \leq 2\sigma^2 mp$ and $\mathbb{E}Y_{m+1,p}^2 \leq 2\sigma^2(n-mp) < 2\sigma^2 p$. Thus, as $n \longrightarrow +\infty$, which implies that $m \longrightarrow +\infty$, it follows

$$|\varphi_n(t) - \varphi_{mp}(t)| \leq \sqrt{2}\,|t|\,\sigma\left(1 - \frac{\sqrt{mp}}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right) \longrightarrow 0.$$

The second term in (14) represents the difference between the joint distribution of the blocks and what we would find if they were independent. To control this term, define $W_{j,p} = \frac{1}{\sqrt{p}} Y_{j,p}$. Taking into account the stationarity of the sequence X_i, the characteristic function of $W_{j,p}$ is $\varphi_p(t)$. As the variables $W_{j,p}$ are transformations of $X_{(j-1)p+1}, \ldots, X_{jp}$, it follows from the definition of L-weak dependence, representing the exponential with the trigonometric functions as done for the proof of Theorem 5.1, that

$$\begin{aligned}
\left|\varphi_{mp}(t) - \varphi_p^m(t)\right| &= \left|\mathbb{E}\left(\exp\left(\frac{it}{\sqrt{m}}\sum_{k=1}^{m} W_{k,p}\right)\right) - \prod_{k=1}^{m}\mathbb{E}\exp\left(\frac{it}{\sqrt{m}} W_{k,p}\right)\right| \\
&\leq \frac{4t^2}{mp}\sum_{\ell=2}^{m-1}\sum_{j=1}^{(\ell-1)p}\sum_{j'=(\ell-1)p+1}^{\ell p}\gamma_{j'-j} \qquad (15)\\
&= \frac{2t^2}{mp}\left(\sum_{j,j'=1}^{mp}\gamma_{|j'-j|} - m\sum_{j,j'=1}^{p}\gamma_{|j'-j|}\right).
\end{aligned}$$

It is easy to verify that

$$\frac{1}{mp}\sum_{j,j'=1}^{mp}\gamma_{|j'-j|} = \sum_{j=1}^{mp-1}(1 - \frac{j}{mp})\gamma_j \longrightarrow D = \sum_{\ell=1}^{\infty}\gamma_\ell < \infty, \qquad (16)$$

which gives us,

$$\limsup_{m\to+\infty}\left|\varphi_{mp}(t) - \varphi_p^m(t)\right| \leq 2t^2\left(D - \frac{1}{p}\sum_{j,j'=1}^{p}\gamma_{|j'-j|}\right).$$

For the third term in (14), the classical Central Limit Theorem for independent random variables implies that

$$\lim_{m\to+\infty}\left|\varphi_p^m(t) - e^{-\frac{t^2\sigma_p^2}{2}}\right| \longrightarrow 0.$$

Concerning the last term in (14), we have $\left|e^{-t^2\sigma_p^2/2} - e^{-t^2\sigma^2/2}\right| \leq \frac{t^2}{2}\left|\sigma_p^2 - \sigma^2\right|$. So, finally we obtain,

$$\limsup_{n\to+\infty}\left|\varphi_n(t) - e^{-\frac{t^2\sigma^2}{2}}\right| \leq \frac{t^2}{2}\left|\sigma_p^2 - \sigma^2\right| + 2t^2\left(D - \frac{1}{p}\sum_{j,j'=1}^{p}\gamma_{|j'-j|}\right).$$

Note that the left hand side above does not depend on p. Allowing now $p \longrightarrow +\infty$ and taking into account that $\lim_{p\to+\infty}\sigma_p^2 = \sigma^2$, it follows that

$$\limsup_{n\to+\infty}\left|\varphi_n(t) - e^{-\frac{t^2\sigma^2}{2}}\right| = 0.$$

■

Remark 5.3. *Versions of the Central Limit Theorem obviously exist for the associated or quasi-associated dependence structures that are embedded in the L-weak dependence. Theorem 5.2 above corresponds exactly to Theorem 4.1 in Oliveira* [21]*, as the convergence of the coefficients is, for the L-weak dependence, included in the definition. A necessary* *and sufficient condition for the CLT in the case of strictly stationary associated random fields, hence in a more general framework than the one considered here, is given in Theorem 1 in Bulinski* [2]*, expressed in terms of the uniform integrability of a suitable normalization of* S_n *using sums of covariances between the variables.*

For quasi-associated random variables, the CLT is proved in Corollary 6 in Bulinski and Suquet [5]*, under assumptions that are, in fact, somewhat weaker than the ones we used in Theorem 5.2. Indeed, Corollary 6 in* [5] *proves the CLT assuming a Lindeberg like condition, which is seen to be a consequence of the convergence of the dependence coefficients, as noted in their Remark 8.*

Remark 5.4. *Concerning the weak dependence defined by Dedecker et al.* [6]*, the CLT above should be compared with their Theorem 7.1 (see Doukhan and Winterberger* [9] *for the original statement), where the existence of moments of order strictly larger than 2 is required, along with a suitable decrease rate on the* κ *dependence coefficients:* $\kappa(n) = O(n^{-k})$ *from some* $k > 2$*.*

In order to prove a functional version of Theorem 5.2, we need a moment inequality, extending Lemma 4.2 in Doukhan and Winterbeger [9] (this result may be found as Lemma 4.3 in Dedecker et al. [6]) to L-weak dependent variables.

Lemma 5.5. *Let the sequence of random variables* X_n*,* $n \geq 1$*, be centered, L-weakly dependent, strictly stationary and satisfies* $\mathbb{E}\left|X_1\right|^{2+\zeta} < \infty$ *for some* $\zeta > 0$*, and (2). Let* $\eta > 0$ *and* $\delta \in (0, \min(B(\zeta,\eta),1))$*, where* $B(\zeta,\eta) \leq \zeta$ *is described below. If the L-weak dependence coefficients* γ_k*,* $k \geq 1$*, are decreasing such that* $\gamma_k = O(k^{-r})$*, for some* $r > 3 + \min(0, \frac{\delta^2+\delta(1-\zeta)+2}{2(\zeta-\delta)})$*, then there exists* $C > 0$ *such that* $\mathbb{E}S_n^{2+\delta} \leq Cn^{1+\delta/2}$*.*

Proof. (Sketch) The proof follows the same arguments as for Lemma 4.2 in [9], so we will only highlight the differences. The initial construction referring to the induction step and truncation is exactly the same as in the proof of Lemma 4.2 in Doukhan and Winterberger [9]. This leads to a bound of order $n^m T^{2+\delta-m} + \mathrm{Cov}((S_n^T)^2, (1+\left|S_{2n+q}^T - S_{n+q}^T\right|)^\delta)$, where $T > 0$ is the truncation parameter and $S_n^T = g_T(X_1) + \cdots + g_T(X_n)$, where g_T is the truncating function we defined earlier in Subsection 4.2. It follows from the L-weak dependence that

$$\mathrm{Cov}((S_n^T)^2, (1+\left|S_{2n+q}^T - S_{n+q}^T\right|)^\delta) \leq nT \sum_{i=1}^{n} \sum_{j=n+q}^{2n+q} \gamma_{j-i} \leq n^2 T v(q).$$

From here onwards, the arguments are exactly the same as in the proof of Lemma 4.2 in [9], taking into account that $v(q) = O(q^{-r+1})$. For the bound on δ, we find $B(\zeta,\eta) = \frac{1}{2}\left(\zeta - 1 - 2\eta + \sqrt{(\zeta-1-2\eta)^2 - 8(1-\zeta\eta)}\right)$. ∎

We now prove a functional version of Theorem 5.2, giving sufficient conditions for the convergence in distribution of the partial sums process:

$$\xi_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} X_j, \qquad 0 \leq t \leq 1. \tag{17}$$

Theorem 5.6. *Assume the conditions of Lemma 5.5 are satisfied. Then* $\xi_n(t)$*,* $n \geq 1$*, converges in distribution, in the Skorokhod space* $\mathbb{D}[0,1]$*, to* σW*, where* W *is the standard Brownian motion.*

Proof. The proof follows the usual arguments to prove the convergence with respect to the Skorokhod topology: prove the convergence of the finite dimensional distributions and the tightness of the sequence. The convergence of the one dimensional marginal distributions of ξ_n follows directly from Theorem 5.2. Now, choose k points such that $0 = u_0 \leq u_1 < u_2 < \cdots < u_k \leq 1$. We shall prove the asymptotic normality of the random vector

$$H(u_1, \ldots, u_k) = \frac{1}{\sqrt{n}} \left(\xi_n(u_1), \xi_n(u_2) - \xi_n(u_1), \ldots, \xi_n(u_k) - \xi_n(u_{k-1})\right).$$

Note that, due to the stationarity, it follows again from Theorem 5.2 that each coordinate of $H(u_1, \ldots, u_k)$ is asymptotically centered normal with variance $(u_s - u_{s-1})\sigma^2$, $s = 1, \ldots, k$. We now compare the characteristic function of the random vector with the product of the characteristic functions of its margins. In the sequel, denote $T = \max_{s=1,\ldots,k} |t_s|$. From the definition of L-weak dependence, reasoning as for the decomposition (15), taking into account that $\| \cos(\sum_j t_j X_j)\|_L = \max_{j=1,\ldots,k} |t_j|$, we get that, for every $t_1, \ldots, t_k \in \mathbb{R}$,

$$\begin{aligned} &\left| \mathbb{E} \exp\left(\frac{i}{\sqrt{n}} \sum_{s=1}^{k} t_s \left(\xi_n(u_s) - \xi_n(u_{s-1})\right)\right) - \prod_{s=1}^{k} \mathbb{E} \exp\left(\frac{it_s}{\sqrt{n}} \left(\xi_n(u_s) - \xi_n(u_{s-1})\right)\right)\right| \\ &\qquad \leq \frac{4kT^2}{n} \sum_{s=2}^{k-1} \sum_{j=1}^{\lfloor nu_{s-1}\rfloor} \sum_{j'=\lfloor nu_{s-1}\rfloor+1}^{\lfloor nu_s\rfloor} \gamma_{j'-j} \\ &\qquad = \frac{2T^2}{n} \left(\sum_{j,j'=1}^{\lfloor nu_k\rfloor} \gamma_{|j'-j|} - \sum_{s=1}^{k} \sum_{j,j'=\lfloor nu_{s-1}\rfloor+1}^{\lfloor nu_s\rfloor} \gamma_{|j'-j|} \right). \end{aligned}$$

Note that our assumption on the decrease rate of the γ_j coefficients implies the convergence of the corresponding series. So, defining D as in (16), the above expression is easily seen to converge to $2T^2D(u_k - u_1 - (u_2 - u_1) - \cdots - (u_k - u_{k-1})) = 0$, hence we have proved the asymptotic normality of $H(u_1, \ldots, u_k)$.

To complete the proof, we need to prove the tightness of the sequence. This is now achieved as in the proof of Theorem 8.1 in Dedecker et al. [6]. Taking into account the moment inequality proved in Lemma 5.5, we have that (ξ_n, λ), where λ is the one dimensional Lebesgue measure is of class $\mathcal{C}(1+\delta, 2+2\delta)$ defined in Bickel and Wichura [1], so tightness follows from their Theorem 3. ■

This result complements Theorem 5 in Doukhan and Louhichi [8] and Theorems 2.1 and 2.2 in Doukhan and Winterberger [9], where different weak dependence structures were considered.

It is still possible to prove the result concerning the convergence of the empirical process, again somehow in a similar way it is done in Doukhan and Louhichi [8]. For this later result, in [8] the dependence coefficient considered is always larger than our generalized Cox-Grimmett coefficients, so that their result implies directly the corresponding one for L-weakly dependent variables. We include a sketch of proof corresponding to the adaptation of the proof of Theorem 6 in [8], and mention some related issues concerning the particular case of associated random variables.

Theorem 5.7. *Let X_n, $n \geq 1$, be L-weakly dependent and strictly stationary random variables uniformly distributed on* $[0,1]$. *If the L-weak dependence coefficients γ_k, $k \geq 1$, satisfy $\gamma_k = O(k^{-15/2-\delta})$, for some $\delta > 0$, then $\zeta_n(t) = \sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n} \mathbb{I}_{[0,t]}(X_j) - t\right)$, $t \in [0,1]$, $n \geq 1$, converges in distribution in the Skorokhod space $\mathbb{D}[0,1]$ to a centered Gaussian process indexed by $[0,1]$ with covariance operator*

$$\Gamma(s,t) = \sum_{k=1}^{+\infty} \operatorname{Cov}\left(\mathbb{I}_{[0,s]}(X_1), \mathbb{I}_{[0,t]}(X_k)\right).$$

Proof. (Sketch) The proof follows the arguments used to prove Theorem 6 in Doukhan and Louhichi [8], based on the large-block-small-block variant of the classical Bernstein decomposition method. First, we bound the covariances between transformations of the variables by indicator functions by means of the covariances of the original variables. Reproducing the arguments in the proof of Corollary 2.36 in Oliveira [21] and adapting the indices in the proof of Theorem 5.1, we find that there exists a constant $C>0$, such that $\mathrm{Cov}(\mathbb{I}_{[0,t]}(X_j),\mathbb{I}_{[0,t]}(X_k))\leq C\gamma_{|k-j|}^{1/3}$. This means that, following now the argument for the proof of Proposition 2 in Doukhan and Louhichi [8], the tightness in $\mathbb{D}[0,1]$ of the sequence ζ_n follows if $\gamma_k=O(k^{-15/2-\delta})$, for some $\delta>0$. To complete the proof, we need to argue the convergence of the finite dimensional distributions. For this purpose, we follow the final part of the proof of Theorem 6 in [8] step by step, constructing sequences p_n, q_n both converging to $+\infty$, such that $\frac{p_n}{n}\longrightarrow 0$ and $\frac{q_n}{p_n}\longrightarrow 0$, in order to apply their Lemma 11. With respect to our dependence coefficients we find, instead of the upper bound (4.15) in Doukhan and Louhichi [8], the bound $\gamma_{q_n}^{1/3}$, hence the construction of the auxiliary sequences holds, and the theorem follows. ■

Remark 5.8. *We expressed our results with respect to the dependence coefficients regarding the original sequence of random variables. We highlight this point as the statement of Proposition 2 in Doukhan and Louhichi* [8] *refers to the coefficients with respect to the family of variables obtained by transformation through indicator functions, requiring the multiplication of the exponent by 3, as described in Doukhan and Louhichi* [8] *in the comments that follow their Proposition 2.*

Remark 5.9. *One final note about the decrease rate of the dependence coefficients and the particular case of associated variables. First, note that the convergence rate for the dependence coefficients is determined by the need to prove the tightness of the sequence* ζ_n *and also by the bound for covariances of transformations through indicator functions by means of the dependence coefficients of the original variables. This explains why the exponent* $\frac{1}{3}$ *appears, likewise what we could reproduce for L-weak dependent variables. For the convergence of the empirical process with underlying positively associated variables, this was at the origin of the decrease rates proposed by Yu* [29]*, Shao and Yu* [25] *or Oliveira and Suquet* [22]*. For this particular dependence structure, the exponent in the upper bound has recently been improved by Demichev* [7]*, showing that this exponent may be arbitrarily close, but never equal, to* $\frac{1}{2}$ *(see Corollary 1.1 in Demichev* [7]*). This improvement means that the proof methodology used above implies the convergence of the uniform empirical process with underlying associated variables if* $\mathrm{Cov}(X_1,X_k)=O(k^{-5-\delta})$*, for some* $\delta>0$*. However, this is still weaker than the condition given by Louhichi* [17] *for underlying associated variables, who only requires that* $\mathrm{Cov}(X_1,X_k)=O(k^{-4-\delta})$*, for some* $\delta>0$*.*

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