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# Generalised enriched categories: exponentiation and injectivity

Tese no âmbito do Programa Interuniversitário de Doutoramento em Matemática, orientada pela Professora Doutora Maria Manuel Clementino e apresentada ao Departamento de Matemática da Faculdade de Ciências e Tecnologia da Universidade de Coimbra.

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Abstract

Among the classical solutions to the problem of non-cartesian closedness of the category Top of topological spaces and continuous maps, in this thesis we are interested in *compactly generated spaces*, *equilogical spaces*, and *quasi-topological spaces*; working with generalised enriched categories, which allow for a unified treatment of a range of categories from Topology and Analysis (e.g., ordered, metric, topological, and approach spaces), we generalise these three concepts from Top to $(\mathbb{T},\mathsf{V})$-Cat.

In order to do so, we start by studying the relation between injective and exponentiable $(\mathbb{T},\mathsf{V})$-spaces, and by proving that $(\mathbb{T},\mathsf{V})$-Cat is a weakly locally cartesian closed category. Then we introduce the category $(\mathbb{T},\mathsf{V})$-Equ of *equilogical* $(\mathbb{T},\mathsf{V})$*-spaces* and its morphisms, which we prove to be a cartesian closed category. Moreover, we study a generalised relation between equilogical $(\mathbb{T},\mathsf{V})$-spaces and the regular and exact completions of $(\mathbb{T},\mathsf{V})$-Cat, culminating in the fact that $(\mathbb{T},\mathsf{V})$-Equ is a quasitopos.

We finish by carrying the concepts of $\mathscr{C}$-generated spaces and quasi-topological spaces into $(\mathbb{T},\mathsf{V})$-Cat. We prove that $\mathscr{C}$-generated $(\mathbb{T},\mathsf{V})$-spaces form a fully coreflective cartesian closed subcategory of $(\mathbb{T},\mathsf{V})$-Cat; examples of such spaces include *compactly generated* $(\mathbb{T},\mathsf{V})$*-spaces* and *Alexandroff* $(\mathbb{T},\mathsf{V})$*-spaces*. For the latter, we make some discussions towards a generalisation of the equivalence between Alexandroff topological spaces and ordered sets. Concerning quasi-$(\mathbb{T},\mathsf{V})$-spaces, they form the category Qs$(\mathbb{T},\mathsf{V})$-Cat which we prove to be cartesian closed and topological over the category Set of sets and maps. We also generalise to $(\mathbb{T},\mathsf{V})$-Cat an interesting relation between quasi-topological spaces and compactly generated spaces.

**Keywords:** generalised enriched category, exponentiation, injectivity, (weak) cartesian closedness, exact completion, equilogical space, C-generated space, quasi-topological space.

# Resumo

Dentre as soluções clássicas para o problema da categoria Top dos espaços topológicos e aplicações contínuas não ser cartesiana fechada, nesta tese estamos interessados em *espaços compactamente gerados*, *espaços equilógicos*, e *espaços quasi-topológicos*; trabalhando com categorias enriquecidas generalizadas, que permitem um tratamento unificado de uma gama de categorias da Topologia e da Análise (e.g., espaços ordenados, métricos, topológicos e de aproximação), generalizamos estes três conceitos de Top para $(\mathbb{T},\mathsf{V})$-Cat.

Para tal finalidade, começamos por estudar a relação entre os $(\mathbb{T},\mathsf{V})$-espaços injectivos e exponenciáveis, e por provar que $(\mathbb{T},\mathsf{V})$-Cat é uma categoria fracamente localmente cartesiana fechada. Em seguida, introduzimos a categoria $(\mathbb{T},\mathsf{V})$-Equ dos $(\mathbb{T},\mathsf{V})$-*espaços equilógicos* e seus morfismos, que provamos ser uma categoria cartesiana fechada. Ademais, estudamos uma relação generalizada entre os $(\mathbb{T},\mathsf{V})$-espaços equilógicos e os completamentos regular e exato de $(\mathbb{T},\mathsf{V})$-Cat, culminando no fato de que $(\mathbb{T},\mathsf{V})$-Equ é um quasitopos.

Por fim, transportamos os conceitos de espaços $\mathscr{C}$-gerados e espaços quasi-topológicos para $(\mathbb{T},\mathsf{V})$-Cat. Provamos que os $(\mathbb{T},\mathsf{V})$-espaços $\mathscr{C}$-gerados formam uma subcategoria plena coreflectiva cartesiana fechada de $(\mathbb{T},\mathsf{V})$-Cat; exemplos de tais espaços incluem $(\mathbb{T},\mathsf{V})$-*espaços compactamente gerados* e $(\mathbb{T},\mathsf{V})$-*espaços de Alexandroff*. Para os últimos, fazemos algumas considerações que direcionam a uma generalização da equivalência entre os espaços topológicos de Alexandroff e os conjuntos ordenados. Quanto aos quasi-$(\mathbb{T},\mathsf{V})$-espaços, eles formam a categoria Qs$(\mathbb{T},\mathsf{V})$-Cat, a qual provamos ser cartesiana fechada e topológica sobre a categoria Set dos conjuntos e aplicações. Generalizamos também para $(\mathbb{T},\mathsf{V})$-Cat uma relação interessante entre espaços quasi-topológicos e espaços compactamente gerados.

**Palavras-chave:** categoria enriquecida generalizada, exponenciação, injectividade, fechamento cartesiano (fraco), completamento exato, espaço equilógico, espaço C-gerado, espaço quasi-topológico.

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# Introduction

The problem of exponentiability of topological spaces can be traced back to Fox in [Fox45], who discusses that certain solution for this problem has been known for years before his publication. Namely, it was known that every locally compact space is exponentiable. In fact, under the assumption of Hausdorff separation, these concepts are equivalent [Mic68]. According to Fox, the question was motivated by homotopy theory: for the unit interval, exponentiability implies that homotopies are equivalent to paths in the set of continuous maps. Fox mentions that was Hurewicz who proposed him the problem. Years later, Day and Kelly gave a characterization of exponentiable topological spaces by means of preservation of quotient maps [DK70]. It is well-known that this is the case because the problem of exponentiability is about the existence of an adjoint to a functor, which, in the particular case of topological spaces, always preserves disjoint sums. Another characterization is the following: a topological space is exponentiable if, and only if, its lattice of open sets is *continuous*; this fact and much more detailed history on exponentiability in Top can be found in [Isb86].

Since not every topological space is exponentiable, that is, the category Top is not cartesian closed, it is an interesting problem to find such a category of topological spaces. A classical approach is to work on *compactly generated spaces*, which are proposed in [Ste67] as convenient for algebraic topology. The earliest references of compactly generated spaces are [Kel55] and [Gal50], and in the latter the author attributes the definition of the notion to Hurewicz. Compactly generated spaces are fully coreflective in Top, what is also shown in [Mac71], and, in fact, they form the *coreflective hull* of compact Hausdorff spaces in Top. Hence the question of cartesian closedness of this subcategory fits the general approach of [Nel78]; we also refer to [Day72]. In the present thesis, we are particularly interested in the approach of [ELS04], which is centered in the idea of generating classes of topological spaces.

An alternative approach to the problem is to enlarge Top to a cartesian closed category. In [Sco96], and later in [BBS04], the category Equ of *equilogical spaces* and equivalence classes of equivariant maps is introduced as such an enlargement, consisting essentially of joining topological spaces with equivalence relations. Equ is directly seen to be cartesian closed by its equivalence with a similar category based on injective separated topological spaces, which can be seen as *algebraic lattices*. Moreover, the category of equilogical spaces relates to the exact completion $\mathsf{Top}_{\mathrm{ex}}$ of Top, as studied in [Ros99]. The fact that Top is *weakly (locally) cartesian closed*, which is further studied in [CR00], implies that $\mathsf{Top}_{\mathrm{ex}}$ is a (locally) cartesian closed category. It is proved in [Ros98] that Equ is equivalent to the regular completion $\mathsf{Top}_{\mathrm{reg}}$ of Top, which proves to be a *quasitopos* by the results of [Men00].

Yet another enlargement of Top are Spanier's *quasi-topological spaces* that can be found in [Spa63]. The author mentions that the main motivation behind *quasi-topologies* is to endow the set of the so-called *quasi-continuous* maps between *quasi-spaces* with a natural suitable quasi-topology, so that the category QsTop of quasi-topological spaces and quasi-continuous maps is cartesian closed. QsTop is convenient for homotopy theory, and although its size has an illegitimacy proved in [HR83], this category has been explored in recent works, for instance, [DM12, Pet15]. The archetype of quasi-topological spaces appears in [XE13] and [Dub79, DE06], and in the last two references a general approach for such spaces using the notion of *Grothendieck topologies* is presented.

The main goal of the present thesis is to carry the concepts discussed in the previous three paragraphs from Top to $(\mathbb{T},\mathsf{V})$-Cat. For details on the history of the $(\mathbb{T},\mathsf{V})$-setting we refer to the book [HST14]. We limit ourselves to comment on the principal well known facts. Manes proved that compact Hausdorff topological spaces are Eilenberg-Moore algebras for the ultrafilter monad $\mathbb{U}$ [Man69]. Later on, by relaxing the axioms of an algebra, Barr presented topological spaces as *relational algebras* for $\mathbb{U}$ [Bar70]; this monad is called there the *triple of compact Hausdorff spaces*. Joining this approach with Lawvere's description of generalised metric spaces [Law73], Clementino and Tholen combined a monad $\mathbb{T}$ and a monoidal-closed category $\mathsf{V}$ creating the $(\mathbb{T},\mathsf{V})$-setting [CT03]. We are interested in this thesis in the particular case when $\mathsf{V}$ is a quantale, so that the setting provides a unified treatment of a range of categories as the ones of ordered, (probabilistic) metric, ultrametric, (bi)topological, approach, and multi-ordered spaces. A *lax extension* of the monad $\mathbb{T}$ to the order-enriched category $\mathsf{V}$-Rel is always assumed to exist, and in order to develop our results, we restrict ourselves to the lax extensions given by means of a $\mathbb{T}$-algebra structure on $\mathsf{V}$, as introduced by Hofmann [Hof07], so we work on the setting of *strict topological theories*. Those extensions are characterized in [CT14] as the *algebraic lax extensions*. We observe that our conditions are stronger than the ones used in [HST14].

We organize this thesis as follows. The first chapter is devoted to the background. We recall some facts about topological functors, and present the $(\mathbb{T},\mathsf{V})$-setting, exposing the main classes of spaces that we are dealing with in the remaining chapters, namely, separated, injective, exponentiable (including a criterion for exponentiability in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$), and compact and Hausdorff $(\mathbb{T},\mathsf{V})$-spaces. We also highlight some topological aspects of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ that are going to be needed. In the second chapter we prove that, under suitable conditions, injective $(\mathbb{T},\mathsf{V})$-spaces are exponentiable in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, and that $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is weakly (locally) cartesian closed. These results are part of [CHR20]. The third chapter is concerned with equilogical $(\mathbb{T},\mathsf{V})$-spaces and their relation with the regular and exact completions $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathrm{reg}}$ and $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathrm{ex}}$ of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, respectively. The results of Chapter II play an important role in this chapter, whose final achievement is to prove, using results of [Men00], that the category $(\mathbb{T},\mathsf{V})$-$\mathsf{Equ}$ of equilogical $(\mathbb{T},\mathsf{V})$-spaces and its morphisms is a *quasitopos*. We follow directly the work in [BBS04], and the results of this chapter are part of [Rib19a]. Finally, in the fourth chapter we present the categories $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathscr{C}}$ of $\mathscr{C}$-generated $(\mathbb{T},\mathsf{V})$-spaces and $(\mathbb{T},\mathsf{V})$-continuous maps, and $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ of quasi-$(\mathbb{T},\mathsf{V})$-spaces and quasi-$(\mathbb{T},\mathsf{V})$-continuous maps. The main examples of $\mathscr{C}$-generated spaces are the compactly generated spaces, when $\mathscr{C}$ is the class of compact and Hausdorff $(\mathbb{T},\mathsf{V})$-spaces, and the Alexandroff $(\mathbb{T},\mathsf{V})$-spaces, when $\mathscr{C}$ is the singleton containing the Sierpiński $(\mathbb{T},\mathsf{V})$-space; the second class leads to investigating a generalisation of the well-known fact that the category of Alexandroff topological spaces is equivalent to the category of ordered sets. Among other properties, we prove that $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathscr{C}}$ is coreflective in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathscr{C}}$ and $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ are cartesian closed categories, and $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is topological over Set. We establish, in our level of generality, the relationship between quasi-topological spaces and compactly generated spaces studied by Day in [Day68], whose work we follow closely.

# Chapter I

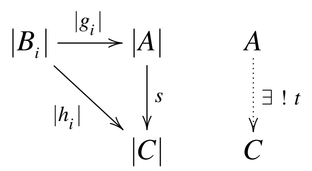
# Categories of lax algebras and some of their topological aspects

The following chapter is thought mainly as a source for references and to fix notation for the thesis. All concepts and facts presented in this chapter can be found in the literature, and we provide references for such.

We start by commenting on two needed facts about topological functors, and then we move on to the background on the category $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$, which is the central object of study of the thesis. Despite its rich theory, we focus on the framework needed to present our results, namely, the concepts of injectivity, (weak) exponentiability, compactness, and (Hausdorff) separation.

## 1 A comment on topological functors

For a category $\mathsf{A}$, a family $(g_i\colon B_i \to A)_{i\in I}$ of morphisms of $\mathsf{A}$ is called an $\mathsf{A}$-*sink*, or simply a *sink*. Let $\mathsf{X}$ be a category and $|\text{-}|\colon \mathsf{A} \to \mathsf{X}$ be a functor. The $\mathsf{A}$-sink $(g_i)_{i\in I}$ is said to be $|\text{-}|$-*final* if, for every $\mathsf{A}$-sink $(h_i\colon B_i \to C)_{i\in I}$, and every morphism $s\colon |A| \to |C|$ of $\mathsf{X}$ such that $s\cdot|g_i| = |h_i|$, for each $i \in I$, there exists a unique morphism $t\colon A \to C$ of $\mathsf{A}$ such that $|t| = s$, and $t\cdot g_i = h_i$, for each $i \in I$.




The dual concept is that of a $|\text{-}|$-*initial source*.

**Definition 1.0.1** The functor $|\text{-}|: \mathsf{A} \to \mathsf{X}$ is *topological* if every $\mathsf{X}$-sink $(f_i : |B_i| \to X)_{i\in I}$ admits a $|\text{-}|$-*final lifting*, that is, there exists a $|\text{-}|$-final $\mathsf{A}$-sink $(g_i : B_i \to A)_{i\in I}$ such that $|g_i| = f_i$, for each $i \in I$.

Equivalently, the functor $|\text{-}|: \mathsf{A} \to \mathsf{X}$ is topological if every $\mathsf{X}$-source $(f_i : X \to |B_i|)_{i\in I}$ admits an $|\text{-}|$-*initial lifting* [AHS90, Theorem 21.9]. Recall that for $X \in \mathsf{X}$, $|\text{-}|^{-1}(X) = \{A \in \mathsf{A} \mid |A| = X\}$ is called the *fibre* of $X$.

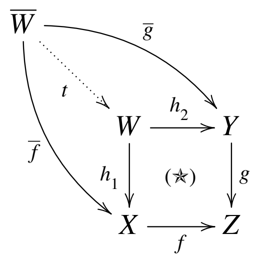
**Definition 1.0.2** The functor $|\text{-}|: \mathsf{A} \to \mathsf{X}$ is *fibre-small* if the fibre of every object of $\mathsf{X}$ is a set (rather than a proper class).

Throughout $|\text{-}|: \mathsf{A} \to \mathsf{X}$ will be a fibre-small forgetful functor, whence, by [AHS90, Proposition 21.34], one only needs to consider *small* sinks in Definition 1.0.1, that is, when $I$ is a set. In this case, each $|\text{-}|$-final lifting of an $\mathsf{X}$-sink $(g_j : |B_j| \to X)_{j\in J}$, for any $J$, is the $|\text{-}|$-final lifting of a small (sub)sink $(g_i : |B_i| \to X)_{i\in I}$, $I \subseteq J$.

# 2 The $(\mathbb{T}, \mathsf{V})$ setting

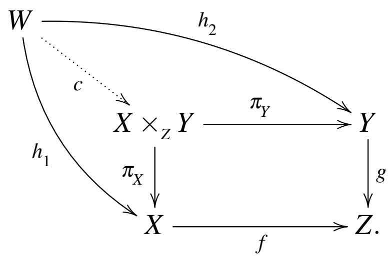
## 2.1 The variables $\mathbb{T}$ and $\mathsf{V}$

The first of our variables is a non-trivial monad $\mathbb{T} = (T, m, e)\colon \mathsf{Set} \to \mathsf{Set}$ satisfying a suitable condition that we present next. As usual, for maps $f\colon X \to Z$ and $g\colon Y \to Z$, we say that the square $(\star)$ below is a *weak pullback*, or a *Beck-Chevalley* square, *(BC)-square* for short, if for each maps $\overline{f}\colon \overline{W} \to X$ and $\overline{g}\colon \overline{W} \to Y$ such that $f \cdot \overline{f} = g \cdot \overline{g}$, there exists a (not necessarily unique) map $t\colon \overline{W} \to W$ such that $h_1 \cdot t = \overline{f}$ and $h_2 \cdot t = \overline{g}$.




(I.1)

Assuming the Axiom of Choice, so that every epimorphism splits in Set, one can see that $(\star)$ is a weak pullback precisely when the canonical map $c\colon W \to X \times_Z Y$ is surjective, where the inner rectangle in the diagram below is a pullback.

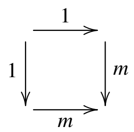



**Definition 2.1.1** A monad $\mathbb{T}=(T,m,e)\colon \mathsf{Set}\to\mathsf{Set}$ satisfies the *Beck-Chevalley condition*, (BC) for short, if the functor $T$ preserves (BC)-squares, and the naturality diagrams of $m$ are (BC)-squares.

From [Man02] we have the following notion:

**Definition 2.1.2** A functor $T$ is *taut* if it preserves *inverse images*, that is, if it preserves pullbacks along monomorphisms.

A morphism $m$ is monic if, and only if, the diagram




is a (BC)-square. Hence if $\mathbb{T}$ satisfies (BC), then the functor $T$ preserves monomorphisms. Moreover, a (BC)-square (I.1) with $g$ monic is a pullback, whence if $\mathbb{T}$ satisfies (BC), then $T$ is taut. For details on the latter facts see [CHJ14].

**Examples 2.1.3** The following monads are going to be used in the thesis:
(1) the *identity monad* $\mathbb{I}=(\mathrm{Id},1,1)$, where $\mathrm{Id}\colon\mathsf{Set}\to\mathsf{Set}$ is the identity functor, and the components of the natural transformation 1 are identity maps;
(2) the *ultrafilter monad* $\mathbb{U}=(U,m,e)$, where $U\colon\mathsf{Set}\to\mathsf{Set}$ is the ultrafilter functor that assigns to each set $X$ its set of ultrafilters $UX$, and to each map $f\colon X\to Y$ the map $Uf\colon UX\to UY$, $\mathfrak{x}\mapsto Uf(\mathfrak{x})$, where $B\in Uf(\mathfrak{x})$ if, and only if, $f^{-1}(B)\in\mathfrak{x}$. The $X$-component of the multiplication is $m_X\colon U^2X\to UX$, where, for each $\mathfrak{X}\in U^2X$, $A\in m_X(\mathfrak{X})$ if, and only if, $\{\mathfrak{x}\in UX\mid A\in\mathfrak{x}\}\in\mathfrak{X}$; and the $X$-component of the unit is $e_X\colon X\to UX$, $x\mapsto\dot{x}$, where $\dot{x}$ is the principal ultrafilter generated by $x$, that is, $A\in\dot{x}$ if, and only if, $x\in A$;
(3) $\mathbb{M}=(-\times M,m,e)$, where $M=(M,*,\alpha_M)$ is a monoid. For each set $X$, $(-\times M)X=X\times M$, and, for each map $f\colon X\to Y$, $(-\times M)f=f\times 1_M\colon X\times M\to Y\times M$, $(x,a)\mapsto(f(x),a)$. The multiplication and

unit are given by $m_X : X\times M\times M \to X\times M$, $(x,a,b)\mapsto (x,a*b)$, and $e_X : X\to X\times M$, $x\mapsto (x,\alpha_M)$, respectively;

(4) the *list monad*, also called the *free-monoid* monad or *word* monad, $\mathbb{L}=(L,m,e)$, where, for each set $X$,

$$LX=\{(x_1,x_2,\ldots,x_p)\mid p\in\mathbb{N},\ p\geq 1,\ x_i\in X,\ 1\leq i\leq p\}\cup\{()\},$$

with $()$ the *empty list*. For a map $f\colon X\to Y$,

$$Lf\colon LX\to LY,\ ()\mapsto (),\ (x_1,x_2,\ldots,x_p)\mapsto (f(x_1),f(x_2),\ldots,f(x_p)).$$

The $X$-component of the multiplication is given by $m_X : L^2X\to LX$,

$$((x^1_1,x^1_2,\ldots,x^1_{p_1}),\ldots,(x^q_1,x^q_2,\ldots,x^q_{p_q}))\mapsto (x^1_1,x^1_2,\ldots,x^1_{p_1},\ldots,x^q_1,x^q_2,\ldots,x^q_{p_q}),$$

and the $X$-component of the unit is given by $e_X : X\to LX$, $x\mapsto (x)$.

We present the second variable of the $(\mathbb{T},\mathsf{V})$-setting:

**Definition 2.1.4** A *unital commutative quantale* $\mathsf{V}=(\mathsf{V},\otimes,k)$ is a complete lattice that is a monoid for the tensor operation $\otimes$, which is commutative with unit $k$, so that, for each $u,v,w\in\mathsf{V}$,

$$(u\otimes v)\otimes w=u\otimes(v\otimes w),\quad k\otimes u=u\otimes k=u,\quad u\otimes v=v\otimes u.$$

Furthermore, for all $v\in\mathsf{V}$, the function $v\otimes - = -\otimes v\colon \mathsf{V}\to\mathsf{V}$ is a *sup-map*, that is, it preserves arbitrary suprema.

Consequently, for each $v\in\mathsf{V}$, there exists a right adjoint $\hom(v,-)\colon\mathsf{V}\to\mathsf{V}$, so that, for each $u,w\in\mathsf{V}$,

$$u\otimes v\leq w\iff u\leq\hom(v,w). \tag{I.2}$$

For each $v,w\in\mathsf{V}$,

$$\hom(v,w)=\bigvee\{u\in\mathsf{V}\mid u\otimes v\leq w\}. \tag{I.3}$$

Let us denote the bottom and the top elements of the complete lattice $\mathsf{V}$ by $\bot$ and $\top$, respectively. Using the formula (I.3), one calculates, for each $v \in \mathsf{V}$,

$$\begin{aligned} \hom(v,\top) &= \bigvee\{u \in \mathsf{V} \mid u\otimes v \leq \top\} = \top, \\ \hom(k,v) &= \bigvee\{u \in \mathsf{V} \mid u\otimes k = u \leq v\} = v, \\ \hom(\bot,v) &= \bigvee\{u \in \mathsf{V} \mid u\otimes \bot = \bot \leq v\} = \top. \end{aligned} \tag{I.4}$$

Notice that the first equality follows directly from the fact that $\hom(v,-)$ is a right adjoint, hence it preserves the terminal object $\top$ of $\mathsf{V}$.

Throughout all quantales considered are going to be *Heyting algebras*, so that, for each $u \in \mathsf{V}$, the map $u\wedge -\colon \mathsf{V}\to\mathsf{V}$ has a right adjoint. Some additional conditions on the quantales are going to be often needed.

**Definition 2.1.5** A quantale $\mathsf{V} = (\mathsf{V},\otimes,k)$ is said to be
(1) *integral* when $k = \top$;
(2) *lean* when for each $u,v \in \mathsf{V}$, $(u\vee v = \top \ \& \ u\otimes v = \bot) \implies (u = \top \text{ or } v = \top)$.

Another fundamental condition on the quantale $\mathsf{V}$ used in [CT03] is *complete distributivity*. Let $\mathrm{Dn}(\mathsf{V}) = \{A \subseteq \mathsf{V} \mid A \text{ is a down-set}\}$, that is, $A \in \mathrm{Dn}(\mathsf{V})$ if, and only if, for each $v \in \mathsf{V}$, if there exists $a \in A$ such that $v \leq a$, then $v \in A$. We have a map $\downarrow\colon \mathsf{V}\to\mathrm{Dn}(\mathsf{V})$, $v \mapsto \downarrow v = \{u \in \mathsf{V} \mid u \leq v\}$, which is monotone when $\mathrm{Dn}(\mathsf{V})$ is ordered by inclusion, and since $\mathsf{V}$ is a complete lattice, we have an adjunction $\bigvee \dashv \downarrow\colon \mathsf{V}\to\mathrm{Dn}(\mathsf{V})$. One says that $\mathsf{V}$ is *completely distributive* if the map $\bigvee$ has a left adjoint: $\Downarrow\dashv \bigvee\colon \mathrm{Dn}(\mathsf{V})\to\mathsf{V}$. The existence of the map $\Downarrow$ implies the existence of a *totally below* relation $\ll$ on $\mathsf{V}$:

$$u \ll v \iff \forall A \subseteq \mathsf{V}\left(v \leq \bigvee A \implies \exists\, z \in \mathsf{V}\ (u \leq z)\right),$$

so that, for each $u,v,z \in \mathsf{V}$, if $u \ll v \leq z$, then $u \ll z$, and $v \leq \bigvee\{w \in \mathsf{V} \mid w \ll v\}$. Moreover, each element $v$ of $\mathsf{V}$ is $\ll$*-atomic*, that is, for each down-set $A \subseteq \mathsf{V}$, $v \ll \bigvee A$ implies $v \in A$. Complete distributivity can be characterized by the existence of such a relation [HST14, II-Proposition 1.11.1], and can be defined for any complete lattice. For details on these definitions we refer to [HST14, II-1.10, II-1.11, III-1.2].

**Examples 2.1.6** The following quantales are going to be used in the thesis:

(1) the *two-chain* $2 = \{\bot, \top\}$, with $\bot < \top$, $\otimes = \wedge$, and $k = \top$. The operation hom has the same value as "implication", where $\bot = \mathit{false}$ and $\top = \mathit{true}$, so that

$$\hom(\bot,\bot) = \hom(\bot,\top) = \hom(\top,\top) = \top \quad \& \quad \hom(\top,\bot) = \bot;$$

(2) the extended real half line $[0,\infty]$, with the order $\geq$, $\otimes = +$ the usual addition, and $k = 0$. This quantale is denoted by $\mathsf{P}_{+} = ([0,\infty]^{\mathrm{op}}, +, 0)$. The right adjoint of the tensor is given by *truncated subtraction*: for each $u, v \in [0,\infty]$,

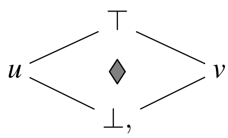
$$\hom(u,v) = v \ominus u = \begin{cases} v - u, \text{ if } v \geq u \\ 0, \text{ otherwise;} \end{cases}$$

(3) the extended real half line with the order $\geq$, $\otimes = \max$ the maximum, and $k = 0$. This one is denoted by $\mathsf{P}_{\max} = ([0,\infty]^{\mathrm{op}}, \max, 0)$; some authors denote it by $\mathsf{P}_{\vee}$. In this case, the operation hom is given by: for each $u, v \in [0,\infty]$,

$$\hom(u,v) = u \varovee v = \begin{cases} v, \text{ if } u < v, \\ 0, \text{ otherwise;} \end{cases}$$

(4) for the complete lattice $[0,1]$ with the order $\leq$, and the tensor being the ordinary multiplication $*$, then $k = 1$, and the quantale $([0,1], *, 1)$ is isomorphic to $\mathsf{P}_{+}$ via the map $[0,1] \to [0,\infty]$, $u \mapsto -\ln(u)$, with $-\ln(0) = \infty$. Moreover, for the tensor being the infimum $\wedge$, $([0,1], \wedge, 1)$ is isomorphic to $\mathsf{P}_{\max}$. Another operation that we can consider on $[0,1]$ is the *Łukasiewicz tensor* $\odot$ given by, for each $u, v \in [0,1]$, $u \odot v = \max(0, u + v - 1)$. For this tensor product, for each $u, v \in [0,1]$, $\hom(u,v) = u \circledast v = \min(1, 1 - u + v)$. Let us denote this quantale by $\mathsf{P}_{1} = ([0,1], \odot, 1)$;

(5) the *diamond lattice* $2^2 = \{\bot, u, v, \top\}$ [HST14, II-Exercise 1.H], with the order as in the diagram




where $u$ and $v$ are incomparable elements. The tensor product is the infimum $\wedge$, hence $k = \top$. Using the formula (I.3) we calculate: $\hom(u,\bot) = \hom(u,v) = v$, $\hom(v,\bot) = \hom(v,u) = u$, and

$$\forall\, \alpha \in 2^2,\ \hom(\bot,\alpha) = \hom(\alpha,\top) = \hom(\alpha,\alpha) = \top,\ \hom(\top,\alpha) = \alpha;$$

(6) the quantale of *distribution functions* [HR13, CH17]:

$$\Delta = \{f\colon [0,\infty] \to [0,1] \mid f \text{ is monotone and } f(x) = \bigvee_{y<x} f(y)\},$$

with pointwise order, tensor product given by $(f \otimes g)(x) = \bigvee_{y+z\leq x} f(y)\cdot g(z)$, and $k = f_{0,1}\colon [0,\infty] \to [0,1]$, where

$$f_{0,1}(x) = \begin{cases} 0, \text{ if } x = 0, \\ 1, \text{ otherwise.} \end{cases}$$

For each $f,g \in \Delta$, $\hom(f,g) = \bigvee\{h \in \Delta \mid \forall r,s,t \in [0,\infty],\ (r+s \leq t \implies f(r)\cdot h(s) \leq g(t))\}$.

**Definition 2.1.7** A *frame* $\mathsf{V}$ is a complete lattice satisfying the infinite distributive law: for each $a, b_i \in \mathsf{V}$, $i \in I$, $a \wedge (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \wedge b_i)$.

Setting $\otimes = \wedge$, every frame $\mathsf{V}$ is an integral quantale, which is a Heyting algebra; this is the case of our examples (1), (3), and (5). The quantales of examples (2), (4), and (6) are also integral and Heyting algebras. All examples are completely distributive, and the quantales in (1), (2), (3), and (4) are lean.

For an example of a quantale which is not integral, consider the powerset $\mathscr{P}M$ of a commutative monoid $M = (M, *, \alpha_M)$. The order is given by inclusion and the tensor product is defined by the multiplication: $A \otimes B = \{a * b \mid a \in A,\ b \in B\}$, for each $A, B \in \mathscr{P}M$. Hence the unit is $k = \{\alpha_M\}$ and the top element of $\mathscr{P}M$ is $\top = M$.

## 2.2 $(\mathbb{T},\mathsf{V})$-spaces and $(\mathbb{T},\mathsf{V})$-continuous maps

Let $\mathsf{V}$ be a quantale. A $\mathsf{V}$-*relation*, or $\mathsf{V}$-*matrix*, is a map $r\colon X \times Y \to \mathsf{V}$, hereinafter denoted by $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$. Any function $f\colon X \to Y$ can be seen as a $\mathsf{V}$-relation $f\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ with

$$f(x,y) = \begin{cases} k, \text{ if } f(x) = y, \\ \bot, \text{ otherwise.} \end{cases} \tag{I.5}$$

For $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and $s\colon Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$, the *relational composition* $s \cdot r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ is given by: for each $(x,z)$ in $X \times Z$,

$$s \cdot r(x,z) = \bigvee_{y\in Y} r(x,y) \otimes s(y,z). \tag{I.6}$$

Observe that for $\mathsf{V} = 2$, this composition has its usual meaning:

$$s\cdot r(x,z) = \top \iff \exists\, y \in Y;\; r(x,y) = s(y,z) = \top.$$

For each set $X$, the identity $\mathsf{V}$-relation is given by the identity map $1_X : X \to X$, and the order of $\mathsf{V}$ induces an order between $\mathsf{V}$-relations: for $r,t\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$,

$$r \le t \iff \forall (x,y) \in X\times Y,\; r(x,y) \le t(x,y). \tag{I.7}$$

These data define the 2-category $\mathsf{V}$-$\mathsf{Rel}$ whose objects are sets, morphisms are $\mathsf{V}$-relations, and 2-cells are given by the order (I.7). There exists an *involution* given by transposition: for each $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $r^\circ\colon Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$ is defined by: for each $(y,x) \in Y\times X$, $r^\circ(y,x) = r(x,y)$; moreover, for $s\colon Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ and $t\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, we have $(s\cdot r)^\circ = r^\circ\cdot s^\circ$ and $r\le t$ if, and only if, $r^\circ \le t^\circ$.

It is useful to observe that each map $f\colon X \to Y$ satisfies

$$1_X \le f^\circ\cdot f \qquad \& \qquad f\cdot f^\circ \le 1_Y, \tag{I.8}$$

and the first inequality is an equality provided that $f$ is an injective map, while the second inequality is an equality provided that $f$ is a surjective map.

Next let $\mathbb{T} = (T,m,e)\colon \mathsf{Set} \to \mathsf{Set}$ be a monad. Throughout we assume that $T\colon \mathsf{Set} \to \mathsf{Set}$ admits a lax extension to the category $\mathsf{V}$-$\mathsf{Rel}$, also denoted by $T\colon \mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{V}\text{-}\mathsf{Rel}$, so that

**(E1)** for each $r,s\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, and $t\colon Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$, $r\le s$ implies $Tr \le Ts$, and $T(t\cdot r) \le Tt\cdot Tr$;

**(E2)** $T$ commutes with involution: for each $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $T(r^\circ) = (Tr)^\circ = Tr^\circ$;

**(E3)** $m$ and $e$ become *oplax transformations*: for each $\mathsf{V}$-relation $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$,

$$\begin{array}{ccccc} X & \xrightarrow{e_X} & TX & \xleftarrow{m_X} & T^2X \\ {\scriptstyle r}\big\downarrow & \le & \big\downarrow{\scriptstyle Tr} & \ge & \big\downarrow{\scriptstyle T^2r} \\ Y & \xrightarrow[e_Y]{} & TY & \xleftarrow[m_Y]{} & T^2Y; \end{array}$$

**(E4)** $T$ is *flat*, that is, for each set $X$, $T1_X = 1_{TX}$.

Hence we have a *lax extension* of $T$ in the sense of [CT03], and a *lax monad* on $\mathsf{V}$-$\mathsf{Rel}$ in the sense of [CH04].

$(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is then defined as the category of Eilenberg-Moore lax algebras of that lax monad. The objects, which are called $(\mathbb{T},\mathsf{V})$-*categories*, or $(\mathbb{T},\mathsf{V})$-*spaces*, are pairs $(X,a)$, where $X$ is a set, and $a\colon TX {\longrightarrow\!\!\!\!\!\!\!+}\; X$ is a $\mathsf{V}$-relation satisfying lax diagrams of *reflexivity* and *transitivity*:

$$\begin{array}{ccccc} X & \xrightarrow{e_X} & TX & \xleftarrow{Ta} & T^2X \\ & \searrow^{1_X} \;\; \leq & \downarrow a & \leq & \downarrow m_X \\ & & X & \xleftarrow[a]{} & TX. \end{array} \tag{I.9}$$

Componentwise this translates as:

**(R)** for all $x\in X$, $k\leq a(e_X(x),x)$;

**(T)** for all $\mathfrak{X}\in T^2X$, $\mathfrak{x}\in TX$, $x\in X$, $Ta(\mathfrak{X},\mathfrak{x})\otimes a(\mathfrak{x},x)\leq a(m_X(\mathfrak{X}),x)$.

When referring to the $\mathsf{V}$-relation $a\colon TX {\longrightarrow\!\!\!\!\!\!\!+}\; X$ itself, one calls it a $(\mathbb{T},\mathsf{V})$-*structure* on $X$. A morphism from $(X,a)$ to $(Y,b)$ is a map $f\colon X\to Y$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ a\downarrow & \leq & \downarrow b \\ X & \xrightarrow[f]{} & Y; \end{array} \tag{I.10}$$

$f$ is called a $(\mathbb{T},\mathsf{V})$-*functor*, or a $(\mathbb{T},\mathsf{V})$-*continuous map*. Componentwise, $f\colon (X,a)\to(Y,b)$ is a $(\mathbb{T},\mathsf{V})$-continuous map if, and only if, for all $\mathfrak{x}\in TX$, $x\in X$, $a(\mathfrak{x},x)\leq b(Tf(\mathfrak{x}),f(x))$. When the diagram (I.10) is *strictly commutative*, that is, when $\leq$ is actually $=$, $f$ is said to be *fully faithful*. For a $(\mathbb{T},\mathsf{V})$-space $(X,a)$, each subset $S\subseteq X$ can be endowed with a *subspace* $(\mathbb{T},\mathsf{V})$-structure:

$$a_S = i_S^\circ\cdot a\cdot Ti_S\colon TS {\longrightarrow\!\!\!\!\!\!\!+}\; S, \tag{I.11}$$

where $i_S\colon S\hookrightarrow X$ is the inclusion map; $i_S\colon (S,a_S)\to(X,a)$ is fully faithful:

$$\begin{array}{ccc} TS & \xrightarrow{Ti_S} & TX \\ a_S\downarrow & & \downarrow a \\ S & \xrightarrow[i_S]{} & X. \end{array}$$

A pair $(X,a)$, $X$ a set and $a\colon TX {\longrightarrow\!\!\!\!\!\!\!+}\; X$ a $\mathsf{V}$-relation satisfying condition **(R)**, and not necessarily satisfying condition **(T)**, is called a $(\mathbb{T},\mathsf{V})$-*graph*. Denoting by $(\mathbb{T},\mathsf{V})$-$\mathsf{Gph}$ the category of $(\mathbb{T},\mathsf{V})$-graphs and $(\mathbb{T},\mathsf{V})$-continuous maps, we have the following [CT03, CHT03]:

**Proposition 2.2.1** $(\mathbb{T},\mathsf{V})$-$\mathsf{Gph}$ *is a quasitopos, and* $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ *is fully epireflective in* $(\mathbb{T},\mathsf{V})$-$\mathsf{Gph}$.

**Examples 2.2.2** Let us consider the following lax extensions:

(1) the identity monad $\mathbb{I} = (\mathrm{Id}, 1, 1)$ identically extended to $\mathsf{V}$-$\mathsf{Rel}$; then $(\mathbb{I}, \mathsf{V})$-$\mathsf{Cat}$ is the category of $\mathsf{V}$-spaces and $\mathsf{V}$-continuous maps, denoted by $\mathsf{V}$-$\mathsf{Cat}$. For the quantales of Examples 2.1.6 we have:

• $2$-$\mathsf{Cat} \cong \mathsf{Ord}$ is the category of (pre-)ordered spaces and monotone maps;

• $\mathsf{P}_+$-$\mathsf{Cat} \cong \mathsf{Met}$ is the category of Lawvere's generalised metric spaces and non-expansive maps [Law73];

• $\mathsf{P}_{\max}$-$\mathsf{Cat} \cong \mathsf{UltMet}$ is the full subcategory of $\mathsf{Met}$ of *ultrametric spaces* [HST14, III-Exercise 2.B];

• $\mathsf{P}_1$-$\mathsf{Cat} \cong \mathsf{BMet}$ is the full subcategory of $\mathsf{Met}$ of the *bounded-by-*1 *metric spaces* [HN18, Examples 2.3(3c)];

• $2^2$-$\mathsf{Cat} \cong \mathsf{BiRel}$ is the category of sets and birelations [HST14, III-Examples 1.1.1(3)];

• $\Delta$-$\mathsf{Cat} \cong \mathsf{ProbMet}$ is the category of probabilistic metric spaces and $\Delta$-functors [HR13];

(2) for the quantales (1) to (5) in Examples 2.1.6, the ultrafilter monad $\mathbb{U} = (U, m, e)$ with the *Barr extension* to $\mathsf{V}$-$\mathsf{Rel}$ given by, for each $r\colon X \rightarrow\!\!\!\!\!\!+\; Y$, $\mathfrak{x} \in UX$, $\mathfrak{y} \in UY$, $Ur(\mathfrak{x}, \mathfrak{y}) = \bigwedge_{\substack{A \in \mathfrak{x} \\ B \in \mathfrak{y}}} \bigvee_{\substack{x \in A \\ y \in B}} r(x, y)$. In particular, we have:

• $(\mathbb{U}, 2)$-$\mathsf{Cat} \cong \mathsf{Top}$ is the category of topological spaces and continuous functions [Bar70];

• $(\mathbb{U}, \mathsf{P}_+)$-$\mathsf{Cat} \cong \mathsf{App}$ is the category of Lowen's approach spaces and contractions [Low97, CT03, CH03];

• $\mathsf{P}_{\max}$-$\mathsf{Cat} \cong \mathsf{NA}$-$\mathsf{App}$ is the full subcategory of $\mathsf{App}$ of non-Archimedean approach spaces [CVO17, Hof14];

• $(\mathbb{U}, 2^2)$-$\mathsf{Cat} \cong \mathsf{BiTop}$ is the category of bitopological spaces and bicontinuous maps [HST14, III-Exercise 2.D];

• $(\mathbb{U}, \Delta)$-$\mathsf{Cat} \cong \mathsf{ProbApp}$ is the category of probabilistic approach spaces and contractive maps [LT17, Jäg15];

(3) for $\mathbb{M} = (- \times M, m, e)$ and each quantale $\mathsf{V}$, the extension of $(- \times M)$ to $\mathsf{V}$-$\mathsf{Rel}$ given by, for each $r\colon X \rightarrow\!\!\!\!\!\!+\; Y$, $(x, a) \in X \times M$, $(y, b) \in Y \times M$,

$$(r \times M)((x,a),(y,b)) = \begin{cases} r(x,y), \text{ if } a = b, \\ \bot, \text{ otherwise} \end{cases}$$

[CHR20]. For the quantale 2, the category $(\mathbb{M}, 2)$-$\mathsf{Cat}$ is thought as *M-labelled ordered sets* and *equivariant maps* [HST14, V-Section 1.4];

(4) for the list monad $\mathbb{L} = (L, m, e)$ and each quantale $\mathsf{V}$, $L\colon \mathsf{V}$-$\mathsf{Rel} \to \mathsf{V}$-$\mathsf{Rel}$ is given by, for each

$r\colon X {\longrightarrow\!\!\!\!\!\!\mapsto} Y$, $(x_1,\dots,x_n)\in LX$, $(y_1,\dots,y_m)\in LY$,

$$Lr((x_1,\dots,x_n),(y_1,\dots,y_m)) = \begin{cases} r(x_1,y_1)\otimes\cdots\otimes r(x_n,y_n), \text{ if } m=n,\\ \bot, \text{ otherwise.}\end{cases}$$

We observe that $\overline{L}\colon \mathsf{V}\text{-}\mathsf{Rel}\to\mathsf{V}\text{-}\mathsf{Rel}$ given by, for each $r\colon X {\longrightarrow\!\!\!\!\!\!\mapsto} Y$, $(x_1,\dots,x_n)\in LX$, $(y_1,\dots,y_m)\in LY$,

$$\overline{L}r((x_1,\dots,x_n),(y_1,\dots,y_m)) = \begin{cases} r(x_1,y_1)\wedge\cdots\wedge r(x_n,y_n), \text{ if } m=n,\\ \bot, \text{ otherwise,}\end{cases}$$

also gives a lax extension of $L$ [Bas17, CH04]. For the particular case of $\mathsf{V}=2$, $(\mathbb{L},2)\text{-}\mathsf{Cat}\cong\mathsf{MultiOrd}$ is the category of multi-ordered sets and their morphisms [HST14, V-Section 1.4], while $(\mathbb{L},\mathsf{V})\text{-}\mathsf{Cat}$ can be thought, more generally, as *multi*-$\mathsf{V}$-*ordered* spaces and their morphisms.

We gather those examples in the following table.

| V \ $\mathbb{T}$ | $\mathbb{I}$ | $\mathbb{U}$ | $\mathbb{M}$ | $\mathbb{L}$ |
|---|---|---|---|---|
| 2 | Ord | Top | $(\mathbb{M},2)$-Cat | MultiOrd |
| $\mathsf{P}_+$ | Met | App | $(\mathbb{M},\mathsf{P}_+)$-Cat | $(\mathbb{L},\mathsf{P}_+)$-Cat |
| $\mathsf{P}_{\max}$ | UltMet | NA-App | $(\mathbb{M},\mathsf{P}_{\max})$-Cat | $(\mathbb{L},\mathsf{P}_{\max})$-Cat |
| $\mathsf{P}_1$ | BMet | $(\mathbb{U},\mathsf{P}_1)$-Cat | $(\mathbb{M},\mathsf{P}_1)$-Cat | $(\mathbb{L},\mathsf{P}_1)$-Cat |
| $2^2$ | BiRel | BiTop | $(\mathbb{M},2^2)$-Cat | $(\mathbb{L},2^2)$-Cat |
| $\Delta$ | ProbMet | ProbApp | $(\mathbb{M},\Delta)$-Cat | $(\mathbb{L},\Delta)$-Cat |

(I.12)

In order to highlight its topological character, we choose to use the terms $(\mathbb{T},\mathsf{V})$-*spaces* and $(\mathbb{T},\mathsf{V})$-*continuous maps* to refer to the objects and to the morphisms of $(\mathbb{T},\mathsf{V})$-Cat, respectively. Furthermore, in order to keep the text simpler, when there is no ambiguity, we will drop the prefix $(\mathbb{T},\mathsf{V})$ and refer to them simply as *spaces* and *continuous maps*.

## 2.3 A fundamental adjunction

The following adjunction is to be used in Subsection 2.7 and in Chapters III and IV; for details we refer to [CT03, CCH15] and [HST14, III-3.4, 3.5, 3.6].

For each $(\mathbb{T},\mathsf{V})$-space $(X,a)$, define $A_e(X,a)=(X,a\cdot e_X)$, with $e_X: X\to TX$ the $X$-component of the natural transformation $e\colon \mathrm{Id}_{\mathsf{Set}}\to T$. For each $\mathsf{V}$-space $(X,a_0)$, define $A^\circ(X,a_0)=(X,a_0^\#)$, with

$a_0^\# = e_X^\circ \cdot Ta_0$. On morphisms, both $A_e$ and $A^\circ$ are the identity. These are well-defined functors, and we have an adjunction:

$$\mathsf{V}\text{-}\mathsf{Cat} \underset{A_e}{\overset{A^\circ}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}. \tag{I.13}$$

Furthermore, $A_e$ is an instance of *algebraic functors* [HST14, III-3.4], which is induced by the natural transformation $e\colon I \to T$. Concerning the quantale part of $(\mathbb{T},\mathsf{V})$-Cat, we have the *change-of-base functors* [HST14, III-3.5]. In summary, consider a *lax homomorphism* of quantales $\mu\colon \mathsf{V} \to \mathsf{W}$, which is an order preserving map such that $\mu(u) \otimes_{\mathsf{W}} \mu(v) \leq \mu(u \otimes_{\mathsf{V}} v)$ and $k_{\mathsf{W}} \leq \mu(k_{\mathsf{V}})$, for each $u, v \in \mathsf{V}$. Then $\mu$ induces a lax functor $\mu\colon \mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{W}\text{-}\mathsf{Rel}$ assigning to each $r\colon X \times Y \to \mathsf{V}$ the composite $\mu \cdot r\colon X \times Y \to \mathsf{W}$. Now, assuming that the monad $\mathbb{T}$ has lax extensions $\mathbb{T}_{\mathsf{V}}$ and $\mathbb{T}_{\mathsf{W}}$ to V-Rel and to W-Rel, respectively, and that $\mu$ is *compatible* with such extensions, that is, the diagram

$$\begin{array}{ccc} \mathsf{V}\text{-}\mathsf{Rel} & \xrightarrow{T_{\mathsf{V}}} & \mathsf{V}\text{-}\mathsf{Rel} \\ {\scriptstyle\mu}\downarrow & \leq & \downarrow{\scriptstyle\mu} \\ \mathsf{W}\text{-}\mathsf{Rel} & \xrightarrow[T_{\mathsf{W}}]{} & \mathsf{W}\text{-}\mathsf{Rel} \end{array}$$

is lax commutative, then $\mu$ induces the *change-of-base functor* $B_\mu\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to (\mathbb{T},\mathsf{W})\text{-}\mathsf{Cat}$. Moreover, we have the following result.

**Proposition 2.3.1** [HST14, III-Proposition 3.5.1] *If* $\mu\colon \mathsf{V} \to \mathsf{W}$ *and* $\rho\colon \mathsf{W} \to \mathsf{V}$ *are lax homomorphisms of quantales which are compatible with the lax extensions of* $\mathbb{T}$ *to* V-Rel *and* W-Rel*, then*

$$\mu \dashv \rho \implies B_\mu \dashv B_\rho.$$

For the particular cases of the monads $\mathbb{I}$ and $\mathbb{U}$, and the quantales 2 and $\mathsf{P}_+$, adjunction (I.13) and adjunctions between change-of-base functors are depicted in the vertical and horizontal arrows, respectively, of the following diagram:

$$\begin{array}{ccc} \mathsf{Top} & \underset{\hookrightarrow}{\overset{\leftarrow}{\bot}} & \mathsf{App} \\ \uparrow\dashv\downarrow & & \uparrow\dashv\downarrow \\ \mathsf{Ord} & \underset{\hookrightarrow}{\overset{\leftarrow}{\bot}} & \mathsf{Met}, \end{array} \tag{I.14}$$

where the hook-arrows are full embeddings, the solid and dotted diagrams are commutative, and the two full embeddings from Ord to App coincide. Next we describe these adjunctions.

• Ord *to* Met. To each ordered set $(X,\leq)$ one assigns the metric space $(X,d_{\leq})$, where, for each $x,x'\in X$,

$$d_{\leq}(x,x')=\begin{cases}0, & \text{if } x\leq x'\\ \infty, & \text{otherwise.}\end{cases}$$

The left adjoint assigns to each $(X,d)$ the ordered set $(X,\leq_d)$, where, for each $x,x'\in X$, $x\leq_d x'$ if, and only if, $d(x,x')<\infty$. The change-of-base functors are induced by the quantale homomorphisms $\iota\colon 2\to \mathsf{P}_+$, $\top\mapsto 0$, $\bot\mapsto\infty$, and $o\colon \mathsf{P}_+\to 2$ given by $o(v)=\top$ if, and only if, $v<\infty$, for each $v\in[0,\infty]$.

• Ord *to* Top. To each ordered set $(X,\leq)$ one assigns the topological space $(X,\tau_{\leq})$, with $\tau_{\leq}$ generated by the basis $\{\downarrow x\mid x\in X\}$, that is, $\tau_{\leq}$ is the *Alexandroff topology*. Its left adjoint assigns to each topological space $(X,\tau)$ the space $(X,\leq_{\tau})$, where $\leq_{\tau}$ is the dual of the *specialization order* [GHK+80, II-Definition 3.6], that is, $x\leq y$ if, and only if, $\dot{x}\to y$, where $\to$ denotes the ultrafilter convergence determined by $\tau$.

**Remark 2.3.2** Let us recall that a topological space is called *Alexandroff* if arbitrary intersections of open sets are open. It is well known that Alexandroff topological spaces are precisely the spaces in the image of Ord by $A_0$ [HST14, II-5.10.5, III-3.4.3(1)]. This paradigm is going to be explored in Subsection 8.4.

• Met *to* App. A metric space $(X,d)$ induces the approach space $(X,\delta_d)$, where, for each $x'\in X$, $A\in\mathscr{P}X$, $\delta_d(x',A)=\inf\{d(x,x')\mid x\in A\}$. The right adjoint of this embedding assigns $(X,d_\delta)$ to each $(X,\delta)$ in App, where, for each $x,x'\in X$, $d_\delta(x,x')=\sup\{\delta(x',A)\mid x\in A\in\mathscr{P}X\}$.

• Top *to* App. A topological space $(X,\tau)$ induces the approach space $(X,\delta_\tau)$, where, for each $x'\in X$, $A\in\mathscr{P}X$,

$$\delta_\tau(x',A)=\begin{cases}0, & \text{if } A\in\mathfrak{x}\text{, for some } \mathfrak{x}\in UX \text{ with } \mathfrak{x}\to x'\\ \infty, & \text{otherwise.}\end{cases}$$

To describe the left adjoint consider an approach space $(X,\delta)$, then form the pseudo-topological space $(X,\to)$ [Cho48], where the convergence relation $\to$ between ultrafilters in $UX$ and points of $X$ is given by

$$\mathfrak{x} \to x \iff \sup\{\delta(x,A) \mid A \in \mathfrak{x}\} < \infty;$$

to this pseudo-topological space apply the reflector to Top [HLCS91] (see also [HST14, III-Exercise 3.D]) obtaining $(X, \tau_\delta)$, where $A \subseteq X$ is open if, and only if, for each $\mathfrak{x} \in UX$, $x \in X$, whenever $\mathfrak{x} \to x$ and $x \in A$, then $A \in \mathfrak{x}$. The left adjoint assigns to $(X, \delta)$ the topological space $(X, \tau_\delta)$. In this case, the quantale homomorphism $\iota\colon 2 \to \mathsf{P}_+$ is compatible with the lax extesions of $\mathbb{U}$, but the same is not true for $o\colon \mathsf{P}_+ \to 2$. Nonetheless, $B_\iota\colon \mathsf{Top} \to \mathsf{App}$ has the left adjoint just described.

## 2.4 Some topological aspects of $(\mathbb{T},\mathsf{V})$-Cat

The forgetful functor $|\text{-}|\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{Set}$ is topological [CH03, CT03] and fibre-small: for each set $X$, a $(\mathbb{T},\mathsf{V})$-structure $a$ on $X$ is an element of $\mathsf{V}\text{-}\mathsf{Rel}(TX,X) = \mathsf{Set}(TX \times X, \mathsf{V})$. In particular, this implies the following (see [AHS90, Proposition 21.12, Proposition 21.13, Proposition 21.14, Corollary 21.17] and [HST14, III- Section 3.1]):

**(TA1)** $(\mathbb{T},\mathsf{V})$-Cat *is complete, cocomplete, well-powered, and co-well-powered;*

**(TA2)** $(\mathbb{T},\mathsf{V})$-Cat *has* (Epi, RegMono)*-factorizations, which form a stable factorization system*: a $(\mathbb{T},\mathsf{V})$-continuous map is an epimorphism if, and only if, it is a surjective map; in Set surjective maps are stable under pullback, and the forgetful functor from $(\mathbb{T},\mathsf{V})$-Cat to Set preserves pullbacks. We observe that (RegEpi, Mono) is also a factorization system for $(\mathbb{T},\mathsf{V})$-Cat, which is not stable; for instance, in Top regular epimorphisms are not stable under pullback.

**(TA3)** $(\mathbb{T},\mathsf{V})$-Cat *is closed under regular monomorphisms, or embeddings, in* $(\mathbb{T},\mathsf{V})$-Gph, that is, whenever $f\colon (X,a) \to (Y,b)$ is a regular monomorphism of $(\mathbb{T},\mathsf{V})$-graphs, and $(Y,b)$ is a $(\mathbb{T},\mathsf{V})$-space, $(X,a)$ is also a $(\mathbb{T},\mathsf{V})$-space. This follows from **(TA2)**, Proposition 2.2.1, and [HST14, II-Proposition 5.1.3].

**(TA4)** *The forgetful functor* $|\text{-}|\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{Set}$ *has a fully faithful left adjoint and a fully faithful right adjoint, which are embeddings.* The left adjoint assigns to each set $X$ the *discrete* $(\mathbb{T},\mathsf{V})$-space $(X, e_X^\circ)$, while the right adjoint, which for future purposes we denote by $\nabla$, assigns to each set $X$ the *indiscrete* $(\mathbb{T},\mathsf{V})$-space $\nabla X = (X, \top)$; both functors leave morphisms unchanged [HST14, III-Section 3.2].

**(TA5)** *For each epimorphism* $f\colon (X,a) \to (Y,b)$*,* $f \times 1_Z\colon (X,a) \times (Z,c) \to (Y,b) \times (Z,c)$ *is an epimorphism in* $(\mathbb{T},\mathsf{V})$-Cat. This follows from **(TA2)** and the fact that $f \times 1_Z$ is the pullback of $f$ along the product projection $\pi_Y\colon (Y,b) \times (Z,c) \to (Y,b)$.

**(TA6)** *If $T$ is taut, then $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is infinitely extensive.* This is proved in [MST06]; we include the proof here for completeness. In order to do so, we need an auxiliary definition and lemma [CH12].

**Definition 2.4.1** A $(\mathbb{T},\mathsf{V})$-continuous map $f\colon (X,a)\to(Y,b)$ is *open* if $f^\circ\cdot b\le a\cdot(Tf)^\circ$.

The map $f\colon (X,a)\to(Y,b)$ is $(\mathbb{T},\mathsf{V})$-continuous if, and only if, $a\cdot(Tf)^\circ\le f^\circ\cdot b$, whence $f$ is open if, and only if, $a\cdot(Tf)^\circ=f^\circ\cdot b$.

**Lemma 2.4.2** *For a family $((X_i,a_i))_{i\in I}$ of $(\mathbb{T},\mathsf{V})$-spaces the following assertions are equivalent:*
*(i) $(X,a)$ is the coproduct of $((X_i,a_i))_{i\in I}$ in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$;*
*(ii) $X$ is the coproduct of $(X_i)_{i\in I}$ in $\mathsf{Set}$ and, for each $i\in I$, the coproduct inclusion $\iota_i\colon (X_i,a_i)\hookrightarrow(X,a)$ is open.*

*Proof of* **(TA6)**. By [CLW93, Proposition 2.14], it suffices to show that coproducts in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ are disjoint and universal, that is, stable under pullback. Since the forgetful functor $|\text{-}|\colon(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to\mathsf{Set}$ preserves colimits, the first condition is satisfied. In order to check universality of coproducts, let $(X,a)$ be the coproduct of the family $((X_i,a_i))_{i\in I}$ of $(\mathbb{T},\mathsf{V})$-spaces, and let $f\colon(Y,b)\to(X,a)$ be a $(\mathbb{T},\mathsf{V})$-continuous map; by Lemma 2.4.2, we must prove that, for each $i\in I$, the pullback of $\iota_i$ along $f$:

$$\begin{array}{ccc} P_i & \xrightarrow{f_i} & Y \\ {\scriptstyle h_i}\downarrow & & \downarrow{\scriptstyle f} \\ X_i & \xrightarrow[\iota_i]{} & X \end{array} \tag{I.15}$$

is an open map $f_i\colon(P_i,p_i)\to(Y,b)$, that is, $p_i\cdot(Tf_i)^\circ=f_i^\circ\cdot b$. Firstly, since the diagram (I.15) is a pullback, it is a (BC)-square, hence, because $T$ satisfies (BC), the diagram

$$\begin{array}{ccc} TP_i & \xrightarrow{Tf_i} & TY \\ {\scriptstyle Th_i}\downarrow & & \downarrow{\scriptstyle Tf} \\ TX_i & \xrightarrow[T\iota_i]{} & TX \end{array}$$

is a (BC)-square, what is equivalent to $Th_i\cdot(Tf_i)^\circ=(T\iota_i)^\circ\cdot Tf$ [HST14, III-Lemma 1.11.1]. Secondly, we recall that $p_i$ is the $|\text{-}|$-initial $(\mathbb{T},\mathsf{V})$-structure with respect to $h_i$ and $f_i$, and calculate:

$$\begin{aligned}
p_i\cdot(Tf_i)^\circ &= \big((h_i^\circ\cdot a_i\cdot Th_i)\wedge(f_i^\circ\cdot b\cdot Tf_i)\big)\cdot(Tf_i)^\circ \qquad \text{(by [CH03, Theorem 4.5])}\\
&\overset{*}{=} (f_i^\circ\cdot b)\wedge(h_i^\circ\cdot a_i\cdot Th_i\cdot(Tf_i)^\circ) \;=\; (f_i^\circ\cdot b)\wedge(h_i^\circ\cdot a_i\cdot(T\iota_i)^\circ\cdot Tf)\\
&= (f_i^\circ\cdot b)\wedge(h_i^\circ\cdot\iota_i^\circ\cdot a\cdot Tf) \qquad \text{(because } \iota_i \text{ is } |\text{-}|\text{-initial)}\\
&= (f_i^\circ\cdot b)\wedge(f_i^\circ\cdot f^\circ\cdot a\cdot Tf) \overset{**}{=} f_i^\circ\cdot b,
\end{aligned}$$

where in $\stackrel{*}{=}$ we use a weak version of *Freyd's Modular Law* proven to be satisfied in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ in [MST06, Remark 6], and $\stackrel{**}{=}$ follows from $(\mathbb{T},\mathsf{V})$-continuity of $f$, since $b \leq f^\circ \cdot a \cdot Tf$ implies $f_i^\circ \cdot b \leq f_i^\circ \cdot f^\circ \cdot a \cdot Tf$. □

Furthermore, as a requirement for Chapter IV, we need to provide conditions under which every constant map between $(\mathbb{T},\mathsf{V})$-spaces is continuous. These conditions are fairly restrictive, as shown by the following characterization.

**Lemma 2.4.3** *In* $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ *the following conditions are equivalent.*
*(i) Every constant map* $f\colon (X,a) \to (Y,b)$ *between* $(\mathbb{T},\mathsf{V})$*-spaces is continuous.*
*(ii) For* $1 = \{*\}$ *a singleton, if* $(1,c)$ *is a* $(\mathbb{T},\mathsf{V})$*-space, then, for each* $\mathfrak{z} \in T1$, $c(\mathfrak{z},*) = \top$.
*(iii)* $k = \top$ *and* $T1 = 1$.

*Proof.* (i)⇔(ii) Let $(1,c)$ be a $(\mathbb{T},\mathsf{V})$-space. The identity map $1_1\colon (1,\top) \to (1,c)$ is constant, so it is continuous by hypothesis, whence $\top \leq c$. Conversely, each constant map $f\colon (X,a) \to (Y,b)$, $x \mapsto y_0$, admits the factorization $(X,a) \xrightarrow{f} (1,b_1) \xrightarrow{i_1} (Y,b)$ , where $1 = \{y_0\} \subseteq Y$ is endowed with the subspace $(\mathbb{T},\mathsf{V})$-structure $b_1$. By hypothesis, $b_1 = \top$, hence $f\colon (X,a) \to (1,b_1)$ is continuous, and so is the composite $f\colon (X,a) \to (Y,b)$.
(ii)⇔(iii) Consider the discrete $(\mathbb{T},\mathsf{V})$-space $(1,e_1^\circ)$. By hypothesis, for each $\mathfrak{z} \in T1$, $e_1^\circ(\mathfrak{z},*) = \top$. Then, for $\mathfrak{z} = e_1(*)$, $k = e_1^\circ(e_1(*),*) = \top$. Moreover, $e_1^\circ(\mathfrak{z},*) = \top = k$ if, and only if, $\mathfrak{z} = e_1(*)$, whence $T1 = \{e_1(*)\}$ is a singleton. For $\mathsf{V}$ integral and $T1 = 1$, one readily checks condition (ii). □

Therefore, under these conditions – $\mathsf{V}$ integral and $T1 = 1$ – $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is a topological category in the sense of [Her74]: there exist initial structures with respect to the forgetful functor, which is fibre-small, and there exists precisely one structure on a singleton set.

## 2.5 Strict topological theories or algebraic lax extensions

In the forthcoming Chapter III, when studying *weak exponentiability* in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, we will use the *Yoneda embedding* for $(\mathbb{T},\mathsf{V})$-spaces, whose fundamental ingredient is provided by the setting of *strict topological theories*, as defined in [Hof07]. The main idea is that the extension $T$ to $\mathsf{V}$-$\mathsf{Rel}$ is determined by a $\mathbb{T}$-algebra structure map $\xi\colon T\mathsf{V} \to \mathsf{V}$, and in [CT14] such extensions are characterized as the *algebraic lax extensions*.

More precisely, we will assume that for a monad $\mathbb{T}$ and a quantale $\mathsf{V}$ satisfying the conditions of Subsection 2.1 (usually $\mathsf{V}$ does not need to be a Heyting Algebra, although we assume this here), there exists a map $\xi: T\mathsf{V} \to \mathsf{V}$ such that the following diagrams are commutative:

$$\begin{array}{ccccc} \mathsf{V} & \xrightarrow{e_{\mathsf{V}}} & T\mathsf{V} & \xleftarrow{T\xi} & T^2\mathsf{V} \\ & \searrow^{1_{\mathsf{V}}} & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle m_{\mathsf{V}}} \\ & & \mathsf{V} & \xleftarrow[\xi]{} & T\mathsf{V} \end{array} \qquad \begin{array}{ccccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\otimes)} & T\mathsf{V} & \xleftarrow{T(k)} & T1 \\ {\scriptstyle \langle \xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle !} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\otimes]{} & \mathsf{V} & \xleftarrow[k]{} & 1, \end{array} \tag{I.16}$$

where $\pi_1$ and $\pi_2$ are the product projections $\mathsf{V}\times\mathsf{V} \to \mathsf{V}$, and $\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle$ is the unique map such that the diagram below is commutative.

$$\begin{array}{ccccc} & & T(\mathsf{V}\times\mathsf{V}) & & \\ & \swarrow^{\xi\cdot T\pi_1} & \downarrow{\scriptstyle \langle \xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle} & \searrow^{\xi\cdot T\pi_2} & \\ \mathsf{V} & \xleftarrow[\pi_1]{} & \mathsf{V}\times\mathsf{V} & \xrightarrow[\pi_2]{} & \mathsf{V} \end{array}$$

The lax extension of $T: \mathsf{Set} \to \mathsf{Set}$ to $\mathsf{V}\text{-}\mathsf{Rel}$ is given by, for each $r: X \nrightarrow Y$, $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$,

$$Tr(\mathfrak{x},\mathfrak{y}) = \bigvee \{\xi \cdot T\vec{r}(\mathfrak{w}) \mid \mathfrak{w} \in T(X\times Y),\ T\pi_X(\mathfrak{w}) = \mathfrak{x},\ T\pi_Y(\mathfrak{w}) = \mathfrak{y}\}, \tag{I.17}$$

where $\pi_X$ and $\pi_Y$ are the product projections from $X \times Y$ into $X$ and $Y$, respectively [Hof07, Definition 3.4]. We adopt the notation $\vec{r}$ used in [CT14] to distinguish between the map from $X\times Y$ to $\mathsf{V}$ and the $\mathsf{V}$-relation $X \nrightarrow Y$, so that $T\vec{r}: T(X\times Y) \to T\mathsf{V}$. This extension satisfies conditions **(E1)** to **(E4)** of Subsection 2.2 [Hof07, Theorem 3.5].

In this context, $\mathsf{V}$ has a $(\mathbb{T},\mathsf{V})$-structure given by the composite

$$T\mathsf{V} \xrightarrow{\xi} \mathsf{V} \xrightarrow{\mathrm{hom}} \mathsf{V}, \qquad \mathrm{hom}_\xi : T\mathsf{V} \nrightarrow \mathsf{V} \tag{I.18}$$

[Hof07, Corollary 5.2(b)]. For the particular case of $\mathbb{T} = \mathbb{U}$ and $\mathsf{V} = 2$, the space $(\mathsf{V}, \mathrm{hom}_\xi)$ is the Sierpiński space $\mathbb{S} = (2, \{\emptyset, 2, \{\bot\}\})$ in $\mathsf{Top}$, and due to this we call $(\mathsf{V}, \mathrm{hom}_\xi)$ the *Sierpiński* $(\mathbb{T},\mathsf{V})$*-space*. This space will be employed in Subsections 2.7 and 8.4.

Next we identify the maps which generate extensions from items (1) to (4) of Examples 2.2.2; more details can be found in [Hof07, Theorem 3.3], [Hof14, Examples 1.4], [CHR20, Examples 7.7], and in the comment after [CHR20, Theorem 7.10].

**Examples 2.5.1** (1) For the identity monad $\mathbb{I} = (\mathrm{Id}, 1, 1)$ and any quantale $\mathsf{V}$, $\xi$ is the identity map $1_{\mathsf{V}} : \mathsf{V} \to \mathsf{V}$. The Sierpiński $\mathsf{V}$-space is $(\mathsf{V}, \hom)$, described for our quantales in Examples 2.1.6.

(2) Considering the ultrafilter monad $\mathbb{U} = (U, m, e)$ and a completely distributive complete lattice $\mathsf{V}$ with $\otimes = \wedge$, we have $\xi : U\mathsf{V} \to \mathsf{V}$, $\mathfrak{v} \mapsto \bigwedge_{A \in \mathfrak{v}} \bigvee A$. In particular, we have:

• for $\mathsf{V}$ finite, every ultrafilter is principal, hence the map $e_{\mathsf{V}} : \mathsf{V} \to U\mathsf{V}$ is surjective, and the equality $\xi \cdot e_{\mathsf{V}} = 1_{\mathsf{V}}$ of (I.16) is equivalent to $\xi = e_{\mathsf{V}}^{\circ}$ by (I.8); this is the case for $\mathsf{V} = 2$ and $\mathsf{V} = 2^2$;

• for $\mathsf{V} = \mathsf{P}_{\max}$, we have $\xi : U[0, \infty] \to [0, \infty]$, $\mathfrak{v} \mapsto \inf\{u \in [0, \infty] \mid [0, u] \in \mathfrak{v}\}$.

For $\mathsf{V} = \mathsf{P}_{+}$ the same $\xi$ defined above fulfills the conditions needed. Analogously, for $\mathsf{V} = \mathsf{P}_{1}$, we have $\xi : U[0, 1] \to [0, 1]$, $\mathfrak{v} \mapsto \sup\{u \in [0, 1] \mid [u, 1] \in \mathfrak{v}\}$. For the quantale $\Delta$, the map $\xi$ defined for a completely distributive complete lattice provides a topological theory [Hof07, Theorem 3.3], however, it is an open question whether this topological theory is strict.

(3) For the monad $\mathbb{M} = (- \times M, m, e)$ and each quantale $\mathsf{V}$, define $\xi = \pi_{\mathsf{V}} : \mathsf{V} \times M \to \mathsf{V}$ as the first product projection, so that, for each $(v, a) \in \mathsf{V} \times M$, $\xi(v, a) = v$.

(4) For the list monad $\mathbb{L} = (L, m, e)$ and any quantale $\mathsf{V}$, $\xi : L\mathsf{V} \to \mathsf{V}$ is such that $\xi(()) = k$, and, for each $n \in \mathbb{N}$, $(v_1, \dots, v_n) \in L\mathsf{V}$, $\xi(v_1, \dots, v_n) = v_1 \otimes \dots \otimes v_n$.

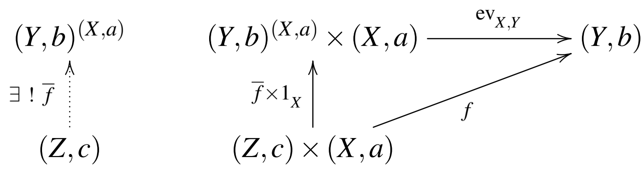
Our Table (I.12) of examples is replaced by the following one.

| $\mathsf{V}$ \ $\mathbb{T}$ | $\mathbb{I}$ | $\mathbb{U}$ | $\mathbb{M}$ | $\mathbb{L}$ |
|---|---|---|---|---|
| $2$ | Ord | Top | $(\mathbb{M}, 2)$-Cat | MultiOrd |
| $\mathsf{P}_{+}$ | Met | App | $(\mathbb{M}, \mathsf{P}_{+})$-Cat | $(\mathbb{L}, \mathsf{P}_{+})$-Cat |
| $\mathsf{P}_{\max}$ | UltMet | NA-App | $(\mathbb{M}, \mathsf{P}_{\max})$-Cat | $(\mathbb{L}, \mathsf{P}_{\max})$-Cat |
| $\mathsf{P}_{1}$ | BMet | $(\mathbb{U}, \mathsf{P}_{1})$-Cat | $(\mathbb{M}, \mathsf{P}_{1})$-Cat | $(\mathbb{L}, \mathsf{P}_{1})$-Cat |
| $2^2$ | BiRel | BiTop | $(\mathbb{M}, 2^2)$-Cat | $(\mathbb{L}, 2^2)$-Cat |
| $\Delta$ | ProbMet | | $(\mathbb{M}, \Delta)$-Cat | $(\mathbb{L}, \Delta)$-Cat |

(I.19)

## 2.6 A sufficient condition for exponentiability in $(\mathbb{T}, \mathsf{V})$-Cat

A space $(X, a)$ is *exponentiable* if the functor $(X, a) \times - : (\mathbb{T}, \mathsf{V})\text{-Cat} \to (\mathbb{T}, \mathsf{V})\text{-Cat}$ has a right adjoint $(-)^{(X,a)} : (\mathbb{T}, \mathsf{V})\text{-Cat} \to (\mathbb{T}, \mathsf{V})\text{-Cat}$. For each space $(Y, b)$, the image $(Y, b)^{(X,a)}$ is called an *exponential*. In terms of the counit ev, standing for *evaluation*, of the adjunction $((X, a) \times -) \dashv (-)^{(X,a)}$, for each space $(Y, b)$, every continuous map $f : (Z, c) \times (X, a) \to (Y, b)$, with $(Z, c) \in (\mathbb{T}, \mathsf{V})$-Cat, factors uniquely though the universal map $\mathrm{ev}_{X,Y} : (Y, b)^{(X,a)} \times (X, a) \to (Y, b)$ as $f = \mathrm{ev}_{X,Y} \cdot (\overline{f} \times 1_X)$, where $\overline{f}\colon (Z,c)\to (Y,b)^{(X,a)}$ is called the *transpose* of $f$.




By Proposition 2.2.1, each $(\mathbb{T},\mathsf{V})$-space is exponentiable as a $(\mathbb{T},\mathsf{V})$-graph, since every quasitopos is *cartesian closed*, that is, all objects are exponentiable. In fact, a quasitopos is always *locally cartesian closed* [Pen77, Definition 2.5], i.e., the *slice categories* are cartesian closed, implying that the category itself is cartesian closed.

In $(\mathbb{T},\mathsf{V})$-$\mathsf{Gph}$, the exponential $(Y,b)^{(X,a)}$ of $(\mathbb{T},\mathsf{V})$-spaces $(Y,b)$ and $(X,a)$ is given by the set

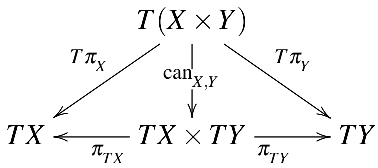
$$Y^X=\{h\colon (X,a)\times(1,e_1^\circ)\to(Y,b)\mid h \text{ is a } (\mathbb{T},\mathsf{V})\text{-continuous map}\}$$

endowed with the largest $\mathsf{V}$-relation $b^a\colon T(Y^X)\,{-\!\!\!\rightarrow}\, Y^X$ making the evaluation map $\mathrm{ev}_{X,Y}\colon Y^X\times X\to Y$, $(h,x)\mapsto h(x)$, $(\mathbb{T},\mathsf{V})$-continuous, where $h(x)$ stands for $h(x,*)$, with $1=\{*\}$. Then $b^a$ is reflexive, and it is given by

$$b^a(\mathfrak{p},h)=\bigvee\{v\in\mathsf{V}\mid \forall\mathfrak{q}\in(T\pi_{Y^X})^{-1}(\mathfrak{p}),\ \forall x\in X,\ a(T\pi_X(\mathfrak{q}),x)\wedge v\le b(T\mathrm{ev}_{X,Y}(\mathfrak{q}),h(x))\},\qquad \text{(I.20)}$$

for each $\mathfrak{p}\in T(Y^X)$, $h\in Y^X$, where $\pi_{Y^X}$ and $\pi_X$ are the product projections from $Y^X\times X$ into $Y^X$ and $X$, respectively [CHT03]. Since our quantales are Heyting algebras, the supremum above is a maximum. If, for each $(\mathbb{T},\mathsf{V})$-space $(Y,b)$, the $(\mathbb{T},\mathsf{V})$-graph $(Y^X,b^a)$ is transitive, and consequently a $(\mathbb{T},\mathsf{V})$-space, then $(X,a)$ is exponentiable in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$. In order to state the sufficient condition for such fact, we fix some notation and present an auxiliary result. This sufficient condition generalises [Hof06, Theorem 4.3] and [Hof07, Theorem 6.5].

For sets $X$ and $Y$, let $\mathrm{can}_{X,Y}=\langle T\pi_X,T\pi_Y\rangle\colon T(X\times Y)\to TX\times TY$ be the unique map such that the following diagram is commutative, where $\pi_X,\pi_Y,\pi_{TX}$, and $\pi_{TY}$ are product projections.




(I.21)

For $\mathsf{V}$-relations $r\colon X \mathrel{\mapstochar\!\!\rightarrow} X'$ and $s\colon Y \mathrel{\mapstochar\!\!\rightarrow} Y'$, let us set the $\mathsf{V}$-relation:

$$r \owedge s\colon X \times Y \mathrel{\mapstochar\!\!\rightarrow} X' \times Y', \quad (r \owedge s)((x,y),(x',y')) = r(x,x') \wedge s(y,y');$$

observe that $r \owedge s = (\pi_{X'}^{\circ} \cdot r \cdot \pi_X) \wedge (\pi_{Y'}^{\circ} \cdot s \cdot \pi_Y)$. For $(\mathbb{T},\mathsf{V})$-spaces $(X,a)$ and $(Y,b)$, their product is $(X \times Y, a \times b)$, where $a \times b\colon T(X \times Y) \mathrel{\mapstochar\!\!\rightarrow} X \times Y$ is given by: for each $\mathfrak{w} \in T(X \times Y)$, $(x,y) \in X \times Y$, $a \times b(\mathfrak{w},(x,y)) = a(T\pi_X(\mathfrak{w}),x) \wedge b(T\pi_Y(\mathfrak{w}),y) = (a \owedge b) \cdot \mathrm{can}_{X,Y}(\mathfrak{w},(x,y))$. Therefore,

$$a \times b = (a \owedge b) \cdot \mathrm{can}_{X,Y}. \tag{I.22}$$

**Lemma 2.6.1** [CHR20, Proposition 7.4] *If the diagram below is lax commutative,*

$$\begin{array}{ccc} T(\mathsf{V} \times \mathsf{V}) & \xrightarrow{T(\wedge)} & T\mathsf{V} \\ {\scriptstyle \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle}\downarrow & \leq & \downarrow{\scriptstyle \xi} \\ \mathsf{V} \times \mathsf{V} & \xrightarrow[\wedge]{} & \mathsf{V} \end{array} \tag{I.23}$$

*then, for each* $\mathsf{V}$*-relations* $r\colon X \mathrel{\mapstochar\!\!\rightarrow} X'$ *and* $s\colon Y \mathrel{\mapstochar\!\!\rightarrow} Y'$*, the following diagram is commutative.*

$$\begin{array}{ccc} T(X \times Y) & \xrightarrow{\mathrm{can}_{X,Y}} & TX \times TY \\ {\scriptstyle T(r \owedge s)}\downarrow & & \downarrow{\scriptstyle (Tr) \owedge (Ts)} \\ T(X' \times Y') & \xrightarrow[\mathrm{can}_{X',Y'}]{} & TX' \times TY' \end{array}$$

*Proof.* We first notice that the inequality $\mathrm{can}_{X',Y'} \cdot T(r \owedge s) \leq ((Tr) \owedge (Ts)) \cdot \mathrm{can}_{X,Y}$ is always true, since this is equivalent to $T(r \owedge s) \leq \mathrm{can}_{X',Y'}^{\circ} \cdot ((Tr) \owedge (Ts)) \cdot \mathrm{can}_{X,Y}$, and

$$\begin{aligned} T(r \owedge s) &= T((\pi_{X'}^{\circ} \cdot r \cdot \pi_X) \wedge (\pi_{Y'}^{\circ} \cdot s \cdot \pi_Y)) \\ &\leq T(\pi_{X'}^{\circ} \cdot r \cdot \pi_X) \wedge T(\pi_{Y'}^{\circ} \cdot s \cdot \pi_Y) \\ &= (T\pi_{X'}^{\circ} \cdot Tr \cdot T\pi_X) \wedge (T\pi_{Y'}^{\circ} \cdot Ts \cdot T\pi_Y) \\ &= (\mathrm{can}_{X',Y'}^{\circ} \cdot \pi_{TX'}^{\circ} \cdot Tr \cdot \pi_{TX} \cdot \mathrm{can}_{X,Y}) \wedge (\mathrm{can}_{X',Y'}^{\circ} \cdot \pi_{TY'}^{\circ} \cdot Ts \cdot \pi_{TY} \cdot \mathrm{can}_{X,Y}) \\ &= \mathrm{can}_{X',Y'}^{\circ} \cdot ((\pi_{TX'}^{\circ} \cdot Tr \cdot \pi_{TX}) \wedge (\pi_{TY'}^{\circ} \cdot Ts \cdot \pi_{TY})) \cdot \mathrm{can}_{X,Y} \\ &= \mathrm{can}_{X',Y'}^{\circ} \cdot (Tr \owedge Ts) \cdot \mathrm{can}_{X,Y}. \end{aligned}$$

For the converse inequality, we start by observing that since $T$ preserves weak pullbacks, for maps $f\colon A\to X$ and $g\colon B\to Y$, the diagram

$$\begin{array}{ccc} T(A\times B) & \xrightarrow{T(f\times g)} & T(X\times Y) \\ {\scriptstyle \mathrm{can}_{A,B}}\downarrow & & \downarrow{\scriptstyle \mathrm{can}_{X,Y}} \\ TA\times TB & \xrightarrow[Tf\times Tg]{} & TX\times TY \end{array}$$

is a weak pullback. We wish to prove that, for each $\mathfrak{w}\in T(X\times Y)$, $\mathfrak{x}'\in TX'$, $\mathfrak{y}'\in TY'$,

$$Tr(T\pi_X(\mathfrak{w}),\mathfrak{x}')\wedge Ts(T\pi_Y(\mathfrak{w}),\mathfrak{y}')\leq\bigvee\{T(r\oslash s)(\mathfrak{w},\mathfrak{w}')\mid \mathfrak{w}'\in T(X'\times Y'),\ \mathrm{can}_{X',Y'}(\mathfrak{w}')=(\mathfrak{x}',\mathfrak{y}')\}.$$

By (I.17),

$$Tr(\mathfrak{x},\mathfrak{x}')=\bigvee\{\xi\cdot T\vec{r}(\mathfrak{w}_1)\mid \mathfrak{w}_1\in T(X\times X'),\ T\pi_X(\mathfrak{w}_1)=\mathfrak{x},\ T\pi_{X'}(\mathfrak{w}_1)=\mathfrak{x}'\}$$

and

$$Ts(\mathfrak{y},\mathfrak{y}')=\bigvee\{\xi\cdot T\vec{s}(\mathfrak{w}_2)\mid \mathfrak{w}_2\in T(Y\times Y'),\ T\pi_Y(\mathfrak{w}_2)=\mathfrak{y},\ T\pi_{Y'}(\mathfrak{w}_2)=\mathfrak{y}'\},$$

where $(\mathfrak{x},\mathfrak{y})=\mathrm{can}_{X,Y}(\mathfrak{w})$, and, because $\mathsf{V}$ is a Heyting algebra, one concludes that

$$Tr(\mathfrak{x},\mathfrak{x}')\wedge Ts(\mathfrak{y},\mathfrak{y}')=\bigvee_{\substack{\mathrm{can}_{X,X'}(\mathfrak{w}_1)=(\mathfrak{x},\mathfrak{x}')\\ \mathrm{can}_{Y,Y'}(\mathfrak{w}_2)=(\mathfrak{y},\mathfrak{y}')}}(\xi\cdot T\vec{r}(\mathfrak{w}_1)\wedge\xi\cdot T\vec{s}(\mathfrak{w}_2))\,.$$

By our first observation, the following diagram is a weak pullback:

$$\begin{array}{ccc} T(X\times X'\times Y\times Y') & \xrightarrow{T(\pi_X\times\pi_Y)} & T(X\times Y) \\ {\scriptstyle \mathrm{can}_{X\times X',Y\times Y'}}\downarrow & & \downarrow{\scriptstyle \mathrm{can}_{X,Y}} \\ T(X\times X')\times T(Y\times Y') & \xrightarrow[T\pi_X\times T\pi_Y]{} & TX\times TY. \end{array}$$

Hence, for each $(\mathfrak{w}_1,\mathfrak{w}_2)\in T(X\times X')\times T(Y\times Y')$ such that $T\pi_X\times T\pi_Y(\mathfrak{w}_1,\mathfrak{w}_2)=(\mathfrak{x},\mathfrak{y})=\mathrm{can}_{X,Y}(\mathfrak{w})$, there exists $\mathfrak{v}\in T(X\times X'\times Y\times Y')$ such that

$$\mathrm{can}_{X\times X',Y\times Y'}(\mathfrak{v})=(\mathfrak{w}_1,\mathfrak{w}_2)\quad\&\quad T(\pi_X\times\pi_Y)(\mathfrak{v})=\mathfrak{w}. \tag{I.24}$$

Furthermore, $\mathrm{can}_{\mathsf{V},\mathsf{V}} \cdot T(\vec{r} \times \vec{s}) = (T\vec{r} \times T\vec{s}) \cdot \mathrm{can}_{X\times X',Y\times Y'}$, whence, by hypothesis, for such a $\mathfrak{v} \in T(X \times X' \times Y \times Y')$ we have

$$\begin{aligned}
\xi \cdot T(\wedge)\cdot T(\vec{r}\times\vec{s})(\mathfrak{v}) \;&\geq\; \wedge\cdot\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle \cdot T(\vec{r}\times\vec{s})(\mathfrak{v}) \;=\; \wedge\cdot(\xi\times\xi)\cdot \mathrm{can}_{\mathsf{V},\mathsf{V}}\cdot T(\vec{r}\times\vec{s})(\mathfrak{v})\\
&=\; \wedge\cdot(\xi\times\xi)\cdot(T\vec{r}\times T\vec{s})\cdot \mathrm{can}_{X\times X',Y\times Y'}(\mathfrak{v}) \;=\; \wedge\cdot(\xi\times\xi)(T\vec{r}(\mathfrak{w}_1), T\vec{s}(\mathfrak{w}_2))\\
&=\; \xi\cdot T\vec{r}(\mathfrak{w}_1)\wedge \xi\cdot T\vec{s}(\mathfrak{w}_2).
\end{aligned}$$

Finally, using that $X \times Y \times X' \times Y' \cong X \times X' \times Y \times Y'$, one can see that $\wedge \cdot (\vec{r} \times \vec{s}) = \overrightarrow{r \owedge s}$. Moreover, each $\mathfrak{v} \in T(X \times X' \times Y \times Y')$ satisfying (I.24) determines a $\mathfrak{w}' = T(\pi_{X'} \times \pi_{Y'})(\mathfrak{v}) \in T(X' \times Y')$ such that $\mathrm{can}_{X',Y'}(\mathfrak{w}') = (\mathfrak{x}', \mathfrak{y}')$, hence, by (I.17) we can conclude:

$$\begin{aligned}
Tr(\mathfrak{x},\mathfrak{x}')\wedge Ts(\mathfrak{y},\mathfrak{y}') \;&\leq\; \bigvee_{\substack{\mathrm{can}_{X\times X',Y\times Y'}(\mathfrak{v})=(\mathfrak{w}_1,\mathfrak{w}_2)\\ T(\pi_X\times\pi_Y)(\mathfrak{v})=\mathfrak{w}}} \xi\cdot T(\wedge)\cdot T(\vec{r}\times\vec{s})(\mathfrak{v})\\
&\leq\; \bigvee_{\mathrm{can}_{X',Y'}(\mathfrak{w}')=(\mathfrak{x}',\mathfrak{y}')} T(r\owedge s)(\mathfrak{w},\mathfrak{w}').
\end{aligned}$$

□

**Remark 2.6.2** By [CHR20, Remark 7.5], the inequality

$$\begin{array}{ccc}
T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{\;T(\wedge)\;} & T\mathsf{V}\\
{\scriptstyle\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle}\downarrow & \geq & \downarrow{\scriptstyle\xi}\\
\mathsf{V}\times\mathsf{V} & \xrightarrow[\;\wedge\;]{} & \mathsf{V}
\end{array}$$

is always true. Let us verify that, for some of our Examples 2.5.1, (I.23) is (strictly) commutative. This is trivial for $\mathbb{T} = \mathbb{I}$ and $\xi = 1_{\mathsf{V}}$. For any $\mathbb{T}$, when $\mathsf{V}$ is a frame, so that $\otimes = \wedge$, this follows from (I.16). For $\mathbb{U}$ and $\mathsf{P}_+$, since the map $\xi$ is the same as for $\mathsf{P}_{\max}$, we can conclude the commutativity. For $\mathbb{U}$ and $\mathsf{P}_1$, to verify the commutativity of

$$\begin{array}{ccc}
U([0,1]\times[0,1]) & \xrightarrow{\;U(\wedge)\;} & U[0,1]\\
{\scriptstyle\langle\xi\cdot U\pi_1,\xi\cdot U\pi_2\rangle}\downarrow & \leq & \downarrow{\scriptstyle\xi}\\
[0,1]\times[0,1] & \xrightarrow[\;\wedge\;]{} & [0,1],
\end{array}$$

let $\mathfrak{w} \in U([0,1] \times [0,1])$ and fix $\mathfrak{v}_i = U\pi_i(\mathfrak{w})$, $i = 1,2$. Suppose that $\xi \cdot U(\wedge)(\mathfrak{w}) < \xi(\mathfrak{v}_1) \wedge \xi(\mathfrak{v}_2)$. Hence there exists $t \in [0,1]$ with $\xi \cdot U(\wedge)(\mathfrak{w}) < t < \xi(\mathfrak{v}_1) \wedge \xi(\mathfrak{v}_2)$. This means that $[t,1] \times [t,1] \notin \mathfrak{w}$

and, by definition of $\xi$, $[t,1] \in \mathfrak{v}_i$, $i = 1,2$. Then $[t,1] \times [t,1] \in \mathfrak{w}$, a contradiction. For $\mathbb{M}$ and any quantale $\mathsf{V}$, we have:

$$\begin{array}{ccc} \mathsf{V}\times\mathsf{V}\times M & \xrightarrow{\wedge\times 1_M} & \mathsf{V}\times M \\ {\scriptstyle \langle \pi_{\mathsf{V}}\cdot(\pi_1\times 1_M),\pi_{\mathsf{V}}\cdot(\pi_2\times 1_M)\rangle}\downarrow & & \downarrow{\scriptstyle \pi_{\mathsf{V}}} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\wedge]{} & \mathsf{V} \end{array} \qquad ☞ \qquad \begin{array}{ccc} (u,v,a) & \longmapsto & (u\wedge v,a) \\ \downarrow & & \downarrow \\ (u,v) & \longmapsto & u\wedge v. \end{array}$$

Therefore, our table of categories satisfying the hypothesis of Lemma 2.6.1 is the following one.

| $\mathsf{V}$ \ $\mathbb{T}$ | $\mathbb{I}$ | $\mathbb{U}$ | $\mathbb{M}$ | $\mathbb{L}$ |
|---|---|---|---|---|
| 2 | Ord | Top | $(\mathbb{M},2)$-Cat | MultiOrd |
| $\mathsf{P}_+$ | Met | App | $(\mathbb{M},\mathsf{P}_+)$-Cat | |
| $\mathsf{P}_{\max}$ | UltMet | NA-App | $(\mathbb{M},\mathsf{P}_{\max})$-Cat | $(\mathbb{L},\mathsf{P}_{\max})$-Cat |
| $\mathsf{P}_1$ | BMet | $(\mathbb{U},\mathsf{P}_1)$-Cat | $(\mathbb{M},\mathsf{P}_1)$-Cat | |
| $2^2$ | BiRel | BiTop | $(\mathbb{M},2^2)$-Cat | $(\mathbb{L},2^2)$-Cat |
| $\Delta$ | ProbMet | | $(\mathbb{M},\Delta)$-Cat | |

(I.25)

**Theorem 2.6.3** [CHR20, Theorem 3.1] *If for all* $\mathsf{V}$*-relations* $r\colon X \relbar\!\!\mapsto X'$ *and* $s\colon Y \relbar\!\!\mapsto Y'$ *the diagram*

$$\begin{array}{ccc} T(X\times Y) & \xrightarrow{\mathrm{can}_{X,Y}} & TX\times TY \\ {\scriptstyle T(r\oslash s)}\downarrow & & \downarrow{\scriptstyle (Tr)\oslash(Ts)} \\ T(X'\times Y') & \xrightarrow[\mathrm{can}_{X',Y'}]{} & TX'\times TY' \end{array}$$

*is commutative, then the* $(\mathbb{T},\mathsf{V})$*-space* $(X,a)$ *is exponentiable whenever, for each* $\mathfrak{X} \in TTX$, $x \in X$, $u,v \in \mathsf{V}$,

$$\bigvee_{\mathfrak{x}\in TX} (Ta(\mathfrak{X},\mathfrak{x})\wedge u)\otimes(a(\mathfrak{x},x)\wedge v) \geq a(m_X(\mathfrak{X}),x)\wedge(u\otimes v). \tag{I.26}$$

*Proof.* Let $(X,a)$ be a $(\mathbb{T},\mathsf{V})$-space satisfying (I.26). For each $(\mathbb{T},\mathsf{V})$-space $(Y,b)$, we show that the $(\mathbb{T},\mathsf{V})$-graph $(Y^X,b^a)$ is transitive, with $b^a$ as in (I.20), that is, for each $\mathfrak{P} \in TT(Y^X)$, $\mathfrak{p} \in T(Y^X)$, $h \in Y^X$,

$$T(b^a)(\mathfrak{P},\mathfrak{p})\otimes b^a(\mathfrak{p},h) \leq b^a(m_{Y^X}(\mathfrak{P}),h).$$

By the definition of $b^a$ it suffices to show that, for each $\mathfrak{q} \in T(Y^X \times X)$ such that $T\pi_{Y^X}(\mathfrak{q}) = m_{Y^X}(\mathfrak{P})$, and each $x \in X$,

$$(T(b^a)(\mathfrak{P},\mathfrak{p}) \otimes b^a(\mathfrak{p},h)) \wedge a(T\pi_X(\mathfrak{q}),x) \leq b(T\mathrm{ev}_{X,Y}(\mathfrak{q}),h(x)),$$

with $\mathrm{ev}_{X,Y}: Y^X \times X \to Y$ the evaluation map. Since $m$ satisfies (BC), there exists $\mathfrak{Q} \in TT(Y^X \times X)$ such that $TT\pi_{Y^X}(\mathfrak{Q}) = \mathfrak{P}$ and $m_{Y^X \times X}(\mathfrak{Q}) = \mathfrak{q}$, hence $m_X(TT\pi_X(\mathfrak{Q})) = T\pi_X \cdot m_{Y^X \times X}(\mathfrak{Q}) = T\pi_X(\mathfrak{q})$ and we have:

$$\begin{aligned}
&(T(b^a)(\mathfrak{P},\mathfrak{p}) \otimes b^a(\mathfrak{p},h)) \wedge a(T\pi_X(\mathfrak{q}),x)\\
\leq\;& \bigvee_{\mathfrak{x}\in TX} \big(T(b^a)(TT\pi_{Y^X}(\mathfrak{Q}),\mathfrak{p}) \wedge Ta(TT\pi_X(\mathfrak{Q}),\mathfrak{x})\big) \otimes (b^a(\mathfrak{p},h) \wedge a(\mathfrak{x},x)) && \text{(by (I.26))}\\
\leq\;& \bigvee_{\mathfrak{x}\in TX} \bigvee_{\mathfrak{q}\in \mathrm{can}^{-1}_{Y^X,X}(\mathfrak{p},\mathfrak{x})} T(b^a \oslash a)(T\,\mathrm{can}_{Y^X,X}(\mathfrak{Q}),\mathfrak{q}) \otimes (b^a \oslash a)(\mathrm{can}_{Y^X,X}(\mathfrak{q}),(h,x)) && \text{(by hypothesis)}\\
=\;& \bigvee_{\mathfrak{q}\in(T\pi_{Y^X})^{-1}(\mathfrak{p})} T(b^a \times a)(\mathfrak{Q},\mathfrak{q}) \otimes (b^a \times a)(\mathfrak{q},(h,x))\\
\leq\;& \bigvee_{\mathfrak{q}\in(T\pi^{-1}_{Y^X})(\mathfrak{p})} Tb(TT\mathrm{ev}_{X,Y}(\mathfrak{Q}),T\mathrm{ev}_{X,Y}(\mathfrak{q})) \otimes b(T\mathrm{ev}_{X,Y}(\mathfrak{q}),h(x))\\
\leq\;& b(m_Y \cdot TT\mathrm{ev}_{X,Y}(\mathfrak{Q}),h(x)) = b(T\mathrm{ev}_{X,Y}(\mathfrak{q}),h(x)).
\end{aligned}$$

□

When $\otimes = \wedge$, (I.26) gives for all $\mathfrak{X} \in TTX$, $x \in X$, $\bigvee_{\mathfrak{x}\in TX}(Ta(\mathfrak{X},\mathfrak{x}) \wedge a(\mathfrak{x},x)) \geq a(m_X(\mathfrak{X}),x)$, what is equivalent to $a \cdot Ta = a \cdot m_X$. This is the case, for instance, of $(\mathbb{U},2)$-Cat $\cong$ Top, where this condition is equivalent to exponentiability as proved in [Pis99]. Moreover, in Top exponentiable spaces are characterized as the *core-compact* spaces (see [EH02] for a thorough discussion). In [Hof14, Definition 1.8], *core-compact* $(\mathbb{T},\mathsf{V})$*-spaces* are naturally those spaces $(X,a)$ such that $a \cdot Ta = a \cdot m_X$.

## 2.7 Injective and representable $(\mathbb{T},\mathsf{V})$-spaces, and the Yoneda embedding

For $(\mathbb{T},\mathsf{V})$-spaces $(X,a)$ and $(Y,b)$, let us consider the following (pre)order on the set of continuous maps from $(X,a)$ to $(Y,b)$: for each $f,g\colon (X,a) \to (Y,b)$,

$$f \leq g \iff \forall x \in X,\; k \leq b(e_Y(f(x)),g(x)). \tag{I.27}$$

This order was first defined in [CT03], and it is compatible with composition, that is, with this order $(\mathbb{T},\mathsf{V})$-Cat becomes a 2-category.

**Definition 2.7.1** A space $(Y,b)$ is called *separated* if, for all $(X,a)$ in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, the order (I.27) is separated, i.e., it is anti-symmetric.

Under the notation $f \simeq g$ if, and only if, $f \leq g$ and $g \leq f$, $(Y,b)$ is separated whenever, for each $f,g\colon (X,a) \to (Y,b)$, $(X,a) \in (\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, $f \simeq g$ implies $f = g$. In fact, $(Y,b)$ is separated if, and only if, the order on $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}((1,e_1^\circ),(Y,b))$ is separated, and this is equivalent to the following order on the set $Y$ to be separated: for each $y,y' \in Y$,

$$y \leq y' \iff k \leq b(e_Y(y),y'). \tag{I.28}$$

[HST14, III-Proposition 3.3.1].

We observe that for the Sierpiński $(\mathbb{T},\mathsf{V})$-space $(\mathsf{V},\hom_\xi)$, the order (I.28) is the order of $\mathsf{V}$, hence $(\mathsf{V},\hom_\xi)$ is a separated space. Moreover, for the particular case of $(\mathbb{U},2)$-$\mathsf{Cat} \cong \mathsf{Top}$, the order (I.28) is the dual of the *specialization order* that was discussed in Subsection 2.3.

**Remark 2.7.2** *For a separated* $(\mathbb{T},\mathsf{V})$*-space* $(X,a)$*, a map* $f\colon (X,a) \to (Y,b)$ *is fully faithful if, and only if, it is an embedding*: if $f$ is an embedding, then it is $|\text{-}|$-initial, where $|\text{-}|\colon (\mathbb{T},\mathsf{V})$-$\mathsf{Cat} \to \mathsf{Set}$ is the forgetful functor, hence it is fully faithful. Conversely, if $f$ is fully faithful, then it is $|\text{-}|$-initial. Let $x,x' \in X$ such that $f(x) = f(x')$. Then $f(x) \leq f(x')$ and $f(x') \leq f(x)$, whence

$$k \leq b(e_Y(f(x)),f(x')) \leq b(Tf \cdot e_X(x),f(x')) = a(e_X(x),x'),$$

that is, $x \leq x'$, and, analogously, $x' \leq x$. Since $(X,a)$ is separated, we have $x = x'$, hence $f$ is an injective map.

Denoting by $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\text{sep}}$ the full subcategory of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ of separated spaces, $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\text{sep}}$ is closed under mono-sources in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$. Moreover, by [HST14, V-Theorem 2.1.2], we have:

**Proposition 2.7.3** $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\text{sep}}$ *is regular epireflective in* $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$*. For each* $(\mathbb{T},\mathsf{V})$*-space* $(X,a)$*, the projection map* $\eta_X : X \to X/{\sim}$*, where, for each* $x,x' \in X$*,* $x \sim x'$ *if, and only if,* $x \leq x'$ *and* $x' \leq x$*, gives a reflection, with* $X/{\sim}$ *endowed with the* $(\mathbb{T},\mathsf{V})$*-structure* $\tilde{a} = \eta_X \cdot a \cdot (T\eta_X)^\circ$*, which makes* $\eta_X$ *both a* $|\text{-}|$*-final and a* $|\text{-}|$*-initial morphism.*

Let us recall that a continuous map $f\colon (X,a) \to (Y,b)$ is said to be *left adjoint* to a continuous map $g\colon (Y,b) \to (X,a)$, and $g$ is *right adjoint* to $f$, denoted as usual by $f \dashv g$, if, and only if,

$$1_X \leq g \cdot f \qquad \& \qquad f \cdot g \leq 1_Y.$$

Considering the associated V-spaces $A_e(X,a) = (X, a \cdot e_X) = (X, a_0)$ and $A_e(Y,b) = (Y, b \cdot e_Y) = (X, b_0)$, and using [HST14, III-Remark 3.3.4], one can check that

$$f \dashv g \iff \forall x \in X,\ \forall y \in Y,\ b_0(f(x), y) = a_0(x, g(y)). \tag{I.29}$$

**Definition 2.7.4** A space $(Z,c)$ is said to be *injective* if for each fully faithful map $y\colon (X,a) \to (Y,b)$, and each continuous map $f\colon (X,a) \to (Z,c)$, there exists $\hat{f}\colon (Y,b) \to (Z,c)$ continuous such that $\hat{f} \cdot y \simeq f$:

$$\begin{array}{ccc} X & \xrightarrow{\quad y \quad} & Y \\ & \simeq & \\ {\scriptstyle f}\searrow & & \swarrow{\scriptstyle \hat{f}} \\ & Z; & \end{array} \tag{I.30}$$

$\hat{f}$ is called an *extension* of $f$ along $y$.

Observe that considering only separated spaces, injectivity assumes its usual meaning [AHS90, Definition 9.1]. For details on injective spaces we refer to [Hof11].

Let us proceed recalling the class of *representable* $(\mathbb{T},\mathsf{V})$-spaces. We refer the reader to [CCH15, HST14] for more detailed information. Starting with the Set-monad $\mathbb{T} = (T, m, e)$ and its extension to V-Rel, for each V-space $(X, a_0)$, $(TX, Ta_0)$ is a V-space. Each V-continuous map $f\colon (X,a_0) \to (Y,b_0)$ induces a V-continuous map $Tf\colon (TX,Ta_0) \to (TY,Tb_0)$, and, moreover, $e_{(X,a_0)}\colon (X,a_0) \to (TX,Ta_0)$ and $m_{(X,a_0)}\colon (T^2X, T^2a_0) \to (TX,Ta_0)$ are V-continuous maps, because $e$ and $m$ are oplax transformations [Tho09]. Hence $\mathbb{T}$ extends to a monad on V-Cat, that we denote again by $\mathbb{T}$.

Consider the category $(\mathsf{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$ of Eilenberg-Moore $\mathbb{T}$-algebras on V-Cat, that is, the objects of $(\mathsf{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$ are pairs $((X,a_0),\alpha)$, also denoted by $(X,a_0,\alpha)$, where $(X,a_0)$ a V-space, and $(X,\alpha)$ is a $\mathbb{T}$-algebra with $\alpha\colon (TX,Ta_0) \to (X,a_0)$ a V-continuous map. A morphism $f\colon (X,a_0,\alpha) \to (Y,b_0,\beta)$ is both a V-continuous map $f\colon (X,a_0) \to (Y,b_0)$ and a $\mathbb{T}$-homomorphism $f\colon (X,\alpha) \to (Y,\beta)$.

For each $(X,a_0,\alpha) \in (\mathsf{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$, set $K(X,a_0,\alpha) = (X, a_0 \cdot \alpha)$, which is a $(\mathbb{T},\mathsf{V})$-space; each morphism $f\colon (X,a_0,\alpha) \to (Y,b_0,\beta)$ is a $(\mathbb{T},\mathsf{V})$-continuous map $f\colon (X,a_0 \cdot \alpha) \to (Y, b_0 \cdot \beta)$. Furthermore, for each $(\mathbb{T},\mathsf{V})$-space $(X,a)$, the triple $M(X,a) = (TX, Ta \cdot m_X^\circ, m_X)$ belongs to $(\mathsf{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$, and, for each $(\mathbb{T},\mathsf{V})$-continuous map $f\colon (X,a) \to (Y,b)$, $Tf\colon (TX, Ta \cdot m_X^\circ, m_X) \to (TY, Tb \cdot m_Y^\circ, m_Y)$ is a morphism in $(\mathsf{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$. The assignments $K$ and $M$ defined above determine well-defined functors which form a 2-adjunction

$$(\mathsf{V}\text{-}\mathsf{Cat})^{\mathbb{T}} \underset{M}{\overset{K}{\rightleftarrows}} (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}. \quad \top \tag{I.31}$$

Then we have a lifting of $\mathbb{T}$ to a 2-monad on $(\mathbb{T},\mathsf{V})$-Cat, again denoted by $\mathbb{T}$, which is *lax-idempotent*, or of *Kock-Zöberlein type*. Lax-idempotent monads are defined in the setting of 2-categories. However, since we are dealing only with order-enriched categories, we recall the definition in this context:

**Definition 2.7.5** For an order-enriched category $\mathsf{C}$, a 2-monad $\mathbb{T} = (T, m, e)\colon \mathsf{C} \to \mathsf{C}$ is *lax-idempotent* if, for each $A \in \mathsf{C}$, $m_A \dashv e_{TA}$.

This, in particular, implies that a $\mathbb{T}$-algebra structure $\alpha\colon TX \to X$ is left adjoint to $e_X : X \to TX$, that is, $1_{TX} \leq e_X \cdot \alpha$ and $\alpha \cdot e_X \leq 1_X$.

**Definition 2.7.6** A $(\mathbb{T},\mathsf{V})$-space $(X,a)$ is called *representable* whenever the unit $e_X : X \to TX$ has a left adjoint.

Such a left adjoint to $e_X$ is in general only a pseudo-algebra structure on $X$, thus

$$\alpha \cdot e_X \simeq 1_X \quad \& \quad \alpha \cdot T\alpha \simeq \alpha \cdot m_X, \tag{I.32}$$

and we have equalities when $(X,a)$ is separated [Hof14, Remark 2.6]. A useful characterization of representability is given in [Hof14, Proposition 2.7]: *a $(\mathbb{T},\mathsf{V})$-space $(X,a)$ is representable if, and only if, it is core-compact and there exists a map $\alpha\colon TX \to X$ such that $a = a_0 \cdot \alpha$, where* $(X,a_0) = A_e(X,a) = (X, a \cdot e_X)$:

$$TX \xrightarrow{\ \alpha\ } X \xrightarrow{\ a_0\ } X; \qquad a\colon TX \to X \tag{I.33}$$

this map $\alpha$ is precisely the left adjoint to $e_X$.

This setting provides the definition of *duals* in $(\mathbb{T},\mathsf{V})$-Cat. For a V-space $(X,a_0)$, its *dual* V-space is simply $(X,a_0^\circ)$. For a $(\mathbb{T},\mathsf{V})$-space $(X,a)$, consider its image by the functor $M$ in $(\mathsf{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$, $(TX, Ta \cdot m_X^\circ, m_X)$, and take the dual of its underlying V-space, $(TX, m_X \cdot (Ta)^\circ, m_X)$, which belongs to $(\mathsf{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$, since the extension $T$ commutes with involution. Then apply the functor $K$ obtaining

$$X^{\mathrm{op}} = (TX, m_X \cdot (Ta)^\circ \cdot m_X). \tag{I.34}$$

The space $X^{\mathrm{op}}$ is fundamental to the definition of the *Yoneda embedding*, which we recall next.

Firstly, the tensor product $\otimes$ of $\mathsf{V}$ induces a $(\mathbb{T},\mathsf{V})$-structure $c$ on $X\times Y$ defined by

$$c(\mathfrak{w},(x,y)) = a(T\pi_X(\mathfrak{w}),x)\otimes b(T\pi_Y(\mathfrak{w}),y),$$

for each $\mathfrak{w}\in T(X\times Y)$, $(x,y)\in X\times Y$. Setting

$$(X,a)\otimes(Y,b) = (X\times Y,c), \tag{I.35}$$

since we are in the setting of a strict topological theory, a functor $(X,a)\otimes - \colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ is defined [Hof07, Lemma 6.1]; on morphisms $((X,a)\otimes -)f = 1_X\times f$. Clearly this functor does not have a right adjoint for every space $(X,a)$, since this would imply in particular that $\mathsf{Top}$ is cartesian closed. However, this functor does have a right adjoint whenever the $(\mathbb{T},\mathsf{V})$-structure $a\colon TX \relbar\joinrel\mapsto X$ satisfies $a\cdot Ta = a\cdot m_X$ [Hof07, Theorem 6.5]. In particular, every $\mathbb{T}$-algebra is $\otimes$-exponentiable.

The proof of this result on $\otimes$-exponentiability is similar to what is done in the usual exponentiability case described in Subsection 2.6; of course, when $\otimes=\wedge$, they are the same. The set

$$Y^X = \{h\colon (X,a)\otimes(1,e_1^\circ)\to(Y,b) \mid h \text{ is a } (\mathbb{T},\mathsf{V})\text{-continuous map}\}$$

is endowed with the largest $\mathsf{V}$-relation $[a,b]\colon T(Y^X)\relbar\joinrel\mapsto Y^X$ that makes the evaluation map $\mathrm{ev}_{X,Y}\colon Y^X\times X\to Y$, $(h,x)\mapsto h(x)$, $(\mathbb{T},\mathsf{V})$-continuous, with $h(x)$ standing for $h(x,*)$, and $1=\{*\}$. Then $[a,b]$ is reflexive and it is given by

$$[a,b](\mathfrak{p},h) = \bigvee\{v\in\mathsf{V} \mid \forall\mathfrak{q}\in(T\pi_{Y^X})^{-1}(\mathfrak{p}),\ \forall x\in X,\ a(T\pi_X(\mathfrak{q}),x)\otimes v\le b(T\mathrm{ev}_{X,Y}(\mathfrak{q}),h(x))\},$$

for each $\mathfrak{p}\in T(Y^X)$, $h\in Y^X$ [CHT03]. Since $\otimes$ distributes over $\bigvee$, the supremum above is a maximum. Moreover, from the relation (I.2) between $\otimes$ and hom, one obtains

$$[a,b](\mathfrak{p},h) = \bigwedge_{\substack{\mathfrak{q}\in(T\pi_{Y^X})^{-1}(\mathfrak{p})\\ x\in X}} \hom\left(a(T\pi_X(\mathfrak{q}),x),b(T\mathrm{ev}_{X,Y}(\mathfrak{q}),h(x))\right), \tag{I.36}$$

for each $\mathfrak{p}\in T(Y^X)$, $h\in Y^X$. Hence this construction defines a right adjoint $(-)^{(X,a)}$ to the functor $(X,a)\otimes -$ in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Gph}$. When the $(\mathbb{T},\mathsf{V})$-graph structure $[a,b]$ is transitive, for each $(\mathbb{T},\mathsf{V})$-space

$(Y,b)$, then this defines a right adjoint for $(X,a)\otimes -$ in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$. As discussed above, this happens when $a\cdot Ta = a\cdot m_X$ [Hof07, Theorem 6.5].

Now let $(X,a_0,\alpha)$ be an element of $(\mathsf{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$. Considering its image by the functor $K$, $(X,a_0\cdot\alpha)$, and its underlying $\mathbb{T}$-algebra $(X,\alpha)$, we have

$$1_X \leq a_0 \implies \alpha \leq a_0\cdot\alpha, \tag{I.37}$$

whence $Y^{(X,a_0\cdot\alpha)} = \{h\colon (X,a_0\cdot\alpha)\otimes(1,e_1^\circ)\to(Y,b)\mid h \text{ is a } (\mathbb{T},\mathsf{V})\text{-continuous map}\}$ is a subset of $Y^{(X,\alpha)} = \{h\colon (X,\alpha)\otimes(1,e_1^\circ)\to(Y,b)\mid h \text{ is a } (\mathbb{T},\mathsf{V})\text{-continuous map}\}$. Motivated by [Hof13, Lemma 5.2], we prove the following:

**Lemma 2.7.7** *For each $(X,a_0,\alpha)\in(\mathsf{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$ and each $(\mathbb{T},\mathsf{V})$-space $(Y,b)$, the inclusion map $Y^{(X,a_0\cdot\alpha)}\hookrightarrow Y^{(X,\alpha)}$ is an embedding. Consequently, $(X,a_0\cdot\alpha)$ is $\otimes$-exponentiable.*

*Proof.* Since $T$ preserves monomorphisms, we can consider that $T(Y^{(X,a_0\cdot\alpha)})\subseteq T(Y^{(X,\alpha)})$. Furthermore, the inclusion map $Y^{(X,a_0\cdot\alpha)}\hookrightarrow Y^{(X,\alpha)}$ is $(\mathbb{T},\mathsf{V})$-continuous, since it is the exponential of the identity map $(X,\alpha)\to(X,a_0\cdot\alpha)$, which is $(\mathbb{T},\mathsf{V})$-continuous by (I.37). By (I.36), for each $\mathfrak{p}\in T(Y^{(X,a_0\cdot\alpha)})$, $h\in Y^{(X,a_0\cdot\alpha)}$,

$$\begin{aligned}
[\alpha,b](\mathfrak{p},h) &= \bigwedge_{\substack{\mathfrak{q}\in(T\pi_{Y^X})^{-1}(\mathfrak{p})\\ x\in X}} \hom(\alpha(T\pi_X(\mathfrak{q}),x), b(T\mathrm{ev}_{X,Y}(\mathfrak{q}),h(x)))\\
&= \bigwedge_{\mathfrak{q}\in(T\pi_{Y^X})^{-1}(\mathfrak{p})} b(T\mathrm{ev}_{X,Y}(\mathfrak{q}), h\cdot\alpha\cdot T\pi_X(\mathfrak{q})) && (\alpha \text{ is a map, (I.5), (I.4)}),
\end{aligned}$$

and

$$[a_0\cdot\alpha,b](\mathfrak{p},h) = \bigwedge_{\substack{\mathfrak{q}\in(T\pi_{Y^X})^{-1}(\mathfrak{p})\\ x\in X}} \hom(a_0\cdot\alpha(T\pi_X(\mathfrak{q}),x), b(T\mathrm{ev}_{X,Y}(\mathfrak{q}),h(x))).$$

Considering the $\mathsf{V}$-space $(Y,b_0)=(Y,b\cdot e_Y)$, we have

$$\begin{aligned}
h^\circ\cdot b_0\cdot h &= h^\circ\cdot b\cdot e_Y\cdot h\\
&= h^\circ\cdot b\cdot Th\cdot e_X && (e \text{ is natural})\\
&\geq (a_0\cdot\alpha\otimes e_1^\circ)\cdot e_X && (h \text{ belongs to } Y^{(X,a_0\cdot\alpha)})\\
&= a_0\cdot(\alpha\cdot e_X)\\
&= a_0 && (\alpha \text{ is a } \mathbb{T}\text{-algebra structure}).
\end{aligned}$$

Hence $a_0\cdot\alpha(T\pi_X(\mathfrak{q}),x)=a_0(\alpha\cdot T\pi_X(\mathfrak{q}),x)\le b_0(h\cdot\alpha\cdot T\pi_X(\mathfrak{q}),h(x))$, and consequently

$$\begin{aligned} b(T\mathrm{ev}_{X,Y}(\mathfrak{q}),h\cdot\alpha\cdot T\pi_X(\mathfrak{q}))\otimes a_0\cdot\alpha(T\pi_X(\mathfrak{q}),x) & \\ &\le b_0(e_Y\cdot T\mathrm{ev}_{X,Y}(\mathfrak{q}),h\cdot\alpha\cdot T\pi_X(\mathfrak{q}))\otimes b_0(h\cdot\alpha\cdot T\pi_X(\mathfrak{q}),h(x))\\ &\le b(T\mathrm{ev}_{X,Y}(\mathfrak{q}),h(x)) \qquad (b_0 \text{ is transitive}),\end{aligned}$$

what is equivalent to $b(T\mathrm{ev}_{X,Y}(\mathfrak{q}),h\cdot\alpha\cdot T\pi_X(\mathfrak{q}))\le\hom(a_0\cdot\alpha(T\pi_X(\mathfrak{q}),x),b(T\mathrm{ev}_{X,Y}(\mathfrak{q}),h(x)))$, whence $[\alpha,b](\mathfrak{p},h)\le[a_0\cdot\alpha,b](\mathfrak{p},h)$.

Since every $\mathbb{T}$-algebra is $\otimes$-exponentiable, $Y^{(X,\alpha)}$ is a $(\mathbb{T},\mathsf{V})$-space. Hence, by **(TA3)** of Subsection 2.4, $Y^{(X,a_0\cdot\alpha)}$ is a $(\mathbb{T},\mathsf{V})$-space. □

In particular, for every $(\mathbb{T},\mathsf{V})$-space $(X,a)$, its dual $(\mathbb{T},\mathsf{V})$-space $X^{\mathrm{op}}$ in (I.34) is $\otimes$-exponentiable.

**Lemma 2.7.8** *Each $(\mathbb{T},\mathsf{V})$-space $(X,a)$ induces a $(\mathbb{T},\mathsf{V})$-continuous map $a\colon X^{\mathrm{op}}\otimes X\to\mathsf{V}$, where $(\mathfrak{x},x)\mapsto a(\mathfrak{x},x)$.*

*Proof.* We wish to prove that, for each $\mathfrak{w}\in T(TX\times X)$, $(\mathfrak{x},x)\in TX\times X$,

$$a^{\mathrm{op}}(T\pi_{TX}(\mathfrak{w}),\mathfrak{x})\otimes a(T\pi_X(\mathfrak{w}),x)\le\hom_\xi(T\vec{a}(\mathfrak{w}),\vec{a}(\mathfrak{x},x))=\hom(\xi\cdot T\vec{a}(\mathfrak{w}),\vec{a}(\mathfrak{x},x)),$$

where $a^{\mathrm{op}}=m_X\cdot(Ta)^\circ\cdot m_X$ and $T\vec{a}\colon T(TX\times X)\to T(\mathsf{V})$. This inequality is equivalent to

$$a^{\mathrm{op}}(T\pi_{TX}(\mathfrak{w}),\mathfrak{x})\otimes a(T\pi_X(\mathfrak{w}),x)\otimes\xi\cdot T\vec{a}(\mathfrak{w})\le\vec{a}(\mathfrak{x},x).$$

We calculate:

$$\begin{aligned} & a^{\mathrm{op}}(T\pi_{TX}(\mathfrak{w}),\mathfrak{x})\otimes\xi\cdot T\vec{a}(\mathfrak{w})\otimes\vec{a}(T\pi_X(\mathfrak{w}),x)\\ &\le m_X^\circ\cdot Ta\cdot m_X^\circ(\mathfrak{x},T\pi_{TX}(\mathfrak{w}))\otimes Ta(T\pi_{TX}(\mathfrak{w}),T\pi_X(\mathfrak{w}))\otimes\vec{a}(T\pi_X(\mathfrak{w}),x) && \text{(by (I.17))}\\ &\le a\cdot Ta\cdot m_X^\circ\cdot Ta\cdot m_X^\circ(\mathfrak{x},x) && \text{(by (I.6))}\\ &\le a\cdot m_X\cdot m_X^\circ\cdot Ta\cdot m_X^\circ(\mathfrak{x},x) && \text{(by \textbf{(T)} of 2.2)}\\ &\le a\cdot Ta\cdot m_X^\circ(\mathfrak{x},x) && \text{(by (I.8))}\\ &\le a(\mathfrak{x},x) && \text{(by \textbf{(T)} and (I.8)).}\end{aligned}$$

□

**Definition 2.7.9** For a $(\mathbb{T},\mathsf{V})$-space $(X,a)$, its *Yoneda embedding* is the $\otimes$-exponential mate of the $(\mathbb{T},\mathsf{V})$-continuous map $a\colon X^{\mathrm{op}}\otimes X\to\mathsf{V}$, which we denote by $y_X : X\to PX$, with $PX=\mathsf{V}^{X^{\mathrm{op}}}$, so that, for each $x\in X$, $\mathfrak{x}\in TX$, $y_X(x)(\mathfrak{x})=a(\mathfrak{x},x)$.

We observe that this definition generalizes the Yoneda embedding for $\mathsf{V}$-*valued categories* from [Law73].

We finish this subsection recalling important results that are going to be used in the next chapter. Proofs can be found in [CH09] and [Hof11], where the concept of $(\mathbb{T},\mathsf{V})$-*modules*, or $(\mathbb{T},\mathsf{V})$-*distributors*, is employed. We do not explore this concept here. The first result is properly contained and proved in [Hof11, Theorem 2.7].

**Proposition 2.7.10** *The following are equivalent for a* $(\mathbb{T},\mathsf{V})$*-space* $(X,a)$*.*
*(i)* $(X,a)$ *is injective.*
*(ii) The Yoneda embedding* $y_X : X\to PX$ *has a left inverse, that is, there exists a continuous map* $\mathrm{Sup}_X : PX\to X$ *such that* $\mathrm{Sup}_X\cdot y_X=1_X$.
*(iii) The Yoneda embedding* $y_X : X\to PX$ *has a left adjoint* $\mathrm{Sup}_X : PX\to X$.

**Corollary 2.7.11** [Hof14, Proposition 3.8] *Every injective* $(\mathbb{T},\mathsf{V})$*-category is representable.*

**Proposition 2.7.12** [CH09, Corollary 5.2][Hof11, Theorem 2.9] *For every* $(\mathbb{T},\mathsf{V})$*-space* $(X,a)$*,* $PX=\mathsf{V}^{X^{\mathrm{op}}}$ *is an injective separated* $(\mathbb{T},\mathsf{V})$*-space, and the Yoneda embedding* $y_X : X\to PX$ *is fully faithful.*

Therefore, when $(X,a)$ is separated, by Remark 2.7.2, $y_X : X\to PX$ is an embedding, and we conclude:

**Theorem 2.7.13** *Every separated* $(\mathbb{T},\mathsf{V})$*-space embeds into an injective separated* $(\mathbb{T},\mathsf{V})$*-space.*

**Corollary 2.7.14** [HT10, Lemma 4.18] *Under the assumption that* $T1=1$*, the Sierpiński* $(\mathbb{T},\mathsf{V})$*-space* $(\mathsf{V},\hom_\xi)$ *is an injective* $(\mathbb{T},\mathsf{V})$*-space.*

## 2.8 Compact and Hausdorff $(\mathbb{T},\mathsf{V})$-spaces

Let us begin with the particular case of $(\mathbb{U},2)\text{-}\mathsf{Cat}\cong\mathsf{Top}$. In terms of convergence, a topological space $(X,a)$ is *compact* if, and only if, every ultrafilter on $X$ converges to *at least* one point of $X$, and it is *Hausdorff* if, and only if, every ultrafilter on $X$ converges to *at most* one point of $X$. Combining both properties, $(X,a)$ is compact and Hausdorff if, and only if, for all $\mathfrak{x}\in UX$, there exists a unique

$x \in X$ such that $a(\mathfrak{x},x) = \top$. As shown in [HST14, III-Propositions 2.3.1 and 2.3.2], this is equivalent to

$$\underbrace{1_{UX} \leq a^\circ \cdot a}_{\text{compact}} \quad \& \quad \underbrace{a \cdot a^\circ \leq 1_X}_{\text{Hausdorff}}.$$

The notions of compactness and Hausdorff separation are then generalised to $(\mathbb{T},\mathsf{V})$-spaces in [HST14, V-Definition 1.1.1], and as commented by the authors, the work of Manes [Man74] can be considered as a predecessor of this generalisation. Among others, Kamnitzer [Kam74] and Möbus [Möb81] have also studied the concepts.

**Definition 2.8.1** A $(\mathbb{T},\mathsf{V})$-space $(X,a)$ is said to be
(1) *compact* if $1_{TX} \leq a^\circ \cdot a$, or componentwise, if, for all $\mathfrak{x} \in TX$, $k \leq \bigvee_{x\in X} a(\mathfrak{x},x) \otimes a(\mathfrak{x},x)$;
(2) *Hausdorff* if $a \cdot a^\circ \leq 1_X$, i.e., for all $x,y \in X$ and $\mathfrak{x} \in TX$, if $\bot < a(\mathfrak{x},x) \otimes a(\mathfrak{x},y)$, then $x = y$, and $a(\mathfrak{x},x) \otimes a(\mathfrak{x},x) \leq k$.

Under the assumption that $\mathsf{V}$ is integral, the second condition in item (2) holds trivially. Furthermore, adding the condition that $\mathsf{V}$ is lean, by [HST14, V-Proposition 1.2.1], a $(\mathbb{T},\mathsf{V})$-space $(X,a)$ is compact and Hausdorff if, and only if, it is a $\mathbb{T}$-algebra. It follows that:
• *compact Hausdorff* $(\mathbb{T},\mathsf{V})$*-spaces are exponentiable*: for $\mathsf{V}$ integral, condition (I.26) is satisfied by any $\mathbb{T}$-algebra;
• *limits of compact Hausdorff* $(\mathbb{T},\mathsf{V})$*-spaces are compact and Hausdorff* [HST14, V-Theorem 1.2.3];
• under the assumption that $T$ preserves finite coproducts (for examples, the identity monad $\mathbb{I}$, the ultrafilter monad $\mathbb{U}$ [Bör87], and the monad $\mathbb{M}$), *finite coproducts of compact Haudorff* $(\mathbb{T},\mathsf{V})$*-spaces are compact and Hausdorff* [HST14, V-Corollary 1.1.6(2)].

Denoting by $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{CompHaus}}$ the class of compact Hausdorff $(\mathbb{T},\mathsf{V})$-spaces, for future references, we summarize these three facts in the following:

**Proposition 2.8.2** *For* $\mathsf{V}$ *an integral and lean quantale, we have an isomorphism*

$$(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{CompHaus}} \cong \mathsf{Set}^{\mathbb{T}}, \tag{I.38}$$

*which implies that compact Hausdorff* $(\mathbb{T},\mathsf{V})$*-spaces are exponentiable.* $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{CompHaus}}$ *is closed under limits, and if* $T$ *preserves finite coproducts, then it is closed under finite coproducts.*

We can form a sub-table of (I.12) of examples satisfying these hypotheses.

| $\mathsf{V}$ \ $\mathbb{T}$ | $\mathbb{I}$ | $\mathbb{U}$ | $\mathbb{M}$ |
|---|---|---|---|
| $2$ | $\mathsf{Ord}$ | $\mathsf{Top}$ | $(\mathbb{M},2)\text{-}\mathsf{Cat}$ |
| $\mathsf{P}_+$ | $\mathsf{Met}$ | $\mathsf{App}$ | $(\mathbb{M},\mathsf{P}_+)\text{-}\mathsf{Cat}$ |
| $\mathsf{P}_{\max}$ | $\mathsf{UltMet}$ | $\mathsf{NA}\text{-}\mathsf{App}$ | $(\mathbb{M},\mathsf{P}_{\max})\text{-}\mathsf{Cat}$ |
| $\mathsf{P}_1$ | $\mathsf{BMet}$ | $(\mathbb{U},\mathsf{P}_1)\text{-}\mathsf{Cat}$ | $(\mathbb{M},\mathsf{P}_1)\text{-}\mathsf{Cat}$ |

(I.39)

**Examples 2.8.3** (1) For the first column of this table, $\mathsf{V}\text{-}\mathsf{Cat}_{\text{CompHaus}} \cong \mathsf{Set}^{\mathbb{I}} = \mathsf{Set}$, whence compact Hausdorff $\mathsf{V}$-spaces are discrete objects: $(X,1_X)$, $X \in \mathsf{Set}$.
(2) For the second column, $(\mathbb{U},\mathsf{V})\text{-}\mathsf{Cat}_{\text{CompHaus}} \cong \mathsf{Set}^{\mathbb{U}}$. For $\mathsf{Top} \cong (\mathbb{U},2)\text{-}\mathsf{Cat}$, as observed in the begining of the subsection, compact and Hausdorff spaces are those such that every ultrafilter converges to a unique point. Analogously, for $\mathsf{NA}\text{-}\mathsf{App} \cong (\mathbb{U},\mathsf{P}_{\max})\text{-}\mathsf{Cat}$ and $\mathsf{App} \cong (\mathbb{U},\mathsf{P}_+)\text{-}\mathsf{Cat}$, compact Hausdorff (non-Archimedean) approach spaces are *topological approach spaces* [Low97, Chapter 2] induced by a compact Hausdorff topology.
(3) For a space $(X,a) \in (\mathbb{M},2)\text{-}\mathsf{Cat}$, for $(x,b) \in X \times M$, $x' \in X$, writing $x' = b\cdot x$ for $a((x,b),x') = \top$, then $(X,a)$ is compact and Haudorff if, and only if,

$$\forall x \in X,\ \forall b \in M,\ \exists\,!\ x' \in X;\ x' = b\cdot x,$$

and this implies that $a\colon X \times M \to X$ defines an action of $M$ on $X$ [HST14, V-Section 1.4]. Hence $(\mathbb{M},2)\text{-}\mathsf{Cat}_{\text{CompHaus}}$ is equivalent to the topos $\mathsf{M}\text{-}\mathsf{Set}$ which consists of sets with actions of $M$ and *equivariant maps*. For $\mathsf{V} = \mathsf{P}_+,\mathsf{P}_{\max},\mathsf{P}_1$, we have

$$(\mathbb{M},\mathsf{V})\text{-}\mathsf{Cat}_{\text{CompHaus}} \cong \mathsf{Set}^{\mathbb{M}} \cong (\mathbb{M},2)\text{-}\mathsf{Cat}_{\text{CompHaus}} \cong \mathsf{M}\text{-}\mathsf{Set}.$$

# Chapter II

# On injectivity and weak exponentiability in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$

We investigate in this chapter the relation between injectivity and exponentiability of $(\mathbb{T},\mathsf{V})$-spaces, generalising some results of [HR13, Hof13]. Moreover, following the lines of [Ros99], and applying results of [CR00], we prove that $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is weakly (locally) cartesian closed. Most of the results of this chapter can be found in [CHR20].

## 3 Injectivity and exponentiability

Following the techniques of [Hof13], we present conditions under which every injective space is exponentiable in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$.

Firstly, $\mathsf{V}$-$\mathsf{Cat}$ is a monoidal closed category for the tensor defined in (I.35) [Law73]. Thus, when $\otimes = \wedge$, $\mathsf{V}$-$\mathsf{Cat}$ is a cartesian closed category. This is the case of $\mathsf{Ord}$, $\mathsf{UltMet}$, and $\mathsf{BiRel}$. Secondly, from [HR13, Theorem 5.3], the following condition on the quantale $\mathsf{V}$:

$$\forall u,v,w \in \mathsf{V},\ w \wedge (u \otimes v) = \bigvee \{u' \otimes v' \mid u' \leq u,\ v' \leq v,\ u' \otimes v' \leq w\}, \tag{II.1}$$

is equivalent to exponentiability in $\mathsf{V}$-$\mathsf{Cat}$ of every injective $\mathsf{V}$-space. The quantales $\mathsf{P}_+$, $\mathsf{P}_1$, and $\Delta$ satisfy condition (II.1), whence injective spaces are exponentiable in the categories $\mathsf{Met}$, $\mathsf{BMet}$, and $\mathsf{ProbMet}$.

Since we are looking for a general result for $(\mathbb{T},\mathsf{V})$-spaces of injectivity implying exponentiability, condition (II.1) must be one of our hypothesis. Let us provide more background and fix some notation.

Each $(\mathbb{T},\mathsf{V})$-space $(X,a)$ induces a continuous map $a\colon X^{\mathrm{op}}\otimes X\to\mathsf{V}$, so we consider the composite

$$X^{\mathrm{op}}\otimes X\otimes\mathsf{V}\xrightarrow{a\otimes 1_{\mathsf{V}}}\mathsf{V}\otimes\mathsf{V}\xrightarrow{\otimes}\mathsf{V}, \tag{II.2}$$

which is a continuous map, since $\otimes\colon\mathsf{V}\times\mathsf{V}\to\mathsf{V}$ is continuous [Hof11, Proposition 1.4(1)]. From $\otimes$-exponentiability of $X^{\mathrm{op}}$, (II.2) induces a continuous map $\tilde{a}\colon X\otimes\mathsf{V}\to PX$. For each $(x,u)\in X\times\mathsf{V}$, $\mathfrak{x}\in TX$,

$$\tilde{a}(x,u)(\mathfrak{x})=a(\mathfrak{x},x)\otimes u. \tag{II.3}$$

Now let us consider that $(X,a)$ is an injective $(\mathbb{T},\mathsf{V})$-space. Then the Yoneda embedding $y_X\colon X\to PX$ has a left adjoint $\mathrm{Sup}_X\colon PX\to X$. Define the continuous map $\oplus\colon X\otimes\mathsf{V}\to X$ as the composite

$$X\otimes\mathsf{V}\xrightarrow{\tilde{a}}PX\xrightarrow{\mathrm{Sup}_X}X,\qquad \oplus=\mathrm{Sup}_X\cdot\tilde{a} \tag{II.4}$$

and, for each $(x,u)\in X\times\mathsf{V}$, we use the notation $x\oplus u=\oplus(x,u)$. For a fixed element $u$ of $\mathsf{V}$, consider the composite

$$X\xrightarrow{(-,u)}X\otimes\mathsf{V}\xrightarrow{\tilde{a}}PX\xrightarrow{\mathrm{Sup}_X}X,\qquad -\oplus u \tag{II.5}$$

which we denote by $-\oplus u\colon X\to X$, where $(-,u)\colon x\mapsto(x,u)$.

**Lemma 3.0.1** *Let $u\in\mathsf{V}$ be such that the diagram*

$$\begin{array}{ccc} T1 & \xrightarrow{Tu} & T\mathsf{V} \\ {\scriptstyle !}\downarrow & \geq & \downarrow{\scriptstyle \xi} \\ 1 & \xrightarrow[u]{} & \mathsf{V} \end{array}$$

*is lax commutative, where $1=\{*\}$ and $u\colon 1\to\mathsf{V}$, $*\mapsto u$. Then, for every $(\mathbb{T},\mathsf{V})$-space $(X,a)$, the map $(-,u)\colon X\mapsto X\otimes\mathsf{V}$ is $(\mathbb{T},\mathsf{V})$-continuous.*

*Proof.* Consider the maps $!_X\colon X\to 1$ and $T!_X\colon TX\to T1$. By hypothesis, for each $\mathfrak{x}\in TX$,

$$\xi\cdot T(u\cdot !_X)(\mathfrak{x})=\xi\cdot Tu(T!_X(\mathfrak{x}))\leq u\cdot !(T!_X(\mathfrak{x}))=u.$$

We have the equivalences:

$$\xi \cdot T(u\cdot !_X)(\mathfrak{x}) \leq u \iff k \otimes \xi \cdot T(u\cdot !_X)(\mathfrak{x}) \leq u \iff k \leq \hom_\xi(T(u\cdot !_X)(\mathfrak{x}), u)$$

Then, for each $\mathfrak{x} \in TX$, $x \in X$,

$$\begin{aligned} a \otimes \hom_\xi(T(-,u)(\mathfrak{x}),(x,u)) &= a(T\pi_X \cdot T(-,u)(\mathfrak{x}),x) \otimes \hom_\xi(T\pi_{\mathsf{V}} \cdot T(-,u)(\mathfrak{x}),u) \\ &= a(\mathfrak{x},x) \otimes \hom_\xi(T(u\cdot !_X)(\mathfrak{x}),u) \\ &\geq a(\mathfrak{x},x) \otimes k \;=\; a(\mathfrak{x},x), \end{aligned}$$

where $\pi_X$ and $\pi_{\mathsf{V}}$ are the product projections from $X \times \mathsf{V}$ into $X$ and $\mathsf{V}$, respectively. □

Therefore, under the conditions of this lemma, the map $-\oplus u$ is continuous. For each $\mathfrak{x} \in TX$, let us denote $T(-\oplus u)(\mathfrak{x}) = \mathfrak{x} \oplus u$. Then continuity of $-\oplus u\colon (X,a) \to (X,a)$ is expressed as:

$$\forall \mathfrak{x} \in TX,\ \forall x \in X,\ a(\mathfrak{x},x) \leq a(\mathfrak{x} \oplus u, x \oplus u). \tag{II.6}$$

Let us set for each $r\colon X \relbar\joinrel\mapsto Y$ and each element $u \in V$, the $\mathsf{V}$-relation $r \otimes u\colon X \relbar\joinrel\mapsto Y$ given by, for each $(x,y) \in X \times Y$, $r \otimes u(x,y) = r(x,y) \otimes u$. The following condition is to be used:

$$\forall r\colon X \relbar\joinrel\mapsto Y,\ \forall u \in \mathsf{V},\ T(r \otimes u) = Tr \otimes u. \tag{II.7}$$

**Lemma 3.0.2** *Let $u \in \mathsf{V}$ be such that the diagram*

$$\begin{array}{ccc} T1 & \xrightarrow{Tu} & T\mathsf{V} \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \xi} \\ 1 & \xrightarrow[u]{} & \mathsf{V} \end{array} \tag{II.8}$$

*is commutative. Then, for every $\mathsf{V}$-relation $r\colon X \relbar\joinrel\mapsto Y$, $T(r \otimes u) = Tr \otimes u$.*

*Proof.* By (I.17), $T(r \otimes u)(\mathfrak{x},\mathfrak{y}) = \bigvee\{\xi \cdot T\overrightarrow{r \otimes u}(\mathfrak{w}) \mid \mathfrak{w} \in T(X \times Y),\ T\pi_X(\mathfrak{w}) = \mathfrak{x},\ T\pi_Y(\mathfrak{w}) = \mathfrak{y}\}$. Moreover, the map $\overrightarrow{r \otimes u}$ is equal to the following composite, where $!_{\mathsf{V}}\colon \mathsf{V} \to 1$,

$$X \times Y \xrightarrow{\ \vec{r}\ } \mathsf{V} \xrightarrow{\langle 1_{\mathsf{V}}, u\cdot !_{\mathsf{V}}\rangle} \mathsf{V} \times \mathsf{V} \xrightarrow{\ \otimes\ } \mathsf{V}.$$

Hence

$$\begin{array}{rcll} \xi\cdot T\overrightarrow{r\otimes u} & = & \xi\cdot T(\otimes)\cdot T(\langle 1_{\mathsf{V}},u\cdot !_{\mathsf{V}}\rangle)\cdot T\vec{r} & \\ & = & \otimes\cdot\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle\cdot T(\langle 1_{\mathsf{V}},u\cdot !_{\mathsf{V}}\rangle)\cdot T\vec{r} & \text{(by (I.16))} \\ & \stackrel{*}{=} & \otimes\cdot\langle\xi,u\cdot !_{\mathsf{V}}\cdot\xi\rangle\cdot T\vec{r} & \\ & = & \otimes\cdot\langle 1_{\mathsf{V}},u\cdot !_{\mathsf{V}}\rangle\cdot\xi\cdot T\vec{r}, & \end{array}$$

so that, for each $\mathfrak{w}\in T(X\times Y)$, $\xi\cdot T\overrightarrow{r\otimes u}(\mathfrak{w})=\xi\cdot T\vec{r}(\mathfrak{w})\otimes u$; equality $\stackrel{*}{=}$ follows from

$$\pi_1\cdot\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle\cdot T(\langle 1_{\mathsf{V}},u\cdot !_{\mathsf{V}}\rangle)=\xi\cdot T\pi_1\cdot T(\langle 1_{\mathsf{V}},u\cdot !_{\mathsf{V}}\rangle)=\xi\cdot T1_{\mathsf{V}}=\xi,$$

and

$$\begin{array}{rcll} \pi_2\cdot\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle\cdot T(\langle 1_{\mathsf{V}},u\cdot !_{\mathsf{V}}\rangle) & = & \xi\cdot T\pi_2\cdot T(\langle 1_{\mathsf{V}},u\cdot !_{\mathsf{V}}\rangle) & \\ & = & \xi\cdot Tu\cdot T!_{\mathsf{V}} & \\ & = & u\cdot !\cdot T!_{\mathsf{V}} & \text{(by hypothesis)} \\ & = & u\cdot !\cdot\xi, & \end{array}$$

where $\pi_1,\pi_2$ are the first and second product projections $\mathsf{V}\times\mathsf{V}\to\mathsf{V}$, respectively. Since $\otimes$ distributes over arbitrary suprema, by (I.17), we conclude the result. $\square$

**Remark 3.0.3** Let us verify that, for the categories of Table (I.25), diagram (II.8) is commutative, for every $u\in\mathsf{V}$. If $T1=1$, then $T1=\{e_1(*)\}$, hence $\xi\cdot Tu(e_1(*))=\xi\cdot e_{\mathsf{V}}(u(*))=u=u\cdot !(e_1(*))$, and $\xi\cdot Tu=u\cdot !$. For the monad $\mathbb{L}$ and any frame $\mathsf{V}$ seen as quantale, commutativity follows directly from the diagram:

$$\begin{array}{ccc} L1 & \xrightarrow{Lu} & L\mathsf{V} \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \xi} \\ 1 & \xrightarrow[u]{} & \mathsf{V} \end{array} \qquad ☞ \qquad \begin{array}{ccc} (*,\dots,*) & \longmapsto & (u,\dots,u) \\ \downarrow & & \downarrow \\ * & \longmapsto & u=u\wedge\dots\wedge u, \end{array}$$

as well as for the monad $\mathbb{M}$ and any quantale $\mathsf{V}$:

$$\begin{array}{ccc} 1\times M & \xrightarrow{u\times 1_M} & \mathsf{V}\times M \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \pi_{\mathsf{V}}} \\ 1 & \xrightarrow[u]{} & \mathsf{V} \end{array} \qquad ☞ \qquad \begin{array}{ccc} (*,b) & \longmapsto & (u,b) \\ \downarrow & & \downarrow \\ * & \longmapsto & u. \end{array}$$

We are in conditions of proving our main result, and in order to make the proof more direct we demonstrate the following auxiliary result:

**Lemma 3.0.4** *Let $(X,a)$ be an injective $(\mathbb{T},\mathsf{V})$-space, with $a = a_0 \cdot \alpha$ as in (I.33), where $(X,a_0)$ is a $\mathsf{V}$-space, and $\alpha\colon TX \to X$ is a map. The following assertions hold, for every $x,x' \in X$, $\mathfrak{x} \in TX$, and $u \in \mathsf{V}$:*

*(1) $a_0(x \oplus u, x') = \hom(u, a_0(x,x'))$;*

*(2) $a_0(x, x' \oplus u) \geq a_0(x,x') \otimes u$;*

*(3) $a(\mathfrak{x} \oplus u, x) \geq \hom(u, a(\mathfrak{x},x))$;*

*(4) $a(\mathfrak{x}, x \oplus u) \geq a(\mathfrak{x},x) \otimes u$.*

*Moreover, if (II.7) is satisfied, then, for every $\mathfrak{X} \in T^2X$,*

*(5) $Ta(\mathfrak{X}, \mathfrak{x} \oplus u) \geq Ta(\mathfrak{X},\mathfrak{x}) \otimes u$.*

*Proof.* (1) For each $x,x' \in X$, and each $u \in \mathsf{V}$,

$$\begin{aligned}
a_0(x \oplus u, x') &= a_0(\mathrm{Sup}_X \cdot \tilde{a}(x,u), x') && \text{(by definition of } \oplus \text{ (II.4))}\\
&= [\tilde{a}(x,u), y_X(x')] && \text{(because } \mathrm{Sup}_X \dashv y_X \text{ (I.29))}\\
&= \bigwedge_{\mathfrak{y}\in TX} \hom(\tilde{a}(x,u)(\mathfrak{y}), y_X(x')(\mathfrak{y})) && \text{(by definition of the } \mathsf{V}\text{-structure } [\ ,\ ] \text{ [Law73])}\\
&= \bigwedge_{\mathfrak{y}\in TX} \hom(a(\mathfrak{y},x)\otimes u, a(\mathfrak{y},x')) && \text{(by (II.3) and Definition 2.7.9)}\\
&\stackrel{*}{=} \hom(u, a_0(x,x'));
\end{aligned}$$

$\stackrel{*}{=}$ follows from:

$$\begin{aligned}
a(\mathfrak{y},x)\otimes u \otimes \hom(u, a_0(x,x')) &\leq a(\mathfrak{y},x)\otimes a_0(x,x') \;=\; a_0(\alpha(\mathfrak{y}),x)\otimes a_0(x,x')\\
&\leq a_0(\alpha(\mathfrak{y}),x') = a(\mathfrak{y},x'),
\end{aligned}$$

hence, for all $\mathfrak{y} \in TX$, $\hom(u, a_0(x,x')) \leq \hom(a(\mathfrak{y},x)\otimes u, a(\mathfrak{y},x'))$, so that

$$\hom(u, a_0(x,x')) \leq \bigwedge_{\mathfrak{y}\in TX} \hom(a(\mathfrak{y},x)\otimes u, a(\mathfrak{y},x'));$$

and we just observe that, for $\mathfrak{y} = e_X(x)$, by reflexivity of $a$, $a(e_X(x),x) \otimes u \geq k \otimes u = u$, whence, because $\alpha \cdot e_X \simeq 1_X$ (I.32),

$$\begin{aligned}
\hom(a(\mathfrak{y},x)\otimes u, a(\mathfrak{y},x')) &= \hom(a(e_X(x),x)\otimes u, a(e_X(x),x')) \;\leq\; \hom(u, a_0(\alpha\cdot e_X(x),x'))\\
&= \hom(u, a_0(x,x')).
\end{aligned}$$

(2) Since the map $-\oplus u\colon (X,a)\to(X,a)$ is $(\mathbb{T},\mathsf{V})$-continuous, it is a $\mathsf{V}$-continuous map $-\oplus u\colon (X,a_0)\to(X,a_0)$, whence, by item (1), for each $x,x'\in X$, $u\in\mathsf{V}$,

$$a_0(x,x')\otimes u = u\otimes a_0(x,x')\le u\otimes a_0(x\oplus u,x'\oplus u) = u\otimes \hom(u,a_0(x,x'\oplus u))\le a_0(x,x'\oplus u).$$

(3) For each $\mathfrak{x}\in TX$, $x\in X$, $u\in\mathsf{V}$,

$$\begin{array}{rcll}
a(\mathfrak{x}\oplus u,x) & = & a_0(\alpha(\mathfrak{x}\oplus u),x) & \\
 & \ge & a_0(\alpha(\mathfrak{x}\oplus u),\alpha(\mathfrak{x})\oplus u)\otimes a_0(\alpha(\mathfrak{x})\oplus u,x) & \text{(by transitivity of } a_0) \\
 & = & a(\mathfrak{x}\oplus u,\alpha(\mathfrak{x})\oplus u)\otimes a_0(\alpha(\mathfrak{x})\oplus u,x) & \\
 & \ge & a(\mathfrak{x},\alpha(\mathfrak{x}))\otimes a_0(\alpha(\mathfrak{x})\oplus u,x) & \text{(by continuity of } -\oplus u) \\
 & = & a_0(\alpha(\mathfrak{x}),\alpha(\mathfrak{x}))\otimes a_0(\alpha(\mathfrak{x})\oplus u,x) & \\
 & \ge & k\otimes a_0(\alpha(\mathfrak{x})\oplus u,x) \;=\; a_0(\alpha(\mathfrak{x})\oplus u,x) & \text{(by reflexivity of } a_0) \\
 & = & \hom(u,a_0(\alpha(\mathfrak{x}),x)) \;=\; \hom(u,a(\mathfrak{x},x)) & \text{(by item (1)).}
\end{array}$$

(4) By item (2), for each $\mathfrak{x}\in TX$, $x\in X$, $u\in\mathsf{V}$,

$$a(\mathfrak{x},x\oplus u) = a_0(\alpha(\mathfrak{x}),x\oplus u)\ge a_0(\alpha(\mathfrak{x}),x)\otimes u = a(\mathfrak{x},x)\otimes u.$$

(5) Item (4) can be expressed as $a\otimes u\le(-\oplus u)^\circ\cdot a$, hence applying $T$, by (II.7), we obtain

$$Ta\otimes u = T(a\otimes u)\le T((-\oplus u)^\circ\cdot a) = T(-\oplus u)^\circ\cdot Ta,$$

that is, for all $\mathfrak{X}\in T^2X$, $\mathfrak{x}\in TX$, $Ta(\mathfrak{X},\mathfrak{x})\otimes u\le Ta(\mathfrak{X},\mathfrak{x}\oplus u)$. $\square$

**Theorem 3.0.5** *Assume that, for each $u\in\mathsf{V}$, the diagrams*

$$\begin{array}{ccc}
T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\wedge)} & T\mathsf{V} \\
{\scriptstyle\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle}\downarrow & \le & \downarrow{\scriptstyle\xi} \\
\mathsf{V}\times\mathsf{V} & \xrightarrow[\wedge]{} & \mathsf{V}
\end{array}
\qquad\qquad
\begin{array}{ccc}
T1 & \xrightarrow{Tu} & T\mathsf{V} \\
{\scriptstyle !}\downarrow & & \downarrow{\scriptstyle\xi} \\
1 & \xrightarrow[u]{} & \mathsf{V}
\end{array}$$

*are (lax) commutative and that $\mathsf{V}$ satisfies condition (II.1):*

$$\forall u,v,w\in\mathsf{V},\ w\wedge(u\otimes v)=\bigvee\{u'\otimes v'\mid u'\le u,\ v'\le v,\ u'\otimes v'\le w\}.$$

*Then every injective $(\mathbb{T},\mathsf{V})$-space is exponentiable in $(\mathbb{T},\mathsf{V})$-Cat.*

*Proof.* Let $(X,a)$ be an injective $(\mathbb{T},\mathsf{V})$-space with $a=a_0\cdot\alpha$ as in (I.33). By Lemma 2.6.1 and Theorem 2.6.3, it suffices to verify that, for each $\mathfrak{X}\in T^2X$, $x\in X$, $u,v\in\mathsf{V}$,

$$\bigvee_{\mathfrak{x}\in TX}(Ta(\mathfrak{X},\mathfrak{x})\wedge u)\otimes(a(\mathfrak{x},x)\wedge v)\geq a(m_X(\mathfrak{X}),x)\wedge(u\otimes v).$$

By condition (II.1),

$$a(m_X(\mathfrak{X}),x)\wedge(u\otimes v)=\bigvee\{u'\otimes v'\mid u'\leq u,\ v'\leq v,\ u'\otimes v'\leq a(m_X(\mathfrak{X}),x)\};$$

let us consider $u',v'\in\mathsf{V}$ such that $u'\leq u$, $v'\leq v$, and $u'\otimes v'\leq a(m_X(\mathfrak{X}),x)$. Fix $\mathfrak{y}=T\alpha(\mathfrak{X})\oplus u'\in TX$; then

$$\begin{aligned}
Ta(\mathfrak{X},\mathfrak{y})\wedge u &= Ta(\mathfrak{X},T\alpha(\mathfrak{X})\oplus u')\wedge u\\
&\geq (Ta(\mathfrak{X},T\alpha(\mathfrak{X}))\otimes u')\wedge u && \text{(by Lemma 3.0.4 (5))}\\
&= (Ta_0(T\alpha(\mathfrak{X}),T\alpha(\mathfrak{X}))\otimes u')\wedge u\\
&\geq (k\otimes u')\wedge u\ =\ u' && \text{(because } Ta_0 \text{ is reflexive)}
\end{aligned}$$

and

$$\begin{aligned}
a(\mathfrak{y},x)\wedge v &= a(T\alpha(\mathfrak{X})\oplus u',x)\wedge v\\
&\geq \hom(u',a(T\alpha(\mathfrak{X}),x))\wedge v && \text{(by Lemma 3.0.4 (3))}\\
&= \hom(u',a_0(\alpha\cdot T\alpha(\mathfrak{X}),x))\wedge v\\
&= \hom(u',a_0(\alpha\cdot m_X(\mathfrak{X}),x))\wedge v && \text{(because } \alpha\cdot T\alpha\cong\alpha\cdot m_X \text{ (I.32))}\\
&= \hom(u',a(m_X(\mathfrak{X}),x))\wedge v.
\end{aligned}$$

Thus

$$(Ta(\mathfrak{X},\mathfrak{y})\wedge u)\otimes(a(\mathfrak{y},x)\wedge v)\geq u'\otimes(\hom(u',a(m_X(\mathfrak{X}),x))\wedge v).$$

From $v'\leq v$ and $u'\otimes v'\leq a(m_X(\mathfrak{X}),x)$ if, and only if, $v'\leq\hom(u',a(m_X(\mathfrak{X}),x))$, we get $v'\leq\hom(u',a(m_X(\mathfrak{X}),x))\wedge v$, whence $(Ta(\mathfrak{X},\mathfrak{y})\wedge u)\otimes(a(\mathfrak{y},x)\wedge v)\geq u'\otimes v'$. □

**Remark 3.0.6** By Remark 2.6.2 and Remark 3.0.3, and since the quantales of Examples 2.1.6 satisfy condition (II.1), we conclude that in the categories of Table (I.25) injective spaces are exponentiable.

## 4 $(\mathbb{T},\mathsf{V})$-Cat is weakly (locally) cartesian closed

We start by investigating the *weak cartesian closedness* of $(\mathbb{T},\mathsf{V})$-Cat, generalising the results of [Ros99] for Top. Let C be a category with finite products.

**Definition 4.0.1** For objects $X,Y$ of $\mathsf{C}$, a *weak exponential* with base $Y$ and exponent $X$ consists of an object $\ll X,Y \gg$ and a morphism $\mathrm{ev}_{X,Y}\colon \ll X,Y \gg \times X \to Y$, the *evaluation morphism*, such that every morphism $f\colon Z\times X \to Y$ of $\mathsf{C}$ factors, not necessarily in a unique way, as $\mathrm{ev}_{X,Y}\cdot(\overline{f}\times 1_X) = f$, for some $\overline{f}\colon Z \to \ll X,Y \gg$, a *transpose* of $f$.

$$\begin{array}{ccc} \ll X,Y \gg & \qquad & \ll X,Y \gg \times X \xrightarrow{\ \mathrm{ev}_{X,Y}\ } Y \\ \uparrow{\scriptstyle \exists\,\overline{f}} & & \uparrow{\scriptstyle \overline{f}\times 1_X} \nearrow{\scriptstyle f} \\ Z & & Z\times X \end{array}$$

$\mathsf{C}$ is *weakly cartesian closed* if, for all objects $X,Y$ of $\mathsf{C}$, there exists a weak exponential with base $Y$ and exponent $X$.

Recall from Proposition 2.7.3 that the subcategory $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathrm{sep}}$ of separated $(\mathbb{T},\mathsf{V})$-spaces is fully reflective in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, and each reflection $\eta_X\colon (X,a)\to(X/{\sim},\tilde{a})$ is $|\text{-}|$-initial, where $\tilde{a} = \eta_X\cdot a\cdot(T\eta_X)^\circ$ and $|\text{-}|\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to\mathsf{Set}$ is the forgetful functor. As outlined in [Ros99], we first prove the following:

**Proposition 4.0.2** *The reflector* $R\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{sep}}$ *preserves finite products.*

*Proof.* The terminal object $(1,\top)$ of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is separated. Let $(X,a)$, $(Y,b)$ be $(\mathbb{T},\mathsf{V})$-spaces, and for simplicity let us denote $R(X,a)=(RX,\tilde{a})$ and $R(Y,b)=(RY,\tilde{b})$. Since $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathrm{sep}}$ is closed under limits, the product $(RX\times RY,d)$ is separated, with $d=\tilde{a}\times\tilde{b}$. Then the morphism $\eta_X\times\eta_Y\colon (X\times Y,c)\to(RX\times RY,d)$, $c=a\times b$, factors uniquely through the reflection $\eta_{X,Y}$, that is, there exists a unique morphism $t\colon R(X\times Y)\to RX\times RY$ such that $t\cdot\eta_{X\times Y}=\eta_X\times\eta_Y$.

$$\begin{array}{ccc} (X\times Y,c) & \xrightarrow{\ \eta_{X\times Y}\ } & (R(X\times Y),\tilde{c}) \\ & {\scriptstyle \eta_X\times\eta_Y}\searrow & \downarrow{\scriptstyle t} \\ & & (RX\times RY,d) \end{array} \tag{II.9}$$

Then $t$ is a bijection: for each $([x],[y])\in RX\times RY$, $t[(x,y)] = t\cdot\eta_{X\times Y}(x,y) = \eta_X\times\eta_Y(x,y) = ([x],[y])$, with $[(x,y)]\in R(X\times Y)$, so $t$ is surjective; if $t[(x,y)]=t[(x',y')]$, for $[(x,y)],[(x',y')]\in R(X\times Y)$, then $([x],[y])=([x'],[y'])$ is equivalent to $x\sim x'$ and $y\sim y'$, hence

$$c(e_{X\times Y}(x,y),(x',y')) = a(T\pi_X\cdot e_{X\times Y}(x,y),x')\wedge b(T\pi_Y\cdot e_{X\times Y}(x,y),y') = a(e_X(x),x')\wedge b(e_Y(y),y')\geq k,$$

that is, $(x,y)\leq(x',y')$, and, by the same argument, $(x',y')\leq(x,y)$, i.e., $(x,y)\sim(x',y')$, so $t$ is injective. Next we prove that $t$ is $|\text{-}|$-initial. Firstly, since $\eta_X$ and $\eta_Y$ are $|\text{-}|$-initial, so is the morphism

$\eta_X \times \eta_Y : X \times Y \to RX \times RY$. Secondly, assuming the Axiom of Choice, so that the functor $T$ preserves surjections, we have for each $\mathfrak{z} \in T(R(X \times Y))$, $(x,y) \in X \times Y$,

$$\begin{aligned}
\tilde{c}(\mathfrak{z},[(x,y)]) &= \eta_{X\times Y}\cdot c\cdot (T\eta_{X\times Y})^\circ(\mathfrak{z},[(x,y)]) && \text{(by definition of } \tilde{c})\\
&= \bigvee_{\substack{T\eta_{X\times Y}(\mathfrak{w})=\mathfrak{z}\\ \eta_{X\times Y}(z,w)=[(x,y)]}} c(\mathfrak{w},(z,w))\\
&= \bigvee_{\substack{T\eta_{X\times Y}(\mathfrak{w})=\mathfrak{z}\\ \eta_{X\times Y}(z,w)=[(x,y)]}} d(T(\eta_X\times\eta_Y)(\mathfrak{w}),\eta_X\times\eta_Y(z,w)) && \text{(because } \eta_X\times\eta_Y \text{ is } |\text{-}|\text{-initial)}\\
&= \bigvee_{\substack{T\eta_{X\times Y}(\mathfrak{w})=\mathfrak{z}\\ \eta_{X\times Y}(z,w)=[(x,y)]}} d(Tt\cdot T\eta_{X\times Y}(\mathfrak{w}),t\cdot\eta_{X\times Y}(z,w)) && \text{(by (II.9))}\\
&= d(Tt(\mathfrak{z}),t[(x,y)]).
\end{aligned}$$

Since $t$ is $|\text{-}|$-initial and a bijection, it is an isomorphism. □

Therefore, by [Sch84, Theorem 1.2], we conclude that the existing exponentials of separated $(\mathbb{T},\mathsf{V})$-spaces are separated. This fact is to be used in the next result.

**Theorem 4.0.3** *If every injective $(\mathbb{T},\mathsf{V})$-space is exponentiable, then $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathsf{sep}}$ is weakly cartesian closed.*

*Proof.* Let $(X,a),(Y,b)$ be separated $(\mathbb{T},\mathsf{V})$-spaces, and consider the Yoneda embeddings $y_X : X \to PX$ and $y_Y : Y \to PY$. Since $PX$ and $PY$ are injective and separated, by hypothesis, they are exponentiable, and we can form the exponential $PY^{PX} = \langle PX,PY\rangle$ which is again separated. The underlying set of $\langle PX,PY\rangle$ consists of all $(\mathbb{T},\mathsf{V})$-continuous maps from $PX \times (1,e_1^\circ)$ to $PY$, and the evaluation map is given by $\mathrm{ev}\colon \langle PX,PY\rangle \times PX \to PY$, $(\varphi,\mathfrak{w}) \mapsto \varphi(\mathfrak{w})$, where the set $PX \times (1,e_1^\circ)$ is identified with $PX$. Let us define

$$\ll X,Y \gg = \{\varphi\colon PX \times (1,e_1^\circ) \to PY \mid \varphi(y_X(X)) \subseteq y_Y(Y)\},$$

and endow this set with the initial $(\mathbb{T},\mathsf{V})$-structure with respect to the inclusion map $i_{X,Y}\colon \ll X,Y \gg \hookrightarrow \langle PX,PY\rangle$. Since $y_Y : Y \to PY$ is an injective map, there exists a unique map $\widetilde{\mathrm{ev}}\colon \ll X,Y \gg \times X \to Y$ such that the composite

$$\ll X,Y \gg \times X \xrightarrow{i_{X,Y}\times y_X} \langle PX,PY\rangle \times PX \xrightarrow{\mathrm{ev}} PY$$

factors through the Yoneda embedding $y_Y : Y \to PY$, that is, such that the diagram

$$\begin{array}{ccc} \ll X,Y \gg \times X & \xrightarrow{\widetilde{\mathrm{ev}}} & Y \\ {\scriptstyle i_{X,Y}\times y_X}\big\downarrow & & \big\downarrow{\scriptstyle y_Y} \\ \langle PX,PY\rangle \times PX & \xrightarrow[\mathrm{ev}]{} & PY \end{array}$$

is commutative. Let us verify that this defines a weak exponential in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathrm{sep}}$. For each separated $(\mathbb{T},\mathsf{V})$-space $(Z,c)$, and each $(\mathbb{T},\mathsf{V})$-continuous map $f\colon Z\times X \to Y$, since $PY$ is injective and separated, there exists an extension $f'\colon Z\times PX \to PY$ of $y_Y\cdot f\colon Z\times X\to PY$ along the embedding $1_Z\times y_X : Z\times X \to Z\times PX$, so that the diagram

$$\begin{array}{ccc} Z\times X & \xrightarrow{1_Z\times y_X} & Z\times PX \\ & {\scriptstyle y_Y\cdot f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & PY & \end{array}$$

is commutative. Factorize $f'\colon Z\times PX\to PY$ through the universal map $\mathrm{ev}\colon \langle PX,PY\rangle\times PX \to PY$:

$$\begin{array}{ccc} \langle PX,PY\rangle \times PX & \xrightarrow{\mathrm{ev}} & PY. \\ {\scriptstyle \overline{f}\times 1_{PX}}\big\uparrow & \nearrow{\scriptstyle f'} & \\ Z\times PX & & \end{array}$$

For each $z\in Z$, $\overline{f}(z)\colon PX\to PY$ is such that, for each $x\in X$,

$$\overline{f}(z)(y_X(x)) = \mathrm{ev}(\overline{f}(z),y_X(x)) = f'(z,y_X(x)) = y_Y(f(z,x)),$$

that is, $\overline{f}(z)(y_X(X))\subseteq y_Y(Y)$, whence $\overline{f}(z)\in\ll X,Y\gg$. Hence the map $\overline{f}$ corestricts to a map $\tilde{f}\colon Z\to\ll X,Y\gg$, which is continuous, since $\ll X,Y\gg$ has the initial structure with respect to the inclusion map $i_{X,Y}$. Then the following diagram is commutative.

$$\begin{array}{ccc} \ll X,Y \gg \times X & \xrightarrow{\widetilde{\mathrm{ev}}} & Y \\ {\scriptstyle \tilde{f}\times 1_X}\big\uparrow & \nearrow{\scriptstyle f} & \\ Z\times X & & \end{array}$$

□

**Theorem 4.0.4** *If* $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathrm{sep}}$ *is weakly cartesian closed, then so is* $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$.

*Proof.* Let $(X,a),(Y,b),(Z,c)$ be spaces, and, for each continuous map $f\colon (Z,c)\times(X,a)\to(Y,b)$, consider its image $Rf\colon RZ\times RX\cong R(Z\times X)\to RY$ by the reflector $R\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{sep}}$. By hypothesis, $Rf$ factorizes through the weak evaluation as $Rf=\widetilde{\mathrm{ev}}\cdot(\overline{Rf}\times 1_{RX})$ in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{sep}}$:

$$\ll RX,RY\gg\times RX \xrightarrow{\ \widetilde{\mathrm{ev}}\ } RY,\qquad RZ\times RX\xrightarrow{\ \overline{Rf}\times 1_{RX}\ }\ll RX,RY\gg\times RX,\qquad RZ\times RX\xrightarrow{\ Rf\ }RY.$$

Define $Z_f=Z/{\sim}$, where, for each $z,z'\in Z$,

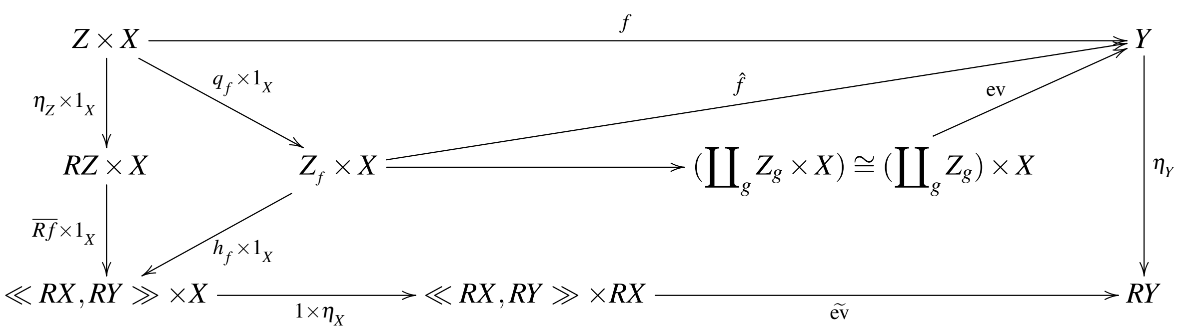
$$z\sim z' \iff \forall x\in X, f(z,x)=f(z',x)\ \&\ \overline{Rf}(\eta_Z(z))=\overline{Rf}(\eta_Z(z')),$$

and $\eta_Z\colon Z\to RZ$ denotes the reflection. Endow the set $Z_f$ with the final $(\mathbb{T},\mathsf{V})$-structure with respect to the projection map $q_f\colon Z\to Z_f$; we have induced maps

$$\begin{array}{rcl} h_f\colon Z_f & \longrightarrow & \ll RX,RY\gg \\ {[z]} & \longmapsto & \overline{Rf}(\eta_Z(z)) \end{array}\qquad \&\qquad \begin{array}{rcl} \hat{f}\colon Z_f\times X & \longrightarrow & Y \\ ([z],x) & \longmapsto & f(z,x), \end{array}$$

which are well-defined by definition of $\sim$. Composing $h_f$ and $\hat{f}$ with $q_f$ and $\eta_Y$, respectively, we obtain $h_f\cdot q_f=\overline{Rf}\cdot\eta_Z$ and $\eta_Y\cdot\hat{f}=\widetilde{\mathrm{ev}}\cdot(h_f\times\eta_X)$, and since $q_f$ is a final morphism and $\eta_Y$ is an initial one, we conclude that $h_f$ and $\hat{f}$ are continuous maps.

The cardinality of $Z_f$ is bounded by the cardinality of the set $|\ll RX,RY\gg|\times|Y|^{|X|}$, since one can define an injective map $Z_f\to|\ll RX,RY\gg|\times|Y|^{|X|}$, $[z]\mapsto(\overline{Rf}(\eta_Z(z)),f(z,-))$, and therefore there is only a set of possible $(\mathbb{T},\mathsf{V})$-spaces $Z_f$. We form the coproduct $\coprod_g Z_g$, and, by extensivity of $(\mathbb{T},\mathsf{V})$-Cat, the morphisms $\hat{g}\colon Z_g\times X\to Y$ induce the $(\mathbb{T},\mathsf{V})$-continuous map $\mathrm{ev}\colon(\coprod_g Z_g)\times X\cong\coprod_g(Z_g\times X)\to Y$. The weak exponential is given by $\ll X,Y\gg=\coprod_g Z_g$ and by the evaluation map ev just defined.




$\square$

Let us proceed with *weak local cartesian closedness* of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$. For each space $(X,a)$, the *slice category* $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}/(X,a)$ has as objects continuous maps $f\colon (Y,b)\to(X,a)$, $(Y,b)\in(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, and a morphism $h\colon f\to g$ in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}/(X,a)$, with $g\colon (Z,c)\to(X,a)$ a continuous map, is a continuous map $h\colon (Y,b)\to(Z,c)$ such that the following triangle is commutative:

$$\begin{array}{ccc} (Y,b) & \xrightarrow{\quad h \quad} & (Z,c) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & (X,a). & \end{array}$$

Naturally, $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is *weakly locally cartesian closed* if, for every space $(X,a)$, the slice category $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}/(X,a)$ is weakly cartesian closed. In order to prove that $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ satisfies this property, we use an auxiliary category that is described in [CR00]. Let us denote the full subcategory of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ of separated and injective spaces by $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\text{sep,inj}}$.

**Definition 4.0.5** The category $\mathscr{F}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{sep,inj}})$ has as objects triples $((X,a),A,\sigma\colon A\to X)$, where $(X,a)$ is a separated injective $(\mathbb{T},\mathsf{V})$-space, $A$ is a set, and $\sigma$ is a function; a morphism

$$f\colon ((X,a),A,\sigma\colon A\to X)\to((Y,b),B,\sigma'\colon B\to Y)$$

is a map $f\colon A\to B$ such that there exists a $(\mathbb{T},\mathsf{V})$-continuous map $g\colon (X,a)\to(Y,b)$ making the square below commutative.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ X & \xrightarrow[g]{} & Y \end{array}$$

Following the techniques of [CR00, BCRS98], we prove:

**Proposition 4.0.6** *If injective $(\mathbb{T},\mathsf{V})$-spaces are exponentiable in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, then $\mathscr{F}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{sep,inj}})$ is weakly locally cartesian closed.*

*Proof.* For an object $((X,a),A,\sigma\colon A\to X)$, we prove that $\mathscr{F}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{sep,inj}})/((X,a),A,\sigma)$ is a weakly cartesian closed category. Consider the objects $f\colon ((Y,b),B,\beta\colon B\to Y)\to((X,a),A,\sigma)$ and $g\colon ((Z,c),C,\gamma\colon C\to Z)\to((X,a),A,\sigma)$ of the slice category. Since $\mathsf{Set}/A$ is a cartesian closed category, we can form the exponential with base $g\colon C\to A$ and exponent $f\colon B\to A$. This consists of a map $g^f\colon E\to A$ and a map $\mathrm{ev}_{f,g}\colon E\times_A B\to C$, which makes the triangle below commutative and it is universal in $\mathsf{Set}/A$, where $E\times_A B$ denotes the $\mathsf{Set}$-pullback of $g^f$ along $f$.

$$\begin{array}{ccc} E\times_A B & \longrightarrow & B \\ \downarrow & \searrow^{g^f\times f} & \downarrow f \\ E & \xrightarrow[g^f]{} & A \end{array} \qquad\qquad \begin{array}{ccc} E\times_A B & \xrightarrow{\mathrm{ev}_{f,g}} & C \\ & {\scriptstyle g^f\times f}\searrow \quad \swarrow{\scriptstyle g} & \\ & A & \end{array}$$

Then $E=\bigcup\limits_{a\in A}\big(\{a\}\times\{h\colon f^{-1}(a)\to g^{-1}(a)\}\big)\subseteq A\times C^B$, where $C^B$ denotes the set of maps from $B$ to $C$, for each $(a,h)\in E$, $g^f(a,h)=a$, and, for each $((a,h),b)\in E\times_A B$, $\mathrm{ev}_{f,g}((a,h),b)=h(b)$. Now consider the following pullback diagram

$$\begin{array}{ccc} D & \xrightarrow{\overline{\delta}} & E \\ & & \big\downarrow i_E \\ \delta\big\downarrow & & A\times C^B \\ & & \big\downarrow \sigma\times\gamma^B \\ X\times Z^Y & \xrightarrow[1_X\times(\text{-})^\beta]{} & X\times Z^B, \end{array}$$

where $i_E$ is the inclusion map, $Z^Y$ denotes the exponential in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, for each $t\in Z^Y$, $(\text{-})^\beta(t)=t\cdot\beta$, and, for each $s\in C^B$, $\gamma^B(s)=\gamma\cdot s$. We claim that the weak exponential with base $g$ and exponent $f$ in $\mathscr{F}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{sep,inj}})/((X,a),A,\sigma)$ is given by

$$\ll f,g\gg=\pi_A\cdot\overline{\delta}\colon\,(X\times Z^Y,D,\delta)\to(X,A,\sigma),$$

where $X\times Z^Y$ is endowed with the obvious $(\mathbb{T},\mathsf{V})$-structure, and the (weak) evaluation

$$\mathrm{ev}\colon\,(X\times Z^Y\times Y,D\times_A B,\delta\times_A\beta)\to(Z,C,\gamma)$$

is given by the composition

$$\underbrace{D\times_A B\hookrightarrow D\times B\xrightarrow{\overline{\delta}\times 1_B}E\times B\hookrightarrow A\times C^B\times B\xrightarrow{1_A\times\mathrm{ev}_{B,C}}A\times C\xrightarrow{\pi_C}C}_{\mathrm{ev}},$$

where $\delta\times_A\beta$ is the restriction of $\delta\times\beta$ to $D\times_A B$, which denotes the $\mathsf{Set}$-pullback of $\ll f,g\gg$ along $f$. Notice that $\ll f,g\gg\times f\colon\,(X\times Z^Y\times Y,D\times_A B,\delta\times_A\beta)\to(X,A,\sigma)$ in the slice category is given by the diagonal map in the following pullback rectangle

$$\begin{array}{ccc} D\times_A B & \longrightarrow & B \\ \downarrow & \searrow^{\ll f,g\gg\times f} & \downarrow{\scriptstyle f} \\ D & \xrightarrow[\ll f,g\gg]{} & A. \end{array}$$

To prove our claim, we first observe that $\ll f,g \gg$ and ev are indeed morphisms of $\mathscr{F}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{sep,inj}})$, for the following diagrams are commutative:

$$\begin{array}{ccc} D & \xrightarrow{\ll f,g\gg} & A \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\sigma} \\ X\times Z^Y & \xrightarrow[\pi_X]{} & X \end{array} \qquad \begin{array}{ccc} D\times_A B & \xrightarrow{\text{ev}} & C \\ {\scriptstyle\delta\times_A\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ X\times Z^Y\times Y & \xrightarrow[\pi_Z\cdot(1_X\times \text{ev}_{Y,Z})]{} & Z. \end{array}$$

Moreover, ev is a morphism from $\ll f,g \gg \times f$ to $g$ in $\mathscr{F}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{sep,inj}})/((X,a),A,\sigma)$, since the triangle below is commutative.

$$\begin{array}{ccc} D\times_A B & \xrightarrow{\text{ev}} & C \\ & {\scriptstyle\ll f,g\gg\times f}\searrow \quad \swarrow{\scriptstyle g} & \\ & A & \end{array}$$

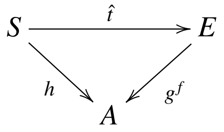
Finally, let $h\colon ((W,s),S,\mu)\to(X,A,\sigma)$ be an object and $t\colon h\times f\to g$ be a morphism in the slice category. Then there exists a continuous map $h'\colon W\to X$ such that $\sigma\cdot h = h'\cdot\mu$, and, moreover, $t\colon (W\times Y, S\times_A B, \mu\times_A\beta)\to(Z,C,\gamma)$ is a morphism of $\mathscr{F}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{sep,inj}})$ such that $g\cdot t = h\times f$; there exists also a continuous map $t'\colon W\times Y\to Z$ such that $\gamma\cdot t = t'\cdot(\mu\times_A\beta)$, so the following

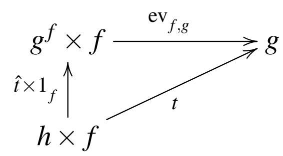
$$\begin{array}{ccc} S & \xrightarrow{h} & A \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\sigma} \\ W & \xrightarrow[h']{} & X \end{array} \qquad \begin{array}{ccc} S\times_A B & \xrightarrow{t} & C \\ & {\scriptstyle h\times f}\searrow \quad \swarrow{\scriptstyle g} & \\ & A & \end{array} \qquad \begin{array}{ccc} S\times_A B & \xrightarrow{t} & C \\ {\scriptstyle\mu\times_A\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ W\times Y & \xrightarrow[t']{} & Z \end{array}$$

are commutative diagrams. We wish to define $\bar{t}\colon h\to\ll f,g\gg$ such that $\text{ev}\cdot(\bar{t}\times 1_f) = t$. Let $\bar{t}\colon S\to D$ be the unique map such that the diagram

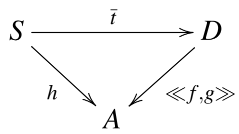
$$\begin{array}{ccccc} S & & \xrightarrow{\hat{t}} & & \\ & \searrow^{\bar{t}} & & & \\ & & D & \xrightarrow{\bar{\delta}} & E \\ & {\scriptstyle\langle\sigma\cdot h,t''\cdot\mu\rangle}\searrow & {\scriptstyle\delta}\downarrow & & \downarrow \\ & & X\times Z^Y & \longrightarrow & X\times Z^B \end{array}$$

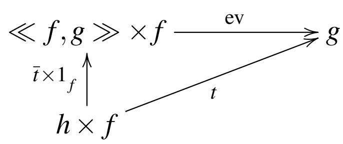
is commutative, where $\hat{t}\colon S \to E$ is the unique morphism from $h$ to $g^f$ in $\mathsf{Set}/A$ satisfying $\mathrm{ev}_{f,g}\cdot(\hat{t}\times 1_f) = t$, that is, the triangles




are commutative, and $t''\colon W \to Z^Y$ is the exponential mate of $t'\colon W\times Y \to Z$. Now $\delta\cdot\bar{t} = \langle \sigma\cdot h, t''\cdot\mu\rangle = \langle h'\cdot\mu, t''\cdot\mu\rangle = \langle h', t''\rangle\cdot\mu$, with $\langle h', t''\rangle\colon W \to X\times Z^Y$ a continuous map, proves that $\bar{t}\colon (W,S,\mu) \to (X\times Z^Y, D, \delta)$ is a morphism of $\mathscr{F}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{sep,inj}})$. Moreover, the following triangles are commutative.

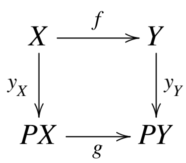



□

**Theorem 4.0.7** *The categories* $(\mathbb{T},\mathsf{V})$-Cat *and* $\mathscr{F}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{sep,inj}})$ *are equivalent.*

*Proof.* Define the functor $G\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to \mathscr{F}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{sep,inj}})$ by, for each $(\mathbb{T},\mathsf{V})$-space $(X,a)$, $G(X,a) = ((PX,\hat{a}), X, y_X\colon X \to PX)$, where $y_X\colon X\to PX$ is the Yoneda embedding and $\hat{a} = [a^{\mathrm{op}}, \hom_\xi]$ is the $(\mathbb{T},\mathsf{V})$-structure on $PX$. For each continuous map $f\colon X\to Y$, since $PY$ is injective and separated and $y_X$ is fully faithful, there exists a continuous map $g\colon PX\to PY$ extending $y_Y\cdot f$ along $y_X$, so the square below is commutative.



(II.10)

Whence set $Gf = f$. Then $G$ is trivially faithful, and to demonstrate that it is full, let $f\colon X\to Y$ be a map such that diagram (II.10) is commutative, for some continuous map $g\colon PX\to PY$. We calculate:

$$
\begin{aligned}
a &= y_X^\circ\cdot\hat{a}\cdot Ty_X \;\leq\; y_X^\circ\cdot g^\circ\cdot\hat{b}\cdot Tg\cdot Ty_X && (y_X \text{ is fully faithful and } g \text{ is continuous})\\
&= (g\cdot y_X)^\circ\cdot\hat{b}\cdot T(g\cdot y_X) \;=\; (y_Y\cdot f)^\circ\cdot\hat{b}\cdot T(y_Y\cdot f) && (f \text{ satisfies (II.10)})\\
&= f^\circ\cdot y_Y^\circ\cdot\hat{b}\cdot Ty_Y\cdot Tf \;=\; f^\circ\cdot b\cdot Tf && (y_Y \text{ is fully faithful}),
\end{aligned}
$$

that is, the map $f\colon (X,a) \to (Y,b)$ is $(\mathbb{T},\mathsf{V})$-continuous. To establish essential surjectivity for $G$, let $((X,a),A,\sigma\colon A \to X)$ in $\mathscr{F}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{sep,inj}})$. Endow the set $A$ with the $|\text{-}|$-initial $(\mathbb{T},\mathsf{V})$-structure $a_\sigma$ with respect to $\sigma\colon A \to (X,a)$, so that $\sigma\colon (A,a_\sigma) \to (X,a)$ becomes fully faithful, and $G(A,a_\sigma) = ((PA,\hat{a}_\sigma),A,y_A\colon A \to PA)$. Now the identity map $1_A\colon A \to A$ is a morphism from $((X,a),A,\sigma\colon A \to X)$ to $G(A,a_\sigma)$ and vice versa. Indeed, since $PA$ is injective and separated and $\sigma$ is fully faithful, there exists a continuous map $g_1\colon X \to PA$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{1_A} & S \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle y_A} \\ X & \xrightarrow[g_1]{} & PA \end{array}$$

is commutative, and since $X$ is an injective and separated space and $y_A$ is fully faithful, there exists a continuous map $g_2\colon PA \to X$ such that the diagram below is commutative.

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle y_A}\downarrow & & \downarrow{\scriptstyle\sigma} \\ PA & \xrightarrow[g_2]{} & X \end{array}$$

□

**Corollary 4.0.8** *If injective* $(\mathbb{T},\mathsf{V})$*-spaces are exponentiable in* $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$*, then* $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ *is weakly locally cartesian closed.*

**Remark 4.0.9** By Remark 3.0.6, the categories of Table (I.25) are weakly locally cartesian closed.

Weak (local) cartesian closure is studied in terms of *weak (dependent) simple products*; for details on that we also refer to [Emm18] and [AR19].

# Chapter III

# Equilogical $(\mathbb{T},\mathsf{V})$-spaces

In this chapter we introduce the category of equilogical $(\mathbb{T},\mathsf{V})$-spaces, carrying from $\mathsf{Top}$ to $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ the category $\mathsf{Equ}$ of equilogical spaces and (equivalence classes of) equivariant maps, which was first presented in [Sco96, BBS04]. We study its main features, and, in particular, its relation with exact and regular completions [CV98].

Throughout we assume that injective $(\mathbb{T},\mathsf{V})$-spaces are exponentiable in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, so that, as proved in Chapter II, $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is weakly (locally) cartesian closed. The results of this chapter can be found in [Rib19a].

## 5 The category of equilogical $(\mathbb{T},\mathsf{V})$-spaces

**Definition 5.0.1** The category $(\mathbb{T},\mathsf{V})$-$\mathsf{Equ}$ of *equilogical* $(\mathbb{T},\mathsf{V})$-*spaces* is defined as follows:

• objects are pairs $\mathscr{X}=\langle (X,a),\equiv_X\rangle$, where $(X,a)$ is a $(\mathbb{T},\mathsf{V})$-space and $\equiv_X$ is an equivalence relation on the set $X$;

• a morphism from $\mathscr{X}=\langle (X,a),\equiv_X\rangle$ to $\mathscr{Y}=\langle (Y,b),\equiv_Y\rangle$ is an equivalence class $[f]$ of a $(\mathbb{T},\mathsf{V})$-continuous map $f\colon (X,a)\to(Y,b)$ which is *equivariant*, i.e., that preserves the equivalence relation: for each $x,x'\in X$, $x\equiv_X x'$ implies $f(x)\equiv_Y f(x')$, with the equivalence relation on maps defined by: for each equivariant $(\mathbb{T},\mathsf{V})$-continuous maps $f,g\colon (X,a)\to(Y,b)$,

$$f\equiv_{X\to Y} g \iff \forall x,x'\in X\ (x\equiv_X x' \implies f(x)\equiv_Y g(x')). \tag{III.1}$$

Indeed, $\equiv_{X\to Y}$ is an equivalence relation: for equivariant continuous maps $f,g,h\colon (X,a)\to(Y,b)$, for each $x,x'\in X$, if $x\equiv_X x'$, then $f(x)\equiv_Y f(x')$, hence $f\equiv_{X\to Y} f$, so $\equiv_{X\to Y}$ is reflexive; if $f\equiv_{X\to Y} g$, then, for $x \equiv_X x'$, $g(x) \equiv_Y g(x') \equiv_Y f(x) \equiv_Y f(x')$, so $g \equiv_{X\to Y} f$, and $\equiv_{X\to Y}$ is symmetric; if $f \equiv_{X\to Y} g$ and $g \equiv_{X\to Y} h$, then, for $x \equiv_X x'$, $f(x) \equiv_Y g(x') \equiv_Y g(x) \equiv_Y h(x')$, thus $f \equiv_{X\to Y} h$, and $\equiv_{X\to Y}$ is transitive.

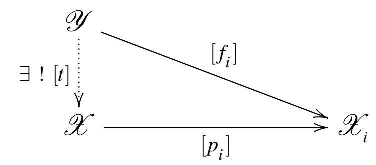
Composition of classes $[f]\colon \mathscr{X} \to \mathscr{Y}$ and $[g]\colon \mathscr{Y} \to \mathscr{Z}$ is given by $[g]\cdot[f] = [g\cdot f]$, which is well-defined: the composite of equivariant continuous maps is equivariant and continuous, and if $f \equiv_{X\to Y} f'$ and $g \equiv_{Y\to Z} g'$, then, for $x \equiv_X x'$, $f(x) \equiv_Y f'(x')$ implies $g(f(x)) \equiv_Z g'(f'(x'))$, that is, $g\cdot f \equiv_{X\to Z} g'\cdot f'$.

**Theorem 5.0.2** $(\mathbb{T},\mathsf{V})$-Equ *is complete, cocomplete, regular well-powered, and regular co-well-powered.*

*Proof.* Given a family $\left(\mathscr{X}_i = \left\langle (X_i,a_i),\equiv_{X_i}\right\rangle\right)_{i\in I}$ of equilogical $(\mathbb{T},\mathsf{V})$-spaces, its product is given by $\mathscr{X} = \langle (X,a),\equiv_X\rangle$, where $(X,a) = \prod_{i\in I}(X_i,a_i)$ is a product in $(\mathbb{T},\mathsf{V})$-Cat, so $X = \prod_{i\in I} X_i$ in Set and $a$ is the $|\text{-}|$-initial $(\mathbb{T},\mathsf{V})$-structure with respect to the product projections $p_j\colon \prod X_i \to X_j$, $j\in I$ , and $x \equiv_X x'$ if, and only if, for all $i\in I$, $p_i(x) \equiv_{X_i} p_i(x')$.

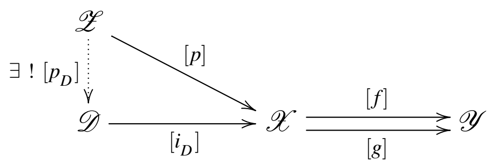
The product projections are equivariant continuous maps, and for a family $([f_i]\colon \mathscr{Y} \to \mathscr{X}_i)_{i\in I}$ of morphisms in $(\mathbb{T},\mathsf{V})$-Equ, with $\mathscr{Y} = \langle (Y,b),\equiv_Y\rangle$, by the universal property in $(\mathbb{T},\mathsf{V})$-Cat there exists a unique continuous map $t\colon (Y,b)\to(X,a)$ such that $p_i\cdot t = f_i$, for all $i\in I$. Now $t$ is equivariant, for if $y \equiv_Y y'$, then $f_i(y) \equiv_{X_i} f_i(y')$ implies $p_i\cdot t(y) \equiv_{X_i} p_i\cdot t(y')$, for all $i\in I$, whence $t(y) \equiv_X t(y')$.

By the composition law, $[p_i]\cdot[t] = [f_i]$, for all $i\in I$, and if $[t']\colon \mathscr{Y}\to\mathscr{X}$ is a morphism with the same property, then, for $y \equiv_Y y'$, we get $p_i(t'(y)) \equiv_{X_i} f_i(y') = p_i(t(y'))$, for all $i\in I$, whence $t'(y) \equiv_X t(y')$, that is, $t' \equiv_{X\to Y} t$.



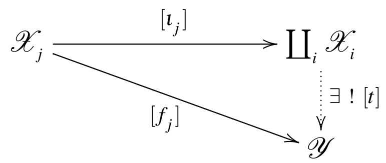

For equalizers, let $[f],[g]\colon \mathscr{X}\to\mathscr{Y}$ be morphisms in $(\mathbb{T},\mathsf{V})$-Equ. Then consider the set $D = \{x\in X \mid f(x) \equiv_Y g(x)\}$ endowed with the subspace $(\mathbb{T},\mathsf{V})$-structure $a_D = i_D^\circ\cdot a\cdot Ti_D$, where $i_D\colon D\to X$ is the inclusion map. This defines the equilogical $(\mathbb{T},\mathsf{V})$-space $\mathscr{D} = \langle (D,a_D),\equiv_D\rangle$, where $\equiv_D$ is the restriction of $\equiv_X$ to $D$.

The map $i_D$ is equivariant and continuous, and if $d \equiv_{\mathscr{D}} d'$, then $f(d) \equiv_Y g(d) \equiv_Y g(d')$, hence $f\cdot i_D(d) \equiv_Y g\cdot i_D(d')$, that is, $f\cdot i_D \equiv_{D\to Y} g\cdot i_D$, and $[f]\cdot[i_D] = [g]\cdot[i_D]$. If $[p]\colon \mathscr{Z}\to\mathscr{X}$ is a morphism in $(\mathbb{T},\mathsf{V})$-Equ such that $[f]\cdot[p] = [g]\cdot[p]$, then, for each $z\in Z$, $z\equiv_Z z$ implies $f(p(z)) \equiv_Y g(p(z))$, hence $p(z)\in D$, and $p$ corestricts to $D$ as $p_D\colon Z\to D$, so that $i_D\cdot p_D = p$. Therefore $p_D$ is an equivariant continuous map, and $[p_D]\colon \mathscr{Z} \to \mathscr{D}$ is the unique morphism such that $[i_D]\cdot[p_D]=[p]$.



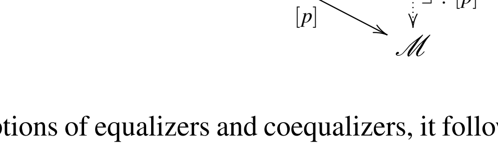

For the coproduct of a family $(\mathscr{X}_i)_{i\in I}$ in $(\mathbb{T},\mathsf{V})$-Equ, first form the coproduct of the underlying $(\mathbb{T},\mathsf{V})$-spaces in $(\mathbb{T},\mathsf{V})$-Cat, which has as underlying set the disjoint union $\dot{\bigcup}_{i\in I} X_i = \bigcup_{i\in I}(X_i\times\{i\})$. Now define $(x,j)\equiv_{\amalg}(x',l)$ if, and only if, $j=l$ and $x\equiv_{X_j}x'$. This determines an equilogical $(\mathbb{T},\mathsf{V})$-space $\coprod_i \mathscr{X}_i$, with the coproduct inclusions $\iota_j\colon X_j \hookrightarrow \dot{\bigcup}_{i\in I} X_i$ being equivariant continuous maps.

For a family of morphisms $[f_i]\colon \mathscr{X}_i \to \mathscr{Y}$ in $(\mathbb{T},\mathsf{V})$-Equ, $i\in I$, $\mathscr{Y}=\langle (Y,b),\equiv_Y\rangle$, by the universal property in $(\mathbb{T},\mathsf{V})$-Cat there exists a unique continuous map $t\colon (\dot{\bigcup} X_i,a)\to(Y,b)$ such that $t\cdot\iota_j=f_j$, for all $j\in I$. The map $t$ is equivariant, for if $(x,j)\equiv_{\amalg}(x',l)$, then $j=l$ and $x\equiv_{X_j}x'$, so $f_j(x)\equiv_Y f_j(x')$, whence $t(x,j)=t\cdot\iota_j(x)\equiv_Y t\cdot\iota_j(x')=t(x',j)$. One can check that $[t]\colon \coprod_i\mathscr{X}_i\to\mathscr{Y}$ is the unique morphism satisfying $[t]\cdot[\iota_j]=[f_j]$, for each $j\in I$.




For coequalizers let us consider morphisms $[f],[g]\colon \mathscr{X}\to\mathscr{Y}$ in $(\mathbb{T},\mathsf{V})$-Equ. We form on $Y$ the least equivalence relation $\equiv_Z$ that contains both $\equiv_Y$ and the set of pairs $\{(f(x),g(x))\mid x\in X\}$. Then define the equilogical $(\mathbb{T},\mathsf{V})$-space $\mathscr{Z}=\langle (Y,b),\equiv_Z\rangle$.

The identity map $1_Y\colon \mathscr{Y}\to\mathscr{Z}$ is equivariant and continuous. If $[p]\colon \mathscr{Y}\to\mathscr{M}=\langle (M,l),\equiv_M\rangle$ is a morphism in $(\mathbb{T},\mathsf{V})$-Equ such that $[p]\cdot[f]=[p]\cdot[g]$, then $p\colon (Y,b)\to(M,l)$ is an equivariant continuous map, because, for all $x\in X$, $p(f(x))\equiv_M p(g(x))$; then we obtain a unique morphism $[p]\colon \mathscr{Z}\to\mathscr{M}$ such that $[p]\cdot[1_Y]=[p]$.




From the previous descriptions of equalizers and coequalizers, it follows that $(\mathbb{T},\mathsf{V})$-Equ is regular well-powered and regular co-well-powered. □

As observed in [BBS04], in the previous proof representatives for the classes of equivariant maps are already chosen, making use of the Axiom of Choice.

In general $(\mathbb{T},\mathsf{V})$-Equ is neither well-powered nor co-well-powered, as observed in [BBS04] for topological spaces. A morphism $[m]\colon \mathscr{X} \to \mathscr{Y}$ is a monomorphism in $(\mathbb{T},\mathsf{V})$-Equ if, and only if,

$$\forall x, x' \in X, (x \equiv_X x' \iff m(x) \equiv_Y m(x')); \tag{III.2}$$

hence any continuous map from $(X,a)$ to $(Y,b)$, with $\mathscr{Y} = \langle (Y,b), \equiv_Y \rangle \in (\mathbb{T},\mathsf{V})$-Equ, becomes a monomorphism if we endow $X$ with the equivalence relation defined by (III.2). Analogously, $[e]\colon \mathscr{X} \to \mathscr{Y}$ is an epimorphism in $(\mathbb{T},\mathsf{V})$-Equ if, and only if,

$$y \equiv_Y y' \iff \exists\, x, x' \in X; (x \equiv_X x' \;\&\; y \equiv_Y e(x) \equiv_Y e(x') \equiv_Y y'); \tag{III.3}$$

whence each surjective continuous map from $(X,a)$ to $(Y,b)$, with $\mathscr{X} = \langle (X,a), \equiv_X \rangle \in (\mathbb{T},\mathsf{V})$-Equ, becomes an epimorphism if we endow $Y$ with the equivalence relation defined by (III.3).

# 6 The category of partial equilogical $(\mathbb{T},\mathsf{V})$-spaces

In order to directly prove the cartesian closedness of $(\mathbb{T},\mathsf{V})$-Equ, as in [BBS04] we define an auxiliary category that is based on injective $(\mathbb{T},\mathsf{V})$-spaces.

**Definition 6.0.1** The category $(\mathbb{T},\mathsf{V})$-PEqu of *partial equilogical* $(\mathbb{T},\mathsf{V})$*-spaces* consists of:
• objects are pairs $\mathscr{X} = \langle (X,a), \equiv_X \rangle$, where $(X,a)$ is an injective $(\mathbb{T},\mathsf{V})$-space and $\equiv_X$ is a *partial equivalence relation* on $X$, that is, $\equiv_X$ is symmetric, transitive, and not necessarily reflexive;
• morphisms are equivalence classes of equivariant $(\mathbb{T},\mathsf{V})$-continuous maps between the underlying $(\mathbb{T},\mathsf{V})$-spaces, with the equivalence relation on maps defined by (III.1).

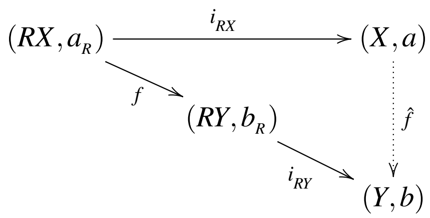
Let us recall that, for a separated $(\mathbb{T},\mathsf{V})$-space $(X,a)$, the Yoneda embedding $y_X : X \to PX$ is fully faithful and an injective map, with $PX$ an injective separated $(\mathbb{T},\mathsf{V})$-space. This fact is necessary to establish an equivalence between the categories of equilogical and partial equilogical $(\mathbb{T},\mathsf{V})$-spaces. We then restrict ourselves to separated $(\mathbb{T},\mathsf{V})$-spaces, considering that the underlying $(\mathbb{T},\mathsf{V})$-spaces of the (partial) equilogical $(\mathbb{T},\mathsf{V})$-spaces of Definitions 5.0.1 and 6.0.1 are separated, and denote the resulting categories by $(\mathbb{T},\mathsf{V})$-$\mathsf{Equ}_{\mathrm{sep}}$ and $(\mathbb{T},\mathsf{V})$-$\mathsf{PEqu}_{\mathrm{sep}}$, respectively.

**Theorem 6.0.2** $(\mathbb{T},\mathsf{V})$-$\mathsf{Equ}_{\mathrm{sep}}$ *and* $(\mathbb{T},\mathsf{V})$-$\mathsf{PEqu}_{\mathrm{sep}}$ *are equivalent.*

*Proof.* Define a "restriction" functor $R\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{PEqu}_{\mathrm{sep}} \to (\mathbb{T},\mathsf{V})\text{-}\mathsf{Equ}_{\mathrm{sep}}$ which assigns to each partial equilogical $(\mathbb{T},\mathsf{V})$-space $\mathscr{X} = \langle (X,a), \equiv_X \rangle$ the (sub)space $R\mathscr{X} = \langle (RX, a_R), \equiv_{RX} \rangle$, where $RX = \{x \in X \mid x \equiv_X x\}$, so the relation $\equiv_{RX}$, which is the restriction of $\equiv_X$ to $RX$, is reflexive, and $a_R$ is the $|\text{-}|$-initial $(\mathbb{T},\mathsf{V})$-structure with respect to the inclusion map $i_{RX}\colon RX \hookrightarrow X$.

For a morphism $[f]\colon \mathscr{X} \to \mathscr{Y}$, with $\mathscr{Y} = \langle (Y,b), \equiv_Y \rangle$, since $f\colon X \to Y$ is equivariant, $f(RX) \subseteq RY$, so $R$ assigns to $[f]$ the class $[Rf]\colon R\mathscr{X} \to R\mathscr{Y}$, where $Rf\colon RX \to RY$ is the (co)restriction of $f$. If $[f] = [g]$, then clearly $[Rf] = [Rg]$, so $R$ is well-defined, and compositions and identities are preserved.

Let $[f],[g]\colon \mathscr{X} \to \mathscr{Y}$ be morphisms of $(\mathbb{T},\mathsf{V})\text{-}\mathsf{PEqu}_{\mathrm{sep}}$ such that $[Rf] = [Rg]$. Whenever $x \equiv_X x'$, then $x' \equiv_X x \equiv_X x'$ implies $x' \equiv_X x'$, that is, $x' \in RX$, and, by the same argument, $x \in RX$, hence $x \equiv_{RX} x'$, what implies that $f(x) = Rf(x) \equiv_{RY} Rg(x') = g(x')$, and this is equivalent to $f(x) \equiv_Y g(x')$, since $f(x), g(x') \in RY$, thus $[f] = [g]$, and $R$ is faithful. Now let $[f]\colon R\mathscr{X} \to R\mathscr{Y}$ be a morphism in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Equ}_{\mathrm{sep}}$; since $(Y,b)$ is injective and separated, there exists an extension $\hat{f}\colon (X,a) \to (Y,b)$ of $i_{RY} \cdot f$ along the embedding $i_{RX}$, which is equivariant and satisfies $[R\hat{f}] = [f]$, thus $R$ is full.




Finally, for essential surjectivity let $\mathscr{X} = \langle (X,a), \equiv_X \rangle \in (\mathbb{T},\mathsf{V})\text{-}\mathsf{Equ}_{\mathrm{sep}}$ and consider the Yoneda embedding $y_X\colon (X,a) \to (PX, \hat{a})$. Then $(PX,\hat{a}) \in (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{sep,inj}}$ and we define the following partial equivalence relation on $PX$:

$$\varphi \equiv_{PX} \psi \iff \exists\, x, x' \in X; (\varphi = y_X(x), \psi = y_X(x') \;\&\; x \equiv_X x'),$$

that is, two elements of $PX$ are equivalent precisely when they are the images by $y_X$ of equivalent elements of $X$. Since $y_X$ is injective and $\equiv_X$ is reflexive, $\varphi \in R(PX)$ precisely when $\varphi \equiv_{PX} \varphi$, what is equivalent to the existence of a unique $x \in X$ such that $\varphi = y_X(x)$, that is, $\varphi \in y_X(X)$; whence we have a bijection $t\colon (R(PX), \hat{a}_R) \to (X,a)$, which is continuous, since $y_X$ is initial and the composite $y_X \cdot t = i_{R(PX)}\colon (R(PX), \hat{a}_R) \hookrightarrow (PX, \hat{a})$ is continuous; and, moreover, $t$ is equivariant by definition of $\equiv_{PX}$. The corestriction of $y_X$ to its image gives an inverse morphism, so $t$ defines an isomorphism $R\langle (PX,\hat{a}), \equiv_{PX} \rangle = \left\langle (R(PX), \hat{a}_R), \equiv_{R(PX)} \right\rangle \cong \mathscr{X}$. □

Under the assumption that injective spaces are exponentiable in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$, we prove:

**Theorem 6.0.3** $(\mathbb{T},\mathsf{V})\text{-}\mathsf{PEqu}_{\text{sep}}$ *is a cartesian closed category.*

*Proof.* Let $\mathscr{X}=\langle (X,a),\equiv_X\rangle$ and $\mathscr{Y}=\langle (Y,b),\equiv_Y\rangle$ be partial equilogical separated $(\mathbb{T},\mathsf{V})$-spaces. Consider the exponential $(Y,b)^{(X,a)}$ which is separated and injective, and whose underlying set is $Y^X=\{h\colon (X,a)\times(1,e_1^\circ)\to(Y,b)\mid h \text{ is a } (\mathbb{T},\mathsf{V})\text{-continuous map}\}$. Identifying the elements of $Y^X$ with maps from $X$ to $Y$, we endow this set with the partial equivalence relation $\equiv_{X\to Y}$ of (III.1), that we denote here by $\equiv_{Y^X}$, so that, for each $h,h'\in Y^X$,

$$h\equiv_{Y^X} h' \iff \forall x,x'\in X\ (x\equiv_X x' \implies h(x)\equiv_Y h'(x')).$$

Hence the partial equilogical separated $(\mathbb{T},\mathsf{V})$-space $\mathscr{Y}^{\mathscr{X}}=\left\langle (Y,b)^{(X,a)},\equiv_{Y^X}\right\rangle$ is defined, and the evaluation map $\mathrm{ev}\colon (Y,b)^{(X,a)}\times(X,a)\to(Y,b)$ is continuous and equivariant:

$$(f,x)\equiv_{Y^X\times X}(f',x') \iff (f\equiv_{Y^X} f' \ \&\ x\equiv_X x') \implies f(x)\equiv_Y f'(x').$$

Furthermore, for each morphism $[f]\colon \mathscr{Z}\times\mathscr{X}\to\mathscr{Y}$, with $\mathscr{Z}=\langle (Z,c),\equiv_Z\rangle\in(\mathbb{T},\mathsf{V})\text{-}\mathsf{PEqu}_{\text{sep}}$, by the universal property in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$, there exists a unique $\overline{f}\colon (Z,c)\to(Y,b)^{(X,a)}$, the transpose of $f\colon (Z,c)\times(X,a)\to(Y,b)$, such that $\mathrm{ev}\cdot(\overline{f}\times 1_X)=f$. Whenever $z\equiv_Z z'$ and $x\equiv_X x'$, we have

$$(z,x)\equiv_{Z\times X}(z',x') \implies f(z,x)\equiv_Y f(z',x') \implies \overline{f}(z)(x)\equiv_Y \overline{f}(z')(x') \implies \overline{f}(z)\equiv_{Y^X}\overline{f}(z'),$$

whence $\overline{f}$ is equivariant, and the morphism $[\overline{f}]\colon \mathscr{Z}\to\mathscr{Y}^{\mathscr{X}}$ is such that $[\mathrm{ev}]\cdot([\overline{f}]\times 1_{\mathscr{X}})=[f]$.

$$\begin{array}{ccc} \mathscr{Y}^{\mathscr{X}}\times\mathscr{X} & \xrightarrow{[\mathrm{ev}]} & \mathscr{Y} \\ {\scriptstyle [\overline{f}]\times 1_{\mathscr{X}}}\big\uparrow & \nearrow_{[f]} & \\ \mathscr{Z}\times\mathscr{X} & & \end{array}$$

Moreover, $[\overline{f}]$ is unique, for if $[f']\colon \mathscr{Z}\to\mathscr{Y}^{\mathscr{X}}$ is a morphism with $[\mathrm{ev}\cdot(f'\times 1_X)]=[f]$, then, for $z\equiv_Z z'$ and $x\equiv_X x'$,

$$\overline{f}(z)(x)=f(z,x)\equiv_Y f(z',x')\equiv_Y f'(z')(x'),$$

hence $\overline{f}(z)\equiv_{Y^X} f'(z')$, and $[\overline{f}]=[f']$. □

By Theorem 6.0.2, under the assumption that injective spaces are exponentiable in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$, $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Equ}_{\text{sep}}$ is a cartesian closed category. Observe that the proof of Theorem 6.0.3 can be applied to the category $(\mathbb{T},\mathsf{V})$-$\mathsf{PEqu}$ without requiring separation. Next we discuss the presentation of equilogical $(\mathbb{T},\mathsf{V})$-spaces as modest sets of assemblies, what corresponds to [BBS04, Section 4].

**Definition 6.0.4** The category $\mathsf{Assm}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$ of *assemblies over injective spaces* is defined as follows:

• objects are triples $(A,(X,a),E_A)$, where $A$ is a set, $(X,a)$ is an injective space, and $E_A : A \to \mathscr{P}X$, with $\mathscr{P}X$ the powerset of $X$, is a function such that, for each $a \in A$, $E_A(a) \neq \emptyset$. The elements of $E_A(a)$ are called *realizers* for $a$;

• a morphism $f\colon (A,(X,a),E_A) \to (B,(Y,b),E_B)$ is a map $f\colon A \to B$ for which there exists a continuous map $g\colon (X,a) \to (Y,b)$ such that, for all $a \in A$, $g(E_A(a)) \subseteq E_B(f(a))$; $g$ is called a *realizer* for $f$, or one says that $g$ *tracks* $f$.

**Definition 6.0.5** An assembly over an injective space $(A,(X,a),E_A)$ is called a *modest set* if, for every $a,a' \in A$, if $a \neq a'$, then $E_A(a) \cap E_A(a') = \emptyset$. The full subcategory of $\mathsf{Assm}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$ of the modest sets is denoted by $\mathsf{Mdst}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$.

As in the particular case of $\mathsf{Top}$, we derive some properties of the categories just defined, which we state next as propositions.

**Proposition 6.0.6** *The categories* $\mathsf{Mdst}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$ *and* $\mathsf{Assm}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$ *have finite limits which are preserved by the inclusion* $\mathsf{Mdst}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}}) \hookrightarrow \mathsf{Assm}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$.

*Proof.* The terminal object is $(1,(1,\top),E_1)$, where $1 = \{*\}$ is a singleton, $(1,\top)$ is the terminal object in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, and $E_1(*) = 1$. Indeed, for each assembly $(A,(X,a),E_A)$ there exists a unique map $!_A : A \to 1$, $a \mapsto *$, which is a morphism, since the unique $(\mathbb{T},\mathsf{V})$-continuous map $!_X : X \to 1$ satisfies, for each $a \in A$, $!_X(E_A(a)) = 1 = E_1(*) = E_1(!_A(a))$; $(1,(1,\top),E_1)$ is a modest set.

For assemblies $(A,(X,a),E_A)$ and $(B,(Y,b),E_B)$, the binary product is $(A\times B,(X\times Y,a\times b),E_{A\times B})$, where, for each $(a,b) \in A\times B$, $E_{A\times B}(a,b) = E_A(a)\times E_B(b)$. Indeed, the product projections $\pi_A$ and $\pi_B$ from $A\times B$ into $A$ and $B$, respectively, satisfy

$$\pi_X(E_{A\times B}(a,b)) = E_A(a) = E_A(\pi_A(a,b)) \qquad \& \qquad \pi_Y(E_{A\times B}(a,b)) = E_B(b) = E_B(\pi_B(a,b)),$$

where $\pi_X$ and $\pi_Y$ are the product projections from $X\times Y$ into $X$ and $Y$, respectively; the universal property of the product follows from the respective universal property in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$. Moreover, if

$(A,(X,a),E_A)$ and $(B,(Y,b),E_B)$ are modest sets, then so is their product, since

$$\begin{aligned} E_{A\times B}(a,b)\cap E_{A\times B}(a',b') &= (E_A(a)\times E_B(b))\cap(E_A(a')\times E_B(b')) \\ &= (E_A(a)\cap E_A(a'))\times(E_B(b)\cap E_B(b')). \end{aligned}$$

For equalizers take $f,g\colon (A,(X,a),E_A)\to(B,(Y,b),E_B)$ in $\mathsf{Assm}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$. Define the assembly $(A',(X,a),E_{A'})$, where $A'=\{a\in A\mid f(a)=g(a)\}$ and $E_{A'}$ is the restriction of $E_A$ to $A'$. Then the inclusion map $i_{A'}\colon A'\hookrightarrow A$ is tracked by the identity $1_X$, and satisfies $f\cdot i_{A'}=g\cdot i_{A'}$. One can readily check the universal property and that if $(A,(X,a),E_A)$ is modest, then so is $(A',(X,a),E_{A'})$. □

From the description of equalizers given in this proposition, we conclude:

**Corollary 6.0.7** *The regular subobjects of an assembly $(A,(X,a),E_A)$ are in bijective correspondence with the powerset $\mathscr{P}A$ of $A$.*

Once more using the hypothesis that injective spaces are exponentiable, we prove the following:

**Proposition 6.0.8** $\mathsf{Mdst}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$ *and* $\mathsf{Assm}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$ *are cartesian closed categories, and the inclusion* $\mathsf{Mdst}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})\hookrightarrow\mathsf{Assm}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$ *preserves exponentials.*

*Proof.* For assemblies over injective spaces $(A,(X,a),E_A)$ and $(B,(Y,b),E_B)$, define the assembly $(C,(Y,b)^{(X,a)},E_C)$, where $(Y,b)^{(X,a)}$ is the exponential in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$,

$$C=\{f\colon A\to B\mid \exists\, g\colon (X,a)\to(Y,b)\ (\mathbb{T},\mathsf{V})\text{-continuous; } g \text{ tracks } f\},$$

and, for each $f\in C$, $E_C(f)=\{g\in(Y,b)^{(X,a)}\mid g\text{ tracks }f\}$. Then the evaluation map $\mathrm{ev}_{A,B}\colon C\times A\to B$ is tracked by the evaluation map $\mathrm{ev}_{X,Y}\colon Y^X\times X\to Y$ in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$: for each $(f,a)\in C\times A$,

$$\begin{aligned} y\in\mathrm{ev}_{X,Y}(E_{C\times A}(f,a)) &\iff y\in\mathrm{ev}_{X,Y}(E_C(f)\times E_A(a)) && \text{(by definition of } E_{C\times A})\\ &\iff \exists\, h\in E_C(f),\exists\, x\in E_A(a);\ y=\mathrm{ev}_{X,Y}(h,x) \\ &\implies y=h(x)\in E_B(f(a))=E_B(\mathrm{ev}_{A,B}(f,a)) && \text{(because } h \text{ tracks } f). \end{aligned}$$

If $(B,(Y,b),E_B)$ is a modest set, then so is $(C,(Y,b)^{(X,a)},E_C)$. Indeed, if $f,f'\colon A\to B$ are tracked by the same $g\colon (X,a)\to(Y,b)$, then, for each $a\in A$, take $x\in E_A(a)\neq\emptyset$. Thus $g(x)\in E_B(f(a))\cap E_B(f'(a))$, whence $f(a)=f'(a)$, and $f=f'$. □

**Proposition 6.0.9** $\mathsf{Mdst}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$ *is reflective in* $\mathsf{Assm}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$.

*Proof.* Define the reflector $R\colon \mathsf{Assm}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}}) \to \mathsf{Mdst}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$ assigning to each $(A,(X,a),E_A)$ the triple $(A/{\sim},(X,a),E_{A/\sim})$, where $a \sim a'$ if, and only if, $E_A(a)\cap E_A(a') \neq \emptyset$, and $E_{A/\sim}([a]) = \bigcup\limits_{a'\in[a]} E_A(a')$. On morphisms, $R$ assigns to each $f\colon (A,(X,a),E_A) \to (B,(Y,b),E_B)$ the map $Rf\colon A/{\sim}\to B/{\sim}$, $[a]\mapsto [f(a)]$. If $a\sim a'$, then for $x\in E_A(a)\cap E_A(a')$ and any $g$ tracking $f$ we have $g(x)\in E_B(f(a))\cap E_B(f(a'))\neq\emptyset$, whence $f(a)\sim f(a')$, and $Rf$ is well-defined; moreover, for such $g$,

$$g(E_{A/\sim}([a])) = g(\bigcup_{a'\in[a]} E_A(a')) \subseteq \bigcup_{a'\in[a]} g(E_A(a')) \subseteq \bigcup_{a'\in[a]} E_B(f(a')) \subseteq \bigcup_{b'\in[f(a)]} E_B(b') = E_{B/\sim}(Rf[a]),$$

that is, $g$ tracks $Rf$. Let us assume that there exists $x\in E_{A/\sim}([a])\cap E_{A/\sim}([a'])$. Then there exists $a_1,a_2\in A$ such that $a_1\sim a$, $a_2\sim a'$, and $x\in E_A(a_1)\cap E_A(a_2)\neq\emptyset$, what implies that $a_1\sim a_2$, whence $a\sim a'$, i.e., $[a]=[a']$, and $(A/{\sim},(X,a),E_{A/\sim})$ is a modest set. Hence $R$ is a well-defined functor.

Each reflection is given by the projection map $p_A\colon A\to A/{\sim}$, which is tracked by the identity map $1_X$; each morphism $f\colon (A,(X,a),E_A)\to(B,(Y,b),E_B)$, with $(B,(Y,b),E_B)$ a modest set, factors uniquely through $p_A$ as $\tilde f\colon (A/{\sim},(X,a),E_{A/\sim})\to(B,(Y,b),E_B)$, $[a]\mapsto f(a)$, which is tracked by any realizer of $f$.

$$\begin{array}{ccc}
(A,(X,a),E_A) & \xrightarrow{\quad p_A\quad} & (A/{\sim},(X,a),E_{A/\sim}) \\
 & \searrow^{f} & \Big\downarrow{\scriptstyle \exists\,!\,\tilde f} \\
 & & (B,(Y,b),E_B)
\end{array}$$

□

**Proposition 6.0.10** $\mathsf{Mdst}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$ *and* $\mathsf{Assm}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$ *are regular categories.*

*Proof.* Given morphisms $f\colon (A,(X,a),E_A)\to(C,(Z,c),E_C)$ and $g\colon (B,(Y,b),E_B)\to(C,(Z,c),E_C)$ in any of the categories, the pullback of $f$ along $g$ is given by

$$\begin{array}{ccc}
(P,(X,a)\times(Y,b),E_P) & \xrightarrow{\quad \pi_B\quad} & (B,(Y,b),E_Y) \\
{\scriptstyle \pi_A}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\
(A,(X,a),E_A) & \xrightarrow[\quad f\quad]{} & (C,(Z,c),E_C),
\end{array}$$

where $P=\{(a,b)\in A\times B \mid f(a)=g(b)\}$ is the pullback of $f$ along $g$ in $\mathsf{Set}$, and, for each $(a,b)\in P$, $E_P(a,b)=E_A(a)\times E_B(b)$.

Monomorphisms and epimorphisms of assemblies coincide with injective and surjective functions, respectively. Hence it follows that, together with monomorphisms, epimorphisms are stable under pullback. Each morphism $f\colon (A,(X,a),E_A) \to (B,(Y,b),E_B)$ admits an image factorization as

$$\begin{array}{ccc} (A,(X,a),E_A) & \xrightarrow{\qquad\qquad f \qquad\qquad} & (B,(Y,b),E_B), \\ & {}_{f_e}\searrow \qquad\qquad \nearrow_{f_m} & \\ & (A/{\sim},(X,a),E_{A/\sim}) & \end{array}$$

with $a \sim a'$ if, and only if, $f(a) = f(a')$, and, for each $a \in A$, $E_{A/\sim}([a]) = \bigcup_{a' \in [a]} E_A(a')$, $f_e(a) = [a]$, $f_m([a]) = f(a)$; the map $f_e$ is tracked by the identity $1_X$, while $f_m$ is tracked by any realizer of $f$. $\square$

Next we establish, in our level of generality, the relation between partial equilogical spaces and modest sets.

**Theorem 6.0.11** $(\mathbb{T},\mathsf{V})$-$\mathsf{PEqu}$ *and* $\mathsf{Mdst}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}})$ *are equivalent.*

*Proof.* Define $F\colon \mathsf{Mdst}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{inj}}) \to (\mathbb{T},\mathsf{V})\text{-}\mathsf{PEqu}$ by $F(A,(X,a),E_A) = \langle (X,a), \equiv_X \rangle$, where

$$x \equiv_X x' \iff \exists\, a \in A;\ (x,x' \in E_A(a)), \tag{III.4}$$

and on morphisms $F$ assigns to each $f\colon (A,(X,a),E_A) \to (B,(Y,b),E_B)$ the equivalence class of a realizer $g\colon (X,a) \to (Y,b)$ for $f$, with the equivalence relation defined in (III.1); $\equiv_X$ is a partial equivalence relation, and if $g$ is a realizer for $f$, then

$$x \equiv_X x' \implies (x,x' \in E_A(a)) \implies (g(x),g(x') \in E_B(f(a))),$$

for some $a \in A$, that is, $g(x) \equiv_Y g(x')$, and $g$ is an equivariant $(\mathbb{T},\mathsf{V})$-continuous map. If $g'$ is another realizer for $f$, then the same argument leads us to $g(x) \equiv_Y g'(x')$ whenever $x \equiv_X x'$, whence the definition is independent from the choice of realizer; compositions and identities are preserved, so that $F$ is a well-defined functor.

To see that $F$ is full, let $g\colon (X,a) \to (Y,b)$ be a $(\mathbb{T},\mathsf{V})$-continuous map which is equivariant with respect to $\equiv_X$ and $\equiv_Y$ given by (III.4). For each $a \in A$, let $x \in E_A(a) \neq \emptyset$. Then $x \equiv_X x$ implies $g(x) \equiv_Y g(x)$, that is, there exists $b \in B$ such that $g(x) \in E_B(b)$. Now set $f(a) = b$, which is uniquely determined, since $(B,(Y,b),E_B)$ is a modest set. Hence we have a map $f\colon A \to B$, which, by definition, is tracked by $g$. For faithfulness, as observed in the proof of Proposition 6.0.8, two maps tracked by the same realizer must be equal. Finally, let $\langle (X,a), \equiv_X \rangle$ be a partial equilogical $(\mathbb{T},\mathsf{V})$-space. Define $(A,(X,a),E_A)$ by $A = \{x \in X \mid x \equiv_X x\}/{\equiv_X}$, and $E_A([x]) = [x] \subseteq \mathscr{P}X$. Hence

$F(A,(X,a),E_A) = \langle (X,a), \equiv_X \rangle$, since for each $x_1, x_2 \in X$, $x_1 \equiv_X x_2$ if, and only if, there exists $[x] \in A$ such that $x_1, x_2 \in [x]$. $\square$

This proof remains valid if we replace $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{inj}}$ with $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{sep,inj}}$, hence

$$\mathsf{Mdst}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{sep,inj}}) \cong (\mathbb{T},\mathsf{V})\text{-}\mathsf{PEqu}_{\text{sep}} \cong (\mathbb{T},\mathsf{V})\text{-}\mathsf{Equ}_{\text{sep}}.$$

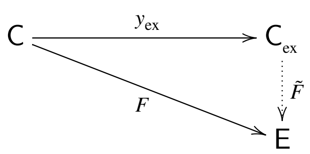
Moreover, Propositions 6.0.6, 6.0.8 and 6.0.10 remain valid under the extra assumption on separation, so we can conclude that $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Equ}_{\text{sep}}$ is finitely complete and cartesian closed, as we have already established, and, furthermore, that it is a regular category.

# 7 Equilogical $(\mathbb{T},\mathsf{V})$-spaces and exact completion

Motivated by [Sco96], among other results, Rosický proved in [Ros99] that the category of equilogical spaces can be obtained as a full reflective subcategory of the exact completion of Top, which is a cartesian closed category, what follows also from his result that Top is weakly cartesian closed. In this section we investigate these facts for equilogical $(\mathbb{T},\mathsf{V})$-spaces.

Let us recall that a category is said to be *exact* in the sense of Barr [Bar71] if it is finitely complete, regular epimorphisms are stable under pullback, every kernel pair admits a coequalizer, and every internal equivalence relation is a kernel pair. The first three conditions define a *regular* category.

For a category $\mathsf{C}$ with finite limits, its (free) exact completion $\mathsf{C}_{\text{ex}}$ is defined in [CM82] as an exact category, in the sense of Barr, with an embedding $y_{\text{ex}} : \mathsf{C} \to \mathsf{C}_{\text{ex}}$ which is a *lex* functor, that is, preserves finite limits, and, moreover, it is universal among lex functors into exact categories in the folllowing sense: for each lex functor $F : \mathsf{C} \to \mathsf{E}$, with $\mathsf{E}$ an exact category, there exists a unique exact functor $\tilde{F} : \mathsf{C}_{\text{ex}} \to \mathsf{E}$ such that the triangle:



(III.5)

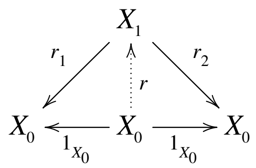
is commutative.

As explained by the authors, denoting the 2-category of exact categories and exact functors by Ex and the 2-category of left exact categories and left exact functors by Lex, the construction of $\mathsf{C}_{\text{ex}}$ defines a left biadjoint to the inclusion $\mathsf{Ex} \hookrightarrow \mathsf{Lex}$. The word “free” for the exact completion is then justified, since the latter inclusion of categories "forgets" all the exactness properties except for finite completeness. We refer to [Car95, CV98] for details.

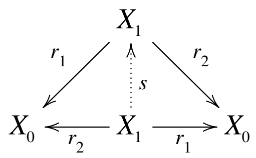
Following [CM82], we can describe the exact completion $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{ex}}$ by:

• objects are *pseudo equivalence relations* on $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$, that is, parallel pairs of $(\mathbb{T},\mathsf{V})$-continuous maps $X_1 \underset{r_2}{\overset{r_1}{\rightrightarrows}} X_0$ satisfying

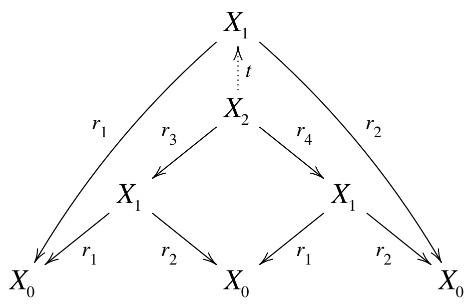
(1) *reflexivity*: there exists a continuous map $r\colon X_0 \to X_1$ such that $r_1 \cdot r = 1_{X_0} = r_2 \cdot r$;




(2) *symmetry*: there exists a continuous map $s\colon X_1 \to X_1$ such that $r_1 \cdot s = r_2$ and $r_2 \cdot s = r_1$;

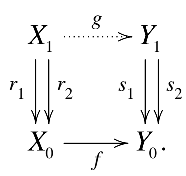



(3) *transitivity*: for $r_3, r_4\colon X_2 \to X_1$ a pullback of $r_1, r_2$, there exists a continuous map $t\colon X_2 \to X_1$ that makes the diagram

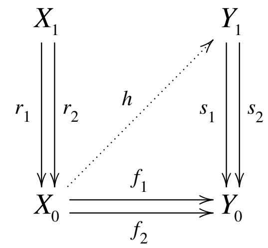



commutative.

• A morphism from $X_1 \underset{r_2}{\overset{r_1}{\rightrightarrows}} X_0$ to $Y_1 \underset{s_2}{\overset{s_1}{\rightrightarrows}} Y_0$ is an equivalence class $[f]$ of a continuous map $f\colon X_0 \to Y_0$ such that there exists $g\colon X_1 \to Y_1$ continuous satisfying $f \cdot r_i = s_i \cdot g$, $i = 1,2$.



Here two maps $f_1, f_2 \colon X_0 \to Y_0$ are related if, and only if, there exists $h\colon X_0 \to Y_1$ continuous such that $f_i = s_i \cdot h$, $i = 1,2$.



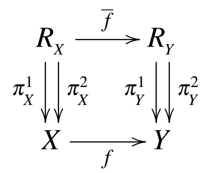

A pseudo equivalence relation $X_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} X_0$ is an (internal) *equivalence relation* if the pairing morphism $\langle r_1, r_2\rangle \colon X_1 \to X_0 \times X_0$ is a monomorphism, that is, a subobject of $X_0 \times X_0$.

As already observed, $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ has a stable factorization system $(\mathrm{Epi}, \mathrm{RegMono})$, and following the lines of [BCRS98] we consider the full subcategory of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathrm{ex}}$ of those pseudo equivalence relations $X_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} X_0$ such that the morphism $\langle r_1, r_2\rangle \colon X_1 \to X_0 \times X_0$ is a regular monomorphism. Denoting this subcategory by $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}, \mathrm{RegMono})$, we have:

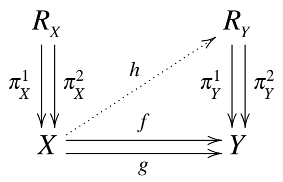
**Lemma 7.0.1** *$(\mathbb{T},\mathsf{V})$-$\mathsf{Equ}$ and $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}, \mathrm{RegMono})$ are equivalent.*

*Proof.* Each equilogical $(\mathbb{T},\mathsf{V})$-space $\mathscr{X} = \langle (X,a), \equiv_X\rangle$ induces the pseudo equivalence relation $R_X \overset{\pi_X^1}{\underset{\pi_X^2}{\rightrightarrows}} X$, where $R_X = \{(x,x') \in X \times X \mid x \equiv_X x'\}$, and $\pi_X^1, \pi_X^2$ are the restrictions to $R_X$ of the respective product projections $X \times X \to X$. Then $\langle \pi_1, \pi_2\rangle = i_{R_X} \colon R_X \hookrightarrow X \times X$ is a regular monomorphism when $R_X$ is endowed with the $|\text{-}|$-initial $(\mathbb{T},\mathsf{V})$-structure with respect to $i_{R_X}$.

For an equilogical $(\mathbb{T},\mathsf{V})$-space $\mathscr{Y} = \langle (Y,b), \equiv_Y\rangle$, a $(\mathbb{T},\mathsf{V})$-continuous map $f\colon (X,a) \to (Y,b)$ is equivariant if, and only if, $(x,x') \in R_X$ implies $(f(x), f(x')) \in R_Y$, and this defines a continuous map $\overline{f}\colon R_X \to R_Y$, which is the (co)restriction of $f \times f$ to $R_X$ and $R_Y$, that makes the diagram




commutative, so that $\pi_Y^i \cdot \overline{f} = f \cdot \pi_X^i$, $i = 1,2$. Moreover, two equivariant maps $f, g\colon (X,a) \to (Y,b)$ are related in $(\mathbb{T},\mathsf{V})$-$\mathsf{Equ}$ if, and only if, $(x,x') \in R_X$ implies $(f(x), g(x')) \in R_Y$. This defines a continuous map $h\colon X \to R_Y$, which is the corestriction of $\langle f,g\rangle$ to $R_Y$, that makes the diagram




commutative, so that $\pi_Y^1 \cdot h = f$, $\pi_Y^2 \cdot h = g$. Hence the class $[f]$ is the same in both categories, and $[f]$ is a morphism in $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$.

This defines a fully faithful functor $F\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Equ} \to \mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$, assigning to each equilogical $(\mathbb{T},\mathsf{V})$-space $\mathscr{X}$ the pseudo equivalence relation $R_X \overset{\pi_X^1}{\underset{\pi_X^2}{\rightrightarrows}} X$ , and leaving morphisms unchanged. To see that $F$ is essentially surjective, let $X_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} X_0$ be a pseudo equivalence relation with $\langle r_1,r_2\rangle\colon X_1 \to X_0 \times X_0$ a regular monomorphism. Then $\left\langle (X_0,a_0),\equiv_{X_0}\right\rangle$, where

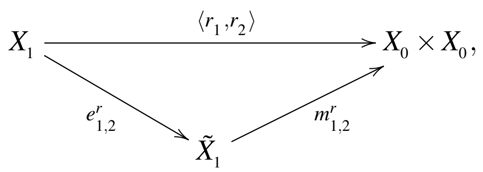
$$x_0 \equiv_{X_0} x_0' \iff \exists\, x_1 \in X_1;\ (r_1(x_1) = x_0\ \&\ r_2(x_1) = x_0'),$$

is an equilogical $(\mathbb{T},\mathsf{V})$-space, and, furthermore, since $\langle r_1,r_2\rangle$ is a regular monomorphism, the identity morphism $[1_{X_0}]$ is an isomorphism between $R_{X_0} \overset{\pi_{X_0}^1}{\underset{\pi_{X_0}^2}{\rightrightarrows}} X_0$ and $X_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} X_0$ . □

Furthermore, as a particular instance of [BCRS98, Theorem 4.3] we prove the following:

**Proposition 7.0.2** $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$ *is a reflective subcategory of* $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{ex}}$*, and the reflector preserves finite products and change of base in the codomain.*

*Proof.* Let us define $R\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{ex}} \to \mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$ in the following way: for each $X_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} X_0$ in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{ex}}$, factorize the pairing morphism $\langle r_1,r_2\rangle$ in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ as an epimorphism followed by a regular monomorphism




and set $R\left( X_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} X_0 \right) = \left( \tilde{X}_1 \overset{\tilde{r}_1}{\underset{\tilde{r}_2}{\rightrightarrows}} X_0 \right)$, where $\tilde{r}_1 = \pi_{X_0}^1 \cdot m^r_{1,2}$, $\tilde{r}_2 = \pi_{X_0}^2 \cdot m^r_{1,2}$; this factorization is unique up to isomorphism, and $\langle \tilde{r}_1,\tilde{r}_2\rangle = m^r_{1,2}$ is a regular monomorphism.

Let $[f]\colon (X_1,X_0,r_1,r_0)\to(Y_1,Y_0,s_1,s_2)$ be a morphism in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathrm{ex}}$, so there exists a continuous $g\colon X_1\to Y_1$ such that $s_i\cdot g=f\cdot r_i$, $i=1,2$. Since $e^r_{1,2}\colon X_1\to\tilde{X}_1$ is an epimorphism, there exists a unique map $\tilde{g}\colon\tilde{X}_1\to\tilde{Y}_1$ such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\;g\;} & Y_1 \\ {\scriptstyle e^r_{1,2}}\downarrow & & \downarrow{\scriptstyle e^s_{1,2}} \\ \tilde{X}_1 & \xrightarrow[\tilde{g}]{} & \tilde{Y}_1 \end{array}$$

is commutative; for $i=1,2$, $\tilde{s}_i\cdot\tilde{g}\cdot e^r_{1,2}=\pi^i_{Y_0}\cdot m^s_{1,2}\cdot e^s_{1,2}\cdot g=\pi^i_{Y_0}\cdot\langle s_1,s_2\rangle\cdot g=s_i\cdot g=f\cdot r_i=f\cdot\tilde{r}_i\cdot e^r_{1,2}$, that is, $\tilde{s}_i\cdot\tilde{g}=f\cdot\tilde{r}_i$, so that the diagram

$$\begin{array}{ccc} \tilde{X}_1 & \overset{\tilde{g}}{\dashrightarrow} & \tilde{Y}_1 \\ {\scriptstyle \tilde{r}_1}\downarrow\downarrow{\scriptstyle \tilde{r}_2} & & {\scriptstyle \tilde{s}_1}\downarrow\downarrow{\scriptstyle \tilde{s}_2} \\ X_0 & \xrightarrow[f]{} & Y_0 \end{array}$$

is commutative, and this implies that $\tilde{g}$ is continuous, because $\tilde{Y}_1$ has the $|\text{-}|$-initial $(\mathbb{T},\mathsf{V})$-structure with respect to $\langle\tilde{s}_1,\tilde{s}_2\rangle$. Hence we have a morphism $[f]\colon(\tilde{X}_1,X_0,\tilde{r}_1,\tilde{r}_0)\to(\tilde{Y}_1,Y_0,\tilde{s}_1,\tilde{s}_2)$, and the functor $R$ leaves the morphisms unchanged. The restriction of $R$ to $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$ is the identity functor. Since the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\;e^r_{1,2}\;} & \tilde{X}_1 \\ {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} & & {\scriptstyle \tilde{r}_1}\downarrow\downarrow{\scriptstyle \tilde{r}_2} \\ X_0 & \xrightarrow[1_{X_0}]{} & X_0 \end{array}$$

is commutative, the identity map $1_{X_0}$ induces a morphism $[1_{X_0}]\colon(X_1,X_0,r_1,r_2)\to(\tilde{X}_1,X_0,\tilde{r}_1,\tilde{r}_2)$ which serves as the coreflection: for each morphism $[f]$ with $(Y_1,Y_0,s_1,s_2)\in\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$, the following diagram is commutative.

$$\begin{array}{ccc} (X_1,X_0,r_1,r_2) & \xrightarrow{\;[1_{X_0}]\;} & (\tilde{X}_1,X_0,\tilde{r}_1,\tilde{r}_2) \\ & {\scriptstyle [f]}\searrow & \vdots\;{\scriptstyle \exists\,!\,[f]} \\ & & (Y_1,Y_0,s_1,s_2) \end{array}$$

Let us proceed proving that $R$ preserves finite products and change of base in the codomain.

*Finite Products* The terminal object of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathrm{ex}}$ is given by $(\{(*,*)\},\top)\rightrightarrows(\{*\},\top)$, which belongs to $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$. For pseudo equivalence relations $X_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} X_0$ and $Y_1 \overset{s_1}{\underset{s_2}{\rightrightarrows}} Y_0$, we form the binary product $X_1 \times Y_1 \overset{r_1\times s_1}{\underset{r_2\times s_2}{\rightrightarrows}} X_0 \times Y_0$. Applying the functor $R$ we obtain

$$R\left( X_1 \times Y_1 \overset{r_1\times s_1}{\underset{r_2\times s_2}{\rightrightarrows}} X_0 \times Y_0 \right) = (\widetilde{X_1 \times Y_1}, X_0 \times Y_0, \widetilde{r_1 \times s_1}, \widetilde{r_2 \times s_2}),$$

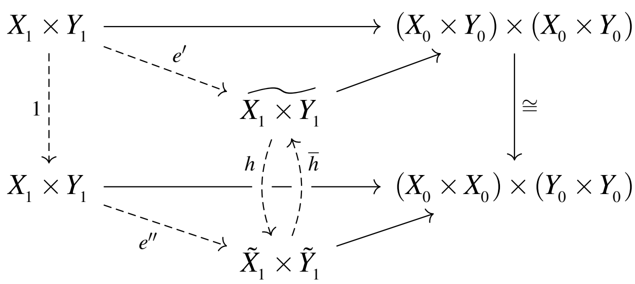
and

$$R\left( X_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} X_0 \right) \times R\left( Y_1 \overset{s_1}{\underset{s_2}{\rightrightarrows}} Y_0 \right) = (\tilde{X}_1 \times \tilde{Y}_1, X_0 \times Y_0, \tilde{r}_1 \times \tilde{s}_1, \tilde{r}_2 \times \tilde{s}_2).$$

Now we just observe that the identity map $1_{X_0\times Y_0}$ induces the intended isomorphism. This follows from commutativity of the diagram

$$\begin{array}{ccccc}
\widetilde{X_1 \times Y_1} & \xrightarrow{\quad h \quad} & \tilde{X}_1 \times \tilde{Y}_1 & \xrightarrow{\quad \overline{h} \quad} & \widetilde{X_1 \times Y_1} \\
{\scriptstyle \widetilde{r_1\times s_1}}\downdownarrows{\scriptstyle \widetilde{r_2\times s_2}} & & {\scriptstyle \tilde{r}_1\times\tilde{s}_1}\downdownarrows{\scriptstyle \tilde{r}_2\times\tilde{s}_2} & & {\scriptstyle \widetilde{r_1\times s_1}}\downdownarrows{\scriptstyle \widetilde{r_2\times s_2}} \\
X_0 \times Y_0 & \xrightarrow[1_{X_0\times Y_0}]{} & X_0 \times Y_0 & \xrightarrow[1_{X_0\times Y_0}]{} & X_0 \times Y_0,
\end{array}$$

where $h$ and $\overline{h}$ are defined so that the dashed squares (and consequently the diagram) below




are commutative, where $e', e''$ are epimorphisms.

*Change of base in the codomain* Let us consider morphisms $[f]\colon (X_1, X_0, r_1, r_0) \to (Z_1, Z_0, t_1, t_2)$ and $[g]\colon (Y_1, Y_0, s_1, s_2) \to (Z_1, Z_0, t_1, t_2)$ in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{ex}}$, with $(Z_1, Z_0, t_1, t_2)$ in $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$. For each $z_0, z_0' \in Z_0$, let us set

$$z_0 \sim_{Z_0} z_0' \iff \exists\, z_1 \in Z_1;\ (t_1(z_1) = z_0\ \&\ t_2(z_1) = z_0').$$

Then the pullback of $[f]$ along $[g]$ is given by

$$\begin{array}{ccc} (X_1 *_{Z_1} Y_1, X_0 *_{Z_0} Y_0, r_1 \times s_1, r_2 \times s_2) & \xrightarrow{[\pi_{Y_0}]} & (Y_1, Y_0, s_1, s_2) \\ {\scriptstyle [\pi_{X_0}]}\downarrow & & \downarrow{\scriptstyle [g]} \\ (X_1, X_0, r_1, r_0) & \xrightarrow[{[f]}]{} & (Z_1, Z_0, t_1, t_2), \end{array} \tag{III.6}$$

where $X_0 *_{Z_0} Y_0 = \{(x_0, y_0) \in X_0 \times Y_0 \mid f(x_0) \sim_{Z_0} g(y_0)\}$, and

$$X_1 *_{Z_1} Y_1 = \{(x_1, y_1) \in X_1 \times Y_1 \mid f \cdot r_1(x_1) \sim_{Z_0} g \cdot s_2(y_1) \ \& \ f \cdot r_2(x_1) \sim_{Z_0} g \cdot s_1(y_1)\},$$

with $\pi_{X_0}, \pi_{Y_0}$ restrictions to $X_0 *_{Z_0} Y_0$ of the product projections from $X_0 \times Y_0$ into $X_0$ and $Y_0$, respectively. Observe that $X_1 *_{Z_1} Y_1 \overset{r_1 \times s_1}{\underset{r_2 \times s_2}{\rightrightarrows}} X_0 *_{Z_0} Y_0$ are well-defined continuous maps, for they are the (co)restrictions of the continuous maps $X_1 \times Y_1 \overset{r_1 \times s_1}{\underset{r_2 \times s_2}{\rightrightarrows}} X_0 \times Y_0$ to $X_1 *_{Z_1} Y_1$ and $X_0 *_{Z_0} Y_0$, which are endowed with $|\text{-}|$-initial $(\mathbb{T},\mathsf{V})$-structures with respect to inclusion maps. Applying the reflector $R$ to (III.6) we obtain

$$\begin{array}{ccc} (\widetilde{X_1 *_{Z_1} Y_1}, X_0 *_{Z_0} Y_0, \widetilde{r_1 \times s_1}, \widetilde{r_2 \times s_2}) & \xrightarrow{[\pi_{Y_0}]} & (\tilde{Y}_1, Y_0, \tilde{s}_1, \tilde{s}_2) \\ {\scriptstyle [\pi_{X_0}]}\downarrow & & \downarrow{\scriptstyle [g]} \\ (\tilde{X}_1, X_0, \tilde{r}_1, \tilde{r}_0) & \xrightarrow[{[f]}]{} & (Z_1, Z_0, t_1, t_2), \end{array}$$

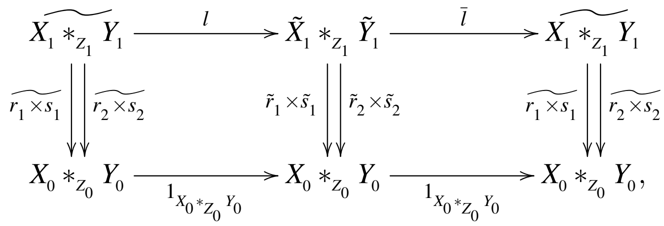
and pulling back $[f]\colon (\tilde{X}_1, X_0, \tilde{r}_1, \tilde{r}_0) \to (Z_1, Z_0, t_1, t_2)$ along $[g]\colon (\tilde{Y}_1, Y_0, \tilde{s}_1, \tilde{s}_2) \to (Z_1, Z_0, t_1, t_2)$ we get

$$\begin{array}{ccc} (\tilde{X}_1 *_{Z_1} \tilde{Y}_1, X_0 *_{Z_0} Y_0, \tilde{r}_1 \times \tilde{s}_1, \tilde{r}_2 \times \tilde{s}_2) & \xrightarrow{[\pi_{Y_0}]} & (\tilde{Y}_1, Y_0, \tilde{s}_1, \tilde{s}_2) \\ {\scriptstyle [\pi_{X_0}]}\downarrow & & \downarrow{\scriptstyle [g]} \\ (\tilde{X}_1, X_0, \tilde{r}_1, \tilde{r}_0) & \xrightarrow[{[f]}]{} & (Z_1, Z_0, t_1, t_2). \end{array}$$

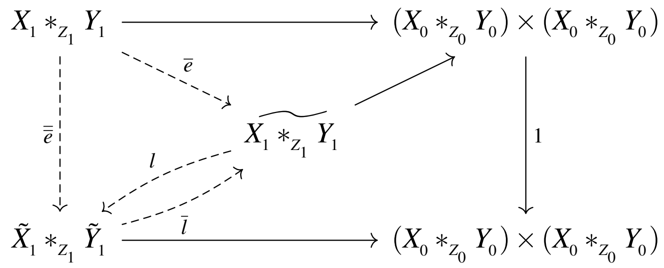
One can see then that the identity map $1_{X_0 *_{Z_0} Y_0}$ induces an isomorphism

$$\left( \widetilde{X_1 *_{Z_1} Y_1} \overset{\widetilde{r_1 \times s_1}}{\underset{\widetilde{r_2 \times s_2}}{\rightrightarrows}} X_0 *_{Z_0} Y_0 \right) \cong \left( \tilde{X}_1 *_{Z_1} \tilde{Y}_1 \overset{\tilde{r}_1 \times \tilde{s}_1}{\underset{\tilde{r}_2 \times \tilde{s}_2}{\rightrightarrows}} X_0 *_{Z_0} Y_0 \right),$$

what follows from the commutativity of the diagram



where $l$ and $\bar{l}$ are defined so that the dashed triangles below



are commutative, where $\overline{e},\overline{\overline{e}}$ are epimorphisms. □

Conditions under which the exact completion of a category is cartesian closed were studied by Rosický:

**Theorem 7.0.3** [Ros99, Theorem 1, Lemma 4] *Let* $\mathsf{C}$ *be a complete, infinitely extensive, well-powered, and weakly cartesian closed category, in which every morphism factorizes as a regular epimorphism followed by a monomorphism. Moreover, assume that for any regular epimorphism* $f\colon A \to A'$ *of* $\mathsf{C}$, $f \times 1_B\colon A \times B \to A' \times B$ *is an epimorphism, for each* $B \in \mathsf{C}$. *Then the exact completion* $\mathsf{C}_{\mathrm{ex}}$ *is a cartesian closed category.*

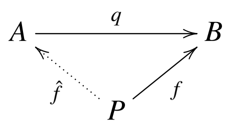
**Corollary 7.0.4** *The exact completion* $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathrm{ex}}$ *is a cartesian closed category.*

*Proof.* This follows from the properties **(TA1)**, **(TA2)**, **(TA5)**, and **(TA6)** of Subsection 2.4, and by Theorems 4.0.3 and 4.0.4, since we are assuming that injective $(\mathbb{T},\mathsf{V})$-spaces are exponentiable in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$. □

Since $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$ is fully reflective in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathrm{ex}}$ and the reflector preserves finite products, by [Sch84, Theorem 1.2], $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$ is a cartesian closed category. Therefore, by Lemma 7.0.1, $(\mathbb{T},\mathsf{V})$-$\mathsf{Equ}$ is a cartesian closed category.

Let us recall that an object $P$ of a category $\mathsf{C}$ is called *(regular) projective* if for each (regular) epimorphism $q\colon A \to B$ and each morphism $f\colon P \to B$ in $\mathsf{C}$, there exists a morphism $\hat{f}\colon P \to A$ in $\mathsf{C}$

such that $q \cdot \hat{f} = f$.



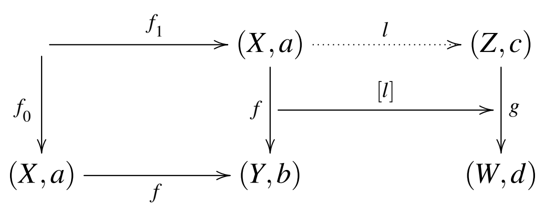
For a finitely complete category $\mathsf{C}$, the exact completion $\mathsf{C}_{ex}$ gives a category with "honest" limits which has $\mathsf{C}$ as a *projective cover* [CV98]. This means that $\mathsf{C}$ is the category of *regular projectives* of $\mathsf{C}_{ex}$ and each object of $\mathsf{C}_{ex}$ is *covered* by a regular projective, that is, there exists a regular epimorphism whose codomain is the object and the domain is a regular projective [CV98, Definition 2]. Locally cartesian closedness of exact completions was studied by Carboni and Rosolini:

**Theorem 7.0.5** [CR00, Theorem 3.3] *Suppose* $\mathsf{P}$ *is the category of regular projectives of a category* $\mathsf{C}$*, and that every object of* $\mathsf{C}$ *is covered by an object of* $\mathsf{P}$*. Then* $\mathsf{P}$ *is weakly locally cartesian closed if, and only if,* $\mathsf{C}$ *is locally cartesian closed.*

By Corollary 4.0.8, we conclude that $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{ex}$ is a locally cartesian closed category. Since $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$ is fully reflective in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{ex}$ and the reflector preserves change of base in the codomain, by [HST14, III-Corollary 4.6.2], and conclude that $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$ is a locally cartesian closed category. By Lemma 7.0.1, $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Equ}$ is a locally cartesian closed category.

As discussed in [Ros98], the *free regular completion* $\mathsf{C}_{reg}$ of a category $\mathsf{C}$ can be obtained as the full subcategory of $\mathsf{C}_{ex}$ of those pseudo equivalence relations which are kernel pairs in $\mathsf{C}$. We recall from [CV98] that $\mathsf{C}_{reg}$ is a regular category with an embedding $y_{reg}\colon \mathsf{C} \to \mathsf{C}_{reg}$ which preserves finite limits, and has a similar universal property as the one in (III.5), but with respect to regular categories instead of exact ones.

Furthermore, following [Car95, Section 5], we can describe the regular completion $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{reg}$ by: objects are $(\mathbb{T},\mathsf{V})$-continuous maps $f\colon (X,a) \to (Y,b)$, and a morphism from $f\colon (X,a) \to (Y,b)$ to $g\colon (Z,c) \to (W,d)$ is an equivalence class $[l]$ of a $(\mathbb{T},\mathsf{V})$-continuous map $l\colon (X,a) \to (Z,c)$ such that $g \cdot l \cdot f_0 = g \cdot l \cdot f_1$, where $f_0, f_1$ form the kernel pair of $f$;



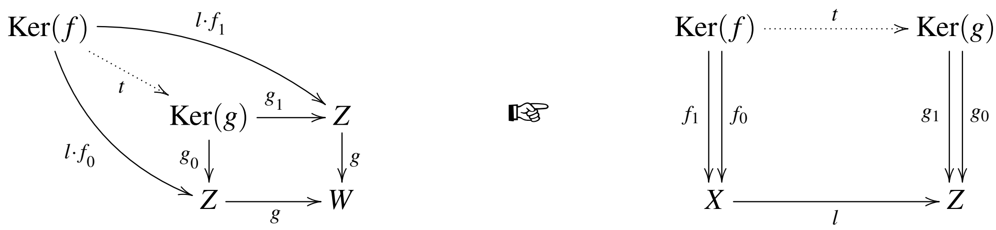
two such arrows $l$ and $m$ are equivalent if $g\cdot l = g\cdot m$. It is immediate to prove the following:

**Lemma 7.0.6** $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{reg}}$ *and* $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$ *are equivalent.*

*Proof.* Define $F\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{reg}} \to \mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$ by

$$\begin{array}{ccc} (f\colon (X,a)\to (Y,b)) & \longmapsto & (\mathrm{Ker}(f),X,f_0,f_1) \\ \big\downarrow{\scriptstyle [l]} & & \big\downarrow{\scriptstyle [l]} \\ (g\colon (Z,c)\to (W,d)) & \longmapsto & (\mathrm{Ker}(g),Z,g_0,g_1). \end{array}$$

Indeed, for each continuous map $f$, the kernel pair $(\mathrm{Ker}(f),X,f_0,f_1)$ is a (pseudo) equivalence relation, where $\langle f_0,f_1\rangle\colon \mathrm{Ker}(f)\to X\times X$ is a regular monomorphism. Moreover, for a continuous map $l\colon X\to Z$, if $g\cdot l\cdot f_0 = g\cdot l\cdot f_1$, then, by the universal property of the pullback, there exists a unique continuous map $t\colon \mathrm{Ker}(f)\to\mathrm{Ker}(g)$ such that $g_i\cdot t = l\cdot f_i$, $i=0,1$, so that $l$ is an equivariant map in $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$.




Conversely, if there exists $t\colon \mathrm{Ker}(f)\to\mathrm{Ker}(g)$ such that $g_i\cdot t = l\cdot f_i$, $i=0,1$, then

$$g\cdot l\cdot f_0 = g\cdot g_0\cdot t = g\cdot g_1\cdot t = g\cdot l\cdot f_1.$$

Hence $F$ is well-defined and easily seen to be fully faithful. For essential surjectivity, we just observe that each object $X_1 \underset{r_2}{\overset{r_1}{\rightrightarrows}} X_0$ of $\mathrm{PER}((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat},\mathrm{RegMono})$ is the kernel pair of the projection map $p_{X_0}\colon X_0\to X_0/{\sim}$, where

$$x_0\sim x_0' \iff \exists\, x_1\in X_1;\ (r_1(x_1)=x_0\ \&\ r_2(x_1)=x_0'),$$

and $X_0/{\sim}$ is endowed with the $|\text{-}|$-final $(\mathbb{T},\mathsf{V})$-structure with respect to $p_{X_0}$. □

By Lemma 7.0.1, $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Equ}$ is (equivalent to) the regular completion of $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$. This is discussed in [Ros98] (see also [Men00]) for the particular case of $\mathsf{Top}$. Furthermore, conditions under which regular completions are quasitoposes were studied in [Men00].

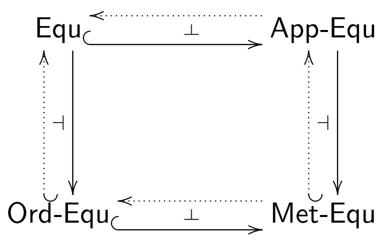
By **(TA1)** and **(TA7)** of Subsection 2.4 and by Corollary 4.0.8, $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is complete, infinitely extensive, and weakly locally cartesian closed, since we are assuming that injective spaces are exponentiable. Moreover, by **(TA5)**, $\mathsf{Set}$ is a *mono-localization* of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, that is, there exists an adjunction $|\text{-}| \dashv \nabla \colon \mathsf{Set} \to (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$, where $\nabla$ is an embedding and the forgetful functor $|\text{-}| \colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{Set}$, which is trivially faithful, preserves finite limits. Hence, by [Men00, Corollary 8.4.2], $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\text{reg}}$ is a quasitopos, hence we conclude:

**Theorem 7.0.7** $(\mathbb{T},\mathsf{V})$-$\mathsf{Equ}$ *is a quasitopos.*

**Remark 7.0.8** By Remark 4.0.9, the categories of Table (I.25) are such that their exact completions are locally cartesian closed categories and their categories of equilogical spaces are quasitoposes.

For example, in the simplest case of $\mathsf{Ord}$, an object of $\mathsf{Ord}$-$\mathsf{Equ}$, or an equilogical ordered space, is an ordered set $(X,\leq)$ with an equivalence relation $\equiv_X$ on $X$; $(X,\leq)$ is separated if, and only if, it is anti-symmetric; then the objects of $\mathsf{Ord}\text{-}\mathsf{Equ}_{\text{sep}}$ are partially ordered sets with equivalence relations on the underlying sets. Furthermore, a partial equilogical separated ordered space, i.e., an object of $\mathsf{Ord}\text{-}\mathsf{PEqu}_{\text{sep}}$, is a complete lattice (an injective partially ordered set [AHS90, Examples 9.3]) with a partial equivalence relation on its underlying set. By the same argument, the objects of $\mathsf{Mdst}(\mathsf{Ord}_{\text{sep,inj}})$ are triples $(A,(X,\leq),E_A)$, where $A$ is a set, $E_A : A \to \mathscr{P}X$ is a map such that, for each $a \in A$, $E_A(a) \neq \emptyset$, and $(X,\leq)$ is a complete lattice.

Moreover, the adjunctions described in diagram (I.14) lift to adjunctions in the respective categories of equilogical spaces:




• $\mathsf{Ord}$-$\mathsf{Equ}$ is fully embedded in $\mathsf{Met}$-$\mathsf{Equ}$ as the equilogical metric spaces $\langle (X,d),\equiv_X\rangle$ such that there exists an order $\leq$ on $X$ with $d = d_{\leq}$;

• $\mathsf{Ord}$-$\mathsf{Equ}$ is fully embedded in $\mathsf{Equ}$ as the equilogical spaces $\langle (X,\tau),\equiv_X\rangle$ for which there exists an order $\leq$ on $X$ such that $\tau = \tau_{\leq}$, hence they are the equilogical Alexandroff spaces;

• $\mathsf{Met}$-$\mathsf{Equ}$ is fully embedded in $\mathsf{App}$-$\mathsf{Equ}$ as equilogical metric approach spaces;

• $\mathsf{Equ}$ is fully embedded in $\mathsf{App}$-$\mathsf{Equ}$ as equilogical topological approach spaces.

# Chapter IV

# Compactly generated $(\mathbb{T},\mathsf{V})$-spaces and quasi-$(\mathbb{T},\mathsf{V})$-spaces

In this final chapter we introduce and study compactly generated spaces, and, more generally, $\mathscr{C}$-generated spaces, and quasi-spaces in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$. We generalise both concepts from $\mathsf{Top}$ to $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, following closely the lines of [ELS04] and [Day68], respectively. The results of this chapter can be found in [Rib19b], where some examples and results are further developed.

## 8 $\mathscr{C}$-generated $(\mathbb{T},\mathsf{V})$-spaces

Let $\mathscr{C}$ denote a class of $(\mathbb{T},\mathsf{V})$-spaces which contains at least one non-empty element. For instance, $\mathscr{C}$ is considered afterwards in this section as the class of compact Hausdorff $(\mathbb{T},\mathsf{V})$-spaces, as the singleton set containing the Sierpiński $(\mathbb{T},\mathsf{V})$-space, as the class of injective $(\mathbb{T},\mathsf{V})$-spaces, and as the one of exponentiable $(\mathbb{T},\mathsf{V})$-spaces, respectively. Throughout we will make use of the (topological) forgetful functor $|\text{-}|\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{Set}$.

### 8.1 The category $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathscr{C}}$

**Definition 8.1.1** Each element of $\mathscr{C}$ is called a *generating space*. A $(\mathbb{T},\mathsf{V})$-continuous map from a generating space to a $(\mathbb{T},\mathsf{V})$-space $(X,a)$ is called a *probe over* $(X,a)$, or simply a *probe*. For a $(\mathbb{T},\mathsf{V})$-space $(X,a)$, the $|\text{-}|$-final $(\mathbb{T},\mathsf{V})$-structure $a^c$ on $X$ with respect to all probes over $(X,a)$ is called the *$\mathscr{C}$-generated structure* on $(X,a)$. A $(\mathbb{T},\mathsf{V})$-space $(X,a)$ is called *$\mathscr{C}$-generated* if $a = a^c$. The full subcategory of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ of $\mathscr{C}$-generated $(\mathbb{T},\mathsf{V})$-spaces is denoted by $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathscr{C}}$.

Therefore, $(X,a^c)$ is such that a map $t\colon (X,a^c)\to(Y,b)$, with $(Y,b)\in(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$, is $(\mathbb{T},\mathsf{V})$-continuous if, and only if, for every probe $p\colon C\to(X,a)$, the composite

$$C \xrightarrow{\quad p \quad} (X,a) \xrightarrow{\quad t \quad} (Y,b)$$

is $(\mathbb{T},\mathsf{V})$-continuous. It follows immediately that, for each $(\mathbb{T},\mathsf{V})$-space $(X,a)$, the identity map $1_X\colon (X,a^c)\to(X,a)$ is $(\mathbb{T},\mathsf{V})$-continuous, that is, $a^c\le a$. Moreover, each element $(D,d)$ of $\mathscr{C}$ is $\mathscr{C}$-generated: the identity map $1_D\colon (D,d)\to(D,d)$ is a probe, hence $d\le d^c$, and since $d^c\le d$, then $d^c=d$. The next step is to verify coreflectiveness of $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}}$ in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$.

**Lemma 8.1.2** *For a $\mathscr{C}$-generated space $(X,a)$, a space $(Y,b)$, and a map $f\colon X\to Y$, the following are equivalent:*
*(i) $f\colon (X,a)\to(Y,b^c)$ is $(\mathbb{T},\mathsf{V})$-continuous;*
*(ii) $f\colon (X,a)\to(Y,b)$ is $(\mathbb{T},\mathsf{V})$-continuous.*

*Proof.* For (i)⇒(ii) observe that $f\colon (X,a)\to(Y,b)$ is given by the composite

$$(X,a) \xrightarrow{\quad f \quad} (X,b^c) \xrightarrow{\quad 1_Y \quad} (Y,b).$$

Conversely, suppose that $f\colon (X,a)\to(Y,b)$ is continuous. Then, for each probe $p\colon C\to(X,a)$, $f\cdot p\colon C\to(Y,b)$ is continuous, hence a probe over $(Y,b)$. By $|\text{-}|$-finality of $b^c$, $f\cdot p\colon C\to(Y,b^c)$ is continuous, and therefore $f\colon (X,a)\to(Y,b^c)$ is continuous because $(X,a)$ is $\mathscr{C}$-generated. □

**Lemma 8.1.3** *For each space $(X,a)$, $(X,a^c)$ is a $\mathscr{C}$-generated space.*

*Proof.* As observed above, each probe $p\colon C\to(X,a)$ is a probe over $(X,a^c)$, and, by the same reasoning, it is a probe over $(X,(a^c)^c)$. Hence, for each probe $p\colon C\to(X,a)$, the composite $1_X\cdot p=p\colon C\to(X,(a^c)^c)$ is a continuous map, whence $1_X\colon (X,a^c)\to(X,(a^c)^c)$ is a continuous map.

$$\begin{array}{ccc} C & \xrightarrow{\quad p \quad} & (X,a) \\ & \searrow^{p} & \downarrow{\scriptstyle 1_X} \\ & & (X,(a^c)^c) \end{array} \qquad\qquad \begin{array}{c} (X,a^c) \\ \downarrow{\scriptstyle 1_X} \\ (X,(a^c)^c) \end{array}$$

Therefore, $a^c\le(a^c)^c$. Since $(a^c)^c\le a^c$, we conclude the result. □

**Proposition 8.1.4** $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}}$ *is coreflective in* $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$.

*Proof.* For each $(\mathbb{T},\mathsf{V})$-space $(Y,b)$, $(Y,b^c)\in(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}}$ and the identity map $1_Y\colon (Y,b^c)\to(Y,b)$ is continuous. Furthermore, every continuous map $f\colon (X,a)\to(Y,b)$, with $(X,a)$ a $\mathscr{C}$-generated space, factorizes through $1_Y$:

$$\begin{array}{ccc} (Y,b^c) & \xrightarrow{\;1_Y\;} & (Y,b). \\ {\scriptstyle f}\uparrow & \nearrow{\scriptstyle f} & \\ (X,a) & & \end{array}$$

The coreflector from $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ to $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}}$ assigns to each space $(X,a)$ the space $(X,a^c)$, and to each continuous map $f\colon (X,a)\to(Y,b)$ the continuous map $f\colon (X,a^c)\to(Y,b^c)$; the coreflections are given by identity maps.

$$(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}} \underset{\mathscr{C}}{\overset{\mathrm{Inc}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$$

□

This coreflection is shown for $\mathsf{Top}$ also in [Mac71]. As a corollary of this proposition, since $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ is complete and cocomplete, so is $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}}$. At this point we recall a fact about topological functors, and some conditions that will be needed throughout. Firstly, since the forgetful functor $|\text{-}|\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to\mathsf{Set}$ is fibre-small, the $|\text{-}|$-final lifting of a sink is the $|\text{-}|$-final lifting of a small (sub)sink. Secondly, we must assure that constant maps are continuous, hence, by Lemma 2.4.3, $\mathsf{V}$ must be integral and $T1=1$. We restrict the available examples to a sub-table of (I.39).

| V \ $\mathbb{T}$ | $\mathbb{I}$ | $\mathbb{U}$ |
|---|---|---|
| 2 | Ord | Top |
| $\mathsf{P}_+$ | Met | App |
| $\mathsf{P}_{\max}$ | UltMet | NA-App |
| $\mathsf{P}_1$ | BMet | $(\mathbb{U},\mathsf{P}_1)$-Cat |

(IV.1)

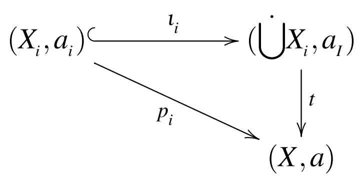
For those examples, $\mathsf{V}$ is lean and finite coproducts are preserved by $\mathbb{T}$, so we guarantee that all the properties from Subsection 2.8 needed afterwards for compact Hausdorff $(\mathbb{T},\mathsf{V})$-spaces are valid. We can now characterize $\mathscr{C}$-generated spaces as colimits of elements of $\mathscr{C}$.

**Proposition 8.1.5** *(1) Coproducts and coequalizers of $\mathscr{C}$-generated spaces are $\mathscr{C}$-generated.*

*(2) A space is $\mathscr{C}$-generated if, and only if, it is a colimit of elements of $\mathscr{C}$.*

*Proof.* (1) The inclusion functor $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}} \hookrightarrow (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ is a left adjoint, hence it preserves colimits.

(2) By item (1), colimits of generating spaces, which are $\mathscr{C}$-generated, are $\mathscr{C}$-generated. Let $(X,a)$ be a $\mathscr{C}$-generated space. Then $a = a^c$ is the $|\text{-}|$-final $(\mathbb{T},\mathsf{V})$-structure with respect to the sink $(p_i\colon (X_i,a_i) \to (X,a))_{i\in I}$ of probes over $(X,a)$ and $I$ can be considered a set, rather than a proper class, of indexes. Form the coproduct $(\dot{\bigcup}_{i\in I} X_i, a_I)$ in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$. By its universal property, the family of continuous maps $(p_i)_{i\in I}$ induces a continuous map $t\colon (\dot{\bigcup}_{i\in I} X_i, a_I) \to (X,a)$, with $t\cdot\iota_i = p_i$, for all $i\in I$, where $\iota_i$ is the coproduct inclusion of $X_i$ into $\dot{\bigcup}_{i\in I} X_i$.



We verify that $t$ is a regular epimorphism, or, equivalently, that it is a $|\text{-}|$-final surjection. Let $s\colon X\to Y$ be a map such that $s\cdot t$ is continuous. By the universal property of the coproduct, this is equivalent to $s\cdot t\cdot\iota_i$ being continuous, for all $i\in I$. Hence $s\cdot p_i$ is continuous, for all $i\in I$, and it follows that $s$ is continuous, since $(X,a)$ is $\mathscr{C}$-generated; therefore $t$ is a $|\text{-}|$-final morphism. For surjectivity of $t$, for each $x_0\in X$, consider a constant map $x_0\colon C_0\to(X,a)$, $c_0\mapsto x_0$, for some non-empty $C_0\in\mathscr{C}$, that we are assuming to exist. If we add these constant maps to the sink of probes, then $a^c$ continues to be $|\text{-}|$-final and the sink is still indexed by a set. Hence, without loss of generality, one can consider that these constant maps are already indexed by $I$. Therefore, for each $x\in X$, $x = p_i(x_i) = t(\iota_i(x_i))$, for some $i\in I$, $x_i\in X_i$. □

A thorough study of regular epimorphisms in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ is done in [Hof05].

## 8.2 $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}}$ is cartesian closed

Since a space is $\mathscr{C}$-generated if, and only if, it is a colimit of generating spaces, then every coreflective subcategory of $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ containing $\mathscr{C}$ must contain $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}}$. Moreover, since $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}}$ is coreflective in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ and contains $\mathscr{C}$, we conclude that $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}}$ is the *coreflective hull* of $\mathscr{C}$ in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$.

Hence the question of cartesian closedness of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathscr{C}}$ fits into the goals of [Nel78]. In this reference, the condition **(CEP)** used below was established. However, we follow the arguments of [ELS04], providing a direct approach to this subject.

**Definition 8.2.1** A map $f\colon (X,a)\to(Y,b)$ is *$\mathscr{C}$-continuous* if, for each probe $p\colon C\to(X,a)$, the composite $f\cdot p\colon C\to(Y,b)$ is continuous.

**Remark 8.2.2** We can readily see that the following are equivalent:
(i) $f\colon (X,a)\to(Y,b)$ is a $\mathscr{C}$-continuous map;
(ii) $f\colon (X,a^c)\to(Y,b)$ is a continuous map;
(iii) $f\colon (X,a^c)\to(Y,b^c)$ is a continuous map.
Furthermore, continuity implies $\mathscr{C}$-continuity, and the converse is true for maps defined on $\mathscr{C}$-generated spaces.

**Proposition 8.2.3** *(1)* $(\mathbb{T},\mathsf{V})$*-spaces and $\mathscr{C}$-continuous maps form a category* $\mathscr{C}$-Map.
*(2) For each space $(X,a)$, the identity map $1_X\colon (X,a^c)\to(X,a)$ is an isomorphism in* $\mathscr{C}$-Map.
*(3)* $\mathscr{C}$-Map *and* $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathscr{C}}$ *are equivalent categories.*

*Proof.* (1) Each identity map is continuous, hence $\mathscr{C}$-continuous. Let $f\colon (X,a)\to(Y,b)$ and $g\colon (Y,b)\to(Z,c)$ be $\mathscr{C}$-continuous maps, and consider a probe $p\colon C\to(X,a)$. By $\mathscr{C}$-continuity of $f$, $f\cdot p\colon C\to(Y,b)$ is continuous, hence it is a probe. Thus, by $\mathscr{C}$-continuity of $g$, $(g\cdot f)\cdot p=g\cdot(f\cdot p)$ is a continuous map, whence $g\cdot f$ is a $\mathscr{C}$-continuous map.
(2) $\mathscr{C}$-continuity of $1_X\colon (X,a^c)\to(X,a)$ follows from its continuity. Conversely, $1_X\colon (X,a^c)\to(X,a^c)$ is a continuous map, whence $1_X\colon (X,a)\to(X,a^c)$ is a $\mathscr{C}$-continuous map.
(3) The inclusion $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathscr{C}}\hookrightarrow\mathscr{C}$-Map is fully faithful. For each space $(X,a)$, $(X,a^c)\cong(X,a)$ in $\mathscr{C}$-Map, hence this inclusion is essentially surjective. □

As a corollary we have that $\mathscr{C}$-Map is a complete and cocomplete category.

**Lemma 8.2.4** *Let $(X,a),(Y,b),(Z,c)$ be $(\mathbb{T},\mathsf{V})$-spaces and $f\colon (X\times Y,a\times b)\to(Z,c)$ be a $\mathscr{C}$-continuous map. Then, for each $x\in X$, the map $f_x\colon (Y,b)\to(Z,c)$, $y\mapsto f(x,y)$, is $\mathscr{C}$-continuous.*

*Proof.* The map $f_x$ is the composition of the two $\mathscr{C}$-continuous maps:

$$(Y,b)\xrightarrow{\langle x,1_Y\rangle}(X\times Y,a\times b)\xrightarrow{\ f\ }(Z,c),$$

where $x\colon Y\to X$ is the constant map assigning $x$ to every $y\in Y$. □

This result establishes, for each $\mathscr{C}$-continuous map $f\colon X\times Y\to Z$, a *transpose* for $f$, denoted, as usual, by $\overline{f}\colon X\to\mathscr{C}\text{-Map}(Y,Z)$, $x\mapsto f_x$. We wish to endow $\mathscr{C}\text{-Map}(Y,Z)$ with a $(\mathbb{T},\mathsf{V})$-structure in such a way that $\mathscr{C}$-continuity of $f$ is equivalent to $\mathscr{C}$-continuity of $\overline{f}$. Let us assume that the following condition holds:

**(CEP)** *the elements of $\mathscr{C}$ are exponentiable spaces in* $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$, *and if $C_1,C_2\in\mathscr{C}$, then the binary product $C_1\times C_2$ in* $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ *is a $\mathscr{C}$-generated space.*

In [ELS04, Definition 3.5] the class $\mathscr{C}$ satisfying **(CEP)** is said to be *productive* (see also [Nel78]).

We define, for $(\mathbb{T},\mathsf{V})$-spaces $(Y,b),(Z,c)$, the required $(\mathbb{T},\mathsf{V})$-structure on $\mathscr{C}\text{-Map}(Y,Z)$. For each probe $q_j\colon (Y_j,b_j)\to(Y,b)$, form the exponential $(Z^{Y_j},d_j)$ in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$, which exists by **(CEP)**. Since $\mathsf{V}$ is an integral quantale, this exponential is given by

$$Z^{Y_j}=\{h\colon (Y_j,b_j)\to(Z,c)\mid h \text{ is a } (\mathbb{T},\mathsf{V})\text{-continuous map}\},$$

where the $(\mathbb{T},\mathsf{V})$-structure $d_j$ is the largest one that makes the evaluation map $\mathrm{ev}_{Z,Y_j}\colon Z^{Y_j}\times Y_j\to Z$ $(\mathbb{T},\mathsf{V})$-continuous; then we have a map

$$t_{q_j}\colon \mathscr{C}\text{-Map}(Y,Z)\to(Z^{Y_j},d_j),\quad g\mapsto g\cdot q_j,$$

which is well-defined, for if $g$ is $\mathscr{C}$-continuous, then, by definition, $g\cdot q_j$ is continuous. Consider the source $(t_{q_j}\colon \mathscr{C}\text{-Map}(Y,Z)\to(Z^{Y_j},d_j))_{j\in J}$ and its $|\text{-}|$-initial lifting in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$

$$(t_{q_j}\colon (\mathscr{C}\text{-Map}(Y,Z),d)\to(Z^{Y_j},d_j))_{j\in J},$$

hence $d=\bigwedge_{j\in J} t_{q_j}^\circ\cdot d_j\cdot Tt_{q_j}$. By definition, a map $h\colon (W,l)\to(\mathscr{C}\text{-Map}(Y,Z),d)$, for $(W,l)$ a $(\mathbb{T},\mathsf{V})$-space, is $(\mathbb{T},\mathsf{V})$-continuous if, and only if, for all probes $q_j\colon (Y_j,b_j)\to(Y,b)$, the composite $t_{q_j}\cdot h\colon (W,l)\to(Z^{Y_j},d_j)$ is continuous.

**Lemma 8.2.5** *Let $(X,a),(Y,b),(Z,c)$ be $(\mathbb{T},\mathsf{V})$-spaces. A map $f\colon (X\times Y,a\times b)\to(Z,c)$ is $\mathscr{C}$-continuous if, and only if, $\overline{f}\colon (X,a)\to(\mathscr{C}\text{-Map}(Y,Z),d)$ is $\mathscr{C}$-continuous.*

*Proof.* Assume that $f\colon X\times Y\to Z$ is $\mathscr{C}$-continuous and let $p\colon C\to(X,a)$ be a probe. We must prove that $\overline{f}\cdot p\colon C\to\mathscr{C}\text{-Map}(Y,Z)$ is a continuous map, hence consider a probe $q_j\colon (Y_j,b_j)\to(Y,b)$. There exists a natural bijection $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}(C,Z^{Y_j})\cong(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}(C\times Y_j,Z)$, and, for each $c\in C$, $y_j\in Y_j$,

$$(t_{q_j}\cdot\overline{f}\cdot p(c))(y_j)=\overline{f}\cdot p(c)(q_j(y_j))=f(p(c),q_j(y_j))=f\cdot(p\times q_j)(c,y_j).$$

Whence $t_{q_j} \cdot \overline{f} \cdot p$ corresponds to $f \cdot (p \times q_j) \colon C \times Y_j \to Z$, which is a continuous map, since $f \cdot (p \times q_j)$ is $\mathscr{C}$-continuous by hypothesis, and $C \times Y_j$ is $\mathscr{C}$-generated by **(CEP)**. Therefore $t_{q_j} \cdot \overline{f} \cdot p \colon C \to Z^{Y_j}$ is a continuous map.

Conversely, assume that $\overline{f} \colon (X,a) \to (\mathscr{C}\text{-Map}(Y,Z),d)$ is a $\mathscr{C}$-continuous map and let $p \colon C \to (X \times Y, a \times b)$ be a probe. We wish to prove that $f \cdot p \colon C \to Z$ is a continuous map. Composition with the product projections $\pi_X$ and $\pi_Y$ from $X \times Y$ into $X$ and $Y$, respectively, give us probes $p^X = \pi_X \cdot p \colon C \to X$ and $p^Y = \pi_Y \cdot p \colon C \to Y$. Then, by $\mathscr{C}$-continuity of $\overline{f}$, the composite $\overline{f} \cdot p^X \colon C \to \mathscr{C}\text{-Map}(Y,Z)$ is continuous, whence, by definition of $d$, $t_{p^Y} \cdot \overline{f} \cdot p^X \colon C \to Z^C$ is a continuous map. For each $c \in C$,

$$f \cdot p(c) = f \cdot \langle p^X, p^Y \rangle (c) = \overline{f}(p^X(c))(p^Y(c)) = \left(t_{p^Y} \cdot \overline{f} \cdot p^X(c)\right)(c),$$

from where we conclude that $f \cdot p$ is continuous. □

**Theorem 8.2.6** $\mathscr{C}$-Map *is a cartesian closed category.*

*Proof.* For each spaces $(Y,b),(Z,c)$, considering the evaluation map

$$\mathrm{ev}_{Y,Z} \colon (\mathscr{C}\text{-Map}(Y,Z) \times Y, d \times b) \to (Z,c),$$

then its transpose

$$\overline{\mathrm{ev}}_{Y,Z} = 1_{\mathscr{C}\text{-Map}(Y,Z)} \colon (\mathscr{C}\text{-Map}(Y,Z),d) \to (\mathscr{C}\text{-Map}(Y,Z),d)$$

is a ($\mathscr{C}$-)continuous map, whence $\mathrm{ev}_{Y,Z}$ is a $\mathscr{C}$-continuous map. Moreover, for each $\mathscr{C}$-continuous map $f \colon (X \times Y, a \times b) \to (Z,c)$, there exists a unique $\mathscr{C}$-continuous map $\overline{f} \colon (X,a) \to (\mathscr{C}\text{-Map}(Y,Z),d)$ such that

$$\begin{array}{ccc} \mathscr{C}\text{-Map}(Y,Z) & \qquad & \mathscr{C}\text{-Map}(Y,Z) \times Y \xrightarrow{\ \mathrm{ev}_{Y,Z}\ } Z \\ \exists\,!\,\overline{f} \uparrow & & \overline{f} \times 1_Y \uparrow \qquad \nearrow_{f} \\ X & & X \times Y \end{array}$$

is a commutative diagram. □

Since $\mathscr{C}$-Map $\cong (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}}$, we conclude:

**Corollary 8.2.7** $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}}$ *is a cartesian closed category.*

The exponential of $\mathscr{C}$-generated spaces $(X,a),(Y,b)$ is given by $(\mathscr{C}\text{-Map}((X,a),(Y,b)),d^c)$.

## 8.3 Compactly generated $(\mathbb{T},\mathsf{V})$-spaces

For the first example of $\mathscr{C}$-generated spaces, let us consider the class $\mathscr{C}$ of compact Hausdorff $(\mathbb{T},\mathsf{V})$-spaces. As usual, in this case the $\mathscr{C}$-generated spaces are called *compactly generated*. Under the conditions assumed for this chapter – $\mathsf{V}$ integral and lean, $T1 = 1$ – $\mathscr{C}$ satisfies condition **(CEP)**. Every compactly generated space is a coequalizer of a coproduct of compact Hausdorff spaces, and the full subcategory of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ of compactly generated spaces is cartesian closed.

Let us go through our examples in Table (IV.1). For $\mathsf{V}$-$\mathsf{Cat}$ with $\mathsf{V}$ integral, compact Hausdorff $\mathsf{V}$-spaces are precisely the discrete spaces, and coequalizers and coproducts of discrete spaces are discrete, so that $\mathsf{V}\text{-}\mathsf{Cat}_{\mathscr{C}} \cong \mathsf{Set}$.

For $\mathsf{Top} \cong (\mathbb{U},2)$-$\mathsf{Cat}$, in Subsection 2.8 we have seen that compactness and Hausdorff separation coincide with the classical notions. A space $(X,a)$ is compactly generated if, and only if, it is the quotient of a disjoint sum of compact Haudorff topological spaces, which is equivalent to being a quotient of a locally compact Hausdorff space. Sequential spaces, topological manifolds, and $CW$-complexes are examples of compactly generated topological spaces.

In the categories $\mathsf{NA}$-$\mathsf{App}$ and $\mathsf{App}$, compactly generated (non-Archimedean) approach spaces are the topological approach spaces induced by a compactly generated topology. This follows from the equivalences

$$(\mathbb{U},\mathsf{P}_{\max})\text{-}\mathsf{Cat}_{\mathrm{CompHaus}} \cong (\mathbb{U},\mathsf{P}_{+})\text{-}\mathsf{Cat}_{\mathrm{CompHaus}} \cong \mathsf{Set}^{\mathbb{U}}$$

given by (I.38), and from the fact that the embedding of $\mathsf{Top}$ into $\mathsf{App}$ corestricts to $\mathsf{NA}$-$\mathsf{App}$, and, furthermore, $\mathsf{Top}$ is coreflective in both categories [Low97, CVO17].

Concerning $(\mathbb{U},\mathsf{P}_1)$-$\mathsf{Cat}$, the quantale homomorphism $\iota\colon 2 \to \mathsf{P}_1$, given by $\iota(\bot) = 0$ and $\iota(\top) = 1$, which is *compatible* with the lax extensions of $\mathbb{U}$ to $2$-$\mathsf{Rel}$ and $\mathsf{P}_1$-$\mathsf{Rel}$, induces an embedding $\mathsf{Top} \hookrightarrow (\mathbb{U},\mathsf{P}_1)$-$\mathsf{Cat}$. The homomorphism $\iota$ has a right adjoint $p\colon \mathsf{P}_1 \to 2$, defined by $p(1) = \top$ and $p(u) = \bot$, for $u \neq 1$, which is also compatible with the lax extensions of $\mathbb{U}$. Hence, by Proposition 2.3.1, we have

$$\mathsf{Top} \underset{\longleftarrow}{\overset{\top}{\hookrightarrow}} (\mathbb{U},\mathsf{P}_1)\text{-}\mathsf{Cat}.$$

Therefore, $\mathsf{Top}$ is coreflective in $(\mathbb{U},\mathsf{P}_1)$-$\mathsf{Cat}$, so that $(\mathbb{U},\mathsf{P}_1)$-compactly generated spaces are $(\mathbb{U},\mathsf{P}_1)$-spaces induced by compactly generated topological spaces, since $(\mathbb{U},\mathsf{P}_1)\text{-}\mathsf{Cat}_{\mathrm{CompHaus}} \cong \mathsf{Set}^{\mathbb{U}}$.

## 8.4 Alexandroff $(\mathbb{T},\mathsf{V})$-spaces

It is discussed in [ELS04] that when $\mathscr{C}$ is the singleton set containing the Sierpiński space $\mathbb{S}\in\mathsf{Top}$, then the $\mathscr{C}$-generated spaces are precisely the Alexandroff spaces. Analogously, let $\mathscr{C}$ be the singleton set containing the Sierpiński $(\mathbb{T},\mathsf{V})$-space $(\mathsf{V},\hom_\xi)$; we call the $\mathscr{C}$-generated $(\mathbb{T},\mathsf{V})$-spaces, or $(\mathsf{V},\hom_\xi)$-generated spaces, by *Alexandroff* $(\mathbb{T},\mathsf{V})$-spaces.

A $(\mathbb{T},\mathsf{V})$-space is Alexandroff if, and only if, it is a coequalizer of a coproduct of copies of $(\mathsf{V},\hom_\xi)$. Moreover, by Corollary 2.7.14, the Sierpiński $(\mathbb{T},\mathsf{V})$-space is an injective space. Hence, under the hypotheses of Theorem 3.0.5, which are satisfied by the categories of Table (IV.1), $(\mathsf{V},\hom_\xi)$ is exponentiable in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$. Therefore, in order to assure condition **(CEP)** and establish that Alexandroff spaces form a cartesian closed subcategory of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, we only need to verify whether the binary product $(\mathsf{V}\times\mathsf{V},\hom_\xi\times\hom_\xi)$ is $(\mathsf{V},\hom_\xi)$-generated.

**Proposition 8.4.1** *For* $\mathbb{T}=\mathbb{I}$ *and* $\mathsf{V}$ *integral and totally ordered, the product* $(\mathsf{V}\times\mathsf{V},\hom\times\hom)$ *is a* $(\mathsf{V},\hom)$*-generated space.*

*Proof.* For simplicity, let us fix $d=\hom\times\hom$. We know that $d^c\leq d$, where $d^c$ denotes the Alexandroff $\mathsf{V}$-structure on $\mathsf{V}\times\mathsf{V}$. Hence, it suffices to show that, for each $(u,v),(u',v')\in\mathsf{V}\times\mathsf{V}$, $d^c((u,v),(u',v'))\geq d((u,v),(u',v'))$. Let us consider the cases:

$\boxed{u\leq u'}$ $\mathsf{V}$ is integral, hence $\top\otimes u=u\leq u'$, and this is equivalent to $\top\leq\hom(u,u')$, whence

$$d((u,v),(u',v'))=\hom(u,u')\wedge\hom(v,v')=\top\wedge\hom(v,v')=\hom(v,v').$$

Define the maps $f_u\colon\mathsf{V}\to\mathsf{V}\times\mathsf{V}$, for each $z\in\mathsf{V}$, $f_u(z)=(u,z)$, and $f_{v'}\colon\mathsf{V}\to\mathsf{V}\times\mathsf{V}$, for each $z\in\mathsf{V}$, $f_{v'}(z)=(z,v')$, which are continuous, since constant maps are continuous. Thus

$$d^c((u,v),(u,v'))=d^c(f_u(v),f_u(v'))\geq\hom(v,v'),$$

and $d^c((u,v'),(u',v'))\geq\hom(u,u')=\top$. Transitivity of $d^c$ implies that

$$d^c((u,v),(u',v'))\geq d^c((u,v),(u,v'))\otimes d^c((u,v'),(u',v'))\geq\hom(v,v')\otimes\top=d((u,v),(u',v')).$$

We can treat the case $v\leq v'$ in an analogous way.

$\boxed{u>u'\ \&\ v>v'}$ We fix $\gamma=d((u,v),(u',v'))=\hom(u,u')\wedge\hom(v,v')$. Similar to what is done in the first case, we have $\hom(u',u)\wedge\hom(v',v)=\top$. Injectivity of $(\mathsf{V},\hom)$ implies injectivity of the product $(\mathsf{V}\times\mathsf{V},d)$. Furthermore, the equivalence $\simeq$ in diagram (I.30) implies equality when we consider extensions with codomain $\mathsf{V}\times\mathsf{V}$, because the order of $\mathsf{V}$, and consequently the order of $\mathsf{V}\times\mathsf{V}$, is separated. Consider the subset $\{\gamma,\top\}\subseteq\mathsf{V}$ with the $|\text{-}|$-initial $\mathsf{V}$-structure with respect to the inclusion map. Define

$$f\colon\{\gamma,\top\}\to\mathsf{V}\times\mathsf{V},\quad \gamma\mapsto(u',v'),\quad \top\mapsto(u,v).$$

Since $\gamma\leq\top$, then $\hom(\gamma,\top)=\top$, and $\hom(\gamma,\top)=\hom(u',u)\wedge\hom(v',v)=d(f(\gamma),f(\top))$; by formula (I.3), $\hom(\top,\gamma)=\bigvee\{w\in\mathsf{V}\mid w\otimes\top=w\leq\gamma\}=\gamma$, hence

$$\hom(\top,\gamma)=\hom(u,u')\wedge\hom(v,v')=d(f(\top),f(\gamma)).$$

Thus $f$ is fully faithful (hence continuous), and there exists a continuous map $\hat{f}\colon\mathsf{V}\to\mathsf{V}\times\mathsf{V}$ extending $f$ along the embedding of $\{\gamma,\top\}$ into $\mathsf{V}$:

$$\begin{array}{ccc}\{\gamma,\top\} & \hookrightarrow & \mathsf{V}\\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \hat{f}} & \\ & \mathsf{V}\times\mathsf{V}. & \end{array}$$

Therefore $d^c((u,v),(u',v'))=d^c(\hat{f}(\top),\hat{f}(\gamma))\geq\hom(\top,\gamma)=\gamma=d((u,v),(u',v'))$. □

Therefore, for $\mathbb{T}=\mathbb{I}$ and $\mathsf{V}$ integral and totally ordered, in particular for our examples in Table (IV.1), $\mathscr{C}$ satisfies condition **(CEP)**, whence, by Corollary 8.2.7, Alexandroff spaces form a cartesian closed subcategory of $\mathsf{V}$-$\mathsf{Cat}$. In $\mathsf{Ord}$ every space is Alexandroff. For $\mathsf{UltMet}$, $\mathsf{Met}$, and $\mathsf{BMet}$ Alexandroff spaces are the coequalizers of coproducts of copies of $([0,\infty],\varovee)$, $([0,\infty],\ominus)$, and $([0,1],\circledast)$, respectively (Examples 2.1.6 (2), (3), and (4)).

Let us turn our attention to the ultrafilter monad $\mathbb{U}$. For $(\mathbb{U},2)$-$\mathsf{Cat}\cong\mathsf{Top}$, the binary product $\mathbb{S}\times\mathbb{S}$ of Sierpiński spaces is Alexandroff, since its topology is finite, so that arbitrary intersections of open sets are open. Then we recover the fact that the full subcategory of $\mathsf{Top}$ of Alexandroff spaces is cartesian closed.

We can derive an interesting relation from the adjunction discussed in Subsection 2.3. As observed in Remark 2.3.2, for the particular case of $\mathsf{Ord}$ and $\mathsf{Top}$, Alexandroff topological spaces are precisely the spaces in the image of $A^\circ$, what culminates in the well known fact that the category of Alexandroff spaces is equivalent to $\mathsf{Ord}$. We wish to study whether the same relation between Alexandroff $(\mathbb{T},\mathsf{V})$-

spaces and Alexandroff $\mathsf{V}$-spaces can be established. Then consider the pair of adjoint functors (I.13):

$$\mathsf{V}\text{-}\mathsf{Cat} \underset{A_e}{\overset{A^\circ}{\rightleftarrows}} (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat};\quad (A_e \dashv A^\circ)$$

let us recall that, for each $(\mathbb{T},\mathsf{V})$-space $(X,a)$, $A_e(X,a)=(X,a\cdot e_X)$, and for each $\mathsf{V}$-space $(Y,b_0)$, $A^\circ(Y,b_0)=(Y,b_0^\#)$, with $b_0^\#=e_Y^\circ\cdot Tb_0$; both functors leave the morphisms unchanged.

Firstly, we must have that $(\mathsf{V},\hom_\xi)$ itself is the image by $A^\circ$ of some Alexandroff $\mathsf{V}$-space; naturally, we wish to provide conditions so that

$$(\mathsf{V},\hom_\xi)=A^\circ(\mathsf{V},\hom)=(\mathsf{V},\hom^\#)=(\mathsf{V},e_\mathsf{V}^\circ\cdot T\hom).$$

By (I.17), for each $\mathfrak{v}\in T\mathsf{V}$, $v\in\mathsf{V}$,

$$\begin{array}{rcl} e_\mathsf{V}^\circ\cdot T\hom(\mathfrak{v},v) & = & T\hom(\mathfrak{v},e_\mathsf{V}(v)) \\ & = & \bigvee\{\xi\cdot T(\hom)(\mathfrak{w}) \mid \mathfrak{w}\in T(\mathsf{V}\times\mathsf{V}),\ T\pi_1(\mathfrak{w})=\mathfrak{v},\ T\pi_2(\mathfrak{w})=e_\mathsf{V}(v)\}, \end{array}$$

with $\pi_1,\pi_2$ the first and second product projections from $\mathsf{V}\times\mathsf{V}$ onto $\mathsf{V}$, respectively. Moreover, by [Hof07, Lemma 3.2], the following diagram

$$\begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\hom)} & T\mathsf{V} \\ {\scriptstyle\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle}\downarrow & \geq & \downarrow{\scriptstyle\xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\hom]{} & \mathsf{V} \end{array}$$

is lax commutative. Then, whenever $\mathfrak{w}\in T(\mathsf{V}\times\mathsf{V})$ is such that $T\pi_1(\mathfrak{w})=\mathfrak{v}$ and $T\pi_2(\mathfrak{w})=e_\mathsf{V}(v)$, $\xi\cdot T(\hom)(\mathfrak{w})\leq\hom_\xi(\mathfrak{v},v)$, whence $e_\mathsf{V}^\circ\cdot T\hom(\mathfrak{v},v)\leq\hom_\xi(\mathfrak{v},v)$.

**Theorem 8.4.2** *If the diagram*

$$\begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\hom)} & T\mathsf{V} \\ {\scriptstyle\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle\xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\hom]{} & \mathsf{V} \end{array} \qquad \text{(IV.2)}$$

*is commutative, then $A^\circ$ preserves Alexandroff spaces.*

*Proof.* Commutativity of (IV.2) implies that $A^\circ(\mathsf{V},\hom) = (\mathsf{V},\hom_\xi)$. Let $(X,a_0)$ be an Alexandroff $\mathsf{V}$-space, and put $(X,a) = A^\circ(X,a_0)$. Let $h\colon (X,a) \to (Y,b)$ be a map such that, for every $(\mathbb{T},\mathsf{V})$-continuous map $f\colon (\mathsf{V},\hom_\xi) \to (X,a)$, the composite $h\cdot f$ is $(\mathbb{T},\mathsf{V})$-continuous.

Then, for each $\mathsf{V}$-continuous map $f\colon (\mathsf{V},\hom) \to (X,a_0)$, applying $A^\circ$ we get a $(\mathbb{T},\mathsf{V})$-continuous map $f\colon (\mathsf{V},\hom_\xi) \to (X,a)$. Therefore $h\cdot f\colon (\mathsf{V},\hom_\xi) \to (Y,b)$ is a $(\mathbb{T},\mathsf{V})$-continuous map, and, by the adjunction $A^\circ \dashv A_e$, $h\cdot f\colon (\mathsf{V},\hom) \to (Y, b\cdot e_Y)$ is a $\mathsf{V}$-continuous map. Since $(X,a_0)$ is Alexandroff, $h\colon (X,a_0) \to A_e(Y,b) = (Y,b\cdot e_Y)$ is a $\mathsf{V}$-continuous map, whence $h\colon A^\circ(X,a_0) \to (Y,b)$ is a $(\mathbb{T},\mathsf{V})$-continuous map. □

Next we make use of the map $\mathrm{can}_{X,X}\colon T(X\times X) \to TX \times TX$ defined in (I.21).

**Proposition 8.4.3** *(1) For each Alexandroff $(\mathbb{T},\mathsf{V})$-space $(X,a)$, $(X,a) = A^\circ \cdot A_e(X,a)$.*
*(2) If, for each set $X$, $(e_X \times e_X)^\circ \cdot \mathrm{can}_{X,X} \le e^\circ_{X\times X}$, then, for each $\mathsf{V}$-space $(X,a_0)$, $(X,a_0) = A_e \cdot A^\circ(X,a_0)$.*

*Proof.* (1) Let $(X,a)$ be an Alexandroff $(\mathbb{T},\mathsf{V})$-space. The equality $m_X \cdot Te_X = 1_{TX}$ implies the inequality $Te_X \le m_X^\circ$, and $1_X \le a\cdot e_X$ is equivalent to $e_X^\circ \le a$. By transitivity of $a$ and because $m_X$ is a map, we have $e_X^\circ \cdot Ta\cdot Te_X \le a\cdot Ta\cdot m_X^\circ \le a\cdot m_X \cdot m_X^\circ \le a$.

Conversely, by the adjunction $A^\circ \dashv A_e$, every continuous map $f\colon (\mathsf{V},\hom_\xi) \to (X,a)$ is continuous from $(\mathsf{V},\hom)$ to $(X, a\cdot e_X)$, and applying $A^\circ$ we get a continuous map from $(\mathsf{V},\hom_\xi)$ to $(X, e_X^\circ \cdot Ta\cdot Te_X)$. Hence, since $(X,a)$ is Alexandroff, the identity map $1_X$ is continuous:

$$\begin{array}{ccc} (\mathsf{V},\hom_\xi) & \xrightarrow{\;f\;} & (X,a) \\ & \searrow^{f} & \downarrow{\scriptstyle 1_X} \\ & & (X, e_X^\circ\cdot Ta\cdot Te_X). \end{array}$$

(2) For each set $X$, $\mathfrak{w} \in T(X\times X)$, $(x,x') \in X\times X$,

$$\begin{aligned} e^\circ_{X\times X}(\mathfrak{w},(x,x')) = k \iff\ & e_{X\times X}(x,x') = \mathfrak{w} \implies \big(T\pi_1^X(\mathfrak{w}) = e_X(x) \quad \& \quad T\pi_2^X(\mathfrak{w}) = e_X(x')\big) \\ \iff\ & \mathrm{can}_{X,X}(\mathfrak{w}) = e_X\times e_X(x,x') \iff (e_X\times e_X)^\circ\cdot \mathrm{can}_{X,X}(\mathfrak{w},(x,x')) = k, \end{aligned}$$

that is, $e^\circ_{X\times X} \le (e_X\times e_X)^\circ \cdot \mathrm{can}_{X,X}$. Hence, by hypothesis, this is an equality. Let $(x,x') \in X\times X$, and $\mathfrak{w} \in T(X\times X)$ such that $T\pi_1^X(\mathfrak{w}) = e_X(x)$ and $T\pi_2^X(\mathfrak{w}) = e_X(x')$. Then $e_{X\times X}(x,x') = \mathfrak{w}$, and for each $\mathsf{V}$-space $(X,a_0)$, since $\xi\colon T\mathsf{V} \to \mathsf{V}$ is a $\mathbb{T}$-algebra, we have:

$$\begin{aligned} Ta_0(e_X(x), e_X(x')) &= \bigvee\{\xi\cdot T\vec{a}_0(\mathfrak{w}) \mid \mathfrak{w} \in T(X\times X),\ T\pi_1^X(\mathfrak{w}) = e_X(x),\ T\pi_2^X(\mathfrak{w}) = e_X(x')\} \\ &= \xi\cdot T\vec{a}_0 \cdot e_{X\times X}(x,x') = \xi\cdot e_{\mathsf{V}}\cdot \vec{a}_0(x,x') = \vec{a}_0(x,x'). \end{aligned}$$

Therefore $(X, a_0) = (X, e_X^\circ \cdot Ta_0 \cdot e_X) = A_e \cdot A^\circ(X, a_0)$. $\square$

**Remark 8.4.4** We have proved that every Alexandroff $\mathsf{V}$-space induces, by $A^\circ$, an Alexandroff $(\mathbb{T},\mathsf{V})$-space, and that every Alexandroff $(\mathbb{T},\mathsf{V})$-space is induced by a $\mathsf{V}$-space, namely, by $A_e(X,a)$. It is still an open question under which conditions the image of an Alexandroff $(\mathbb{T},\mathsf{V})$-space by $A_e$ is an Alexandroff $\mathsf{V}$-space. By the characterization of $\mathscr{C}$-generated spaces as colimits, it suffices to determine conditions under which the functor $A_e$ is a left adjoint.

**Proposition 8.4.5** *If* $(e_\mathsf{V} \times e_\mathsf{V})^\circ \cdot \mathrm{can}_{\mathsf{V},\mathsf{V}} \le e^\circ_{\mathsf{V}\times\mathsf{V}}$ *and the following diagrams are (lax) commutative,*

$$\begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\wedge)} & T\mathsf{V} \\ {\scriptstyle \langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle}\downarrow & \le & \downarrow{\scriptstyle \xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\wedge]{} & \mathsf{V} \end{array} \qquad\qquad \begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\hom)} & T\mathsf{V} \\ {\scriptstyle \langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle \xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\hom]{} & \mathsf{V} \end{array}$$

*then, for* $\mathsf{V}$ *integral and totally ordered,* $(\mathsf{V}\times\mathsf{V}, \hom_\xi \times \hom_\xi)$ *is an Alexandroff* $(\mathbb{T},\mathsf{V})$*-space.*

*Proof.* It suffices to show that $(\mathsf{V}\times\mathsf{V}, \hom_\xi\times\hom_\xi) = A^\circ(\mathsf{V}\times\mathsf{V}, \hom\times\hom)$. For each $(u,v),(z,w)$ in $\mathsf{V}\times\mathsf{V}$,

$$\begin{aligned} (\hom_\xi \times \hom_\xi)\cdot e_{\mathsf{V}\times\mathsf{V}}((u,v),(z,w)) &= \hom_\xi \times \hom_\xi(e_{\mathsf{V}\times\mathsf{V}}(u,v),(z,w)) \\ &= \hom_\xi(T\pi_1\cdot e_{\mathsf{V}\times\mathsf{V}}(u,v),z) \wedge \hom_\xi(T\pi_2\cdot e_{\mathsf{V}\times\mathsf{V}}(u,v),w) \\ &= \hom_\xi(e_\mathsf{V}(u),z)\wedge\hom_\xi(e_\mathsf{V}(v),w) \;=\; \hom(u,z)\wedge\hom(v,w) \\ &= \hom\times\hom((u,v),(z,w)), \end{aligned}$$

hence $(\mathsf{V}\times\mathsf{V}, (\hom\times\hom)^\#) = A^\circ(\mathsf{V}\times\mathsf{V},\hom\times\hom) = A^\circ\cdot A_e(\mathsf{V}\times\mathsf{V},\hom_\xi\times\hom_\xi)$; the counit of the adjunction $A^\circ \dashv A_e$ is given by an identity map, so we conclude that $(\hom\times\hom)^\# \le \hom_\xi\times\hom_\xi$. For the converse inequality, since $(\mathsf{V},\hom_\xi) = A^\circ(\mathsf{V},\hom) = (\mathsf{V}, e^\circ_\mathsf{V}\cdot T\hom)$, for each $\mathfrak{w}\in T(\mathsf{V}\times\mathsf{V})$, $(u,v)\in\mathsf{V}\times\mathsf{V}$, we have:

$$\begin{aligned} \hom_\xi\times\hom_\xi(\mathfrak{w},(u,v)) &= \hom_\xi(T\pi_1(\mathfrak{w}),u)\wedge\hom_\xi(T\pi_2(\mathfrak{w}),v) \\ &= e^\circ_\mathsf{V}\cdot T\hom(T\pi_1(\mathfrak{w}),u)\wedge e^\circ_\mathsf{V}\cdot T\hom(T\pi_2(\mathfrak{w}),v) \\ &= T\hom(T\pi_1(\mathfrak{w}),e_\mathsf{V}(u))\wedge T\hom(T\pi_2(\mathfrak{w}),e_\mathsf{V}(v)) \\ &= T\hom\times T\hom(\mathfrak{w}, e_V\times e_\mathsf{V}(u,v)) \\ &= (e_\mathsf{V}\times e_\mathsf{V})^\circ\cdot(T\hom\times T\hom)(\mathfrak{w},(u,v)). \end{aligned}$$

Moreover,

$$\begin{aligned}(e_{\mathsf{V}}\times e_{\mathsf{V}})^{\circ}\cdot(T\hom\times T\hom) &= (e_{\mathsf{V}}\times e_{\mathsf{V}})^{\circ}\cdot(T\hom\owedge T\hom)\cdot\mathrm{can}_{\mathsf{V},\mathsf{V}} && \text{(by (I.22))}\\ &= (e_{\mathsf{V}}\times e_{\mathsf{V}})^{\circ}\cdot\mathrm{can}_{\mathsf{V},\mathsf{V}}\cdot T(\hom\owedge\hom) && \text{(by Lemma 2.6.1)}\\ &\leq e^{\circ}_{\mathsf{V}\times\mathsf{V}}\cdot T(\hom\times\hom) \;=\; (\hom\times\hom)^{\#} && \text{(by hypothesis).}\end{aligned}$$

□

Therefore, under the conditions of this proposition, $\mathscr{C}=\{(\mathsf{V},\hom_{\xi})\}$ satisfies **(CEP)**, hence Alexandroff $(\mathbb{T},\mathsf{V})$-spaces form a cartesian closed subcategory of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$.

**Example 8.4.6** Let us verify that in the category $\mathsf{App}\cong(\mathbb{U},\mathsf{P}_{+})$-$\mathsf{Cat}$ the conditions of Proposition 8.4.5 are satisfied. We recall from Examples 2.5.1(2) that

$$\xi : U[0,\infty]\to[0,\infty],\quad \mathfrak{v}\mapsto\inf\{u\in[0,\infty]\mid[0,u]\in\mathfrak{v}\};$$

lax commutativity of

$$\begin{array}{ccc} U([0,\infty]\times[0,\infty]) & \xrightarrow{\;U(\max)\;} & U[0,\infty]\\ {\scriptstyle\langle\xi\cdot U\pi_1,\,\xi\cdot U\pi_2\rangle}\downarrow & \leq & \downarrow{\scriptstyle\xi}\\ [0,\infty]\times[0,\infty] & \xrightarrow[\;\max\;]{} & [0,\infty]\end{array}$$

is verified in Remark 2.6.2. Let us verify that the diagram

$$\begin{array}{ccc} U([0,\infty]\times[0,\infty]) & \xrightarrow{\;U\vec{\ominus}\;} & U[0,\infty]\\ {\scriptstyle\langle\xi\cdot U\pi_1,\,\xi\cdot U\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle\xi}\\ [0,\infty]\times[0,\infty] & \xrightarrow[\;\vec{\ominus}\;]{} & [0,\infty]\end{array}$$

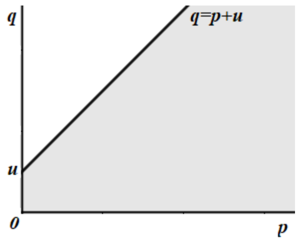
is commutative. By [Hof07, Lemma 3.2], we know that $\vec{\ominus}\cdot\langle\xi\cdot U\pi_1,\xi\cdot U\pi_2\rangle\leq\xi\cdot U\vec{\ominus}$. Let $\mathfrak{w}$ in $U([0,\infty]\times[0,\infty])$ and suppose that

$$\xi(\mathfrak{v}_2)\ominus\xi(\mathfrak{v}_1)<\xi(U\vec{\ominus}(\mathfrak{w}))=\inf\{u\in[0,\infty]\mid[0,u]\in U\vec{\ominus}(\mathfrak{w})\},$$

where $\mathfrak{v}_1=U\pi_1(\mathfrak{w})$ and $\mathfrak{v}_2=U\pi_2(\mathfrak{w})$. Here $[0,u]\in U\vec{\ominus}(\mathfrak{w})$ if, and only if, $(\vec{\ominus})^{-1}([0,u])\in\mathfrak{w}$, so that $\xi(U\vec{\ominus}(\mathfrak{w}))=\inf\{u\in[0,\infty]\mid S_u\in\mathfrak{w}\}$, where the set

$$S_u=(\vec{\ominus})^{-1}([0,u])=\{(p,q)\in[0,\infty]\times[0,\infty]\mid q\ominus p\leq u\}$$

can be depicted as the gray area in the following graphic:



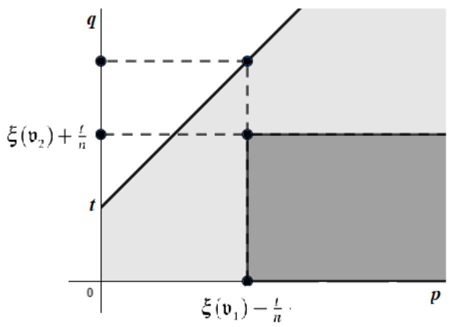

Let $t \in [0,\infty]$ such that $\xi(\mathfrak{v}_2) \ominus \xi(\mathfrak{v}_1) < t < \xi(U\vec{\ominus}(\mathfrak{w}))$; then $S_t \notin \mathfrak{w}$. Since $\xi(\mathfrak{v}_2) < \xi(\mathfrak{v}_1) + t$, there exists $n \in \mathbb{N}$ such that $\xi(\mathfrak{v}_2) + \frac{t}{n} < \xi(\mathfrak{v}_1) + t$. Let us assume that $\xi(\mathfrak{v}_1) > 0$ so that we can choose $\frac{t}{n} < \xi(\mathfrak{v}_1)$. Hence

$$\begin{aligned} \xi(\mathfrak{v}_1) - \tfrac{t}{n} < \xi(\mathfrak{v}_1) \quad &\Rightarrow \quad [0, \xi(\mathfrak{v}_1) - \tfrac{t}{n}] \notin \mathfrak{v}_1 \ \Rightarrow ]\xi(\mathfrak{v}_1) - \tfrac{t}{n}, \infty] \in \mathfrak{v}_1 = U\pi_1(\mathfrak{w}) \\ &\Rightarrow \quad ]\xi(\mathfrak{v}_1) - \tfrac{t}{n}, \infty] \times [0,\infty] \in \mathfrak{w} \end{aligned}$$

and

$$\xi(\mathfrak{v}_2) + \tfrac{t}{n} > \xi(\mathfrak{v}_2) \quad \Rightarrow \quad [0, \xi(\mathfrak{v}_2) + \tfrac{t}{n}] \in \mathfrak{v}_2 = U\pi_2(\mathfrak{w}) \ \Rightarrow \ [0,\infty] \times [0, \xi(\mathfrak{v}_2) + \tfrac{t}{n}] \in \mathfrak{w}.$$

Thus $]\xi(\mathfrak{v}_1) - \frac{t}{n}, \infty] \times [0, \xi(\mathfrak{v}_2) + \frac{t}{n}] \in \mathfrak{w}$, but $]\xi(\mathfrak{v}_1) - \frac{t}{n}, \infty] \times [0, \xi(\mathfrak{v}_2) + \frac{t}{n}] \subseteq S_t$ implies $S_t \in \mathfrak{w}$, a contradiction.




In the case $\xi(\mathfrak{v}_1) = 0$, we have that $\xi(\mathfrak{v}_2) + \frac{t}{n} < t$, whence $[0,\infty] \times [0, \xi(\mathfrak{v}_2) + \frac{t}{n}] \in \mathfrak{w}$, and $[0,\infty] \times [0, \xi(\mathfrak{v}_2) + \frac{t}{n}] \subseteq S_t$, so we obtain a contradiction.

Finally, if $\mathfrak{w} \in U(\mathsf{V} \times \mathsf{V})$ is such that $U\pi_1(\mathfrak{w}) = e_{\mathsf{V}}(u)$ and $U\pi_2(\mathfrak{w}) = e_{\mathsf{V}}(v)$, for $(u,v) \in \mathsf{V} \times \mathsf{V}$, then $\{u\} \in U\pi_1(\mathfrak{w})$ is equivalent to $\pi_1^{-1}(\{u\}) = \{u\} \times \mathsf{V} \in \mathfrak{w}$, and $\{v\} \in U\pi_2(\mathfrak{w})$ is equivalent to $\pi_2^{-1}(\{v\}) = \mathsf{V} \times \{v\} \in \mathfrak{w}$, whence $(\{u\} \times \mathsf{V}) \cap (\mathsf{V} \times \{v\}) = \{(u,v)\} \in \mathfrak{w}$, that is, $\mathfrak{w} = e_{\mathsf{V}\times\mathsf{V}}(u,v)$. Thus $(e_{\mathsf{V}} \times e_{\mathsf{V}})^{\circ} \cdot \mathrm{can}_{\mathsf{V},\mathsf{V}} \leq e_{\mathsf{V}\times\mathsf{V}}$.

Therefore we conclude that Alexandroff approach spaces form a cartesian closed category of App. Furthermore, if $(X,d)$ is an Alexandroff metric space, then the space $A^{\circ}(X,d) = (X,d^{\#})$ given by, for each $\mathfrak{x} \in UX$, $x \in X$, $d^{\#}(\mathfrak{x},x) = \sup\limits_{A \in \mathfrak{x}} \big(\inf\{d(x',x) \mid x' \in A\}\big)$, is an Alexandroff approach space. In terms of approach distances, for each $x \in X$, $A \subseteq X$, $d^{\#}(x,A) = \inf\{d(x',x) \mid x' \in A\}$. For each Alexandroff approach space $(X,a)$, we have $a = d^{\#}$, with $d = a \cdot e_X$, that is, for each $\mathfrak{x} \in UX$, $x \in X$, $a(\mathfrak{x},x) = \sup\limits_{A \in \mathfrak{x}} \big(\inf\{a(e_X(x'),x) \mid x' \in A\}\big)$, or in terms of approach distances: if $(X,\delta)$ is an Alexandroff approach space, then, for each $x \in X$, $A \subseteq X$, $\delta(x,A) = \inf\limits_{x' \in A} \big(\sup\{\delta(x,B) \mid B \in e_X(x')\}\big)$.

For $\mathsf{V} = \mathsf{P}_{\max}$, we have the adjunction $A^{\circ} \dashv A_e \colon \mathsf{NA\text{-}App} \to \mathsf{UltMet}$ between non-Archimedean approach spaces and ultrametric spaces [CVO17, Section 4]. However, it remains an open question whether the diagram below is commutative.

$$\begin{array}{ccc}
U([0,\infty] \times [0,\infty]) & \xrightarrow{\;U\vec{\oslash}\;} & U[0,\infty] \\
{\scriptstyle \langle \xi \cdot U\pi_1,\, \xi \cdot U\pi_2 \rangle} \big\downarrow & & \big\downarrow {\scriptstyle \xi} \\
[0,\infty] \times [0,\infty] & \xrightarrow[\;\vec{\oslash}\;]{} & [0,\infty]
\end{array}$$

## 8.5 Injectively generated $(\mathbb{T},\mathsf{V})$-spaces

We can consider $\mathscr{C}$ as the class of injective spaces. The binary product of injective spaces is injective and, under the conditions of Theorem 3.0.5, so in particular for the categories of Table (IV.1), every element of $\mathscr{C}$ is exponentiable. Therefore, $\mathscr{C}$ satisfies condition **(CEP)** and the full subcategory of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ of $\mathscr{C}$-generated spaces, or *injectively generated* spaces, is cartesian closed.

The class of injective spaces in the usual sense, so that we require equality instead of $\simeq$ in (I.30), also satisfies condition **(CEP)**. Hence we conclude (see [AHS90, Examples 9.3(4)]):

**Lemma 8.5.1** *In* Top*, quotients of disjoint sums of retracts of powers $D^I$, with $I$ a set and $D = (\{0,1,2\},\{\emptyset,\{0,1,2\},\{0,1\}\})$, form a cartesian closed subcategory.*

## 8.6 Exponentiably generated $(\mathbb{T},\mathsf{V})$-spaces

Finally, the largest class $\mathscr{C}$ satisfying condition **(CEP)** is the one of exponentiable $(\mathbb{T},\mathsf{V})$-spaces. Then coequalizers of coproducts of exponentiable spaces form a cartesian closed subcategory of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$.

In the case of Top, exponentiable spaces are the *core-compact* spaces. Exponentiably generated spaces in Top, which are quotients of disjoint sums of core-compact spaces, are then called *core-compactly generated* [ELS04].

For the cartesian closed category Ord, $\mathscr{C} = \mathsf{Ord}_{\mathscr{C}} = \mathsf{Ord}$. For an account on exponentiable metric spaces and exponentiable approach spaces we refer to [CH06] and [HS15], respectively.

# 9 Quasi-$(\mathbb{T},\mathsf{V})$-spaces

In this section we generalise Spanier's quasi-topological spaces [Spa63], following Day's presentation [Day68]. Throughout $\mathscr{C}$ will denote the full subcategory of $(\mathbb{T},\mathsf{V})$-Cat of compact Hausdorff spaces. Let us recall from Subsection 2.8 that, under the conditions that V is integral and lean, $T1 = 1$ and $T$ preserves finite coproducts (hence, in particular, for the categories of Table IV.1), every constant map is continuous in $(\mathbb{T},\mathsf{V})$-Cat, and $\mathscr{C}$ is closed under the formation of finite coproducts, binary products and equalizers.

## 9.1 The category $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-Cat

**Definition 9.1.1** For $C \in \mathscr{C}$ and a finite family of maps $(\alpha_i\colon C_i \to X)_{i\in I}$, with $C_i \in \mathscr{C}$, for each $i \in I$, we say that a map $\alpha\colon C \to X$ is *covered by the family* $(\alpha_i)_{i\in I}$ if there exists a surjective continuous map $\eta\colon \coprod_i C_i \to C$ such that the diagram

$$\begin{array}{ccc} \coprod_i C_i & \xrightarrow{\ \coprod_i \alpha_i\ } & \\ {\scriptstyle\eta}\downarrow & \searrow & \\ C & \xrightarrow[\ \alpha\ ]{} & X \end{array} \tag{IV.3}$$

is commutative, where $\coprod_i C_i$ denotes the coproduct of the family $(C_i)_{i\in I}$ in $(\mathbb{T},\mathsf{V})$-Cat, and $\coprod_i \alpha_i$ is the canonical induced map.

Observe that, by this definition, every map $\alpha\colon C \to X$, $C \in \mathscr{C}$, is covered by itself.

**Definition 9.1.2** A *quasi-$(\mathbb{T},\mathsf{V})$-space* consists of a set $X$ and, for each element $C$ of $\mathscr{C}$, a set $Q(C,X)$ of maps from $C$ to $X$, hereinafter called *admissible maps*, such that the following conditions are satisfied:

**(QS1)** for each $C \in \mathscr{C}$, constant maps belong to $Q(C,X)$;

**(QS2)** for $C_1, C_2 \in \mathscr{C}$, for each continuous map $h\colon C_1 \to C_2$ and each admissible map $\alpha \in Q(C_2,X)$, $\alpha \cdot h \in Q(C_1,X)$;

**(QS3)** if a map $\alpha\colon C \to X$, for $C \in \mathscr{C}$, is covered by a family of admissible maps as in (IV.3), then $\alpha$ is admissible.

A quasi-$(\mathbb{T},\mathsf{V})$-space is denoted by $(X,(Q(C,X))_{C\in\mathscr{C}})$, and when its *quasi-$(\mathbb{T},\mathsf{V})$-structure* $(Q(C,X))_{C\in\mathscr{C}}$

is clear by the context, we will denote it simply by $X$. A *quasi-*$(\mathbb{T},\mathsf{V})$*-continuous* map

$$f\colon (X,(Q(C,X))_{C\in\mathscr{C}}) \to (Y,(Q(C,Y))_{C\in\mathscr{C}}),$$

between quasi-$(\mathbb{T},\mathsf{V})$-spaces is a map $f\colon X\to Y$ such that, for each $C\in\mathscr{C}$ and $\alpha\in Q(C,X)$, $f\cdot\alpha\in Q(C,Y)$; we denote the set of quasi-$(\mathbb{T},\mathsf{V})$-continuous maps from $X$ to $Y$ by $\mathsf{Qs}(X,Y)$.

In **(QS3)** we have an equivalence, since every admissible map is covered by itself. When there is no ambiguity, we might drop the middle term $(\mathbb{T},\mathsf{V})$ and refer to the concepts of Definition 9.1.2 by *quasi-spaces*, *quasi-structures*, and *quasi-continuous* maps. Each identity map is quasi-continuous and composition of quasi-continuous maps is quasi-continuous, so we have a category $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$. We are aware of the size illegitimacy of $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ proved in [HR83] for the particular case of $\mathsf{Top}$, which comes from the fact that its collection of objects do not form a class. However, we still call $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ a category.

Let $(X,a)$ be a $(\mathbb{T},\mathsf{V})$-space, and define, for each $C\in\mathscr{C}$,

$$Q_a(C,X) = \{\alpha\colon (C,c)\to(X,a) \mid \alpha \text{ is } (\mathbb{T},\mathsf{V})\text{-continuous}\}. \tag{IV.4}$$

**Lemma 9.1.3** *For each* $(X,a)\in(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$*,* $(X,(Q_a(C,X))_{C\in\mathscr{C}})$ *is a quasi-*$(\mathbb{T},\mathsf{V})$*-space.*

*Proof.* Every constant map is continuous, so **(QS1)** is satisfied, and so is **(QS2)**, since the composition of continuous maps is continuous. For **(QS3)**, let $\alpha\colon C\to X$, $C\in\mathscr{C}$, be a map covered by a family of admissible maps $(\alpha_i)_{i\in I}$ as in (IV.3). Each map $\alpha_i$ is continuous, then so is the composite $\alpha\cdot\eta = \coprod_i \alpha_i \in Q_a(\coprod_i C_i, X)$. Axiom of Choice granted, we can conclude the following:

$$\begin{aligned}
c &\le c\cdot T\eta\cdot(T\eta)^\circ && (T\eta \text{ is a surjective map})\\
&\le c\cdot T\eta\cdot b^\circ\cdot b\cdot(T\eta)^\circ && ((\textstyle\coprod_i C_i,b) \text{ is compact})\\
&\le c\cdot c^\circ\cdot\eta\cdot b\cdot(T\eta)^\circ && (\eta \text{ is } (\mathbb{T},\mathsf{V})\text{-continuous})\\
&\le \eta\cdot b\cdot(T\eta)^\circ && ((C,c) \text{ is Hausdorff})\\
&\le \eta\cdot(\alpha\cdot\eta)^\circ\cdot a\cdot T(\alpha\cdot\eta)\cdot(T\eta)^\circ && (\alpha\cdot\eta \text{ is } (\mathbb{T},\mathsf{V})\text{-continuous})\\
&= \eta\cdot\eta^\circ\cdot\alpha^\circ\cdot a\cdot T\alpha\cdot T\eta\cdot(T\eta)^\circ && \\
&\le \alpha^\circ\cdot a\cdot T\alpha && (\eta \text{ and } T\eta \text{ are maps}).
\end{aligned}$$

Hence $\alpha$ is continuous, that is, $\alpha\in Q_a(C,X)$. □

We call $(X,(Q_a(C,X))_{C\in\mathscr{C}})$ the quasi-space *associated* with $(X,a)$. If $f\colon (X,a)\to(Y,b)$ is a continuous map in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, then $f\colon (X,(Q_a(C,X))_{C\in\mathscr{C}})\to(Y,(Q_b(C,Y))_{C\in\mathscr{C}})$ is a quasi-continuous map in $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, for if $\alpha\colon C\to X$ is an admissible map, for $C\in\mathscr{C}$, then $\alpha$ continuous implies that $f\cdot\alpha\colon C\to Y$ is continuous, hence admissible. This defines an inclusion of $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ into $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ that, in general, is not full. However, in particular cases, continuous maps and quasi-continuous maps might coincide, as we verify next.

Let $(C,c)$ be an element of $\mathscr{C}$, and consider its associated quasi-structure: for each $B\in\mathscr{C}$, $Q_c(B,C)=(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}(B,C)$. Let $(X,(Q(C,X))_{C\in\mathscr{C}})$ be a quasi-space, and $\alpha\colon C\to X$ be a map. If $\alpha$ is quasi-continuous, then, since $1_C\in Q_c(C,C)$, by definition of quasi-continuity, $\alpha\cdot 1_C=\alpha\in Q(C,X)$. Conversely, if $\alpha\in Q(C,X)$, then, for each $D\in\mathscr{C}$ and each $\beta\in Q_c(D,C)$, by **(QS2)**, $\alpha\cdot\beta\in Q(D,X)$, hence $\alpha$ is quasi-continuous. Therefore, when $C$ is endowed with its associated quasi-structure, $Q(C,X)=\mathsf{Qs}(C,X)$. Furthermore, for the particular case when $X=(D,d)\in\mathscr{C}$, quasi-continuous maps between the associated quasi-spaces of $C$ and $D$ coincide with the admissible maps $Q_d(C,D)$, which are all continuous maps from $C$ to $D$, that is,

$$Q_d(C,D)=\mathsf{Qs}(C,D)=(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}(C,D).$$

This fact extends from elements of $\mathscr{C}$ to $\mathscr{C}$-generated spaces, as we will see in details in Subsection 9.4. Concerning $\mathscr{C}$-generated spaces, we observe the following:

**Lemma 9.1.4** *For each $(\mathbb{T},\mathsf{V})$-space $(X,a)$, the $\mathscr{C}$-generated space $(X,a^c)$ induces the same quasi-space associated with $(X,a)$. Moreover, $a^c$ is the least $(\mathbb{T},\mathsf{V})$-structure on $X$ with this property.*

*Proof.* The first statement follows from the fact that, for each $C\in\mathscr{C}$, $\alpha\colon C\to(X,a)$ is a continuous map if, and only if, $\alpha\colon C\to(X,a^c)$ is a continuous map. Now, if $\overline{a}$ is such a structure, then every probe $\alpha\colon C\to(X,a)$, with $C\in\mathscr{C}$, is a continuous map $\alpha\colon C\to(X,\overline{a})$, whence, by the definition of $\mathscr{C}$-generated structure, the identity map $1_X\colon (X,a^c)\to(X,\overline{a})$ is continuous, that is, $a^c\le\overline{a}$. □

It is proved in [Spa63, Lemma 5.5] that there exist quasi-topological spaces which are not associated with any topological space. We can verify the same for $\mathsf{V}$-$\mathsf{Cat}$ with $\mathsf{V}$ integral: compact and Hausdorff $\mathsf{V}$-spaces are the discrete $\mathsf{V}$-spaces $(C,1_C)$, $C\in\mathsf{Set}$; hence, for $(X,a)$ a $\mathsf{V}$-space, every map $\alpha\colon (C,1_C)\to(X,a)$ is $\mathsf{V}$-continuous. Then the associated quasi-$\mathsf{V}$-structure for $(X,a)$ is given by: for each $C\in\mathsf{Set}$, $Q_a(C,X)=\mathsf{Set}(C,X)$. Therefore, setting, for each $C\in\mathsf{Set}$, $Q'(C,X)=\{\alpha\colon C\to X\mid \alpha \text{ has finite image}\}$, we see that, if $X$ is not finite, then $Q'$ is not associated with $(X,a)$.

In the same way, we can define, for each set $X$, *indiscrete* and *discrete* quasi-structures given by, for each $C \in \mathscr{C}$, $Q_{\text{ind}}(C,X) = \mathsf{Set}(C,X)$ and $Q_{\text{dis}}(C,X) = \{\alpha\colon C \to X \mid \alpha \text{ has finite image}\}$, respectively. One can directly verify that $(X,(Q_{\text{ind}}(C,X))_{C\in\mathscr{C}})$ is a quasi-$(\mathbb{T},\mathsf{V})$-space, and that, for a quasi-$(\mathbb{T},\mathsf{V})$-space $(Y,(Q(C,Y))_{C\in\mathscr{C}})$, each map $f\colon Y \to X$ in $\mathsf{Set}$ is a quasi-continuous map

$$f\colon (Y,(Q(C,Y))_{C\in\mathscr{C}}) \to (X,(Q_{\text{ind}}(C,X))_{C\in\mathscr{C}}).$$

For the discrete quasi-structure, we prove:

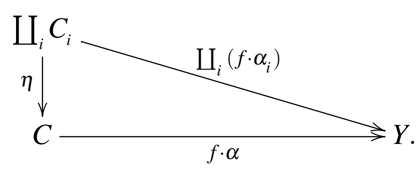
**Lemma 9.1.5** *For each set $X$, $(X,(Q_{\text{dis}}(C,X))_{C\in\mathscr{C}})$ is a quasi-$(\mathbb{T},\mathsf{V})$-space. Moreover, for a quasi-space $(Y,(Q(C,Y))_{C\in\mathscr{C}})$, every map $f\colon X \to Y$ in* $\mathsf{Set}$ *is a quasi-continuous map*

$$f\colon (X,(Q_{\text{dis}}(C,X))_{C\in\mathscr{C}}) \to (Y,(Q(C,Y))_{C\in\mathscr{C}}).$$

*Proof.* **(QS1)** and **(QS2)** follow immediately for $Q_{\text{dis}}$. To verify **(QS3)**, let $\alpha\colon C \to X$, for $C \in \mathscr{C}$, be covered by a (finite) family of admissible maps $(\alpha_i)_{i\in I}$ as in (IV.3). Since the $\alpha_i$'s have finite image, say $\text{card}(\alpha_i(C_i)) = n_i$, $n_i \in \mathbb{N}$, then

$$\text{card}(\alpha(C)) = \text{card}(\alpha\cdot\eta(\coprod_i C_i)) = \text{card}(\coprod_i \alpha_i(\coprod_i C_i)) \leq \sum_i \text{card}(\alpha_i(C_i)) = \sum_i n_i,$$

hence $\alpha$ is admissible. Let $f\colon X \to Y$ be a map and $\alpha \in Q_{\text{dis}}(C,X)$, for some $C \in \mathscr{C}$. Then $\alpha$ has finite image, say $\alpha(C) = \{x_1,\ldots,x_n\}$, $n \in \mathbb{N}$. Define, for each $i \in \{1,\ldots,n\}$, $C_i = \alpha^{-1}(x_i) \subseteq C$, and endow this fibre with the $|\text{-}|$-initial $(\mathbb{T},\mathsf{V})$-structure with respect to the inclusion map into $C$. For each $i \in \{1\ldots,n\}$, $C_i$ belongs to $\mathscr{C}$, because $\mathscr{C}$ is closed under regular monomorphisms. The inclusion maps $C_i \hookrightarrow C$, $i \in \{1\ldots,n\}$, induce a surjective $(\mathbb{T},\mathsf{V})$-continuous map $\eta\colon \coprod_i C_i \to C$. Define, for each $i$, the constant map $\alpha_i\colon C_i \to X$, $c_i \mapsto \alpha(c_i) = x_i$, so that the composite $f\cdot\alpha_i$ is a constant map, hence $f\cdot\alpha_i \in Q(C_i,Y)$. Finally, observe that $f\cdot\alpha\colon C \to Y$ is covered by the family $(f\cdot\alpha_i)_{i\in\{1,\ldots,n\}}$:




Therefore $f\colon (X,(Q_{\text{dis}}(C,X))_{C\in\mathscr{C}}) \to (Y,(Q(C,Y))_{C\in\mathscr{C}})$ is a quasi-continuous map. $\square$

We remark that the indiscrete quasi-$(\mathbb{T},\mathsf{V})$-space $(X,(Q_{\text{ind}}(C,X))_{C\in\mathscr{C}})$ is associated with the indiscrete $(\mathbb{T},\mathsf{V})$-space $(X,\top)$, since every map $f\colon C\to(X,\top)$ is continuous, for $C\in\mathscr{C}$. For discrete quasi-spaces, this is not true in general, as we have seen above for V-Cat. However, for Top and QsTop, it is true that the discrete quasi-topological space is associated with the discrete topological space: if $\alpha\colon C\to X$ is a continuous map, with $C$ compact and Hausdorff and $X$ discrete, then the image $\alpha(C)\subseteq X$ is compact and discrete, hence it is finite.

**Proposition 9.1.6** *The forgetful functor* $|\text{-}|\colon \mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to\mathsf{Set}$ *has left and right adjoints, and it is represented by the singleton quasi-*$(\mathbb{T},\mathsf{V})$*-space.*

*Proof.* The left adjoint to the forgetful functor assigns to each set $X$ the discrete quasi-space $(X,(Q_{\text{dis}}(C,X))_{C\in\mathscr{C}})$, and leaves maps unchanged. Then, for each quasi-space $(Y,(Q(C,Y))_{C\in\mathscr{C}})$,

$$\mathsf{Qs}((X,(Q_{\text{dis}}(C,X))_{C\in\mathscr{C}}),(Y,(Q(C,Y))_{C\in\mathscr{C}}))=\mathsf{Set}(X,Y)=\mathsf{Set}(X,|(Y,(Q(C,Y))_{C\in\mathscr{C}})|).$$

The right adjoint assigns to each set $Y$ the indiscrete quasi-space $(Y,(Q_{\text{ind}}(C,Y))_{C\in\mathscr{C}})$, and each map is assigned to itself. Then, for each quasi-space $(X,(Q(C,X))_{C\in\mathscr{C}})$,

$$\mathsf{Set}(|(X,(Q(C,X))_{C\in\mathscr{C}})|,Y)=\mathsf{Set}(X,Y)=\mathsf{Qs}((X,(Q(C,X))_{C\in\mathscr{C}}),(Y,(Q_{\text{ind}}(C,Y))_{C\in\mathscr{C}})).$$

Since $|\text{-}|$ has a left adjoint, it is represented by the singleton discrete quasi-space, which coincides with the singleton indiscrete quasi-space, and it is given by: for each $C\in\mathscr{C}$, $Q(C,1)=\{!_C\colon C\to 1\}$. □

## 9.2 $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-Cat is topological over Set

Let $(X,(Q(C,X))_{C\in\mathscr{C}})$ be a quasi-space. For each subset $A\subseteq X$, define: for each $C\in\mathscr{C}$, $\alpha\in Q_{\text{sub}}(C,A)$ if, and only if, $i_A\cdot\alpha\in Q(C,X)$, where $i_A\colon A\hookrightarrow X$ is the inclusion map. Let us call it the *sub-quasi-structure*.

**Lemma 9.2.1** *For each quasi-space* $(X,(Q(C,X))_{C\in\mathscr{C}})$ *and each* $A\subseteq X$, $(A,(Q_{\text{sub}}(C,A))_{C\in\mathscr{C}})$ *is a quasi-space, and, moreover, the inclusion map* $i_A$ *from* $A$ *into* $X$ *becomes a quasi-continuous map* $i_A\colon(A,(Q_{\text{sub}}(C,A))_{C\in\mathscr{C}})\to(X,(Q(C,X))_{C\in\mathscr{C}})$, *which is, moreover, a* $|\text{-}|$*-initial morphism, with* $|\text{-}|\colon\mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to\mathsf{Set}$ *the forgetful functor.*

*Proof.* It is straightforward to verify that $(A,(Q_{\text{sub}}(C,A))_{C\in\mathscr{C}})$ is a quasi-space and that the inclusion map $i_A$ is quasi-continuous. Let $f\colon Y\to A$ be a map such that, for a quasi-space $(Y,(Q(C,Y))_{C\in\mathscr{C}})$,

$i_A \cdot f \in \mathsf{Qs}(Y,X)$. Then, for each $\alpha \in Q(C,Y)$, with $C \in \mathscr{C}$, $i_A \cdot f \cdot \alpha \in Q(C,X)$, and, by definition, $f \cdot \alpha \in Q_{\text{sub}}(C,A)$, that is, $f \in \mathsf{Qs}(Y,A)$. $\square$

Now let us consider, for a quasi-space $(X,(Q(C,X))_{C\in\mathscr{C}})$, a surjective map $f\colon X \to Y$, and define: for each $C \in \mathscr{C}$, $\alpha \in Q_{\text{quo}}(C,Y)$ if, and only if, there exists a surjective continuous map $f'\colon C' \to C$, with $C' \in \mathscr{C}$, and an admissible map $\alpha' \in Q(C',X)$ such that the diagram

$$\begin{array}{ccc} C' & \xrightarrow{f'} & C \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{IV.5}$$

is commutative. Let us call it the *quotient quasi-structure*.

**Lemma 9.2.2** *If $f\colon X \to Y$ is a surjective map, for $(X,(Q(C,X))_{C\in\mathscr{C}})$ a quasi-space, then $(Y,(Q_{\text{quo}}(C,Y))_{C\in\mathscr{C}})$ is a quasi-space. Moreover, $f\colon (X,(Q(C,X))_{C\in\mathscr{C}}) \to (Y,(Q_{\text{quo}}(C,Y))_{C\in\mathscr{C}})$ is a quasi-continuous map, which is a $|\text{-}|$-final morphism, with $|\text{-}|\colon \mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{Set}$ the forgetful functor.*

*Proof.* One can readily check **(QS1)** and **(QS3)**. For **(QS2)**, let $\alpha \in Q(C,Y)$, with $C \in \mathscr{C}$, and let $h\colon B \to C$ be a continuous map, with $B \in \mathscr{C}$. By definition, there exists a surjective map $f'\colon C' \to C$ and $\alpha' \in Q(C',X)$, with $C' \in \mathscr{C}$, such that the diagram (IV.5) is commutative. Form the pullback of $f'$ along $h$:

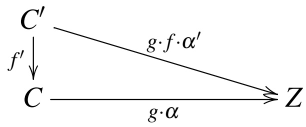
$$\begin{array}{ccc} B\times_C C' & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_{C'}}\downarrow & & \downarrow{\scriptstyle h} \\ C' & \xrightarrow[f']{} & C \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

Then $\pi_B$ is a surjective map, because $f'$ is surjective in $\mathsf{Set}$; $B\times_C C' \in \mathscr{C}$, since $\mathscr{C}$ is closed under binary products and equalizers; and since $\alpha' \in Q(C',X)$ and $\pi_{C'}$ is continuous, $\alpha' \cdot \pi_{C'} \in Q(B\times_C C',X)$. Then $\alpha \cdot h \in Q_{\text{quo}}(B,C)$.

For each $\alpha \in Q(C,X)$, the diagram

$$\begin{array}{ccc} C & \xrightarrow{1_C} & C \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle f\cdot\alpha} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is commutative, hence $f\cdot\alpha\in Q_{\text{quo}}(C,Y)$, and $f\in\mathsf{Qs}(X,Y)$. Let $g\colon Y\to Z$ be a map, for a quasi-space $(Z,(Q(C,Z))_{C\in\mathscr{C}})$, such that $g\cdot f\colon X\to Z\in\mathsf{Qs}(X,Z)$. Then, for each $C\in\mathscr{C}$ and each $\alpha\in Q_{\text{quo}}(C,Y)$, there exist a surjection $f'\colon C'\to C$ and a map $\alpha'\in Q(C',X)$, with $C'\in\mathscr{C}$, as in the commutative diagram (IV.5). Hence $g\cdot f\cdot\alpha'\in Q(C',Z)$, and because $g\cdot f\cdot\alpha'=g\cdot\alpha\cdot f'$, the map $g\cdot\alpha$ is covered by an admissible map, whence $g\cdot\alpha\in Q(C,Z)$, and $g\in\mathsf{Qs}(Y,Z)$.

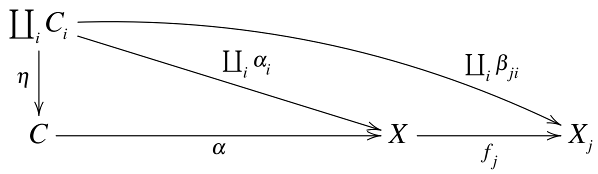



□

**Proposition 9.2.3** *The forgetful functor* $|\text{-}|\colon\mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to\mathsf{Set}$ *is topological.*

*Proof.* For a source $(f_j\colon X\to|(X_j,(Q(C,X_j))_{C\in\mathscr{C}})|)_{j\in J}$ in Set, for quasi-spaces $(X_j,(Q(C,X_j))_{C\in\mathscr{C}})$, $j\in J$, define: for each $C\in\mathscr{C}$, $\alpha\in Q(C,X)$ if, and only if, for all $j\in J$, $f_j\cdot\alpha\in Q(C,X_j)$.

Firstly, $(Q(C,X))_{C\in\mathscr{C}}$ is a quasi-structure: it immediately satisfies **(QS1)** and **(QS2)**. For **(QS3)**, if $\alpha\colon C\to X$, with $C\in\mathscr{C}$, is a map covered by the family $(\alpha_i\colon C_i\to X)_{i\in I}$ of admissible maps, then, for each $j\in J$, $f_j\cdot\alpha$ is covered by the family of maps $(\beta_{ji}=f_j\cdot\alpha_i)_{i\in I}$, which are admissible by definition of $Q(C_i,X)$.




Hence, for each $j\in J$, $f_j\cdot\alpha\in Q(C,X_j)$, so that $\alpha\in Q(C,X)$ by definition.

Secondly, let $(Y,(Q(C,Y))_{C\in\mathscr{C}})$ be a quasi-space, and $t\colon Y\to X$ be a map such that, for all $j\in J$, $f_j\cdot t\in\mathsf{Qs}(Y,X_j)$. Then, for each admissible map $\alpha\in Q(C,Y)$, with $C\in\mathscr{C}$, for all $j\in J$, $f_j\cdot t\cdot\alpha\in Q(C,X_j)$. Hence $t\cdot\alpha\in Q(C,X)$ by definition, and $t\in\mathsf{Qs}(Y,X)$.

Therefore, $(f_j\colon(X,(Q(C,X))_{C\in\mathscr{C}})\to(X_j,(Q(C,X_j))_{C\in\mathscr{C}}))_{j\in J}$ is a $|\text{-}|$-initial lifting for the given source. Uniqueness follows from amnesticity of the forgetful functor $|\text{-}|\colon\mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to\mathsf{Set}$ [AHS90, Definition 5.4(4), Proposition 21.5]. □

In particular, $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-Cat is complete and cocomplete. We describe next its limits and colimits.

*Limits* Consider a small category A and a functor $\mathscr{D}\colon\mathsf{A}\to\mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$. To construct the limit of $\mathscr{D}$, consider first the limit in Set of the composite functor $|\text{-}|\cdot\mathscr{D}\colon\mathsf{A}\to\mathsf{Set}$, with $|\text{-}|$ the forgetful functor from $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ to $\mathsf{Set}$, which we denote by $(\pi_A\colon X\to|\mathscr{D}A|)_{A\in\mathrm{Obj}\mathsf{A}}$. The limit of $\mathscr{D}$ is given by the $|\text{-}|$-initial lifting of this mono-source.

As particular instances, the product of a family $((X_i,(Q(C,X_i))_{C\in\mathscr{C}}))_{i\in I}$ of quasi-spaces is given by the set $\prod_{i\in I}X_i$ endowed with the quasi-structure: for each $C\in\mathscr{C}$,

$$\alpha\in Q(C,\prod_{i\in I}X_i)\iff\forall i\in I,\pi_i\cdot\alpha\in Q(C,X_i),$$

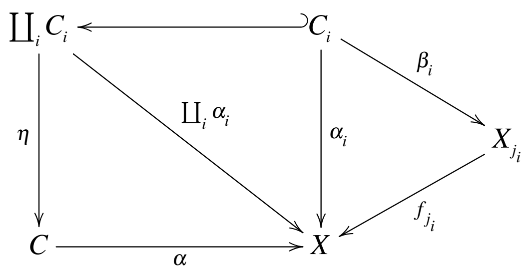
where, for each $i\in I$, $\pi_i$ is the product projection from $\prod_{i\in I}X_i$ to $X_i$; hence the terminal object in $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is given by the singleton set 1 endowed with the quasi-structure:

$$Q(C,1)=\{!_C\colon C\to 1\},\text{ for each } C\in\mathscr{C},$$

that was described in Proposition 9.1.6. For the equalizer of the quasi-continuous maps $f,g\colon X\to Y$, for quasi-spaces $(X,(Q(C,X))_{C\in\mathscr{C}})$ and $(Y,(Q(C,Y))_{C\in\mathscr{C}})$, endow the set $E=\{x\in X\mid f(x)=g(x)\}$ with the sub-quasi-structure with respect to the inclusion map $i_E\colon E\hookrightarrow X$.

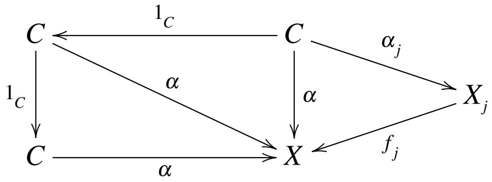
*Colimits* Let $\mathscr{D}\colon\mathsf{A}\to\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ be a diagram, and form the colimit $(\iota_A\colon|\mathscr{D}A|\to X)_{A\in\mathrm{Obj}\mathsf{A}}$ in $\mathsf{Set}$ of the composite functor $|\text{-}|\cdot\mathscr{D}\colon\mathsf{A}\to\mathsf{Set}$, $|\text{-}|\colon\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}\to\mathsf{Set}$ the forgetful functor. The colimit of $\mathscr{D}$ is given by the $|\text{-}|$-final lifting of this epi-sink.

**Lemma 9.2.4** *The $|\text{-}|$-final lifting of an epi-sink $(f_j\colon|(X_j,(Q(C,X_j))_{C\in\mathscr{C}})|\to X)_{j\in J}$ in* $\mathsf{Set}$, *for quasi-spaces $(X_j,(Q(C,X_j))_{C\in\mathscr{C}})$, $j\in J$, is given by $X$ endowed with the quasi-structure defined by: for each $C\in\mathscr{C}$, $\alpha\in Q(C,X)$ if, and only if, $\alpha$ is covered by a family of maps $(\alpha_i\colon C_i\to X)_{i\in I}$ such that, for each $i\in I$, there exists $j_i\in J$ and $\beta_i\in Q(C_i,X_{j_i})$ with $\alpha_i=f_{j_i}\cdot\beta_i$.*

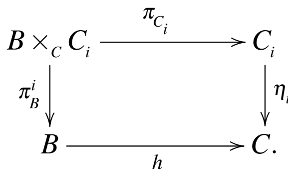



(IV.6)

*Proof.* First we prove that $(X,(Q(C,X))_{C\in\mathscr{C}})$ is a quasi-space. For condition **(QS1)**, consider, for $C\in\mathscr{C}$, a constant map $\alpha\colon C\to X$, $c\mapsto x_0$. Since we are considering an epi-sink, there exist $j\in J$ and $x_j\in X_j$ such that $f_j(x_j)=x_0$. Define $\alpha_j\colon C\to X_j$, $c\mapsto x_j$, which is admissible, since it is a constant map. Then commutativity of the following diagram implies that $\alpha \in Q(C,X)$.



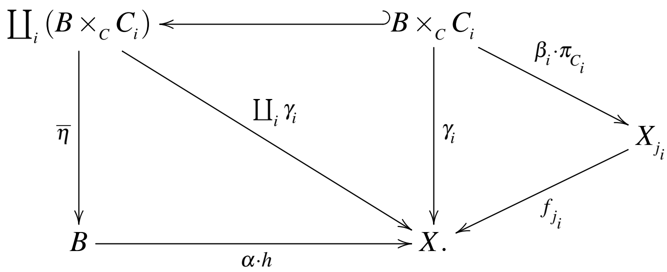

For **(QS2)** let $\alpha \in Q(C,X)$, for $C \in \mathscr{C}$, and $h\colon B \to C$ a continuous map, with $B \in \mathscr{C}$. Then $\alpha$ is covered by a family $(\alpha_i)_{i\in I}$ as in (IV.6). For each $i \in I$, set $\eta_i = \eta \cdot \iota_i$, where $\iota_i\colon C_i \hookrightarrow \coprod_i C_i$ is the coproduct inclusion, and consider the pullback of $h$ along $\eta_i$:



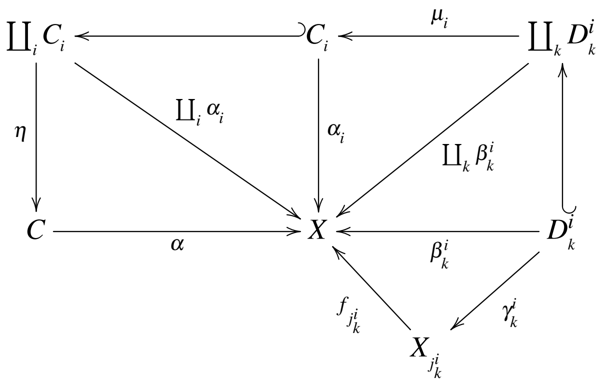

For each $i \in I$, $B \times_C C_i$ belongs to $\mathscr{C}$; moreover, the family of continuous maps $(\pi_B^i\colon B \times_C C_i \to B)_{i\in I}$ induces the surjective continuous map $\overline{\eta}\colon \coprod_i (B \times_C C_i) \to B$: if $b \in B$, then there exists $i \in I$ such that $h(b) = \eta(c_i, i)$, for some $(c_i, i) \in \coprod_i C_i$, because $\eta$ is surjective, hence $h(b) = \eta_i(c_i)$ implies $(b, c_i) \in B \times_C C_i$, and $b = \pi_B^i(b, c_i) = \overline{\eta}((b, c_i), i)$. For each $i \in I$, define $\gamma_i = \alpha_i \cdot \pi_{C_i}\colon B \times_C C_i \to X$, whence, for each $((b, c_i), i) \in \coprod_i (B \times_C C_i)$,

$$\alpha \cdot h \cdot \overline{\eta}((b,c_i),i) = \alpha \cdot h(b) = \alpha \cdot \eta_i(c_i) = \alpha_i \cdot \pi_{C_i}(b,c_i) = \coprod_i \gamma_i((b,c_i),i).$$

Furthermore, for each $i \in I$, $\gamma_i = \alpha_i \cdot \pi_{C_i} = f_{j_i} \cdot \beta_i \cdot \pi_{C_i}$, for some $j_i \in J$, with $\beta_i \in Q(C_i, X_{j_i})$, whence $\beta_i \cdot \pi_{C_i} \in Q(B \times_C C_i, X_{j_i})$, and we can conclude with the commutative diagram



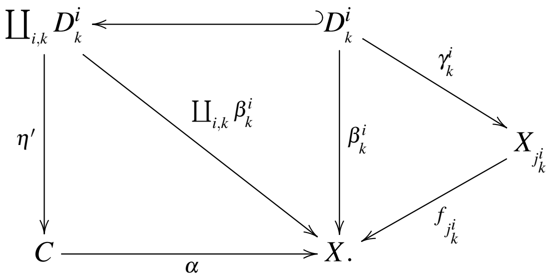

For **(QS3)**, let $\alpha\colon C \to X$, with $C \in \mathscr{C}$, be a map covered by a family $(\alpha_i)_{i\in I}$ of admissible maps as in (IV.3). Then, for each $i \in I$, $\alpha_i$ is covered by a family $(\beta_k^i\colon D_k^i \to X)_{k\in K}$, with $D_k^i \in \mathscr{C}$, of maps such that, for each $k\in K$, there exists $j^i_k\in J$ and $\gamma^i_k\in Q(D^i_k,X_{j^i_k})$ with $\beta^i_k=f_{j^i_k}\cdot\gamma^i_k$.



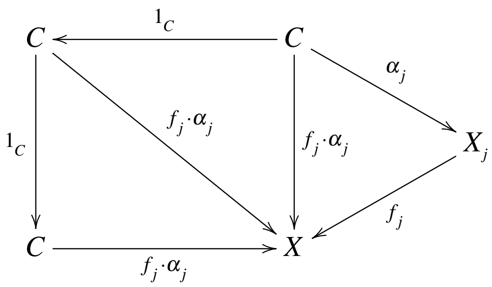
Since $\mu_i$ is a surjective continuous map, so is the induced morphism $\coprod_i\mu_i\colon\ \coprod_i\coprod_k D^i_k\to\coprod_i C_i$, and so is the composite with $\eta$, $\eta'=\eta\cdot\coprod_i\mu_i\colon\ \coprod_{i,k}D^i_k\to C$. Furthermore,

$$\alpha\cdot\eta'=\alpha\cdot\eta\cdot\coprod_i\mu_i=\coprod_i\alpha_i\cdot\coprod_i\mu_i=\coprod_i\alpha_i\cdot\mu_i=\coprod_{i,k}\beta^i_k,$$

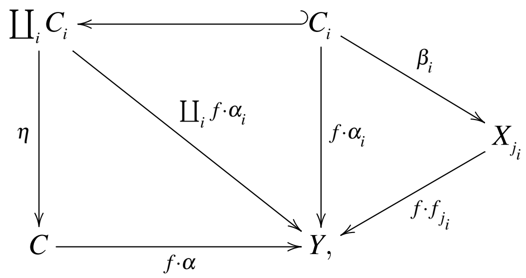
so we can conclude with the commutative diagram



With $X$ endowed with this quasi-structure, for each $j\in J$, the map $f_j\colon X_j\to X$ is quasi-continuous, for if $\alpha_j\in Q(C,X_j)$, for $C\in\mathscr{C}$, then, by definition, $f_j\cdot\alpha_j\in Q(C,X)$, since



is a commutative diagram. Finally, let $f\colon X\to Y$ be a map, for $(Y,(Q(C,Y))_{C\in\mathscr{C}})$ a quasi-space, such that, for all $j\in J$, $f\cdot f_j\colon X_j\to Y$ is a quasi-continuous map. If $\alpha\in Q(C,X)$, for $C\in\mathscr{C}$, then, by definition, $\alpha$ is covered by a family $(\alpha_i)_{i\in I}$ of maps as in (IV.6). Then the commutative diagram

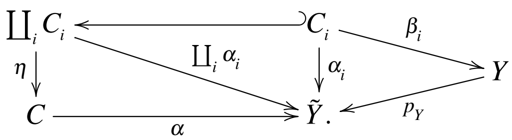



shows that $f\cdot\alpha$ is covered by the family of maps $f\cdot\alpha_i = f\cdot f_{j_i}\cdot\beta_i$, $i\in I$, which are admissible because $f\cdot f_{j_i}$ is a quasi-continuous map and $\beta_i \in Q(C_i, X_{j_i})$. Therefore, $f\cdot\alpha \in Q(C,Y)$, and $f\colon X\to Y$ is a quasi-continuous map. □

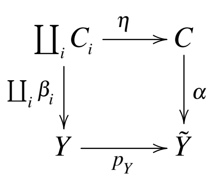
In particular, the coproduct of a family $((X_i,(Q(C,X_i))_{C\in\mathscr{C}}))_{i\in I}$ of quasi-spaces is given by the disjoint union $\dot\bigcup_i X_i$ endowed with the quasi-structure defined in the previous lemma with respect to the epi-sink $(\iota_i\colon X_i \hookrightarrow \dot\bigcup_i X_i)_{i\in I}$. The initial object is the empty set $\emptyset$ endowed with the quasi-structure:

$$Q(C,\emptyset) = \begin{cases} \emptyset, \text{ if } C\neq\emptyset \\ \{1_\emptyset\}, \text{ otherwise.}\end{cases}$$

For coequalizers of quasi-continuous maps $f,g\colon X\to Y$, consider in the set $Y$ the smallest equivalence relation $\sim$ that contains the pairs $(f(x),g(x))$, for $x\in X$. Endow the quotient set $\tilde{Y} = Y/{\sim}$ with the quotient quasi-structure with respect to the projection map $p_Y\colon Y\to\tilde{Y}$. Let us verify that this quasi-structure coincides with the one given by the previous lemma: $\alpha\in Q(C,\tilde{Y})$, for $C\in\mathscr{C}$, if, and only if, $\alpha$ is covered by a family $(\alpha_i)_{i\in I}$ as in (IV.6):




Since the maps $\beta_i$ are admissible from $C_i$ to $Y$, we have $\coprod_i \beta_i \in Q(\coprod_i C_i, Y)$, whence



is a commutative diagram, with $\eta$ a surjective continuous map, so that $\alpha \in Q_{\text{quo}}(C,\tilde{Y})$. Conversely, if $\alpha \in Q_{\text{quo}}(C,\tilde{Y})$, for $C \in \mathscr{C}$, then, by definition, we have a commutative diagram

$$\begin{array}{ccc} C' & \xrightarrow{\eta} & C \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow[p_Y]{} & \tilde{Y}, \end{array}$$

with $\eta : C' \to C$ a surjective continuous map and $\alpha' \in Q(C',Y)$, for $C' \in \mathscr{C}$. Then $\alpha$ is covered by $p_Y \cdot \alpha'$, hence, $\alpha$ belongs to the $|\text{-}|$-final quasi-structure with respect to the projection map $p_Y$.

## 9.3 $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is cartesian closed

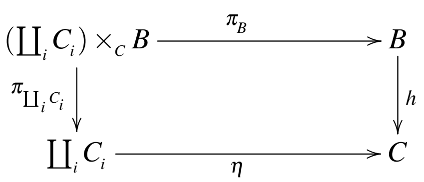
For quasi-spaces $X$ and $Y$, a natural candidate for an exponential is the set $\mathsf{Qs}(X,Y)$ of quasi-continuous maps. Then we have the evalutation map $\text{ev}\colon \mathsf{Qs}(X,Y) \times X \to Y$, that we wish to be a quasi-continuous map, for some quasi-structure on $\mathsf{Qs}(X,Y)$. For such a structure, for each $\gamma \in Q(C, \mathsf{Qs}(X,Y) \times X)$, with $C \in \mathscr{C}$, $\text{ev} \cdot \gamma \in Q(C,Y)$. Consequently, for admissible maps $\beta \in Q(C, \mathsf{Qs}(X,Y))$ and $\alpha \in Q(C,X)$, $\langle \beta, \alpha \rangle : C \to \mathsf{Qs}(X,Y) \times X$ is an admissible map, whence $\text{ev} \cdot \langle \beta, \alpha \rangle$ belongs to $Q(C,Y)$. Under this intuition and considering conditions **(QS1)** to **(QS3)**, we prove:

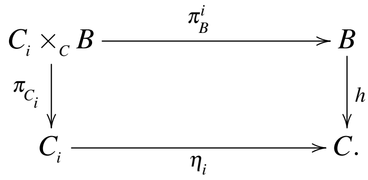
**Lemma 9.3.1** *For quasi-spaces $X$ and $Y$, the following defines a quasi-structure on the set $\mathsf{Qs}(X,Y)$: for each $C \in \mathscr{C}$, $\beta \in Q(C, \mathsf{Qs}(X,Y))$ if, and only if, for each $(\mathbb{T},\mathsf{V})$-continuous map $h\colon B \to C$, for $B \in \mathscr{C}$, and each admissible map $\alpha \in Q(B,X)$, $\text{ev} \cdot \langle \beta \cdot h, \alpha \rangle : B \to Y \in Q(B,Y)$. Moreover, when $\mathsf{Qs}(X,Y)$ is endowed with this quasi-structure, the evaluation map is quasi-continuous.*

*Proof.* Conditions **(QS1)** and **(QS2)** are readily verifiable. For **(QS3)**, let $\beta\colon C \to \mathsf{Qs}(X,Y)$, for $C \in \mathscr{C}$, be a map covered by a family of admissible maps $(\beta_i)_{i \in I}$:

$$\begin{array}{ccc} \coprod_i C_i & \xrightarrow{\coprod_i \beta_i} & \\ {\scriptstyle \eta}\downarrow & & \searrow \\ C & \xrightarrow[\beta]{} & \mathsf{Qs}(X,Y). \end{array}$$

To verify that $\beta$ is admissible, let $h\colon B \to C$ be a continuous map, for $B \in \mathscr{C}$, and $\alpha \in Q(B,X)$. Let us consider, for each $i \in I$, the map $\eta_i = \eta \cdot \iota_i$, with $\iota_i : C_i \hookrightarrow \coprod_i C_i$ the coproduct inclusion. Consider the pullbacks of $\eta$ and $\eta_i$ along $h$:

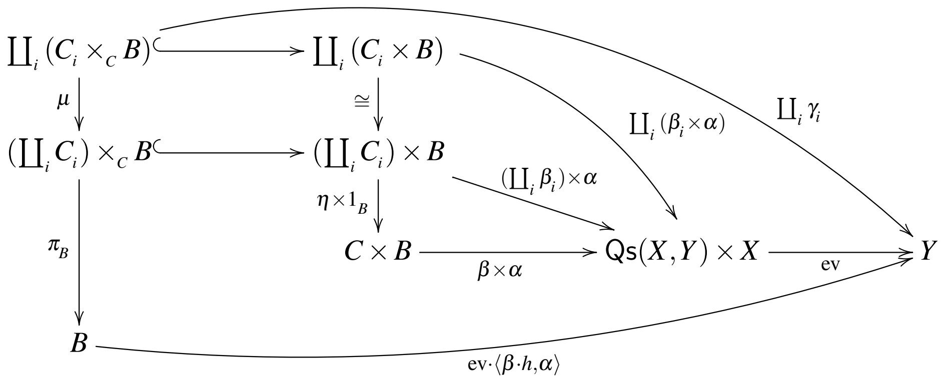




Then we have a surjective continuous map

$$\begin{array}{rccc} \mu\colon & \coprod_i (C_i \times_C B) & \longrightarrow & (\coprod_i C_i) \times_C B \\ & ((c_i,b),i) & \longmapsto & ((c_i,i),b), \end{array}$$

where $\coprod_i (C_i \times_C B) \in \mathscr{C}$. Since $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is distributive, we have $\coprod_i (C_i \times B) \cong (\coprod_i C_i) \times B$, and we assemble the following commutative diagram.



Where, for each $i \in I$, $\gamma_i\colon C_i \times_C B \to Y$ is given by, for each $(c_i,b) \in C_i \times_C B$,

$$\gamma_i(c_i,b) = \beta_i(c_i)(\alpha(b)) = \beta_i(\pi_{C_i}(c_i,b))(\alpha \cdot \pi^i_B(c_i,b)),$$

that is, $\gamma_i = \mathrm{ev} \cdot \left\langle \beta_i \cdot \pi_{C_i}, \alpha \cdot \pi^i_B \right\rangle$, which is an admissible map, because $\beta_i$ is an admissible map, $\pi_{C_i}\colon C_i \times_C B \to C_i$ is a continuous map, and $\alpha$ is an admissible map, so that $\alpha \cdot \pi^i_B$ is an admissible map. Finally, $\mathrm{ev} \cdot \langle \beta \cdot h, \alpha \rangle$ is covered by the family of admissible maps $(\gamma_i)_{i\in I}$: for each $(c_i,b) \in C_i \times_C B$,

$$\begin{aligned} \mathrm{ev} \cdot \langle \beta \cdot h, \alpha \rangle \cdot \pi_B \cdot \mu(c_i,b) &= \beta(h(b))(\alpha(b)) = \beta(\eta_i \cdot \pi_{C_i}(c_i,b))(\alpha \cdot \pi^i_B(c_i,b)) \\ &= \beta_i(\pi_{C_i}(c_i,b))(\alpha \cdot \pi^i_B(c_i,b)) = \gamma_i(c_i,b), \end{aligned}$$

whence $\mathrm{ev} \cdot \langle \beta \cdot h, \alpha \rangle \in Q(B,Y)$. For each admissible map $\gamma \in Q(C, \mathsf{Qs}(X,Y) \times X)$, composing with the product projections $\pi_{\mathsf{Qs}(X,Y)}$ and $\pi_X$ from $\mathsf{Qs}(X,Y) \times X$ into $\mathsf{Qs}(X,Y)$ and $X$, respectively, we get admissible maps $\gamma_1 : C \to \mathsf{Qs}(X,Y)$ and $\gamma_2 : C \to X$. By definition of the quasi-structure on $\mathsf{Qs}(X,Y)$, $\mathrm{ev}\cdot\gamma = \mathrm{ev}\cdot\langle\gamma_1\cdot 1_C,\gamma_2\rangle \in Q(C,Y)$, hence $\mathrm{ev}\colon \mathsf{Qs}(X,Y)\times X \to Y$ is a quasi-continuous map. □

**Theorem 9.3.2** $\mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ *is cartesian closed.*

*Proof.* For quasi-spaces $X$ and $Y$, endow the set $\mathsf{Qs}(X,Y)$ with the quasi-structure defined previously, so that the evaluation map $\mathrm{ev}\colon \mathsf{Qs}(X,Y)\times X \to Y$ is quasi-continuous. For each quasi-continuous map $f\colon Z\times X \to Y$, with $Z$ a quasi-space, there exists a unique map $\overline{f}\colon Z \to \mathsf{Set}(X,Y)$, the *transpose* of $f$, such that $\mathrm{ev}\cdot(\overline{f}\times 1_X) = f$. Let $z\in Z$ and $\alpha\in Q(C,X)$, for $C\in\mathscr{C}$. For each $c\in C$,

$$\overline{f}(z)\cdot\alpha(c) = \overline{f}(z)(\alpha(c)) = \mathrm{ev}\cdot(\overline{f}\times 1_X)(z,\alpha(c)) = f(z,\alpha(c)) = f\cdot\langle z,\alpha\rangle(c),$$

where $z$ denotes the contant (hence admissible) map $z\colon C\to Z$, $c\mapsto z$. Since $\langle z,\alpha\rangle\colon C\to Z\times X$ is an admissible map and $f$ is a quasi-continuous map, we conclude that $\overline{f}(z)\cdot\alpha = f\cdot\langle z,\alpha\rangle$ belongs to $Q(C,Y)$. Hence $\overline{f}(z)\colon X\to Y$ is a quasi-continuous map and we have a corestriction $\overline{f}\colon Z\to\mathsf{Qs}(X,Y)$. Moreover, for each $\gamma\in Q(C,Z)$, for $C\in\mathscr{C}$, let us prove that $\overline{f}\cdot\gamma\in Q(C,\mathsf{Qs}(X,Y))$. Let $h\colon B\to C$ be a continuous map, with $B\in\mathscr{C}$, and $\alpha\in Q(B,X)$. By **(QS2)**, $\gamma\cdot h\in Q(B,Z)$, whence $\langle\gamma\cdot h,\alpha\rangle$ belongs to $Q(B,Z\times X)$ and $f\cdot\langle\gamma\cdot h,\alpha\rangle\in Q(B,Y)$. For each $b\in B$,

$$\mathrm{ev}\cdot\left\langle\overline{f}\cdot\gamma\cdot h,\alpha\right\rangle(b) = \overline{f}\cdot\gamma\cdot h(b)(\alpha(b)) = \overline{f}(\gamma\cdot h(b))(\alpha(b)) = f(\gamma\cdot h(b),\alpha(b)) = f\cdot\langle\gamma\cdot h,\alpha\rangle(b),$$

hence $\overline{f}\colon Z\to\mathsf{Qs}(X,Y)$ is a quasi-continuous map.

$$\begin{array}{ccc} \mathsf{Qs}(X,Y) & \quad & \mathsf{Qs}(X,Y)\times X \xrightarrow{\ \mathrm{ev}\ } Y \\ {\scriptstyle \exists\,!\,\overline{f}}\,\uparrow & & {\scriptstyle \overline{f}\times 1_X}\uparrow \quad \nearrow{\scriptstyle f} \\ Z & & Z\times X \end{array}$$

□

## 9.4 Day's relationship between $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}}$ and $\mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$

As we mentioned in Subsection 9.1, the inclusion of $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ into $\mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ given by Lemma 9.1.3 is not full. However, it is full when restricted to $\mathscr{C}$-generated spaces, and this relationship was studied for the particular case of $\mathsf{Top}$ in [Day68].

By definition, if $(X,a)$ is a $\mathscr{C}$-generated space, then a map $f\colon (X,a)\to(Y,b)$, for $(Y,b)$ a $(\mathbb{T},\mathsf{V})$-space, is $(\mathbb{T},\mathsf{V})$-continuous if, and only if, for each $(\mathbb{T},\mathsf{V})$-continuous map (probe) $\alpha\colon C\to(X,a)$, with $C \in \mathscr{C}$, the composite $f \cdot \alpha \colon C \to Y$ is $(\mathbb{T},\mathsf{V})$-continuous. Considering the associated quasi-spaces $(X, Q_a(C,X)_{C\in\mathscr{C}})$ and $(Y, Q_b(C,X)_{C\in\mathscr{C}})$, a map $f\colon X \to Y$ is $(\mathbb{T},\mathsf{V})$-continuous if, and only if, $f\colon X \to Y$ is quasi-continuous, that is,

$$(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}(X,Y) = \mathsf{Qs}(X,Y). \tag{IV.7}$$

The converse implication is also true: for a $(\mathbb{T},\mathsf{V})$-space $(X,a)$, if (IV.7) is satisfied for every $(\mathbb{T},\mathsf{V})$-space $(Y,b)$, where the spaces are endowed with the associated quasi-structures, then, by definition, $(X,a)$ is $\mathscr{C}$-generated.

Let $\mathscr{C}$-$(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ be the full subcategory of $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ of quasi-spaces associated with $\mathscr{C}$-generated spaces. By (IV.7), $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}} \cong \mathscr{C}\text{-}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$. Moreover, we can prove (see [Day68, Theorem 4.2]):

**Proposition 9.4.1** *$\mathscr{C}$-$(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is reflective in $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$.*

*Proof.* For a quasi-space $(X, (Q(C,X))_{C\in\mathscr{C}})$ consider the sink $(\alpha\colon (C,c) \to X)_{C\in\mathscr{C}, \alpha\in Q(C,X)}$. Take its $|\text{-}|$-final lifting $(\alpha\colon (C,c) \to (X,a_Q))_{C\in\mathscr{C},\alpha\in Q(C,X)}$ in $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, with $|\text{-}|\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{Set}$ the forgetful functor. Consider the quasi-space $(X, (Q_{a_Q}(C,X))_{C\in\mathscr{C}})$ associated with $(X,a_Q)$. For a map $f\colon X \to Y$, with $(Y,b)$ a $(\mathbb{T},\mathsf{V})$-space, if $f\colon (X,a_Q) \to (Y,b)$ is $(\mathbb{T},\mathsf{V})$-continuous, then

$$f\colon (X, (Q_{a_Q}(C,X))_{C\in\mathscr{C}}) \to (Y, (Q_b(C,Y))_{C\in\mathscr{C}})$$

is quasi-continuous. Moreover, the converse implication holds, for if $f$ is a quasi-continuous map when considering the associated quasi-structures, then, for each $C \in \mathscr{C}$ and $\alpha \in Q(C,X)$, $\alpha\colon C \to (X,a_Q)$ is $(\mathbb{T},\mathsf{V})$-continuous, whence $f\cdot\alpha\colon C \to (Y,b)$ is $(\mathbb{T},\mathsf{V})$-continuous, hence $f\colon (X,a_Q) \to (Y,b)$ is $(\mathbb{T},\mathsf{V})$-continuous by definition of $a_Q$.

We have proved that $(X,a_Q)$ satisfies (IV.7), for every $(\mathbb{T},\mathsf{V})$-space $(Y,b)$, so that its associated quasi-space $(X, (Q_{a_Q}(C,X))_{C\in\mathscr{C}})$ belongs to $\mathscr{C}$-$(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$. Each $\alpha \in Q(C,X)$, for $C \in \mathscr{C}$, is a $(\mathbb{T},\mathsf{V})$-continuous map $\alpha\colon C \to (X,a_Q)$, hence it is an admissible map in the associated quasi-structure $Q_{a_Q}$, that is, the identity map $1_X\colon (X, (Q(C,X))_{C\in\mathscr{C}}) \to (X, (Q_{a_Q}(C,X))_{C\in\mathscr{C}})$ is quasi-continuous. Finally, if $f\colon (X, (Q(C,X))_{C\in\mathscr{C}}) \to (Y, (Q_b(C,Y))_{C\in\mathscr{C}})$ is a quasi-continuous map, for $(Y, (Q_b(C,Y))_{C\in\mathscr{C}})$ in $\mathscr{C}$-$(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, then, as we deduced before, $f\colon (X,a_Q) \to (Y,b)$ is a $(\mathbb{T},\mathsf{V})$-continuous map, and consequently $f\colon (X,(Q_{a_Q}(C,X))_{C\in\mathscr{C}}) \to (Y,(Q_b(C,Y))_{C\in\mathscr{C}})$ is a quasi-continuous map.

$$\begin{array}{ccc} (X,Q(C,X)) & \xrightarrow{\;1_X\;} & (X,Q_{a_Q}(C,X)) \\ & \searrow^{f} & \downarrow f \\ & & (Y,Q_b(C,Y)) \end{array}$$

□

Recalling that we have fixed $\mathscr{C}$ as the class of compact Hausdorff $(\mathbb{T},\mathsf{V})$-spaces, we summarize the above in the following diagram.

$$\begin{array}{ccc} (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathscr{C}} & \underset{\top}{\hookrightarrow} & (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \\ \downarrow \cong & & \downarrow \text{non-full} \\ \mathscr{C}\text{-}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} & \underset{\bot}{\hookrightarrow} & \mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \end{array}$$

**Examples 9.4.2** (1) For $\mathsf{V}$-$\mathsf{Cat}$ with $\mathsf{V}$ integral, compact and Hausdorff $\mathsf{V}$-spaces are discrete, hence a quasi-$\mathsf{V}$-space consists of a set $X$ and, for each $(C,1_C)\in\mathscr{C}$, a set $Q(C,X)$ of maps satisfying conditions **(QS1)** to **(QS3)**. In particular, for the quantales 2, $\mathsf{P}_+$, $\mathsf{P}_{\max}$, and $\mathsf{P}_1$ the respective categories of quasi-spaces coincide: $\mathsf{QsOrd} = \mathsf{QsMet} = \mathsf{QsUltMet} = \mathsf{QsBMet}$.
(2) For $(\mathbb{U},\mathsf{V})$-$\mathsf{Cat}$, with $\mathsf{V} = 2,\mathsf{P}_+,\mathsf{P}_{\max},\mathsf{P}_1$, by the same reasoning of item (1), since

$$(\mathbb{U},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{CompHaus}} \cong \mathsf{Set}^{\mathbb{U}},$$

we can conclude that $\mathsf{Qs}(\mathbb{U},\mathsf{P}_1)\text{-}\mathsf{Cat} = \mathsf{QsNA}\text{-}\mathsf{App} = \mathsf{QsApp} = \mathsf{QsTop}$, where $\mathsf{QsTop}$ denotes the category of quasi-topological spaces and quasi-continuous maps [Spa63].

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