

LAX ORTHOGONAL FACTORISATIONS IN MONAD-QUANTALE-ENRICHED CATEGORIES

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Dedicated to Jiří Adámek

ABSTRACT. We show that, for a quantale V and a **Set**-monad \mathbb{T} laxly extended to V -Rel, the presheaf monad on the category of (\mathbb{T}, V) -categories is simple, giving rise to a lax orthogonal factorisation system (LOFS) whose corresponding weak factorisation system has embeddings as left part. In addition, we present presheaf submonads and study the LOFSs they define. This provides a method of constructing weak factorisation systems on some well-known examples of topological categories over **Set**.

1. INTRODUCTION

In 1985 Cassidy-Hébert-Kelly [8] studied orthogonal factorisations systems induced by reflective subcategories, with particular emphasis in the case when the reflection is simple. Among the lax orthogonal factorisation systems (LOFSs), that generalise the orthogonal ones in 2-categories, those arising from simple monads – as defined by the authors of this paper in [14, 15] – have particular relevance. This paper intends to give a systematic way of producing simple monads in (some) topological categories over **Set** using the presheaf monads of (\mathbb{T}, V) -Cat studied in [23, 11]. Given a quantale V and a well-behaved **Set**-monad \mathbb{T} , the category (\mathbb{T}, V) -Cat, of generalised V -enriched categories and their functors, is topological and locally preordered (see [10, 16]). As crucial examples we mention the categories **Ord** of (pre)ordered sets and monotone maps, **Top** of topological spaces and continuous maps, **Met** of Lawvere generalised metric spaces and non-expansive maps [28], and **App** of Lowen approach spaces and non-expansive maps [30]. Equipping the quantale V with a canonical (\mathbb{T}, V) -category structure, one gets naturally a Yoneda Lemma and a well-behaved presheaf monad that was shown to be lax idempotent in [23]. Here we show that it is simple, inducing a lax orthogonal factorisation system which underlies a weak factorisation system having embeddings as left part. (In order to avoid technicalities we restrict ourselves to separated, or skeletal, (\mathbb{T}, V) -categories, so that their hom-sets have an anti-symmetric order.) This encompasses the weak factorisation system in **Ord** studied by Adámek-Herrlich-Rosický-Tholen in [1].

These presheaf monads have interesting simple submonads, namely the ones which have as algebras the Lawvere complete (\mathbb{T}, V) -categories (see [12]), and that gives – as shown by Lawvere in [28] – Cauchy-complete spaces when one takes $\mathbb{T} = \mathbb{Id}$ the identity monad and V the complete half-real line. These also cover, following techniques developed in [11], the weak factorisation systems of **Top**₀ studied in [5], having as left parts embeddings, dense embeddings, flat embeddings and completely flat embeddings.

The examples of LOFSs and weak factorisation systems we consider are the result of a general construction distinct from, and in many ways orthogonal to, the more usual method of cofibrant generation. The construction of cofibrantly generated weak factorisation systems is usually known as Quillen’s small object argument [32]. A version for algebraic factorisation systems

2010 *Mathematics Subject Classification*. 18A32, 18C15, 18C20, 06B35, 54B30.

Key words and phrases. Quantale, monad, enriched category, (\mathbb{T}, V) -category, presheaf monad, injective morphism.

The authors acknowledge partial financial assistance by Centro de Matemática da Universidade de Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

was introduced in [19]. The construction we employ, introduced in [14], gives rise to weak factorisations systems that need not be cofibrantly generated (see [29]).

This paper does not intend to be self-contained. In Section 2 and 3 we present the basic definitions and results on lax orthogonal factorisation systems and on (\mathbb{T}, V) -categories that are needed for this work. For a better understanding of these topics we refer to the papers mentioned there and to the monograph [24]. In Section 4 we study the presheaf monads on (\mathbb{T}, V) -categories and their simplicity. In Section 5 we explore the examples of lax orthogonal factorisation systems induced by these presheaf monads.

2. LAX ORTHOGONAL FACTORISATION SYSTEMS

Throughout we will be working in a category \mathbf{C} enriched in posets, or *Ord-enriched category*, so that each hom-set $\mathbf{C}(X, Y)$ is equipped with an order structure \leq that is preserved by composition: if $f, f' : X \rightarrow Y$, with $f \leq f'$, $g : Y \rightarrow Z$ and $h : W \rightarrow X$, then $g \cdot f \leq g \cdot f'$ and $f \cdot h \leq f' \cdot h$.

2.1. Weak factorisation systems. Given morphisms f, g , we say that f has the *left lifting property with respect to g* , and that g has the *right lifting property with respect to f* , if every commutative square as shown has (a not necessarily unique) diagonal filler.

$$\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow f & \nearrow & \downarrow g \\ \cdot & \xrightarrow{\quad} & \cdot \end{array}$$

A *weak factorisation system* (WFS) in a category is a pair $(\mathcal{L}, \mathcal{R})$ of families of morphisms such that:

- \mathcal{L} consists of those morphisms with the left lifting property with respect to each morphism of \mathcal{R} .
- \mathcal{R} consists of those morphisms with the right lifting property with respect to each morphism of \mathcal{L} .
- Each morphism in the category factors through an element of \mathcal{L} followed by one of \mathcal{R} .

2.2. Algebraic weak factorisation systems. An Ord-functorial factorisation on an Ord-category \mathbf{C} consists of a factorisation $\text{dom} \xrightarrow{\lambda} E \xrightarrow{\rho} \text{cod}$ of the natural transformation $\text{dom} \Rightarrow \text{cod}$ with component at $f \in \mathbf{C}^2$ equal to $f : \text{dom}(f) \rightarrow \text{cod}(f)$, in the category of locally monotone functors $\mathbf{C}^2 \rightarrow \mathbf{C}$. As in the case of functorial factorisations on ordinary categories, an Ord-functorial factorisation can be equivalently described as:

- A copointed endo-Ord-functor $\Phi : L \Rightarrow 1_{\mathbf{C}^2}$ on \mathbf{C}^2 with $\text{dom}(\Phi) = 1$.
- A pointed endo-Ord-functor $\Lambda : 1_{\mathbf{C}^2} \Rightarrow R$ on \mathbf{C}^2 with $\text{cod}(\Lambda) = 1$.

The three descriptions of an Ord-functorial factorisation are related by:

$$\text{dom}(\Lambda_f) = Lf = \lambda_f \quad \text{cod}(\Phi_f) = Rf = \rho_f.$$

An *algebraic weak factorisation system*, abbreviated AWFS, on an Ord-category \mathbf{C} consists of a pair (\mathbb{L}, \mathbb{R}) , where $\mathbb{L} = (L, \Phi, \Sigma)$ is an Ord-comonad and $\mathbb{R} = (R, \Lambda, \Pi)$ is an Ord-monad on \mathbf{C}^2 , such that (L, Φ) and (R, Λ) represent the same Ord-functorial factorisation on \mathbf{C} (i.e., the equalities above hold), fulfilling a distributivity condition which we explain next.

Note that the components of Σ and Π are as follows

$$\Sigma_f = \begin{array}{ccc} \cdot & \xlongequal{\quad} & \cdot \\ \downarrow Lf & \sigma_f & \downarrow L^2f \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \quad \text{and} \quad \Pi_f = \begin{array}{ccc} \cdot & \xrightarrow{\pi_f} & \cdot \\ \downarrow R^2f & & \downarrow Rf \\ \cdot & \xlongequal{\quad} & \cdot \end{array}$$

which together form a natural transformation $\Delta : LR \Rightarrow RL$, with $\Delta_f = (\sigma_f, \pi_f)$ as below.

$$\Delta_f = \begin{array}{ccc} Ef & \xrightarrow{\sigma_f} & ELf \\ \downarrow LRf & \searrow 1 & \downarrow RLf \\ ERf & \xrightarrow{\pi_f} & Ef \end{array}$$

The distributivity axiom requires Δ to be a mixed distributive law between the comonad \mathbb{L} and the monad \mathbb{R} , that reduces to the commutativity of the following diagrams.

$$(a) \quad \begin{array}{ccc} LR^2 & \xrightarrow{\Delta R} & RLR & \xrightarrow{R\Delta} & R^2L & & LR & \xrightarrow{\Delta} & RL \\ \downarrow L\Pi & & & & \downarrow \Pi L & & \downarrow \Sigma R & & \downarrow R\Sigma \\ LR & \xrightarrow{\Delta} & RL & & L^2R & \xrightarrow{L\Delta} & LRL & \xrightarrow{\Delta L} & RL^2 \end{array}$$

Algebraic weak factorisation systems were introduced by Grandis-Tholen in [20] under the name *natural factorisation system*; later, in [19], Garner added to this definition the distributivity conditions we described above.

Each AWFS has an underlying WFS $(\mathcal{L}, \mathcal{R})$, with $\mathcal{L} = \{f \mid f \text{ has an } (L, \Phi)\text{-coalgebra structure}\}$ and $\mathcal{R} = \{g \mid g \text{ has an } (R, \Lambda)\text{-algebra structure}\}$. A coalgebra structure $(1_X, s : Y \rightarrow Ef)$ for $f \in \mathcal{L}$, so that $s \cdot f = Lf$ and $Rf \cdot s = 1_Y$, and an (R, Λ) -algebra structure $(p : Eg \rightarrow Z, 1_W)$ for $g \in \mathcal{R}$, so that $g \cdot p = Rg$ and $p \cdot Lg = 1_Z$, give a natural lifting $d = p \cdot E(u, v) \cdot s$ for a commutative square $v \cdot f = g \cdot u$:

$$(b) \quad \begin{array}{ccccc} X & \xrightarrow{u} & Z & \xlongequal{\quad} & Z \\ & \searrow Lf & & \nearrow p & \\ & & Ef & \xrightarrow{E(u,v)} & Eg \\ & \nearrow s & & \searrow Rg & \\ Y & \xlongequal{\quad} & Y & \xrightarrow{v} & W \end{array}$$

This lifting is unique – so that $(\mathcal{L}, \mathcal{R})$ is an *orthogonal factorisation system* – if, and only if, \mathbb{L} and \mathbb{R} are idempotent. In fact idempotency of \mathbb{L} implies idempotency of \mathbb{R} and vice-versa, as shown in [4].

2.3. Lax orthogonal factorisation systems. Informally, a lax orthogonal factorisation system is an AWFS whose liftings as in (b) have a universal property, as we explain next. First we recall that:

Definition ([27]). An Ord-enriched monad $\mathbb{S} = (S, \eta, \mu)$ is *lax idempotent*, or *Kock-Zöberlein*, if it satisfies any of the following equivalent conditions:

- (i) $S\eta \leq \eta S$;
- (ii) $S\eta \dashv \mu$ (or, equivalently, $S\eta \cdot \mu \leq 1$);
- (iii) $\mu \dashv \eta S$ (or, equivalently, $1 \leq \eta S \cdot \mu$);
- (iv) a morphism $f : SX \rightarrow X$ is an \mathbb{S} -algebra structure for X if, and only if, $f \dashv \eta_X$ with $f \cdot \eta_X = 1_X$.

A lax idempotent Ord-comonad is defined dually.

Lemma. *If $\mathbb{S} = (S, \eta, \mu)$ is a lax idempotent monad on an Ord-category, the following conditions are equivalent, for an object X of \mathbf{C} :*

- (i) X admits an \mathbb{S} -algebra structure;
- (ii) X admits a unique \mathbb{S} -algebra structure;
- (iii) $\eta_X : X \rightarrow SX$ has a right inverse, i.e. X admits an (S, η) -algebra structure;
- (iv) X is a retract of SX ;
- (v) X is a retract of an \mathbb{S} -algebra.

An AWFS (\mathbb{L}, \mathbb{R}) is a *lax orthogonal factorisation system*, abbreviated LOFS, if \mathbb{L} and \mathbb{R} are lax idempotent. These factorisations were introduced by the authors in [14] and further studied in the Ord-enriched categories setting, as used here, in [15].

Corollary. *If (\mathbb{L}, \mathbb{R}) is a LOFS, then its underlying weak factorisation system $(\mathcal{L}, \mathcal{R})$ consists of the class \mathcal{L} of the morphisms admitting a (unique) \mathbb{L} -coalgebra structure and the class \mathcal{R} consists of the morphisms admitting a (unique) \mathbb{R} -algebra structure.*

As for orthogonal factorisation systems, lax idempotency of \mathbb{L} implies lax idempotency of \mathbb{R} and vice-versa. In fact:

Theorem ([15]). (1) *Given an AWFS (\mathbb{L}, \mathbb{R}) on an Ord-category \mathbf{C} , the following conditions are equivalent:*

- (i) (\mathbb{L}, \mathbb{R}) is a LOFS;
- (ii) \mathbb{L} is lax idempotent;
- (iii) \mathbb{R} is lax idempotent.

(2) *Given a domain-preserving Ord-comonad \mathbb{L} and a codomain-preserving Ord-monad \mathbb{R} inducing the same Ord-functorial factorisation $f = Rf \cdot Lf$, the following conditions are equivalent:*

- (i) (\mathbb{L}, \mathbb{R}) is a LOFS.
- (ii) Both \mathbb{L} and \mathbb{R} are lax idempotent.
- (iii) One of \mathbb{L} and \mathbb{R} is lax idempotent and the distributive law axioms (a) hold.

2.4. Lifting operations. Diagram (b) shows that every functorial factorisation system induces a canonical lifting operation from the forgetful Ord-functor $U : (L, \Phi)\text{-Coalg} \rightarrow \mathbf{C}^2$ to the forgetful Ord-functor $V : (R, \Lambda)\text{-Alg} \rightarrow \mathbf{C}^2$, meaning that every commutative diagram

$$(c) \quad \begin{array}{ccc} & \xrightarrow{h} & \\ Ua \downarrow & & \downarrow Vb \\ & \xrightarrow{k} & \end{array}$$

has a canonical diagonal filler $\phi_{a,b}(h, k)$ so that $Vb \cdot \phi_{a,b}(h, k) = k$, $\phi_{a,b}(h, k) \cdot Ua = h$. Those fillers respect both composition and order in a natural way (see [15] for details).

A lifting operation from $U : \mathbf{A} \rightarrow \mathbf{C}^2$ to $V : \mathbf{B} \rightarrow \mathbf{C}^2$ is said to be KZ if, for every commutative diagram (c) and every diagonal filler d , one has $\phi_{a,b}(h, k) \leq d$.

Theorem ([15]). *For an AWFS (\mathbb{L}, \mathbb{R}) on an Ord-category \mathbf{C} , the following conditions are equivalent:*

- (i) (\mathbb{L}, \mathbb{R}) is a LOFS.
- (ii) *The lifting operation from $\mathbb{L}\text{-Coalg} \rightarrow \mathbf{C}^2$ to $\mathbb{R}\text{-Alg} \rightarrow \mathbf{C}^2$ is KZ.*

2.5. Simple monads and their LOFSs. The notion of simple monad we present here, studied in [14, 15], is the Ord-enriched version of simple reflection of [8]. In an Ord-enriched category \mathbf{C} with comma-objects, given an Ord-monad $\mathbb{S} = (S, \eta, \mu)$, we construct a monad \mathbb{R} on \mathbf{C}^2 by considering the comma-object $Kf = Sf \downarrow \eta_Y$ and defining $Rf : Kf \rightarrow Y$ as the second projection. Then Lf is the unique morphism making the following diagram commute.

$$\begin{array}{ccccc} X & & & & \\ & \searrow \eta_X & & & \\ & & Kf & \xrightarrow{-q_f} & SX \\ & \searrow Lf & \downarrow Rf & \geq & \downarrow Sf \\ & & Y & \xrightarrow{\eta_Y} & SY \\ & \searrow f & & & \end{array}$$

The Ord-functorial factorisation $f = Rf \cdot Lf$ defines a copointed endo-Ord-functor $(L, \Phi : L \Rightarrow 1)$, with $\Phi_f = (1_X, Rf)$, and a pointed endo-Ord-functor (R, Λ) , with $\Lambda_f = (Lf, 1_Y)$. Moreover,

(R, Λ) underlies a monad \mathbb{R} on \mathbf{C}^2 whose multiplication $\Pi_f = (\pi_f, 1_Y)$ is defined by the unique morphism π_f given by the universal property of the comma-object:

$$\begin{array}{ccccc}
KRf & \xrightarrow{q_{Rf}} & SKf & \xrightarrow{Sq_f} & SSX \\
& \searrow \pi_f & & & \downarrow \mu_X \\
& & Kf & \xrightarrow{q_f} & SX \\
RRf & \searrow & \downarrow Rf & \geq & \downarrow Sf \\
& & Y & \xrightarrow{\eta_Y} & SY
\end{array}$$

(See [15] for details.)

Lemma. *Given a monad \mathbb{S} on \mathbf{C} , the following conditions are equivalent for a morphism $f : X \rightarrow Y$ in \mathbf{C} :*

- (i) f has an (L, Φ) -coalgebra structure.
- (ii) Sf is a LARI (=left adjoint right inverse¹), that is, it has a right adjoint S^*f such that $S^*f \cdot Sf = 1$.

Proof. (i) \Rightarrow (ii): If $(1_X, s : Y \rightarrow Kf)$ is an (L, Φ) -coalgebra structure for $f : X \rightarrow Y$, then $S^*f := \mu_X \cdot Sq_f \cdot Ss$ is a left inverse of Sf :

$$S^*f \cdot Sf = \mu_X \cdot Sq_f \cdot Ss \cdot Sf = \mu_X \cdot Sq_f \cdot SLf = \mu_X \cdot S\eta_X = 1;$$

and, moreover, it is right adjoint to Sf :

$$Sf \cdot S^*f = Sf \cdot \mu_X \cdot Sq_f \cdot Ss = \mu_Y \cdot SSf \cdot Sq_f \cdot Ss \leq \mu_Y \cdot S\eta_Y \cdot SRf \cdot Ss = 1.$$

(ii) \Rightarrow (i): Let S^*f be a right adjoint left inverse of Sf . By definition of comma-object, from $Sf \cdot (S^*f \cdot \eta_Y) \leq \eta_Y$ there exists a unique $s : Y \rightarrow Kf$ such that $Rf \cdot s = 1_Y$ and $q_f \cdot s = S^*f \cdot \eta_Y$. To conclude that $s \cdot f = Lf$, compose $s \cdot f$ with Rf and q_f :

$$Rf \cdot s \cdot f = f \text{ and } q_f \cdot s \cdot f = S^*f \cdot \eta_Y \cdot f = S^*f \cdot Sf \cdot \eta_X = \eta_X.$$

□

A morphism f in \mathbf{C} such that Sf is a LARI is called an S -embedding. Denote by $\mathbb{S}\text{-Emb}$ the category that has as objects pairs (f, r) of morphisms of \mathbf{C} such that $Sf \dashv r$ with $r \cdot Sf = 1$, and as morphisms $(h, k) : (f, r) \rightarrow (g, s)$ morphisms $(h, k) : f \rightarrow g$ in \mathbf{C}^2 such that $s \cdot Sk = Sh \cdot r$.

Definition. The Ord-monad \mathbb{S} is said to be *simple* if the locally monotone forgetful functor $\mathbb{S}\text{-Emb} \rightarrow \mathbf{C}^2$ has a right adjoint and the induced comonad has underlying functor L and counit Φ .

As shown in [15]:

Proposition. *A lax idempotent monad $\mathbb{S} = (S, \eta, \mu)$ on \mathbf{C} is simple if, and only if, for every morphism $f : X \rightarrow Y$, there is an adjunction*

$$SLf \dashv \mu_X \cdot Sq_f.$$

Theorem. *If \mathbb{S} is a lax idempotent and simple monad, then (\mathbb{L}, \mathbb{R}) is a LOFS. Moreover, the left class \mathcal{L} of the weak factorisation system it induces is the class of S -embeddings.*

Proof. (Sketch of the proof; for details see [15].) Simplicity of \mathbb{S} gives the comonad structure for L needed to define the AWFS.

In order to show that \mathbb{R} is a lax idempotent monad, that is, $R\Lambda_f \leq \Lambda_{Rf} = (LRf, 1_Y)$, we denote $R\Lambda_f$ by $((R\Lambda_f)_1, 1_Y)$ and note that, by definition of R , $R^2f \cdot (R\Lambda_f)_1 = Rf$ and $q_{Rf} \cdot (R\Lambda_f)_1 = SLf \cdot q_f$. Then

$$q_{Rf} \cdot (R\Lambda_f)_1 = \mu_{SKf} \cdot \eta_{S^2Kf} \cdot SLf \cdot q_f = SLf \cdot \mu_X \cdot Sq_f \cdot \eta_{Kf} \leq \eta_{Kf},$$

¹We use the terminology introduced by J.W. Gray in [21].

by simplicity of S . Now, by definition of comma-object and by the equalities $R^2f \cdot LRf = Rf$ and $q_f \cdot LRf = \eta_{Kf}$, it follows that $(R\Lambda_f)_1 \leq LRf$ as claimed.

The last assertion follows from the lemma above. \square

2.6. Submonads of simple monads. Well-behaved submonads of simple monads are simple, as stated below.

Theorem ([15]). *Suppose that $\varphi : \mathbb{S}' \rightarrow \mathbb{S}$ is a monad morphism between Ord-monads whose components are pullback-stable \mathbb{S} -embeddings, and that \mathbb{S} -embeddings are full. If \mathbb{S} is lax idempotent, then \mathbb{S}' is simple whenever \mathbb{S} is so. Moreover, every \mathbb{S}' -embedding is an \mathbb{S} -embedding.*

(Here by *full morphism* in an Ord-category we mean a morphism $f : X \rightarrow Y$ such that, for every $u, v : Z \rightarrow X$, $f \cdot u \leq f \cdot v$ implies $u \leq v$.)

3. (\mathbb{T}, V) -CATEGORIES

3.1. The setting. First we describe the setting where we will be working throughout the paper.

A. V is a *commutative and unital quantale*, that is, a complete lattice equipped with a tensor product \otimes , with unit $k \neq \perp$, and with right adjoint hom . We denote by $V\text{-Rel}$ the bicategory of V -relations, having sets as objects, while morphisms $r : X \dashrightarrow Y$ are V -relations, i.e. maps $r : X \times Y \rightarrow V$; their composition is given by relational composition, that is, for $r : X \dashrightarrow Y$ and $s : Y \dashrightarrow Z$,

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

Every map $f : X \rightarrow Y$ is a V -relation $f : X \times Y \rightarrow V$ with $f(x, y) = k$ if $f(x) = y$ and $f(x, y) = \perp$ elsewhere. This correspondence defines a bijective on objects and faithful pseudofunctor $\mathbf{Set} \rightarrow V\text{-Rel}$. $V\text{-Rel}$ is a locally ordered and locally complete bicategory, with $r \leq s$ if $r(x, y) \leq s(x, y)$, for $r, s : X \dashrightarrow Y$, $x \in X$, $y \in Y$. It has an involution $(\)^\circ : V\text{-Rel} \rightarrow V\text{-Rel}$ assigning to each $r : X \dashrightarrow Y$ the V -relation $r^\circ : Y \dashrightarrow X$ defined by $r^\circ(y, x) = r(x, y)$. For each $r : X \dashrightarrow Y$ both left and right compositions with r preserve suprema, and therefore we have the following adjunctions

$$V\text{-Rel}(Y, Z) \begin{array}{c} \xrightarrow{(\cdot)r} \\ \perp \\ \xleftarrow{(\cdot)\bullet r} \end{array} V\text{-Rel}(X, Z) \quad \text{and} \quad V\text{-Rel}(Z, X) \begin{array}{c} \xrightarrow{r \cdot (\cdot)} \\ \perp \\ \xleftarrow{r \bullet (\cdot)} \end{array} V\text{-Rel}(Z, Y)$$

so that, for every $s : Y \dashrightarrow Z$, $s' : X \dashrightarrow Z$, $t : Z \dashrightarrow X$, $t' : Z \dashrightarrow Y$,

$$s \cdot r \leq s' \iff s \leq s' \bullet r \quad \text{and} \quad r \cdot t \leq t' \iff t \leq r \bullet t'.$$

B. $\mathbb{T} = (T, e, m)$ is a non-trivial \mathbf{Set} -monad that *satisfies (BC)*; that is, T preserves weak pullbacks and every naturality square of m is a weak pullback. We point out that, in particular, the monad \mathbb{T} is *taut* in the sense of Manes [31] (see [13] for details).

C. $\xi : TV \rightarrow V$ is a \mathbb{T} -algebra structure on V such that both $\otimes : V \times V \rightarrow V$ and $k : 1 \rightarrow V$, $* \mapsto k$, are \mathbb{T} -algebra homomorphisms, that is, the following diagrams

$$\begin{array}{ccc} T1 & \xrightarrow{Tk} & TV \\ \downarrow ! & & \downarrow \xi \\ 1 & \xrightarrow{k} & V \end{array} \quad \begin{array}{ccc} T(V \times V) & \xrightarrow{T(\otimes)} & TV \\ \downarrow \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle & & \downarrow \xi \\ V \times V & \xrightarrow{\otimes} & V \end{array}$$

are commutative, and, for all maps $f : X \rightarrow Y$, $\varphi : X \rightarrow V$ and $\psi : Y \rightarrow V$ with $\psi(y) =$

$\bigvee_{x \in f^{-1}(y)} \varphi(x)$ for every $y \in Y$, the following inequality holds

$$\xi \cdot T\psi(y) \leq \bigvee_{\mathfrak{r} \in Tf^{-1}(y)} \xi \cdot T\varphi(\mathfrak{r}),$$

for every $\mathfrak{y} \in TY$. (For alternative descriptions of the latter condition see [22].)

D. Using ξ we define, for each V -relation $r : X \multimap Y$, the V -relation $T_\xi r : TX \multimap TY$ as the composite

$$\begin{array}{ccc} TX \times TY & \xrightarrow{T_\xi r} & V \\ \downarrow \langle T\pi_1, T\pi_2 \rangle^\circ & & \uparrow \xi \\ T(X \times Y) & \xrightarrow{Tr} & TV \end{array}$$

that is, for each $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$,

$$T_\xi r(\mathfrak{x}, \mathfrak{y}) = \bigvee \{ \xi(Tr(\mathfrak{w})) \mid \mathfrak{w} \in T(X \times Y), T\pi_1(\mathfrak{w}) = \mathfrak{x}, T\pi_2(\mathfrak{w}) = \mathfrak{y} \}.$$

This defines a pseudofunctor $T_\xi : V\text{-Rel} \rightarrow V\text{-Rel}$ that extends $T : \text{Set} \rightarrow \text{Set}$, so that $m : T_\xi T_\xi \rightarrow T_\xi$ is a natural transformation while $e : \text{Id}_{V\text{-Rel}} \rightarrow T_\xi$ is an op-lax natural transformation (see [22] for details).

3.2. (\mathbb{T}, V) -categories. Having fixed these data, a (\mathbb{T}, V) -category is a pair (X, a) , where X is a set and $a : TX \multimap X$ is a V -relation such that

$$1_X \leq a \cdot e_X \text{ and } a \cdot Ta \leq a \cdot m_X.$$

Given (\mathbb{T}, V) -categories (X, a) , (Y, b) , a (\mathbb{T}, V) -functor $f : (X, a) \rightarrow (Y, b)$ is a map $f : X \rightarrow Y$ such that

$$f \cdot a \leq b \cdot Tf.$$

We denote the category of (\mathbb{T}, V) -categories and (\mathbb{T}, V) -functors by $(\mathbb{T}, V)\text{-Cat}$. As defined in [16, Section 12], $(\mathbb{T}, V)\text{-Cat}$ is (pre)order-enriched by:

$$f \leq g \text{ if } g \leq b \cdot e_Y \cdot f,$$

for $f, g : (X, a) \rightarrow (Y, b)$. (This structure is in fact inherited from the order-enrichment of $V\text{-Rel}$ as explained in 3.5.) Identifying an element x of X with the (\mathbb{T}, V) -functor $E = (1, e_1^\circ) \rightarrow (X, a)$, $* \mapsto x$, (X, a) becomes (pre)ordered; (X, a) is called *separated*, or *skeletal*, if, for $x, x' \in X$, $x \leq x'$ and $x' \leq x$ implies $x = x'$. The category of separated (\mathbb{T}, V) -categories and (\mathbb{T}, V) -functors will be denoted by $(\mathbb{T}, V)\text{-Cat}_0$.

Examples. Let \mathbb{T} be the identity monad Id and $\xi : V \rightarrow V$ the identity map.

- When $V = 2$, $(\text{Id}, 2)\text{-Cat}$ is the category of (pre)ordered sets and monotone maps.
- Let $V = [0, \infty]_+$ be the complete half-real line $[0, \infty]$ ordered by the greater or equal relation, with $\otimes = +$ and hom the truncated minus, so that $\text{hom}(u, v) = v \ominus u$, which is equal to $v - u$ if $v \geq u$ and 0 otherwise. As shown by Lawvere in [28], $(\text{Id}, [0, \infty]_+)\text{-Cat}$ is the category Met of generalised metric spaces and non-expansive maps.

Let \mathbb{T} be the ultrafilter monad \mathbb{U} and $\xi : TV \rightarrow V$ be defined by $\xi(\mathfrak{x}) = \bigvee \{ v \in V \mid \mathfrak{x} \in T(\uparrow v) \}$.

- When $V = 2$ – as shown by Barr in [3] – $(\mathbb{U}, 2)\text{-Cat}$ is the category Top of topological spaces and continuous maps.
- When $V = [0, \infty]_+$ – as shown in [10] – $(\mathbb{U}, [0, \infty]_+)\text{-Cat}$ is the category of approach spaces and non-expansive maps [30].

3.3. The dual of a (\mathbb{T}, V) -category. When \mathbb{T} is the identity monad, $(\mathbb{T}, V)\text{-Cat}$ is the category $V\text{-Cat}$ of V -categories and V -functors. In $V\text{-Cat}$ there is a natural notion of *dual category*, inducing a functor $D : V\text{-Cat} \rightarrow V\text{-Cat}$, with $D(X, a) = (X, a^\circ)$. To build a dual for a (\mathbb{T}, V) -category we first note that the Set -monad \mathbb{T} can be extended to $V\text{-Cat}$, with $T(X, a) = (TX, Ta)$, and make use of the following adjunction

$$(V\text{-Cat})^{\mathbb{T}} \begin{array}{c} \xrightarrow{N} \\ \xleftarrow{M} \end{array} (\mathbb{T}, V)\text{-Cat}$$

where, for a V -category (X, a) , a \mathbb{T} -algebra structure $\alpha : T(X, a) \rightarrow (X, a)$ and a \mathbb{T} -homomorphism f , $N((X, a), \alpha) = (X, a \cdot \alpha)$ and $Nf = f$; and, for a (\mathbb{T}, V) -category (Y, b) and a (\mathbb{T}, V) -functor g ,

$M(Y, b) = ((TY, Tb \cdot m_Y^\circ), m_Y)$ and $Mg = Tg$. The functor $D : V\text{-Cat} \rightarrow V\text{-Cat}$ lifts to a functor $D : (V\text{-Cat})^{\mathbb{T}} \rightarrow (V\text{-Cat})^{\mathbb{T}}$, with $D((X, a), \alpha) = ((X, a^\circ), \alpha)$, and we define the *dual* $(X, a)^{\text{op}}$ of a (\mathbb{T}, V) -category (X, a) as

$$D \left(\begin{array}{c} \curvearrowright \\ (V\text{-Cat})^{\mathbb{T}} \\ \curvearrowleft \end{array} \right) \begin{array}{c} \xrightarrow{N} \\ \xleftarrow{M} \\ \perp \end{array} (\mathbb{T}, V)\text{-Cat}$$

$NDM(X, a) = (TX, m_X \cdot (Ta)^\circ \cdot m_X)$; that is, denoting its structure by a^{op} ,

$$a^{\text{op}}(\mathfrak{X}, \mathfrak{H}) = \bigvee_{\mathfrak{Q} : m_X(\mathfrak{Q}) = \mathfrak{H}} Ta(\mathfrak{Q}, m_X(\mathfrak{X})),$$

for $\mathfrak{X} \in T^2X$ and $\mathfrak{H} \in TX$ (see [9]).

3.4. V as a (\mathbb{T}, V) -category. As we have in V both a V -categorical structure $\text{hom} : V \dashrightarrow V$ and a \mathbb{T} -algebra structure $\xi : TV \rightarrow V$, which is a V -functor $\xi : (TV, T\text{hom}) \rightarrow (V, \text{hom})$ due to our assumptions, $N((V, \text{hom}), \xi) = (V, \text{hom}_\xi)$ is a (\mathbb{T}, V) -category; this structure has a crucial role in our study, as we will see in the next section.

3.5. (\mathbb{T}, V) -bimodules. Given (\mathbb{T}, V) -categories (X, a) and (Y, b) , a (\mathbb{T}, V) -bimodule (or simply a bimodule) $\psi : (X, a) \dashrightarrow (Y, b)$ is a V -relation $\psi : TX \dashrightarrow Y$ such that $\psi \circ a \leq \psi$ and $b \circ \psi \leq \psi$, where the composition $s \circ r$ of two V -relations $r : TX \dashrightarrow Y$ and $s : TY \dashrightarrow Z$ is given by the *Kleisli convolution* (see [25]), that is

$$s \circ r = s \cdot Tr \cdot m_X^\circ.$$

Under our assumptions bimodules compose, with the (\mathbb{T}, V) -categorical structures as identities for this composition. We denote by $(\mathbb{T}, V)\text{-Mod}$ the category of (\mathbb{T}, V) -categories and (\mathbb{T}, V) -bimodules. $(\mathbb{T}, V)\text{-Mod}$ is locally preordered by the preorder inherited from $V\text{-Rel}$.

Every (\mathbb{T}, V) -functor $f : (X, a) \rightarrow (Y, b)$ induces a pair of bimodules $f_* : (X, a) \dashrightarrow (Y, b)$ and $f^* : (Y, b) \dashrightarrow (X, a)$, defined by $f_* = b \cdot Tf$ and $f^* = f^\circ \cdot b$; that is, $f_*(\mathfrak{r}, y) = b(Tf(\mathfrak{r}), y)$ and $f^*(\mathfrak{H}, x) = b(\mathfrak{H}, f(x))$, for $\mathfrak{r} \in TX$, $\mathfrak{H} \in TY$, $x \in X$ and $y \in Y$. The Kleisli convolution becomes simpler when composing with these bimodules: for any $\varphi : X \dashrightarrow Z$ and $\psi : Z \dashrightarrow X$, $f^* \circ \varphi = f^\circ \cdot \varphi$ and $\psi \circ f_* = \psi \cdot Tf$. It is easy to check that $a \leq f^* \circ f_*$ and $f_* \circ f^* \leq b$, that is, $f_* \dashv f^*$. The (\mathbb{T}, V) -functor f is said to be *fully faithful* when $f^* \circ f_* = a$, or, equivalently, $a(\mathfrak{r}, x) = b(Tf(\mathfrak{r}), f(x))$, for every $\mathfrak{r} \in TX, x \in X$. The local (pre)order on $(\mathbb{T}, V)\text{-Cat}$ corresponds to the local (pre)order on $(\mathbb{T}, V)\text{-Mod}$: for (\mathbb{T}, V) -functors $f, g : (X, a) \rightarrow (Y, b)$,

$$f \leq g \iff f^* \leq g^* \iff f_* \geq g_*.$$

4. THE PRESHEAF MONAD AND ITS SUBMONADS

4.1. The Yoneda Lemma. The tensor product in V defines a tensor product in $(\mathbb{T}, V)\text{-Cat}$, with $(X, a) \otimes (Y, b) = (X \times Y, c)$, where $c(\mathfrak{w}, (x, y)) = a(T\pi_1(\mathfrak{w}), x) \otimes b(T\pi_2(\mathfrak{w}), y)$, for $\mathfrak{w} \in T(X \times Y)$, $x \in X$, $y \in Y$. Its neutral element is $E = (1, e_1^\circ)$. For each (\mathbb{T}, V) -category (X, a) , the functor $X^{\text{op}} \otimes () : (\mathbb{T}, V)\text{-Cat} \rightarrow (\mathbb{T}, V)\text{-Cat}$ has a right adjoint $()^{X^{\text{op}}} : (\mathbb{T}, V)\text{-Cat} \rightarrow (\mathbb{T}, V)\text{-Cat}$.

Proposition ([12]). *For (\mathbb{T}, V) -categories $(X, a), (Y, b)$ and a V -relation $\psi : TX \dashrightarrow Y$, the following conditions are equivalent:*

- (i) $\psi : (X, a) \dashrightarrow (Y, b)$ is a bimodule;
- (ii) $\psi : (X, a)^{\text{op}} \otimes (Y, b) \rightarrow (V, \text{hom}_\xi)$ is a (\mathbb{T}, V) -functor.

Since $a : (X, a) \dashrightarrow (X, a)$ is a bimodule, this result tells us that $a : X^{\text{op}} \otimes X \rightarrow V$ is a (\mathbb{T}, V) -functor, and therefore, from the adjunction $X^{\text{op}} \otimes () \dashv ()^{X^{\text{op}}}$, a induces the *Yoneda (\mathbb{T}, V) -functor*

$$(X, a) \xrightarrow{y_X} V^{X^{\text{op}}}.$$

The following result provides a Yoneda Lemma for (\mathbb{T}, V) -categories.

Theorem ([12]). *Let (X, a) be a (\mathbb{T}, V) -category. For all $\psi \in V^{X^{\text{op}}}$ and all $\mathfrak{x} \in TX$,*

$$\widehat{a}(T y_X(\mathfrak{x}), \psi) = \psi(\mathfrak{x}),$$

where \widehat{a} denotes the (\mathbb{T}, V) -categorical structure on $V^{X^{\text{op}}}$. In particular, y_X is fully faithful.

4.2. The presheaf monad. In order to work in an Ord-enriched category, from now on we restrict ourselves to $(\mathbb{T}, V)\text{-Cat}_0$. We remark that the results of the previous subsection remain valid when we replace $(\mathbb{T}, V)\text{-Cat}$ by $(\mathbb{T}, V)\text{-Cat}_0$. Denoting $V^{X^{\text{op}}}$ by PX , we point out that, via Theorem 4.1,

$$PX = \{\varphi : (X, a) \multimap E \mid \varphi \text{ bimodule}\}.$$

Moreover, the Yoneda (\mathbb{T}, V) -functor y_X turns out to assign to each $x \in X$, that is, to each (\mathbb{T}, V) -functor $x : E \rightarrow X$, the bimodule $x^* : X \multimap E$. Each (\mathbb{T}, V) -functor $f : (X, a) \rightarrow (Y, b)$ induces a (\mathbb{T}, V) -functor $Pf : PX \rightarrow PY$, assigning to $\varphi : X \multimap E$ the bimodule $\varphi \circ f^* : Y \multimap E$, that is $Pf = (\circ) \circ f^*$. This defines an endofunctor P on $(\mathbb{T}, V)\text{-Cat}$. From the adjunction $f_* \dashv f^*$, for every (\mathbb{T}, V) -functor $f : (X, a) \rightarrow (Y, b)$ one gets a right adjoint to Pf , $P^*f = (\circ) \circ f_* : PY \rightarrow PX$. In particular, $P y_X : PX \rightarrow PPX$ has a right adjoint $m_X : PPX \rightarrow PX$, which, together with P and y , defines a lax idempotent monad, the *presheaf monad*. Next we show that this monad is simple. In order to do that we use Proposition 2.5.

Theorem. *The presheaf monad \mathbb{P} on $(\mathbb{T}, V)\text{-Cat}$ is simple.*

Proof. We need to show that, for any (\mathbb{T}, V) -functor $f : (X, a) \rightarrow (Y, b)$, in the diagram below $PLf \dashv m_X \cdot Pq_f$.

$$\begin{array}{ccccc}
 X & \xrightarrow{y_X} & PX & \xrightleftharpoons[\perp]{P y_X} & PPX \\
 \downarrow f & \searrow Lf & \uparrow q_f & \swarrow PLf & \uparrow Pq_f \\
 & & Kf & \xrightarrow{y_{Kf}} & PKf \\
 & \nearrow Rf & & \downarrow Pf & \swarrow PRf \\
 Y & \xrightarrow{y_Y} & PY & &
 \end{array}$$

First we recall that the comma object $(Kf, \tilde{a}) = Pf \downarrow y_Y$ is given by $Kf = \{(\varphi, y) \in PX \times Y \mid Pf(\varphi) \leq y^*\}$, and $\tilde{a}(\mathfrak{w}, (\varphi, y)) = \widehat{a}(Tq_f(\mathfrak{w}), \varphi) \wedge b(TRf(\mathfrak{w}), y)$, where \widehat{a} is the structure on PX . On one hand, as we observed before, PLf has as right adjoint the (\mathbb{T}, V) -functor $P^*Lf = (\circ) \circ (Lf)_*$. On the other hand, $m_X \cdot Pq_f = P^*y_X \cdot Pq_f = (\circ) \circ q_f^* \circ (y_X)_*$. Next we will show that $(Lf)_* = q_f^* \circ (y_X)_* : X \multimap Kf$, which concludes the proof. For each $\mathfrak{x} \in TX$ and $(\varphi, y) \in Kf$,

$$(q_f^* \circ (y_X)_*)(\mathfrak{x}, (\varphi, y)) = \widehat{a}(T y_X(\mathfrak{x}), \varphi) = \varphi(\mathfrak{x}),$$

while

$$(Lf)_*(\mathfrak{x}, (\varphi, y)) = \tilde{a}(TLf(\mathfrak{x}), (\varphi, y)) = \widehat{a}(T y_X(\mathfrak{x}), \varphi) \wedge b(Tf(\mathfrak{x}), y) = \varphi(\mathfrak{x}) \wedge b(Tf(\mathfrak{x}), y).$$

To show that $b(Tf(\mathfrak{x}), y) \geq \varphi(\mathfrak{x})$, so that $(Lf)_*(\mathfrak{x}, (\varphi, y)) = \varphi(\mathfrak{x})$, note that

$$\begin{aligned}
 b(Tf(\mathfrak{x}), y) &= \widehat{b}(T y_X(Tf(\mathfrak{x})), y^*) && \text{(by the Yoneda Lemma)} \\
 &\geq \widehat{b}(T y_X(Tf(\mathfrak{x})), Pf(\varphi)) && \text{(by definition of } Kf\text{)} \\
 &= \widehat{b}(TPf \cdot T y_X(\mathfrak{x}), Pf(\varphi)) \\
 &\geq \widehat{a}(T y(\mathfrak{x}), \varphi) && \text{(because } Pf \text{ is a } (\mathbb{T}, V)\text{-functor).}
 \end{aligned}$$

□

Proposition. (1) *A (\mathbb{T}, V) -functor is a P -embedding if, and only if, it is fully faithful.*
(2) *Fully faithful (\mathbb{T}, V) -functors are pullback stable.*

Proof. (1) If $f : (X, a) \rightarrow (Y, b)$ is a (\mathbb{T}, V) -functor, then Pf has a right adjoint, P^*f . It remains to show that $P^*f \cdot Pf = 1_X$ when f is fully faithful; this means $f^* \cdot f_* = a$, and so, for any bimodule $\varphi : X \dashrightarrow E$,

$$P^*f \cdot Pf(\varphi) = \varphi \circ f^* \circ f_* = \varphi \circ a = \varphi.$$

Conversely, if $P^*f \cdot Pf = 1_{PX}$, then, for any $x \in X$,

$$a(-, x) = x^* = P^*f \cdot Pf(x^*) = x^* \circ f^* \circ f_* = f^\circ \cdot x^\circ \cdot b \cdot Tf = b(Tf(-), f(x)),$$

that is, f is fully faithful.

(2) As in any topological category, (bijective, fully faithful (\mathbb{T}, V) -functors) is an orthogonal factorisation system in $(\mathbb{T}, V)\text{-Cat}$, and therefore fully faithful (\mathbb{T}, V) -functors are pullback-stable. \square

4.3. Presheaf submonads. Let Φ be a class of (\mathbb{T}, V) -bimodules satisfying the conditions:

(S1) Φ is closed under composition.

(S2) For every (\mathbb{T}, V) -functor f , $f^* \in \Phi$.

(S3) For every (\mathbb{T}, V) -bimodule $\psi : X \dashrightarrow Y$, $\psi \in \Phi$ provided that $y^* \circ \psi \in \Phi$ for every $y \in Y$.

We call such a class *saturated*. There is a largest saturated class, of all (\mathbb{T}, V) -bimodules, and a smallest one, $\{f^* \mid f \text{ is a } (\mathbb{T}, V)\text{-functor}\}$. In the last section we will present other saturated classes.

For each (\mathbb{T}, V) -category (X, a) , we define

$$\Phi X = \{\varphi : X \dashrightarrow E \mid \varphi \in \Phi\} \subseteq PX,$$

equipped with the structure \hat{a} inherited from PX , and, to each (\mathbb{T}, V) -functor $f : (X, a) \rightarrow (Y, b)$ we assign

$$\Phi f : \Phi X \rightarrow \Phi Y, \text{ with } \Phi f(\varphi) = \varphi \circ f^*.$$

Since $x^* \in \Phi$ for every $x \in X$, y_X corestricts to ΦX ,

$$X \xrightarrow{y_X^\Phi} \Phi X.$$

Moreover, condition (S3) guarantees that $m_X = P^* y_X$ (co)restricts to $m_X^\Phi : \Phi \Phi X \rightarrow \Phi X$: by the Yoneda Lemma, for all $\varphi \in \Phi X$, $\varphi^* \circ P^* y_X = \varphi \in \Phi$. So, (Φ, y^Φ, m^Φ) is a submonad of P .

Theorem. *If Φ is a saturated class of bimodules, then the monad (Φ, y^Φ, m^Φ) is lax idempotent and simple, and so it defines a lax orthogonal factorisation system.*

Proof. Since fully faithful (\mathbb{T}, V) -functors are pullback-stable and full, and the inclusion $\Phi X \rightarrow PX$ is clearly fully faithful, this result follows directly from Theorem 2.6. \square

5. EXAMPLES: THE INDUCED LOFSS

5.1. General description. Now let us fix a saturated class Φ of (\mathbb{T}, V) -bimodules as in 4.3. The presheaf submonad Φ induces a LOFS $(\mathbb{L}^\Phi, \mathbb{R}^\Phi)$, and consequently a WFS $(\mathcal{L}^\Phi, \mathcal{R}^\Phi)$ where \mathcal{L}^Φ is the class of Φ -embeddings.

Following [12], we say that a (\mathbb{T}, V) -functor f is Φ -dense if $f_* \in \Phi$.

Lemma ([12]). *For a (\mathbb{T}, V) -functor h , the following conditions are equivalent:*

- (i) h is Φ -dense;
- (ii) Φh is a left adjoint;
- (iii) Φh is Φ -dense.

We note that Φh has a right adjoint if and only if the right adjoint P^*h of Ph can be (co)restricted to $\Phi_*h : \Phi Y \rightarrow \Phi X$, which is the case precisely when $h_* \in \Phi$.

Proposition. *For a (\mathbb{T}, V) -functor $h : (X, a) \rightarrow (Y, b)$, the following conditions are equivalent:*

- (i) h belongs to \mathcal{L}^Φ ;
- (ii) h is fully faithful and Φ -dense.

Proof. (i) \Rightarrow (ii): From Theorem 4.3 we know that a Φ -embedding h is fully faithful, and, by definition, Φh is a left adjoint. (ii) \Rightarrow (i): If h is Φ -dense, then Φh has a right adjoint $\Phi^* h$, and so it remains to show that, when $h^* \circ h_* = a$, $\Phi^* h \cdot \Phi h = 1_{PX}$: since $x^* \in \Phi$ for every $x \in X$, the proof follows the arguments used in Proposition 4.2(1). \square

Corollary. *For every (\mathbb{T}, V) -category (X, a) , y_X^Φ is a Φ -embedding.*

The class \mathcal{R}^Φ is the class of morphisms with the right lifting property with respect to morphisms in \mathcal{L}^Φ ; that is, a morphism belongs to \mathcal{R}^Φ if, and only if, it is *injective with respect to the class \mathcal{L}^Φ* . Since $(\mathbb{L}^\Phi, \mathbb{R}^\Phi)$ is a LOFS, these morphisms have the KZ-lifting property with respect to morphisms in \mathcal{L}^Φ . Such morphisms encompass interesting properties, but are usually very difficult to identify.

5.2. Examples: the lari–opfibration LOFS. When

$$\Phi = \{f^* \mid f \text{ is a } (\mathbb{T}, V)\text{-functor}\},$$

then $y_X^\Phi : X \rightarrow \Phi X$, $x \mapsto x^*$, is an isomorphism; that is, the monad (Φ, y^Φ, m^Φ) is isomorphic to the identity monad. Therefore the corresponding LOFS (\mathbb{L}, \mathbb{R}) is the one studied in [15, Examples 4.7, 4.8], and the monad \mathbb{R} is the free (split) opfibration monad on $(\mathbb{T}, V)\text{-Cat}_0$. Then \mathcal{L}^Φ is the class of laris and \mathcal{R}^Φ the class of split opfibrations.

5.3. Examples: the presheaf LOFS. Let us now take the largest saturated class

$$\Phi = PX = \{\varphi : (X, a) \multimap E \mid \varphi \text{ bimodule}\}.$$

From Theorem 2.5 we know that the presheaf monad defines a LOFS (\mathbb{L}, \mathbb{R}) in $(\mathbb{T}, V)\text{-Cat}_0$, and, consequently, a WFS $(\mathcal{L}, \mathcal{R})$, where \mathcal{L} is the class of fully faithful (\mathbb{T}, V) -functors. It is easy to check that they coincide with extremal monomorphisms in $(\mathbb{T}, V)\text{-Cat}_0$, that is, *topological embeddings*. Therefore, from Theorem 2.5 we conclude that, for every quantale V and monad \mathbb{T} in the conditions of 3.1, $(\mathbb{T}, V)\text{-Cat}_0$ has a WFS $(\mathcal{L}, \mathcal{R})$ where \mathcal{L} is the class of embeddings.

When \mathbb{T} is the identity monad and $V = 2$, that is, in the category **Ord** of (anti-symmetric) ordered sets and monotone maps, the morphisms in \mathcal{R} is the class of monotone maps were characterised by Adámek (as mentioned in [35]), as those monotone maps which are *fibre-complete*, *fibrations* and *co-fibrations* (among some other characterisations; see also [1]).

When \mathbb{T} is the identity monad and $V = [0, \infty]_+$, that is, in the category **Met**₀ of separated generalised metric spaces and non-expansive maps, a characterisation of the morphisms in \mathcal{R} is not known. It follows from [2] that a (non-expansive) map $f : X \rightarrow 1$ belongs to \mathcal{R} if, and only if, X is an *hyperconvex metric spaces* (see also [26]).

When $\mathbb{T} = \mathbb{U}$ and $V = 2$, that is, in the category **Top**₀ of T0-spaces and continuous maps, morphisms in \mathcal{R} were studied in a series of papers by Cagliari and Mantovani (see [7] and references there), and characterised in [5, 6]; in [5] they are identified via a way-below relation while [6] gives characterisations that extend those of **Ord** mentioned above.

5.4. Examples: the Lawvere LOFS. The choice of

$$\Phi = \{\psi \in PX \mid \psi \text{ is right adjoint}\}$$

has particular relevance.

When $\mathbb{T} = \mathbb{Id}$ and $V = [0, \infty]_+$, the injective objects in **Met**₀ with respect to Φ -embeddings are the Cauchy-complete metric spaces, that is, a non-expansive map $X \rightarrow 1$ belongs to \mathcal{R}^Φ if and only if X is Cauchy-complete (see [28, 12, 11]). Therefore, the morphisms in \mathcal{R}^Φ are good candidates for a fibrewise notion of Cauchy-completeness. This LOFS was studied in [15]. The morphisms in \mathcal{L}^Φ are the embeddings (=isometries) $f : (X, a) \rightarrow (Y, b)$ such that, for every $y \in Y$, $b(f(-), y) = \lim_n a(-, x_n)$ for some Cauchy sequence (x_n) in X . We point out that the non-expansive maps in \mathcal{R}^Φ do not coincide with Sozubev’s L-complete maps [34]. Indeed,

Sozubeck defines them via an injective property, but his left part – the so called L -equivalences – is a proper subclass of \mathcal{L}^Φ .

When $\mathbb{T} = \mathbb{U}$ and $V = 2$, this choice of Φ gives also an interesting LOFS. As shown in [12], the Φ -algebras for this monad in \mathbf{Top}_0 are the sober spaces. Since sober spaces are also the algebras for the lax idempotent and simple monad of completely prime filters on \mathbf{Top}_0 , the WFS induced by Φ coincides with the WFS studied in [5]; that is, \mathcal{L}^Φ is the class of *completely flat embeddings* and \mathcal{R}^Φ is the class of *fibrewise sober* continuous maps (see [17, 18, 33]).

5.5. Further examples. Using the techniques of [17, 18] and [11, 3.7], one can define saturated classes of (\mathbb{T}, V) -bimodules Φ_0 and Φ_ω so that the left parts of the corresponding WFS are $\mathcal{L}^{\Phi_0} = \{\text{dense embeddings}\}$ and $\mathcal{L}^{\Phi_\omega} = \{\text{flat embeddings}\}$. The simple presheaf submonads they define induce LOFS whose underlying WFS were studied in [5], where \mathcal{R}^{Φ_0} and $\mathcal{R}^{\Phi_\omega}$ give the fibrewise notions of *Scott domains* and *stably compact spaces*.

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