One - Phase Parabolic Free Boundary Problem in Convex Ring

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Abstract

In this paper we study a free boundary problem for the heat equation in a convex ring. Here we prove that under some conditions on initial data, the considered problem has unique solution.

1 Introduction and statement of the problem

Let us be given a domain $\Omega_0 \subset \mathbb{R}^n \times [0,\infty)$ with Lipschitz regular (in time) boundary and with convex time sections for which the set $K_0 := \Omega_0 \cap \{t=0\}$ is not empty, and a compactly supported continuous function $u_0(x), x \in \mathbb{R}^n \setminus K_0$ for which the set $K_1 := \mathrm{supp} u_0 \cup K_0$ is compact and convex. We assume, that Ω_0 expands in time. We are looking for a pair $(u, \Omega_1), \Omega_0 \subset \Omega_1 \subset \mathbb{R}^n \times [0,\infty)$ and $u \in C^{2,1}_{x,t}(\Omega_1 \setminus \overline{\Omega}_0) \cap C(\overline{\Omega}_1 \setminus \Omega_0)$ which is the solution for the following problem:

$$\begin{cases} u_t = \Delta u & \text{in } \Omega_1 \setminus \overline{\Omega}_0 \\ u(x,t) = 1 & \text{on } \Gamma_0 \\ u(x,t) = 0 & \text{on } \Gamma_1 \\ |\nabla u(x,t)| = 1 & \text{on } \Gamma_1 \\ u(x,0) = u_0(x) & \text{in } K_1 \setminus \overline{K}_0 \end{cases} \tag{1.1}$$

where Γ_i is the lateral boundary of Ω_i, $i = 0, 1$. Here the condition on the gradient is to be understood in classical sense, i.e.

$$\lim_{\substack{(x,t)\to(x_0,t_0) \\ (x,t)\in\Omega_1\setminus\overline{\Omega}_0}} |\nabla u(x,t)| = 1 \text{ for every } (x_0,t_0) \in \Gamma_1.$$

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This kind of problem was considered by A. Henrot and H. Shahgholian (see [3], [4]) in 2000 for the elliptic operator. In particular, they consider the following problem: for a given bounded domain $K \subset \mathbb{R}^n$ ($n \geq 2$ and K is convex) one seeks a larger domain Ω such that the gradient of the p-capacitary potential of $\Omega \setminus K$ has a prescribed magnitude on $\partial\Omega$ (the boundary of Ω).

Mathematically the problem, considered by A. Henrot and H. Shahgholian is formulated as follows: given a (not necessarily bounded) convex $K \setminus \mathbb{R}^n$, one looks for a function u and a domain $\Omega(\supset K)$ satisfying, for a given constant $c > 0$,

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega \setminus \overline{K} \\ u(x,t) = 1 & \text{on } \partial K \\ u(x,t) = 0 & \text{on } \partial\Omega \\ |\nabla u(x,t)| = c & \text{on } \partial\Omega, \end{cases}$$

where Δ_p denotes the p-Laplace operator, i.e. $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$. The overdetermined boundary condition $|\nabla u| = c$ is to be understood in the following sense:

$$\liminf_{\Omega \ni y \to x} |\nabla u(y)| = \limsup_{\Omega \ni y \to x} |\nabla u(y)| = c, \text{ for every } x \in \partial\Omega.$$

A. Henrot and H. Shahgholian proved the following (see [3]): if K is convex domain, not necessarily bounded or regular, then there exists a classical solution Ω to the considered free boundary problem with $C^{2,\alpha}$ boundary $\partial\Omega$. Moreover, if K is bounded then the solution Ω is unique.

This kind of result the same others also got for the interior case (when one searches $K \supset \Omega$, [4]).

Later, in 2002, A. Petrosyan considered this kind of problem but now for the parabolic operator. In particular, the following problem was considered by A. Petrosyan (see [5]): find a nonnegative continuous function u in $Q_T = \mathbb{R}^n \times (0,T)$, $T > 0$, such that

$$\begin{cases} \Delta u - u_t = 0 & \text{in } \Omega = \{u > 0\} \\ |\nabla u| = 1 & \text{on } \partial\Omega \cap Q_T \\ u(\cdot, 0) = u_0, \end{cases}$$

with a given nonnegative initial function $u_0 \in C_0(\mathbb{R}^n)$ (here $\Delta = \Delta_x$ and $\nabla = \nabla_x$). A. Petrosyan proved, that under some conditions on u_0, there exists a classical unique solution for considered problem for some T (see [5]).

The purpose of this paper is the following: we'll show, that under some assumptions on initial data, the problem (1.1) has a unique solution for a short time (i.e. for $t < T$ for some $T > 0$). Here we will mainly follow the technique used by A. Henrot and H. Shahgholian in [3], [4] and then extended for a heat equation by A. Petrosyan in [5]. Throughout the paper we will use the following notations:

$\partial_l\Omega =$ the lateral boundary of Ω; $\Omega(t_0) = \Omega \cap \{t = t_0\}$; $\Omega^T = \Omega \cap \{t \leq T\}$; $Q_T = \mathbb{R}^n \times (0,T)$.

2 Subsolutions and Supersolutions

Definition 2.1. *The pair* (u,Ω) *is called a* ***supersolution*** *for* (1.1) *for a short time, if there exists a* $T>0$ *such that* $\Omega\subset\mathbb{R}^n\times[0,T],\Omega_0^T\subset\Omega$ *and the function* $u\in C^{2,1}_{x,t}(\Omega\setminus\overline{\Omega}_0)\cap C(\overline{\Omega}\setminus\Omega_0)$ *satisfies to the following conditions:*

$$
\begin{array}{ll}
(a) & u_t=\Delta u \text{ in } \Omega\setminus\overline{\Omega}_0\\
(b) & u(x,t)=1 \text{ on } \Gamma_0\cap\mathbb{R}^n\times[0,T] \text{ and } u(x,t)=0 \text{ on } \partial_l\Omega\\
(c) & \limsup\limits_{\substack{(x,t)\to(x_0,t_0)\\(x,t)\in\Omega\setminus\overline{\Omega}_0}}|\nabla u(x,t)|\le 1 \text{ for every } (x_0,t_0)\in\partial_l\Omega\\
(d) & u(x,0)\ge u_0(x) \text{ for } x\in K_1\setminus\overline{K}_0
\end{array}
$$

Definition 2.2. *The pair* (u,Ω) *is called a* ***subsolution*** *for* (1.1) *for a short time, if there exists a* $T>0$ *such that* $\Omega\subset\mathbb{R}^n\times[0,T],\Omega_0^T\subset\Omega$ *and the function* $u\in C^{2,1}_{x,t}(\Omega\setminus\overline{\Omega}_0)\cap C(\overline{\Omega}\setminus\Omega_0)$ *satisfies to the following conditions:*

$$
\begin{array}{ll}
(a) & u_t=\Delta u \text{ in } \Omega\setminus\overline{\Omega}_0\\
(b) & u(x,t)=1 \text{ on } \Gamma_0\cap\mathbb{R}^n\times[0,T] \text{ and } u(x,t)=0 \text{ on } \partial_l\Omega\\
(c) & \liminf\limits_{\substack{(x,t)\to(x_0,t_0)\\(x,t)\in\Omega\setminus\overline{\Omega}_0}}|\nabla u(x,t)|\ge 1 \text{ for every } (x_0,t_0)\in\partial_l\Omega\\
(d) & u(x,0)\le u_0(x) \text{ for } x\in K_1\setminus\overline{K}_0
\end{array}
$$

The pair (u,Ω) *is called a* ***strict subsolution*** *for* (1.1) *for a short time, if it is a subsolution for* (1.1) *for a short time, and the sign* $>$ *holds in* (c).

Definition 2.3. *The pair* (u,Ω) *is called a* ***classical solution*** *for* (1.1) *for a short time, if it is supersolution and subsolution for* (1.1) *at the same time for a short time.*

Throughout the paper we will assume that the function u_0 satisfies the following conditions:

$$\Delta u_0(x)=0, x\in K_1\setminus\overline{K}_0 \tag{2.1}$$

$$u_0(x)=1, x\in\partial K_0 \tag{2.2}$$

$$u_0\in C^{0,1}(\overline{K}_1\setminus K_0) \text{ and } \lim_{\substack{x\to x_0\\x\in K_1\setminus\overline{K}_0}}|\nabla u_0(x)|=1 \text{ for all } x_0\in\partial K_1 \tag{2.3}$$

Suppose that the initial function u_0 is *starshaped* with respect to a point x_0 in the following sense:

$$u_0(\lambda x+x_0)\ge u_0(x+x_0), \tag{2.4}$$

for every $\lambda\in(0,1)$ and $x\in\mathbb{R}^n$ such that $\lambda x+x_0$ and $x+x_0$ are in $K_1\setminus K_0$.

Let (u,Ω) be a supersolution of (1.1). Let λ and λ' be two real numbers with $0<\lambda<\lambda'<1$. Define

$$u^\lambda(x,t)=\frac{1}{\lambda'}u(\lambda x,\lambda^2 t) \tag{2.5}$$

The rescaling of variables is taken so that u^λ, like u, satisfies the heat equation in the set $\Omega_\lambda\setminus(\overline{\Omega}_0)_\lambda$, where

$$\Omega_\lambda=\{(x,t):(\lambda x,\lambda^2 t)\in\Omega\}.$$

Lemma 2.4. *Let the initial function* u_0 *satisfy condition* (2.4). *Then every subsolution of* (1.1) *is smaller than every supersolution of* (1.1).

Remark 2.5. *In this lemma and further in the paper we say that a pair* (u', Ω') *is smaller than* (u, Ω)*, if* $\Omega' \subset \Omega$ *and* $u' \leq u$ *in the set where both functions are defined.*

Proof. We follow to the proof of the lemma 2.4 in [5].

Let (u, Ω) be a supersolution and (u', Ω') a subsolution of (1.1). We need to proof only that $\Omega' \subset \Omega$; the inequality $u' \leq u$ will follow from this inclusion by the maximum principle.

In the case when $u \in \mathbf{C}^1(\overline{\Omega} \setminus \Omega_0)$ and $u' \in \mathbf{C}^1(\overline{\Omega'} \setminus \Omega_0)$, the statement can be proved by the Lavrent'ev rescaling method as follows. Suppose

$$\lambda_0 = \sup\{\lambda \in (0,1)/\Omega' \subset \Omega_\lambda\} < 1,$$

where Ω_λ is defined as above. Then $\Omega' \subset \Omega_{\lambda_0}$ and there is a common point $(x_0, t_0) \in \partial\Omega' \cap \partial\Omega_{\lambda_0} \cap Q_T$. Let $\lambda_0 < \lambda_0' < 1$ and u^{λ_0} be as in (2.5). Then $u' \leq u^{\lambda_0}$ in some neighborhood of (x_0, t_0) in Ω'. At the common point (x_0, t_0) this inequality implies $\partial_\nu u'(x_0, t_0) \leq \partial_\nu u^{\lambda_0}(x_0, t_0)$, where ν is the inward spatial normal vector for both $\partial\Omega'$ and $\partial\Omega_{\lambda_0}$ at (x_0, t_0) (recall that we are in $\mathbf{C}^1$ case). This leads to a contradiction, since $\partial_\nu u'(x_0, t_0) = |\nabla u'(x_0, t_0)| \geq 1$ and $\partial_\nu u^{\lambda_0}(x_0, t_0) = |\nabla u^{\lambda_0}(x_0, t_0)| = \frac{\lambda_0}{\lambda_0'} < 1$. Therefore $\lambda_0 = 1$ and $\Omega' \subset \Omega$.

The general case can be reduced to the considered regular case by the following procedure. Let $(\widetilde{u}, \widetilde{\Omega})$ be a subsolution. Choose $0 < \lambda < \lambda' < 1$ close to 1 and regularize $\widetilde{u}$ by setting

$$u(x,t) = (\widetilde{u}^\lambda(x, t+h) - \eta)^+$$

for small $h, \eta > 0$. Analogously regularize a subsolution $(\widetilde{u}', \widetilde{\Omega}')$. Then we will arrive in the considered regular case and can finish the proof by letting first $h, \eta \to 0+$ and then $\lambda \to 1-$. ■

Remark 2.6. *The above lemma leads us to the uniqueness: the problem* (1.1) *has at most one solution.*

3 Classes $\mathcal{B}$ and $\mathcal{D}$

Definition 3.1. *We will say that the supersolution* (u, Ω) *is in class* $\mathcal{B}$*, if* $\Omega(t)$ *is convex and expands in time for all* $t \in [0, T]$ *(*T *is the same quantity appearing in the definition of supersolution) and moreover* $\partial_l\Omega$ *is Lipschitz regular in time.*

Remark 3.2. *The Lipsichtz regularity in time is understood in the following sense: for every* $(x_0, t_0) \in \partial_l\Omega$ *there exists a neighborhood* V *such that*

$$V \cap \Omega = \{x_n > f(x_1, \dots, x_{n-1}, t)\} \cap V, \tag{3.1}$$

for a suitable coordinate system and where f *is a global defined function, uniformly Lipschitz in time. We point out in spatial coordinates* f *can be chosen to be convex, if time sections* $\Omega(t)$ *are convex.*

Proposition 3.3. *The class $\mathcal{B}$ is not empty.*

Proof. Without loss of generality we can assume that $(0, t) \in \Omega_0(t)$ for $0 < t < \infty$. Let us denote by $G(x, t)$ the fundamental solution for the heat equation, i.e.

$$G(x,t) = \begin{cases} (4\pi t)^{\frac{-n}{2}} \cdot e^{\frac{-|x|^2}{4t}}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0. \end{cases}$$

Then we can choose such numbers $\varepsilon, \alpha > 0$ and $C > 0$ that for the function

$$U(x,t) := C \cdot [G(x, t + \varepsilon) - \alpha]$$

we'll have

$$U(x,t) > 1 \text{ for } (x,t) \in \partial_l \Omega_0^T$$

for some fixed T, and

$$U(x, 0) > u_0(x) \text{ in } K_1 \setminus K_0.$$

Denote $\Omega = \{(x,t) : U(x,t) > 0\}$. It is easy to see that Ω is expanding in $0 < t < T$ for some $T > 0$, and has Lipschitz regular (in t) lateral boundary. Then for any point $(x_0, t_0) \in \partial_l \Omega$ we have (since time sections of Ω are balls centered at the origin)

$$|\nabla U(x_0, t_0)| = \frac{\partial U(x_0, t_0)}{\partial r}, \quad r := |x|.$$

Hence, by the choice of constants C and α we can reach to the property

$$|\nabla U(x_0, t_0)| < 1$$

to be satisfied for any $(x_0, t_0) \in \partial_l \Omega^T$.

Now let $u(x,t)$ be the solution to the following Dirichlet problem:

$$\begin{cases} \Delta u - u_t = 0, & \text{in } \Omega^T \setminus \overline{\Omega}_0^T \\ u(x,t) = 1, & \text{on } (x,t) \in \partial_l \Omega_0^T \\ u(x,t) = 0, & \text{on } (x,t) \in \partial_l \Omega^T \\ u(x,0) = u_0(x), & \text{in } (\Omega^T \setminus \Omega_0^T) \cap \{t = 0\} \end{cases}$$

Then by the comparison principle for parabolic equations it follows that $u(x,t) < U(x,t)$ for $(x,t) \in \Omega^T \setminus \overline{\Omega}_0^T$. Hence

$$\limsup_{\substack{(x,t) \to (x_0,t_0) \\ (x,t) \in \Omega^T \setminus \overline{\Omega}_0^T}} |\nabla u(x,t)| \leq \limsup_{\substack{(x,t) \to (x_0,t_0) \\ (x,t) \in \Omega^T \setminus \overline{\Omega}_0^T}} |\nabla U(x,t)| < 1$$

for any $(x_0, t_0) \in \partial_l \Omega^T$. This shows, that the pair $(u, \Omega) \in \mathcal{B}$.■

Proposition 3.4. *The class of strict supersolutions is not empty.*

Proof. Let (u, Ω) be a supersolution of (1.1), and $u^\lambda(x,t)$ be as in (2.5) with $0 < \lambda < \lambda' < 1$. Then we have

$$\begin{cases} (u^\lambda)_t = \Delta u_\lambda & \text{in} \quad \Omega_\lambda \setminus (\overline{\Omega}_0)_\lambda \\ u^\lambda(x,t) = \frac{1}{\lambda'} & \text{on} \quad (\Gamma_0)_\lambda \\ u^\lambda(x,t) = 0 & \text{on} \quad \partial_l \Omega_\lambda \\ u^\lambda(x,0) \geq u_0(x) & \text{in} \quad (K_1)_\lambda \setminus (\overline{K}_0)_\lambda \\ \limsup_{(x,t)\to(x_0,t_0)} |\nabla u^\lambda| < 1 & \text{on} \quad \partial_l \Omega_\lambda. \end{cases}$$

Now let $v^\lambda(x,t)$ be a solution to the following Dirichlet problem:

$$\begin{cases} v^\lambda_t = \Delta v^\lambda & \text{in} \quad \Omega_\lambda \setminus \overline{\Omega}_0 \\ v^\lambda(x,t) = 1 & \text{on} \quad \Gamma_0 \\ v^\lambda(x,t) = 0 & \text{on} \quad \partial_l \Omega_\lambda \\ v^\lambda(x,0) = u_0(x) & \text{in} \quad (K_1)_\lambda \setminus \overline{K}_0, \end{cases}$$

Then by the comparison principle for parabolic equations it follows that $v^\lambda \leq u^\lambda$ for $(x,t) \in \Omega_\lambda \setminus (\overline{\Omega}_0)_\lambda$. Hence

$$\limsup_{(x,t)\to(x_0,t_0)} |\nabla v^\lambda(x,t)| \leq \limsup_{(x,t)\to(x_0,t_0)} |\nabla u^\lambda(x,t)| < 1$$

for all $(x_0,t_0) \in \partial_l \Omega_\lambda$. This shows, that the pair $(v^\lambda, \Omega_\lambda) \in \mathcal{B}$. Moreover, the selection $\lambda < \lambda' < 1$ makes the pair $(v^\lambda, \Omega_\lambda)$ not only a supersolution of (1.1), but also a strict supersolution.■

Definition 3.5. *We'll say that the subsolution (u, Ω) is in class $\mathcal{D}$, if $\Omega(t)$ is convex and expands in time for all $t \in [0,T]$ (T is the same as in the definition of subsolution) and moreover $\partial_l \Omega$ is Lipschitz regular in time.*

Proposition 3.6. *The class $\mathcal{D}$ is not empty.*

Proof. Note, that we have

$$\begin{cases} \Delta u_0(x) = 0 & \text{in } K_1 \setminus \overline{K}_0 \\ u_0(x) = 1 & \text{on } \partial K_0 \\ u_0(x) = 0 & \text{on } \partial K_1 \\ \lim_{\substack{x\to x_0 \\ x\in K_1\setminus \overline{K}_0}} |\nabla u_0(x)| = 1 & \text{for all } x_0 \in \partial K_1, \end{cases}$$

Let us define $v(x,t)$ in the following way:

$$v(x,t) = u_0(x) \ , \ x \in E_1^T \setminus E_0^T, \tag{3.2}$$

where $E_1^T = K_1 \times [0,T)$ and $E_0^T = K_0 \times [0,T)$. Then

$$\begin{cases} v_t = \Delta v & \text{in } E_1^T \setminus \overline{E}_0^T \\ v(x,t) = 0 & \text{on } \partial_l E_1^T \\ v(x,t) = 1 & \text{on } \partial_l E_0^T \\ \lim_{\substack{(x,t)\to(x_0,t_0) \\ (x,t)\in E_1^T\setminus E_0^T}} |\nabla v(x,t)| = 1 & \text{for all } (x_0,t_0) \in \partial_l E_1^T. \end{cases}$$

Let $u(x,t)$ be the solution of the following Dirichlet problem:

$$\begin{cases} u_t = \Delta u & \text{in } E_1^T \setminus \overline{\Omega}_0^T \\ u(x,t) = 0 & \text{on } \partial_l E_1^T \\ u(x,t) = 1 & \text{on } \Gamma_0^T \\ u(x,0) = u_0(x) & x \in K_1 \setminus K_0. \end{cases}$$

Since Ω_0 expands in time, we have $E_0^T \subset \Omega_0^T$ and from the comparison principle we conclude, that

$$v(x,t) \le u(x,t) \text{ in } E_1^T \setminus \overline{\Omega}_0^T.$$

This implies

$$\liminf_{\substack{(x,t)\to(x_0,t_0)\\(x,t)\in E_1^T\setminus\Omega_0^T}} |\nabla u(x,t)| \ge \liminf_{\substack{(x,t)\to(x_0,t_0)\\(x,t)\in E_1^T\setminus\Omega_0^T}} |\nabla v(x,t)| = 1 \text{ for all } (x_0,t_0) \in \partial_l E_1^T.$$

So, for any T the pair (E_1^T, u) belongs to the class $\mathcal{D}$.■

4 The minimal element of $\mathcal{B}$

If the class $\mathcal{B}$ has a minimal element, then it is a good candidate for a classical solution of (1.1). We set

$$\Omega^* = \left(\bigcap_{(u,\Omega)\in\mathcal{B}} \Omega \right)^o, \tag{4.1}$$

where A^o denotes the set of interior points of A. Recalling Proposition 3.6 and Lemma 2.4, we can assist, that Ω^* does not coincide with Ω_0.

Let also u^* be a solution to the following Dirichlet problem:

$$\begin{cases} u_t^* = \Delta u^* & \text{in} \quad \Omega^* \setminus \overline{\Omega}_0 \\ u^*(x,t) = 1 & \text{on} \quad \Gamma_0 \\ u^*(x,t) = 0 & \text{on} \quad \partial_l \Omega^* \\ u^*(x,0) = u_0(x) & \text{in} \quad K_1 \setminus \overline{K}_0. \end{cases} \tag{4.2}$$

In this section we show that under some conditions on u_0 (conditions (2.1)−(2.4)) and for small $T \le T(u_0)$ the pair (u^*, Ω^*) is the minimal element of $\mathcal{B}$ and in fact a classical solution of (1.1). The following lemma plays one of the fundamental roles in our study.

Lemma 4.1. *Let u_0 satisfy* (2.1) − (2.4) *and let Ω^* be given by* (4.1). *Then $\partial\Omega^* \cap Q_\beta$ is Lipschitz regular in time for some $\beta > 0$.*

Proof. Let $(u,\Omega) \in \mathcal{B}$. For small $\varepsilon, h > 0$ let us define

$$w(x,t) = \frac{1}{1-\varepsilon} u((1-\varepsilon)x, (1-\varepsilon)^2(t+h))$$

in $Q_{(1-\varepsilon)^{-2}T-h}$. Now $w(x,t)$ will satisfy the heat equation in the set $\Omega_{1-\varepsilon,h} \setminus \overline{\Omega}^0_{1-\varepsilon,h}$, where

$$\Omega_{1-\varepsilon,h} = \{(x,t) : ((1-\varepsilon)x, (1-\varepsilon)^2(t+h)) \in \Omega\},$$

$$\Omega^0_{1-\varepsilon,h} = \{(x,t) : ((1-\varepsilon)x, (1-\varepsilon)^2(t+h)) \in \Omega_0\}.$$

Let as prove, that $w(x,0) \geq u_0(x)$ in $\overline{\Omega_{1-\varepsilon,h} \setminus \Omega^0_{1-\varepsilon,h}} \cap \{t=0\}$. In view of Proposition 3.6 and (2.4)

$$w(x,0) = \frac{1}{1-\varepsilon} u((1-\varepsilon)x, (1-\varepsilon)^2 h) \geq \frac{1}{1-\varepsilon} v((1-\varepsilon)x, (1-\varepsilon)^2 h) =$$

$$= \frac{1}{1-\varepsilon} u_0((1-\varepsilon)x) \geq u_0(x),$$

where $v(x,t)$ is defined in (3.2).
Now consider the following Dirichlet boundary problem:

$$\begin{cases} \tilde{w}_t = \Delta \tilde{w} & \text{in } (\Omega_{1-\varepsilon,h} \setminus \overline{\Omega}_0) \cap \{t>0\} \\ \tilde{w} = 1 & \text{on } \Gamma_0 \\ \tilde{w} = 0 & \text{on } \partial_l \Omega_{1-\varepsilon,h} \\ \tilde{w} = u_0 & \text{in } \overline{\Omega_{1-\varepsilon,h} \setminus \Omega^0_{1-\varepsilon,h}} \cap \{t=0\}. \end{cases}$$

Then, using comparison principle, we'll obtain $\tilde{w} \leq w$ in $\Omega_{1-\varepsilon,h} \setminus \overline{\Omega}^0_{1-\varepsilon,h}$, and hence,

$$\limsup_{\substack{(x,t)\to(x_0,t_0) \\ (x,t)\in\Omega_{1-\varepsilon,h}\setminus\overline{\Omega}^0_{1-\varepsilon,h}}} |\nabla \tilde{w}| \leq \limsup_{\substack{(x,t)\to(x_0,t_0) \\ (x,t)\in\Omega_{1-\varepsilon,h}\setminus\overline{\Omega}^0_{1-\varepsilon,h}}} |\nabla w| \leq 1,$$

for all $(x_0,t_0) \in \partial_l \Omega_{1-\varepsilon,h}$, and we obtain, that $(\tilde{w}, \Omega_{1-\varepsilon,h}) \in \mathcal{B}$.
Note now, that the time levels of $\Omega_{1-\varepsilon,h}$ are given by the identity

$$\frac{1}{1-\varepsilon}\Omega(t) = \Omega_{1-\varepsilon,h}\left(\frac{t}{(1-\varepsilon)^2} - h\right).$$

Running over all $(u,\Omega) \in \mathcal{B}$, we may conclude therefore, that

$$\frac{1}{1-\varepsilon}\Omega^*(t) \supset \Omega^*\left(\frac{t}{(1-\varepsilon)^2} - h\right). \tag{4.3}$$

Since $\Omega^*(t)$ expands in time, the inclusion (4.3) is not trivial, if

$$\frac{t}{(1-\varepsilon)^2} - h < t.$$

The latter is equivalent to the inequality $t < \frac{h(1-\varepsilon)^2}{\varepsilon(2-\varepsilon)} (= \beta)$. Besides, (4.3) implies also the Lipschitz regularity of $\partial_l \Omega^*$ in time variable.■

For the supersolutions $(u,\Omega) \in \mathcal{B}$ of (1.1) one can replace the gradient condition (c) in Definition 2.1 with

$$\limsup_{\substack{(x,t)\to(x_0,t_0) \\ (x,t)\in\Omega\setminus\overline{\Omega}_0}} \frac{u(x,t)}{d_\Omega(x,t)} \leq 1$$

for every $(x_0,t_0) \in \partial_l \Omega$, where

$$d_\Omega(x,t) = dist(x, \partial\Omega(t)).$$

This is taken care in the next lemma ([5], Lemma 5.1).

Lemma 4.2. *Let Ω be a bounded domain in Q_T such that $\Omega(t)$ are convex for $t \in (0,T)$ and $\partial_l\Omega$ is Lipschitz regular in time. Let also u be a nonnegative function, continuously vanishing on $\partial_l\Omega$, and such that $\Delta u - u_t = 0$ in Ω. Then*

$$\limsup_{\substack{(x,t)\to(x_0,t_0)\\(x,t)\in\Omega\setminus\overline{\Omega}_0}} |\nabla u(x,t)| = \limsup_{\substack{(x,t)\to(x_0,t_0)\\(x,t)\in\Omega\setminus\overline{\Omega}_0}} \frac{u(x,t)}{d_\Omega(x,t)}, \tag{4.4}$$

for every $(x_0,t_0) \in \partial_l\Omega$.

The following lemma is an elliptic counterpart of the lemma above and is proved in a similar way (see [5]).

Lemma 4.3. *Let D be a bounded spatial convex domain and u a nonnegative function in D, continuously vanishing on ∂D and such that $\Delta u = f$ in D with f bounded. Then*

$$\limsup_{D\ni x\to x_0} |\nabla u(x)| = \limsup_{D\ni x\to x_0} \frac{u(x)}{d(x)}, \tag{4.5}$$

where $d(x) = dist(x, \partial D)$.

Let u_0 satisfy (2.1) – (2.4).

Lemma 4.4. *The pair (u^*, Ω^*) is the minimal element of $\mathcal{B}$.*

Proof. The only thing we have to verify now is that (u^*, Ω^*) is a supersolution of (1.1). Let $(u_k, \Omega_k) \in \mathcal{B}$ be such that

(i) $\Omega^* = (\cap_k \Omega_k)^o$;

(ii) the sequence $\{\Omega_k\}$ is decreasing.

(iii) $u_k(x,0) = u_0(x)$.

We can construct such a sequence as follows. First of all, it is easy to prove, using separability of $\mathbb{R}^n$ and convexity property of sets in $\mathcal{B}$, that we can find a sequence $(u_k, \Omega_k) \in \mathcal{B}$ such that $\Omega^* = (\cap_k \Omega_k)^o$. Next, in order to have (ii) we observe the following. Denote $\Omega_{k,m} := \Omega_k \cap \Omega_m$, and let $u_{k,m}$ be the solution of the following Dirichlet problem:

$$\begin{cases} \partial_t u_{k,m} = \Delta u_{k,m} & \text{in} \quad \Omega_{k,m} \setminus \overline{\Omega}_0 \\ u_{k,m}(x,t) = 1 & \text{on} \quad \Gamma_0 \\ u_{k,m}(x,t) = 0 & \text{on} \quad \partial_l \Omega_{k,m} \\ u_{k,m}(x,0) = \min\{u_k(\cdot,0), u_m(\cdot,0)\} & \text{in} \quad (\Omega_{k,m} \setminus \overline{\Omega}_0) \cap \{t=0\}, \end{cases}$$

then

$$u_{k,m}(x,t) \le \min\{u_k(x,t), u_m(x,t)\} \qquad \text{for every} \quad (x,t) \in \Omega_{k,m}.$$

Besides, for the distance functions we will have

$$d_{\Omega_{k,m}}(x,t) = \min\{d_{\Omega_k}(x,t), d_{\Omega_m}(x,t)\} \qquad \text{for every} \quad (x,t) \in \Omega_{k,m} \setminus \overline{\Omega}_0 \ .$$

Therefore, using Lemma 4.2, we can conclude that $(u_{k,m}, \Omega_{k,m}) \in \mathcal{B}$. If now (ii) is not satisfied, we can replace Ω_k by the intersection of all Ω_m with $m \leq k$ and thus to make $\{\Omega_k\}$ decreasing.
Now let $(u_k, \Omega_k) \in \mathcal{B}$ with Ω_k decreasing. Denote by $\tilde{u}_k$ the solution of the following Dirichlet problem:

$$\begin{cases} (\tilde{u}_k)_t = \Delta \tilde{u}_k & \text{in} \quad \Omega_k \setminus \overline{\Omega}_0 \\ \tilde{u}_k = 1 & \text{on} \quad \Gamma_0 \\ \tilde{u}_k = 0 & \text{on} \quad \partial_l \Omega_k \\ \tilde{u}_k(x,0) = u_0(x) & \text{on} \quad \{t=0\} \cap (\overline{\Omega_k \setminus \Omega_0}). \end{cases}$$

We have $u_k \geq \tilde{u}_k$ on the parabolic boundary of $\Omega_k \setminus \Omega_0$, so, by comparison principle, we can deduce that $u_k \geq \tilde{u}_k$ in $\Omega_k \setminus \Omega_0$. It follows that

$$\limsup_{\substack{(x,t)\to(x_0,t_0) \\ (x,t)\in\Omega_k\setminus\overline{\Omega}_0}} |\nabla \tilde{u}_k(x,t)| \leq \limsup_{\substack{(x,t)\to(x_0,t_0) \\ (x,t)\in\Omega_k\setminus\overline{\Omega}_0}} |\nabla u_k(x,t)| \leq 1 \text{ for all } (x_0,t_0) \in \partial_l \Omega_k,$$

and hence $(\tilde{u}_k, \Omega_k) \in \mathcal{B}$ with $\tilde{u}_k(x,0) = u_0(x)$.
So, we can assume from now, that $(u_k, \Omega_k) \in \mathcal{B}$, Ω_k decreases and $u_k(x,0) = u_0(x)$.

Denote now by ω_k the solution of the following Dirichlet problem:

$$\begin{cases} (\omega_k)_t = \Delta \omega_k & \text{in} \quad \Omega_k \setminus \overline{\Omega}_0 \\ \omega_k(x,t) = 1 & \text{on} \quad \Gamma_0 \\ \omega_k(x,t) = 0 & \text{on} \quad \partial_l \Omega_k \\ \omega_k(x,0) = 1 & \text{in} \quad (\Omega_k \setminus \overline{\Omega}_0) \cap \{t=0\}. \end{cases}$$

Let now (w, W) be some subsolution for (1.1). Then, using Lemma 2.4, we can deduce that $u_k(x,t) \geq w(x,t)$ in W and hence:

$$|\nabla u_k(x,t)| \leq |\nabla w(x,t)| \text{ on } \Gamma_0.$$

Let us denote

$$M := \sup_{(x,t)\in\Gamma_0} |\nabla w(x,t)| \text{ and } C := \sup_{x\in K_1\setminus K_0} |\nabla u_0(x)|.$$

We want to prove that

$$|\nabla u_k(x,t)| \leq 1 + (C + M - 1)\omega_k(x,t) \text{ for all } (x,t) \in \Omega_k \setminus \overline{\Omega}_0.$$

To prove this, let us note that $|\nabla u(x,t)|$ is subcaloric in $\Omega_k \setminus \overline{\Omega_0}$ and

$$|\nabla u_k(x,t)| \leq 1 + (C + M - 1)\omega_k(x,t)$$

on parabolic boundary of $\Omega_k \setminus \overline{\Omega}_0$. So the comparison principle applies, and we obtain the desired inequality.

For the next step, observe that since u_k are caloric in $\Omega_k \setminus \overline{\Omega}_0$ and uniformly bounded, a subsequence of $\{u_k\}$ will converge in C^1 norm on compact subsets of $\Omega^* \setminus \overline{\Omega}_0$ to a function u^*. We may assume also that over this subsequence, the corresponding ω_k converge to ω^*, with

$$\begin{cases} \partial_t \omega^* = \Delta \omega^* & \text{in} \quad \Omega^* \setminus \overline{\Omega}_0 \\ \omega^*(x,t) = 1 & \text{on} \quad \Gamma_0 \\ \omega^*(x,t) = 0 & \text{on} \quad \partial_l \Omega^* \\ \omega^*(x,0) = 1 & \text{in} \quad (\Omega^* \setminus \overline{\Omega}_0) \cap \{t=0\}. \end{cases}$$

Then in the limit we will obtain

$$|\nabla u^*(x,t)| \leq 1 + (C + M - 1)\omega^*(x,t)$$

for every (x,t) in $\Omega^* \setminus \overline{\Omega}_0$. As a consequence, (u^*, Ω^*) is in $\mathcal{B}$ and therefore is its minimal element.■

5 Further properties of the minimal element

The method used in this and the next section is due to A. Henrot and H. Shahgholian [3], [4]. For caloric functions this method used in [5].

Definition 5.1. *A point $(x,t) \in \partial_l \Omega$, where $\Omega(t)$ is convex, is said to be extreme, if $x \in \partial\Omega(t)$ is extreme for $\Omega(t)$. The latter means that x is not a convex combination of points on $\partial\Omega(t)$, other than x.*

Lemma 5.2. *Under the conditions* (2.1) – (2.4) *on u_0 the pair (u^*, Ω^*) satisfies*

$$\limsup_{\substack{(x,t)\to(x_0,t_0) \\ (x,t)\in\Omega^*\setminus\overline{\Omega}_0}} |\nabla u^*(x,t)| = 1 \tag{5.1}$$

for every extreme point $(x_0, t_0) \in \partial\Omega^* \cap Q_\lambda$, $\lambda \leq \beta$.

Proof. Let us point out that it is enough to prove the lemma in the case when x_0 is an extremal point of $\partial\Omega^*(t_0)$, which means that there is a spatial supporting hyperplane to $\Omega^*(t_0)$, touching $\partial\Omega^*(t_0)$ at x_0 only. This follows from the fact that the extremal points are dense among the extreme points.

Suppose now $x_0 \in \partial\Omega^*(t_0)$ is an extremal, and that (5.1) is not true. Then, in view of Lemmas 4.2 and 4.1, there exists a (space-time) neighborhood V of (x_0, t_0) and $\alpha > 0$ such that

$$u^*(x,t) \leq (1-\alpha) d_{\Omega^*}(x,t) \tag{5.2}$$

for every $(x,t) \in V \cap \Omega^*$. We may assume additionally that the intersection $V \cap \Omega^*$ is given by (3.1).

Let now Π be a spatial supporting hyperplane to $\Omega^*(t_0)$, such that $\Pi \cap \partial\Omega^*(t_0) = \{x_0\}$. By translation and rotation, we may assume that $x_0 = 0$ and that $\Pi = \{x_n = 0\}$. Moreover, let $\Omega^*(t_0) \subset \{x_n > 0\}$. Using the extremality of (x_0, t_0), it is easy to see that there are $\delta_0 > 0$ and $\eta_0 > 0$ such that

$$\{(x,t) \in \Omega^* : x_n \leq \eta_0 \text{ and } t \in [t_0, t_0 + \delta_0]\} \subset V.$$

Let us consider the function

$$h(t) = -\min_{x\in\Omega^*(t)} x_n,\ t\in[t_0, t_0+\delta_0].$$

In view of Lipschitz regularity of $\partial_l\Omega^*$ in time,

$$h(t) \le L(t-t_0)$$

for $t\in[t_0,t_0+\delta_0]$, where L is the Lipschitz constant of f in t.

Let now $\eta_1\in(0,\eta_0)$ be very small and a constant $C\ge L$ be chosen such that

$$h(t_0+\delta_0)\le C\delta_0-\eta_1.$$

Further, we can find $\delta_1\in(0,\delta_0]$ such that

$$h(t_0+\delta_1)=C\delta_1-\eta_1$$

and

$$h(t)\ge C(t-t_0)-\eta_1$$

for every $t\in[t_0,t_0+\delta_1]$.

Define now a domain $\Omega\subset Q_T$ by giving its time sections as follows

$$\Omega(t)=\begin{cases}\Omega^*(t), & t\in(t_0+\delta_1,T)\\ \Omega^*(t)\cap\{x_n>\eta_1-C(t-t_0)\}, & t\in[t_0,t_0+\delta_1]\\ \Omega(t_0)\cap\Omega^*(t), & t\in(0,t_0).\end{cases}\tag{5.3}$$

Let also u be a solution to the following Dirichlet problem:

$$\begin{cases}\Delta u-u_t=0 & \text{in} \quad \Omega\setminus\overline{\Omega}_0\\ u=0 & \text{on} \quad \partial_l\Omega\\ u=1 & \text{on} \quad \partial_l\Omega_0\\ u=u_0 & \text{in} \quad \Omega\cap\{t=0\}\setminus\overline{K}_0.\end{cases}$$

We claim that if η_1 is small enough, then (u,Ω) is in $\mathcal{B}$. This will lead to a contradiction since Ω^* is not a subset of Ω and the lemma will follow. Since Ω has convex time sections and expands in time, we need to verify only that (u,Ω) is a supersolution of (1.1).

There exists $\varepsilon=\varepsilon(n,L)>0$ such that in a neighborhood of (x_0,t_0) the function

$$w^*(x)=w^*(x;t)=u^*(x,t)+u^*(x,t)^{1+\varepsilon}$$

is subharmonic in x, ([1], Lemma 5). Moreover the size of the neighborhood depends only on n and L. We may suppose V has this property. Next, note that we can take $C\le L+1$ if η_1 is sufficiently small and δ_0 is fixed. This will make the boundary of new constructed Ω $(L+1)$-Lipschitz in time. Therefore we may assume also, that

$$w(x)=w(x;t)=u(x,t)+u(x,t)^{1+\varepsilon}$$

is subharmonic in the neighborhood V.

Now let us prove that (u,Ω) is indeed a supersolution.

Case 1. $\tilde{t} \in (t_0 + \delta_1, T)$. Since $\Omega \subset \Omega^*$, it follows that $u \leq u^*$,so we have

$$\lim_{\substack{(x,t)\to(\tilde{x},\tilde{t})\\(x,t)\in\Omega\setminus\Omega_0}} \frac{u(x,t)}{d_\Omega(x,t)} \leq \lim_{\substack{(x,t)\to(\tilde{x},\tilde{t})\\(x,t)\in\Omega^*\setminus\Omega_0}} \frac{u^*(x,t)}{d_{\Omega^*}(x,t)} \leq 1$$

for every $(\tilde{x},\tilde{t}) \in \partial_l\Omega$, $\tilde{t} > t_0$.

Case 2. $\tilde{t} \in [t_0, t_0 + \delta_1]$. First of all, since $\Omega(t) \subset \Omega^*(t)$ for these t, we have also $u \leq u^*$ there. Let now consider a part $D(t)$ of $\Omega^*(t)$ between the planes: $\Pi_1 = \{x_n = \eta_1 - C(t-t_0)\}$ and $\Pi_0 = \{x_n = \eta_0 - C(t-t_0)\}$. Compare there two functions $w(x) = w(x;t)$ and $l(x) = x_n - (\eta_1 - C(t-t_0))$. On Π_1 both functions are 0. Next

$$l(x) = \eta_0 - \eta_1 \text{ on } \Pi_0$$

To estimate w on Π_0, let us first estimate u on Π_0. Thus, recalling (5.2) we conclude

$$u(x,t) \leq u^*(x,t) \leq (1-\alpha)d_{\Omega^*}(x,t) \leq (1-\alpha)(x_n - L(t-t_0)) \leq (1-\alpha)\eta_0$$

and therefore, if η_0 is small enough, we will obtain

$$w(x) \leq (1-\alpha/2)\eta_0 \text{ on } \Pi_0.$$

Choose now η_1 so small that $(1-\alpha/2)\eta_0 \leq \eta_0 - \eta_1$. Then $w \leq l$ on $\partial D(t)$ and, since w is subharmonic and l is harmonic (linear), we conclude that $w \leq l$ in $D(t)$. Along with $u \leq u^*$ this gives

$$\limsup_{\substack{(x,t)\to(\tilde{x},\tilde{t})\\(x,t)\in\Omega\setminus\overline{\Omega}_0}} \frac{u(x,t)}{d_\Omega(x,t)} \leq 1, \tag{5.4}$$

where $(\tilde{x},\tilde{t}) \in \partial_l\Omega$, t is free to vary within $[t_0, t_0+\delta_1]$.

Case 3. $\tilde{t} \in (0, t_0)$. Since $\Omega(t)$ expand in time, and u_0 is harmonic, considering the time derivative u_t in Ω, we can infer from the maximum principle for the heat equation that $u_t \geq 0$ in Ω. Remembering that $\Omega \subset \Omega^*$, we have $u \leq u^*$. In the case of $(\tilde{x},\tilde{t}) \in \partial_l\Omega \cap \partial_l\Omega^*$, we can write

$$\lim_{\substack{(x,t)\to(\tilde{x},\tilde{t})\\(x,t)\in\Omega\setminus\overline{\Omega}_0}} \frac{u(x,t)}{d_\Omega(x,t)} \leq \lim_{\substack{(x,t)\to(\tilde{x},\tilde{t})\\(x,t)\in\Omega^*\setminus\overline{\Omega}_0}} \frac{u^*(x,t)}{d^*_\Omega(x,t)} \leq 1.$$

The second possibility of $(\tilde{x},\tilde{t}) \in \partial_l\Omega$ is the case $\tilde{t} \in (0,t_0)$ is to be on the cylindrical boundary obtained from vertical movement of $\partial_l\Omega(t_0)$. In this case

$$d_\Omega(x,\tilde{t}) = d_\Omega(x,t_0),$$

and, using $u_t \geq 0$, we obtain $u(x,\tilde{t}) \leq u(x,t_0)$, which leads

$$\lim_{\substack{(x,t)\to(\tilde{x},\tilde{t})\\(x,t)\in\Omega\setminus\overline{\Omega}_0}} \frac{u(x,t)}{d_\Omega(x,t)} \leq \lim_{\substack{(x,t)\to(\tilde{x},t_0)\\(x,t)\in\Omega\setminus\overline{\Omega}_0}} \frac{u(x,t_0)}{d_\Omega(x,t_0)} \leq 1.$$

Summing up, we see that (5.4) holds for all $t \in (0,T)$, and by Lemma 4.2 this implies $(u,\Omega) \in \mathcal{B}$, which is a contradiction.■

6 The main theorem

Theorem 6.1. *Let u_0 satisfy* (2.1) – (2.4)*. Then the minimal element* (u^*, Ω^*) *of $\mathcal{B}$ is a classical solution of* (1.1) *for a short time. Moreover, this classical solution is unique.*

Before the proof of the theorem let us recall some facts, which are proved in [5] (and it is not hard to see that this lemmas are true for convex rings also).

Lemma 6.2. *Let D be a bounded spatial convex domain with C^1 regular boundary, V a neighborhood of ∂D and w a smooth positive subharmonic function in $D \cap V$, continuously vanishing in ∂D. If the level lines $\{w = s\}$ are strictly convex surfaces for $0 < s < s_0$, then the condition*

$$\limsup_{\substack{x \to x_0 \\ x \in D \cap V}} |\nabla w(x)| \geq 1$$

for every extreme point $x_0 \in \partial D$ implies that

$$|\nabla w(x)| \geq 1$$

for every x with $0 < w(x) < s_0$.

Lemma 6.3. *Let D be a bounded spatial convex domain and x_0 a singular point on ∂D, such that there are more than one supporting hyperplanes to D at x_0. Let also V be a neighborhood of x_0 and w a subharmonic function in $D \cap V$, continuously vanishing on $\partial D \cap V$. Then*

$$\lim_{\substack{x \to x_0 \\ x \in D \cap V}} \frac{w(x)}{d(x)} = 0,$$

where $d(x) = dist(x, \partial D)$.

Proof of Theorem 6.1. First observe that we need only to show that (u^*, Ω^*) is a subsolution. The uniqueness follows from the Lemma 2.4 (see the Remark 2.6). For the rest we again follow to the proof of the similar theorem in [5].

Recall that from the Lemma 5.2 we know that

$$\limsup_{\substack{(x,t) \to (x_0,t_0) \\ (x,t) \in \Omega^* \setminus \overline{\Omega}_0}} |\nabla u^*(x,t)| = 1$$

for every extreme point $(x_0, t_0) \in \partial\Omega^* \cap Q_\beta$. Denote by $\mathfrak{R}$ the set of all $t_0 \in (0, \beta)$ such that

$$\limsup_{\substack{x \to x_0 \\ x \in \Omega^*(t_0)}} |\nabla u^*(x, t_0)| = 1$$

for every extreme point $x_0 \in \partial\Omega^*(t_0)$.

Note that the complement $(0, \beta) \setminus \mathfrak{R}$ is a union of a countable family of nowhere dense subsets of $(0, \beta)$ (see [5]).

As in the proof of Lemma 5.2, consider now the function

$$w^*(x) = w^*(x; t) = u^*(x, t) + u^*(x, t)^{1+\varepsilon}.$$

Let $t_0 \in (0, \beta)$. There exists $\varepsilon > 0$, $\delta > 0$ and $s_0 > 0$ such that $w^* = w^*(\cdot; t)$ is subharmonic in a convex ring $D(t) = \{0 < w^*(x,t) < s_0\}$ whenever $t \in (t_0 - \delta, t_0 + \delta)$ ([1], Lemma 5; the fact, that $D(t)$ is a convex ring, can be proved as in [2], Theorem 3).

Now, we point out that if $t \in \mathfrak{R}$, then $\partial\Omega^*(t)$ is C^1 regular. Otherwise there would exist a singular extreme point $x_0 \in \partial\Omega^*(t)$ with

$$\lim_{\substack{x \to x_0 \\ x \in \Omega^*(t)}} |\nabla u^*(x,t)| = 0, \tag{6.1}$$

which contradicts to the definition of $\mathfrak{R}$. Indeed, if $x_0 \in \partial\Omega^*(t)$ is singular then by Lemma 6.3

$$\frac{w^*(x;t)}{d_\Omega(x,t)} \to 0,$$

or, equivalently,

$$\frac{u^*(x,t)}{d_\Omega(x,t)} \to 0$$

as $x \to x_0$, and (6.1) will follow from the Lemma 4.3.
Let now $t \in (t_0 - \delta, t_0 + \delta) \cap \mathfrak{R}$. Then Lemma 6.2 implies that

$$|\nabla w^*(x;t)| \geq 1,$$

if $0 < w^*(x;t) < s_0$ and t is as above. The inequality is extended for all $t \in (t_0 - \delta, t_0 + \delta)$ because of everywhere density of $\mathfrak{R}$ and continuity of $|\nabla w^*(x;t)|$ in Ω^*. Since $|\nabla w^*|$ and ∇u^* are asymptotically equivalent when $u^* \to 0$, we obtain immediately that

$$\liminf_{\substack{(x,t) \to (x_0,t_0) \\ (x,t) \in \Omega^* \setminus \overline{\Omega}_0}} |\nabla u^*(x,t)| \geq 1$$

whenever $x_0 \in \partial\Omega^*(t_0)$. Since t_0 was arbitrary, we conclude, that (u^*, Ω^*) is indeed a classical solution of (1.1) in Q_β.■

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