HOMOGENIZATION OF OBSTACLE PROBLEMS IN ORLICZ-SOBOLEV SPACES

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ABSTRACT. We study the homogenization of obstacle problems in Orlicz-Sobolev spaces for a wide class of monotone operators (possibly degenerate or singular) of the $p(\cdot)$-Laplacian type. Our approach is based on the Lewy-Stampacchia inequalities, which then give access to a compactness argument. We also prove the convergence of the coincidence sets under non-degeneracy conditions.

Keywords: Homogenization, obstacle problem, Orlicz-Sobolev spaces, convergence of coincidence sets.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p : \Omega \to \mathbb{R}$ be measurable and such that

$$1 < \alpha \leq p(x) \leq \beta < \infty \quad \text{a.e. in } \Omega, \tag{1.1}$$

where α and β are constants. The following variable exponent Lebesgue space is an Orlicz space:

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable } \rho(u) := \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

This Orlicz space is a separable reflexive Banach space with the following (Luxembourg) norm:

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0,\ \rho\left(\frac{|u|}{\lambda}\right) \leq 1 \right\}.$$

We define an Orlicz-Sobolev space by

$$W^{1,p(\cdot)}(\Omega) := \left\{ u \in L^{p(\cdot)}(\Omega),\ \nabla u \in \left(L^{p(\cdot)}(\Omega)\right)^n \right\},$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \quad \|\nabla u\|_{L^{p(\cdot)}(\Omega)} = \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p(\cdot)}(\Omega)}.$$

This Orlicz-Sobolev space is also a separable and reflexive Banach space. We also define

$$W_0^{1,p(\cdot)}(\Omega) := \left\{ u \in W_0^{1,1}(\Omega),\ \rho(|\nabla u|) < \infty \right\}.$$

The latter is a Banach space endowed with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} := \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

In this paper we study the periodic homogenization of obstacle problems in Orlicz-Sobolev spaces. We consider

$$a(x, \xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$$

to be a Carathéodory vector function, that is, we assume it is continuous with respect to ξ, for almost every $x \in \mathbb{R}^n$, and that it is measurable with respect to x, for every ξ. Moreover, the functions $a(\cdot, \xi)$ and $p(\cdot)$ are assumed to be periodic with period 1 in each argument $x_1, x_2, \ldots, x_n$. We denote the periodicity cell by Q, i.e. $Q := (0, 1]^n$. Additionally, we assume that the following structural conditions (monotonicity, coercitivity and boundedness) hold:

$$\begin{cases} (a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) > 0, \quad \text{for a.e. } x,\ \xi \neq \eta, \\ a(x,\xi) \cdot \xi \geq C_1 \left(|\xi|^{p(x)} - 1 \right), \\ |a(x,\xi)| \leq C_2 \left(|\xi|^{p(x)-1} + 1 \right), \end{cases} \tag{1.2}$$

where C_1, $C_2 > 0$ are constants. For $\varepsilon > 0$, we define

$$a_\varepsilon(x, \xi) := a\left(\frac{x}{\varepsilon}, \xi\right), \ \ x \in \Omega,\ \xi \in \mathbb{R}^n \tag{1.3}$$

and $p_\varepsilon(x) = p(x/\varepsilon)$. The Orlicz-Sobolev spaces of periodic functions, denoted by $W^{1,p(\cdot)}_{\text{per}}(Q)$, is defined as the set of periodic functions u from $W^{1,1}_{\text{per}}(Q)$ with

$$\int_Q u \, dx = 0 \qquad \text{and} \qquad \int_Q |\nabla u|^{p(x)} \, dx < \infty.$$

For the homogenized functional defined by

$$h(\xi) := \min_{v \in W^{1,p(\cdot)}_{\text{per}}(Q)} \int_Q \frac{|\xi + \nabla v|^{p(x)}}{p(x)} \, dx, \tag{1.4}$$

we introduce also the Orlicz-Sobolev spaces

$$W^h(\Omega) := \left\{ u \in W^{1,1}(\Omega),\ h(\nabla u) \in L^1(\Omega) \right\},$$

$$W_0^h(\Omega) := \left\{ u \in W_0^{1,1}(\Omega),\ h(\nabla u) \in L^1(\Omega) \right\},$$

with the norm, $\|u\|_{W_0^h(\Omega)} := \|\nabla u\|_{L^h(\Omega)}$, and the vector Orlicz space

$$L^h(\Omega) := \left\{ \xi \in \left[L^1(\Omega)\right]^n,\ h(\xi) \in L^1(\Omega) \right\},$$

normed by

$$\|\xi\|_{L^h(\Omega)} := \inf \left\{ \lambda > 0,\ \int_\Omega h\left(\frac{\xi}{\lambda}\right) \leq 1 \right\}.$$

By the properties of h, as it was observed in [24], we have the continuous embeddings

$$L^{\beta}(\Omega) \subset L^{h}(\Omega) \subset L^{\alpha}(\Omega),$$

Assuming that

$$f \text{ and } (A_\varepsilon \psi_\varepsilon - f)^+ \in L^s(\Omega), \tag{1.5}$$

$$\left\| (A_\varepsilon \psi_\varepsilon - f)^+ \right\|_{L^s(\Omega)} \leq C, \tag{1.6}$$

where $C > 0$ is a constant independent of ε and

$$\psi_\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega),\ \psi_0 \in W^h(\Omega),\ \psi_\varepsilon^+ \in W_0^{1,p_\varepsilon(\cdot)}(\Omega),\ \psi_0^+ \in W_0^h(\Omega), \tag{1.7}$$

where $\alpha' = \alpha/(\alpha - 1)$, u^+ is the positive part of u and $s > \frac{n\alpha'}{n+\alpha'}$ if $\alpha < n$, $s > 1$, if $\alpha = n$ and $s = 1$ for $\alpha > n$, we show (Theorem 3.1) that the unique solution $u_\varepsilon \in K_\varepsilon$ of the obstacle problem

$$\int_\Omega a_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla(v - u_\varepsilon)\, dx \geq \int_\Omega f(v - u_\varepsilon)\, dx,\ \ \forall v \in K_\varepsilon, \tag{1.8}$$

where

$$K_\varepsilon := \left\{ v \in W_0^{1,p_\varepsilon(\cdot)}(\Omega),\ v \geq \psi_\varepsilon \ \text{ a.e. in } \ \Omega \right\},$$

converges to the unique solution $u_0 \in K_0$ of the following homogenized obstacle problem

$$\int_\Omega a_0(\nabla u_0) \cdot \nabla(v - u_0)\, dx \geq \int_\Omega f(v - u_0)\, dx,\ \ \forall v \in K_0, \tag{1.9}$$

where

$$K_0 := \left\{ v \in W_0^{h}(\Omega),\ v \geq \psi_0 \ \text{ a.e. in } \ \Omega \right\}.$$

The homogenized operator $a_0 : \mathbb{R}^n \to \mathbb{R}^n$ is given in terms of the weighted average of a as in [24], that is,

$$a_0(\xi) := \int_Q a(x, \xi + \nabla v)\, dx, \tag{1.10}$$

with $v \in W_{\text{per}}^{1,p(\cdot)}(Q)$, such that,

$$\int_Q a(x, \xi + \nabla v) \cdot \nabla \varphi\, dx = 0,\ \ \forall \varphi \in W_{\text{per}}^{1,p(\cdot)}(Q),$$

where Q is the periodicity cell.

Note that, due to the Lavrent'ev effect, if instead of $W_{\text{per}}^{1,p(\cdot)}(Q)$, we take $\varphi \in C_{\text{per}}^\infty(Q)$, we may end up with a different homogenized operator, since in general the space $C_{\text{per}}^\infty(Q)$ is not dense in $W_{\text{per}}^{1,p(\cdot)}(\Omega)$. These homogenized operators, referred to as W and H solutions in [24], respectively, in general may be different, but our results hold for both solutions, with minor modifications for the space framework of the H solutions. Although we prefer to work with W solutions, that is due to the fact that [24, Theorem 3.1] (see Theorem 2.1 below) is true for both types of solutions. Observe that we do not impose any regularity assumption on $p(\cdot)$. However, in the particular case when p is log-Lipschitz continuous, i.e., when for a constant $L > 0$

$$-|p(x) - p(y)| \log |x - y| \leq L, \ \ \forall x, y \in \overline{\Omega}, \ \ |x - y| < 1/2,$$

the notion of W and H solutions coincide (see [10, 14]), since then the smooth functions are dense in the Orlicz-Sobolev space.

Our approach is a development of the classical methods [6, 11] (see also [20, 21, 24]) combined with the Lewy-Stampacchia inequalities in the Orlicz-Sobolev framework, in accordance with [19], which then allows the use of a Rellich-Kondrachov compactness argument.

The result generalizes, in part, that of [5], which covers the case when p is constant (and hence the homogenization is in usual Sobolev spaces). The latter, in turn, implies the case of $p = 2$ obtained in [4]. Nonetheless, we observe that the structural assumptions (1.2) allow us to consider a wider range of monotone operators, which cover these cases and include other interesting quasilinear operators, some of which we list below.

1. If $a(x, \xi) = |\xi|^{p(x)-2}\xi$, we deal with the obstacle problem for the $p(x)$-Laplace operator.
2. We can also consider perturbations of the p-Laplace (p constant) and of the $p(x)$-Laplace operators, taking

$$a(x, \xi) = \gamma(x)|\xi|^{p-2}\xi \ \text{ and } \ a(x, \xi) = \gamma(x)|\xi|^{p(x)-2}\xi$$

   for any non-negative bounded periodic function $\gamma(x)$.
3. It is possible to consider functions which are essentially different from these previous "power like" functions. One general example can be

$$a(x, \xi) = \gamma_1(x)|\xi|^{p(x)-1}\xi \log\left(\gamma_2(x)|\xi| + \gamma_3(x)\right),$$

   where $\gamma_3(x)$, $p(x) > 1$ and $\gamma_1(x)$, $\gamma_2(x) > 0$ a.e. in Ω are bounded periodic functions.

The paper is organized as follows: in Section 2, we state some preliminaries facts, which then serve to prove our main result in Section 3 (Theorem 3.1). In Section 4, we prove the convergence of the coincidence sets (Theorems 4.1 and 4.2).

2. Preliminaries

In this section we give some preliminaries. In particular, we provide the concept of G-convergence of operators in our framework, as well as convergence of sets in Mosco sense. We also recall some results from [22] and [24] for future reference. We start by setting some notations, which will be used throughout the paper: $p_\varepsilon(x) = p(x/\varepsilon)$; $\alpha' = \frac{\alpha}{\alpha-1}$; $\rightharpoonup$ denotes the weak convergence;

$$A_\varepsilon u := -\text{div}\big(a_\varepsilon(x, \nabla u)\big) \ \text{ and } \ A_0 u := -\text{div}\big(a_0(\nabla u)\big),$$

where a_ε is defined by (1.3), and a_0 is defined by (1.10). Next, we define the notion of G-convergence of a_ε to a_0. Observe, that most definitions of G-convergence that can be found in the literature (see, for example, [2, 3, 7, 17]), allow a_0 to depend on x as well, just as a_ε depends. However, in some particular cases, more information can be said about the limiting operator. One example is that of operators with rapidly oscillating "coefficients". Since our assumptions ensure that $a(x, \xi)$ and $p(\cdot)$ are periodic with respect to x in each of the arguments $x_1, x_2, \ldots, x_n$, there is no loss in generality to impose a_0 to be independent of x in the definition of G-convergence, which is more relevant for our purposes.

Definition 2.1. *Consider $a_\varepsilon : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ and $a_0 : \mathbb{R}^n \to \mathbb{R}^n$ as above. We say that a_ε G-converges to a_0 when, considering the unique solution $u_\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega)$ of*

$$-\mathrm{div}\,(a_\varepsilon(x, \nabla u_\varepsilon)) = f,\ \ f \in W_0^{-1,\alpha'}(\Omega)\ \ in\ \ \mathcal{D}'(\Omega)$$

and $u_0 \in W_0^h(\Omega)$ the unique solution of

$$-\mathrm{div}\,(a_0(\nabla u_0)) = f \quad in\ \ \mathcal{D}'(\Omega),$$

there holds:

(1) *$u_\varepsilon \rightharpoonup u_0$ in $W_0^{1,\alpha}(\Omega)$, as $\varepsilon \to 0$;*
(2) *$a_\varepsilon(x, \nabla u_\varepsilon) \rightharpoonup a_0(\nabla u_0)$ in $\left(L^{\beta'}(\Omega)\right)^n$, as $\varepsilon \to 0$.*

Note that the choice of s in (1.5) guarantees, in particular, $f \in W^{-1,\alpha'}(\Omega)$. Additionally, $a(x, \xi)$ is assumed to be continuous with respect to ξ, for almost every $x \in \mathbb{R}^n$.

Next, we state a theorem from [24, Theorem 3.1] that insures the G-convergence of a_ε to a function a_0, as $\varepsilon \to 0$, given explicitly in terms of a. Its proof is based on a compensated compactness argument from [23, 24], which, in the case of $p(\cdot) =$ constant, resembles the well known result of Tartar-Murat (see [16]).

Theorem 2.1. *Let $a(x, \xi)$ be a Carathéodory vector function, which is periodic with respect to x in each argument and satisfy (1.2). Let also p be periodic, measurable and satisfy (1.1). If structural conditions (1.2) hold, then a_ε G-converges to a_0, where a_0 is defined by (1.10). Moreover,*

$$\int_\Omega a_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx \to \int_\Omega a_0(\nabla u_0) \cdot \nabla u_0 \, dx,$$

as $\varepsilon \to 0$.

As it is shown in [24], the vector function $a_0(\xi)$ is strictly monotone, i.e.,

$$(a_0(\xi) - a_0(\eta)) \cdot (\xi - \eta) > 0,\ \ \xi \neq \eta,$$

and coercive, that is,

$$a_0(\xi) \cdot \xi > c_0(h(\xi) - 1),$$

where $c_0 > 0$ is a constant, and the homogenized functional $h(\xi)$ is defined by (1.4). Moreover, h satisfies the so-called Δ_2 condition, [24, Proposition 2.1], which implies that the Orlicz space $L^h(\Omega)$ is reflexive. As it is observed in [24], $h(\xi)$ being defined by (1.4), is convex on $\mathbb{R}^n$ and satisfies the following two-sided estimate:

$$c_1|\xi|^\alpha - 1 \le h(\xi) \le c_2|\xi|^\beta + 1,$$

for a $c_1 > 0$ constant. As a consequence, we have

$$W_0^{1,\beta}(\Omega) \subset W_0^h(\Omega) \subset W_0^{1,\alpha}(\Omega),$$

which implies that

$$K_0 \subset W_0^{1,\alpha}(\Omega).$$

The following result is from [22], and it provides more information on the homogenized functional.

Lemma 2.1. *If u_ε is a sequence uniformly bounded in $W_0^{1,p_\varepsilon(\cdot)}(\Omega)$, such that, $u_\varepsilon \rightharpoonup u_0$ in $W_0^{1,\alpha}(\Omega)$ as $\varepsilon \to 0$, then $h(\nabla u_0) \in L^1(\Omega)$.*

Observe that Lemma 2.1 guarantees that, within G-convergence, the weak limits of u_ε in $W_0^{1,\alpha}(\Omega)$ belong to $W_0^h(\Omega)$, and therefore, if also $u_\varepsilon \in K_\varepsilon$ then $u_0 \in K_0$.

In order to state our main result, we will also need to redefine the Mosco convergence of sets.

Definition 2.2. *The sequence of closed convex sets $K_\varepsilon \subset W_0^{1,p_\varepsilon(\cdot)}(\Omega)$, is said to converge to the set $K_0 \subset W_0^h(\Omega)$ in the Mosco sense, if*

- *for any $v_0 \in K_0$ there exists a sequence $v_\varepsilon \in K_\varepsilon$, such that, $v_\varepsilon \to v_0$ in $W_0^{1,\alpha}(\Omega)$;*
- *weak limits in $W_0^{1,\alpha}(\Omega)$ of any sequence of elements in K_ε, that is uniformly bounded in $W_0^{1,p_\varepsilon(\cdot)}(\Omega)$, belong to K_0.*

Remark 2.1. *Since $W_0^{1,p_\varepsilon(\cdot)}(\Omega)$ is continuously embedded into $W_0^{1,\alpha}(\Omega)$ (see, for example, [10]), then $\psi_\varepsilon \to \psi_0$ in $W^{1,\beta}(\Omega)$ provides $K_\varepsilon \to K_0$ in the Mosco sense, where K_ε and K_0 are as in (1.8) and (1.9) respectively.*

3. Homogenization of the obstacle problem

We are now ready to prove our main result, which states as follows.

Theorem 3.1. *Let $a(x,\xi)$ be a Carathéodory vector function satisfying (1.2) and periodic with respect to x in each argument. Let $p(\cdot)$ be periodic, measurable and satisfying (1.1). Assume further that (1.5)-(1.7) hold. If $K_\varepsilon \to K_0$ in the Mosco sense, then the unique solution of (1.8) converges weakly in $W_0^{1,\alpha}(\Omega)$, as $\varepsilon \to 0$, to the unique solution of (1.9), where a_0 is given by (1.10).*

Proof. We divide the proof into five steps.

Step 1 (*Apriori estimates*). Existence and uniqueness of the solution of (1.8) (and (1.9)) is a classical result (see, for instance, [9, 18, 19]). As in the proof of [5, Theorem 2.3] (see also [18, page 145]), the coercitivity and boundedness assumptions from (1.2) imply that u_ε is bounded in $W_0^{1,p_\varepsilon(\cdot)}(\Omega)$ by a constant depending only from C_1, C_2 but independent of ε. For the details we refer the reader to [12]. As a consequence we obtain that u_ε is bounded also in $W_0^{1,\alpha}(\Omega)$, since $W_0^{1,p_\varepsilon(\cdot)}(\Omega) \subset W_0^{1,\alpha}(\Omega)$. Set

$$\sigma_\varepsilon := a_\varepsilon(x, \nabla u_\varepsilon), \qquad \mu_\varepsilon := -\mathrm{div}\,(a_\varepsilon(x, \nabla u_\varepsilon)) - f. \tag{3.1}$$

The boundedness condition from (1.2) implies that σ_ε and μ_ε are bounded (see [5, 24]), therefore we can extract weakly convergent subsequence (still denoted by ε) from each one of them. Thus, there exist u^*, σ^*, μ^* such that

$$u_\varepsilon \rightharpoonup u^* \quad \text{in } W_0^{1,\alpha}(\Omega) \text{ and } u_\varepsilon \to u^* \quad \text{in } \ L^\alpha(\Omega), \tag{3.2}$$

$$\sigma_\varepsilon \rightharpoonup \sigma^* \quad \text{in } \left(L^{\beta'}(\Omega)\right)^n, \tag{3.3}$$

$$\mu_\varepsilon \rightharpoonup \mu^* \quad \text{in } W^{-1,\beta'}(\Omega). \tag{3.4}$$

Note that

$$\mu^* = -\mathrm{div}\sigma^* - f. \tag{3.5}$$

Moreover, using Lemma 2.1 and since $K_\varepsilon \to K_0$ in the Mosco sense, then

$$u^* \in K_0. \tag{3.6}$$

Step 2 (*Compactness*). Note that our assumptions provide the Lewy-Stampacchia inequalities (see [19]), that is, we have

$$f \le f + \mu_\varepsilon \le (A_\varepsilon \psi_\varepsilon - f)^+ + f,$$

which implies, by a Rellich-Kondrachov compactness argument,

$$\mu_\varepsilon \to \mu^* \quad \text{in } W^{-1,\alpha'}(\Omega). \tag{3.7}$$

Step 3. In this step we prove that $\sigma^* = a_0(\nabla u^*)$, where a_0 is defined by (1.10). To see this, let $w_0 \in \mathcal{D}(\Omega)$ and $w_\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega)$ be the unique solution of

$$\mathrm{div}\,(a_\varepsilon(x, \nabla w_\varepsilon)) = \mathrm{div}\,(a_0(\nabla w_0)) \quad \text{in } \mathcal{D}'(\Omega). \tag{3.8}$$

From Theorem 2.1, we have that a_ε G-converges to a_0, as $\varepsilon \to 0$, where $a_0(\xi)$ is defined by (1.10). In particular,

$$\begin{cases} w_\varepsilon \rightharpoonup w_0 \quad \text{in } W^{1,\alpha}(\Omega) \\ a_\varepsilon(x, \nabla w_\varepsilon) \rightharpoonup a_0(\nabla w_0) \quad \text{in } \left(L^{\beta'}(\Omega)\right)^n. \end{cases} \tag{3.9}$$

Fix now φ such that

$$\varphi \in \mathcal{D}(\Omega), \ \ 0 \le \varphi \le 1. \tag{3.10}$$

From the monotonicity of a_ε one has

$$\int_\Omega \varphi\,(a_\varepsilon(x, \nabla u_\varepsilon) - a_\varepsilon(x, \nabla w_\varepsilon)) \cdot (\nabla u_\varepsilon - \nabla w_\varepsilon)\, dx \ge 0. \tag{3.11}$$

Since $u^* \in K_0$, and $K_\varepsilon \to K_0$ in the Mosco sense, there exists a sequence $\bar{u}_\varepsilon$, such that,

$$\bar{u}_\varepsilon \in K_\varepsilon \quad \text{and} \quad \bar{u}_\varepsilon \to u^* \quad \text{in } W_0^{1,\alpha}(\Omega). \tag{3.12}$$

Next, we write (3.11) as

$$\begin{aligned}\int_\Omega \varphi\sigma_\varepsilon \cdot (\nabla u_\varepsilon - \nabla \bar{u}_\varepsilon) \quad &+ \quad \int_\Omega \varphi\sigma_\varepsilon \cdot \nabla \bar{u}_\varepsilon - \int_\Omega \varphi\sigma_\varepsilon \cdot \nabla w_\varepsilon \\ &- \quad \int_\Omega \varphi a_\varepsilon(x, \nabla w_\varepsilon) \cdot \nabla(u_\varepsilon - w_\varepsilon) \\ &:= \quad I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.13}$$

Since $0 \le \varphi \le 1$ on Ω, and K_ε is convex, then the function $v = \varphi\bar{u}_\varepsilon + (1-\varphi)u_\varepsilon$ can be used as a test function in (1.8), which gives

$$\int_\Omega \sigma_\varepsilon \cdot \nabla\big(\varphi(\bar{u}_\varepsilon - u_\varepsilon)\big) \ge \int_\Omega f\varphi(\bar{u}_\varepsilon - u_\varepsilon) \tag{3.14}$$

and so

$$\begin{aligned} I_1 \quad &= \quad \int_\Omega \sigma_\varepsilon \cdot \nabla\big(\varphi(u_\varepsilon - \bar{u}_\varepsilon)\big) - \int_\Omega (u_\varepsilon - \bar{u}_\varepsilon)\sigma_\varepsilon \cdot \nabla\varphi \\ &\le \quad \int_\Omega f\varphi(u_\varepsilon - \bar{u}_\varepsilon) - \int_\Omega (u_\varepsilon - \bar{u}_\varepsilon)\sigma_\varepsilon \cdot \nabla\varphi. \end{aligned}$$

Since u_ε and $\bar{u}_\varepsilon$ converge to u^* weakly in $W_0^{1,\alpha}(\Omega)$ (and strongly in $L^\alpha(\Omega)$), we obtain

$$\limsup_{\varepsilon\to 0} I_1 \le 0. \tag{3.15}$$

As we know from (3.12), $\bar{u}_\varepsilon \to u^*$ in $W_0^{1,\alpha}(\Omega)$, which gives

$$\lim_{\varepsilon\to 0} I_2 = \int_\Omega \varphi\sigma^* \cdot \nabla u^*. \tag{3.16}$$

Note that

$$I_3 = -\int_\Omega \sigma_\varepsilon \cdot \nabla\left(\varphi w_\varepsilon\right) + \int_\Omega w_\varepsilon\sigma_\varepsilon \cdot \nabla\varphi.$$

From (3.7) and (3.12), we pass to the limit in the first term of I_3. Using (3.3) and (3.12), we pass to the limit also in the second term of I_3, arriving at

$$\lim_{\varepsilon\to 0} I_3 = -\int_\Omega \varphi\sigma^* \nabla w_0. \tag{3.17}$$

Observe that

$$I_4 = -\int_\Omega a_\varepsilon(x, \nabla w_\varepsilon) \cdot \nabla\left(\varphi(u_\varepsilon - w_\varepsilon)\right) + \int_\Omega (u_\varepsilon - w_\varepsilon) a_\varepsilon(x, \nabla w_\varepsilon) \cdot \nabla\varphi,$$

and recalling (3.2) and (3.9) and passing to the limit we obtain

$$\lim_{\varepsilon\to 0} I_4 = -\int_\Omega \varphi a_0(\nabla w_0) \cdot \nabla(u^* - w_0). \tag{3.18}$$

Combining (3.13), (3.15)-(3.18), one has

$$\int_\Omega \varphi(\sigma^* - a_0(\nabla w_0)) \cdot \nabla(u^* - w_0) \geq 0 \quad \text{for } w_0 \in \mathcal{D}(\Omega). \tag{3.19}$$

By density, (3.19) is true also for any w_0 in $W_0^{1,\alpha}(\Omega)$. Consider $w_0 = u^* + t\varphi$, with $t \geq 0$ and $\varphi \in W_0^{1,\alpha}(\Omega)$. Letting $t \to 0$ and using Minty's trick as in [5, page 94] (see also [15]), we conclude

$$\sigma^* = a_0(\nabla u^*). \tag{3.20}$$

Step 4 (*Lower semicontinuity of the energy*). From (3.11) and (3.13) one has

$$\begin{aligned} \int_\Omega \varphi\sigma_\varepsilon \cdot \nabla u_\varepsilon \ &\geq \ \int_\Omega \varphi\sigma_\varepsilon \cdot \nabla w_\varepsilon + \int_\Omega \varphi a_\varepsilon(x, \nabla w_\varepsilon) \cdot \nabla(u_\varepsilon - w_\varepsilon) \\ &= \ -I_3 - I_4. \end{aligned}$$

From (3.17), (3.18) and (3.20) for any $w_0 \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned} &\liminf_{\varepsilon \to 0} \int_\Omega \varphi\sigma_\varepsilon \cdot \nabla u_\varepsilon \\ \geq \ &\int_\Omega \varphi a_0(\nabla u^*) \cdot \nabla w_0 + \int_\Omega \varphi a_0(\nabla w_0) \cdot \nabla(u^* - w_0). \end{aligned} \tag{3.21}$$

Letting w_0 go to u^* in $W_0^{1,\alpha}(\Omega)$, one gets from (3.21)

$$\liminf_{\varepsilon \to 0} \int_\Omega \varphi\sigma_\varepsilon \cdot \nabla u_\varepsilon \geq \int \varphi a_0(\nabla u^*) \cdot \nabla u^*, \tag{3.22}$$

$\forall \varphi \in \mathcal{D}(\Omega)$ such that $0 \leq \varphi \leq 1$.

Step 5. Finally, we claim that u^* is the unique solution u_0 of (1.9).

Let $v_0 \in K_0$ and since $K_\varepsilon \to K_0$ in the Mosco sense, then there is a sequence $\bar{v}_\varepsilon \in K_\varepsilon$ such that $\bar{v}_\varepsilon \to v_0$ in $W_0^{1,\alpha}(\Omega)$. Using $\bar{v}_\varepsilon$ as a test function in (1.8) for $\varphi \in \mathcal{D}(\Omega)$, $0 \leq \varphi \leq 1$, one gets

$$\int_\Omega \sigma_\varepsilon \cdot \nabla \bar{v}_\varepsilon - \int_\Omega f(\bar{v}_\varepsilon - u_\varepsilon) \geq \int_\Omega \sigma_\varepsilon \cdot \nabla u_\varepsilon \geq \int_\Omega \varphi(\sigma_\varepsilon \cdot \nabla u_\varepsilon). \tag{3.23}$$

Recalling (3.22) and passing to the limit in ε in (3.23), we obtain

$$\int_\Omega a_0(\nabla u^*) \cdot \nabla v_0 - \int_\Omega f(v_0 - u^*) \geq \int_\Omega \varphi a_0(\nabla u^*) \cdot \nabla u^*.$$

Letting $\varphi \to 1$ in the last inequality, one gets

$$\int_\Omega a_0(\nabla u^*) \cdot \nabla(v_0 - u^*) - \int_\Omega f(v_0 - u^*) \geq 0, \ \forall v_0 \in K_0.$$

The latter, combined with (3.6), allow us to conclude that u^* coincides with the unique solution u_0 of (1.9) and the whole sequence $u_\varepsilon \rightharpoonup u_0$ in $W_0^{1,\alpha}(\Omega)$. □

Remark 3.1. *One can also show the convergence of the energies. More precisely,*

$$\int_\Omega a_\varepsilon(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,dx \to \int_\Omega a_0(\nabla u_0)\cdot\nabla u_0\,dx. \tag{3.24}$$

Proof. For any $\varphi\in\mathcal{D}(\Omega)$ such that $0\le\varphi\le 1$ from (3.14) we have

$$\int_\Omega \varphi\sigma_\varepsilon\cdot\nabla u_\varepsilon \le \int_\Omega \sigma_\varepsilon\cdot\nabla(\varphi\bar u_\varepsilon) - \int_\Omega u_\varepsilon\sigma_\varepsilon\cdot\nabla\varphi - \int_\Omega f\varphi(\bar u_\varepsilon - u_\varepsilon),$$

which gives

$$\limsup_{\varepsilon\to 0}\int_\Omega \varphi\sigma_\varepsilon\cdot\nabla u_\varepsilon \le \int_\Omega a_0(\nabla u_0)\cdot\nabla u_0. \tag{3.25}$$

The latter, combined with (3.22), implies

$$\sigma_\varepsilon\cdot\nabla u_\varepsilon \to a_0(\nabla u_0)\cdot\nabla u_0 \quad \text{in } \mathcal{D}'(\Omega).$$

Since $K_\varepsilon\to K_0$ in the Mosco sense, then taking $v_0=u_0$ in (3.23), we get

$$\begin{aligned}\int_\Omega a_0(\nabla u_0)\cdot\nabla u_0 &\ge \limsup_{\varepsilon\to 0}\int_\Omega \sigma_\varepsilon\cdot\nabla u_\varepsilon\\ &\ge \liminf_{\varepsilon\to 0}\int_\Omega \sigma_\varepsilon\cdot\nabla u_\varepsilon\\ &\ge \int_\Omega \varphi a_0(\nabla u_0)\cdot\nabla u_0,\end{aligned}$$

and letting $\varphi\to 1$, we obtain (3.24). □

Remark 3.2. *If in (1.8) we have f_ε instead of f and $f_\varepsilon \rightharpoonup f$ in $L^s(\Omega)$, then the conclusion of the Theorem 3.1 still holds.*

Remark 3.3. *Since there are Lewy-Stampacchia inequalities also for the two obstacles problem (see [19]), the Theorem 3.1 can be extended for two obstacles problems with similar assumptions.*

4. Convergence of the coincidence sets

In this section, using the Lewy-Stampacchia inequalities, we prove a stability result for the coincidence sets as it was done, for example, in Theorem 6:6.1 in [18].

Theorem 4.1. *Let the conditions of Theorem 3.1 hold. If, as $\varepsilon\to 0$,*

$$u_\varepsilon-\psi_\varepsilon\to u_0-\psi_0 \quad in \quad L^1(\Omega), \tag{4.1}$$

$$(A_\varepsilon\psi_\varepsilon - f)^+\to(A_0\psi_0-f)^+ \quad in \quad L^1(\Omega), \tag{4.2}$$

$$A_\varepsilon u_\varepsilon\to A_0 u_0 \quad in \quad \mathcal{D}'(\Omega), \tag{4.3}$$

$$\int_S d(A_0\psi_0-f)\neq 0,\ \forall S\subset\Omega \quad such\ that \quad |S|>0, \tag{4.4}$$

and

$$A_0u_0-f=(A_0\psi_0-f)\chi_0 \quad a.e.\ in \quad \Omega, \tag{4.5}$$

where χ_0 is the characteristic function of the set $I_0 := \{u_0 = \psi_0\}$, then the coincidence sets $I_\varepsilon := \{u_\varepsilon = \psi_\varepsilon\}$ converge in measure, i.e.,

$$\chi_\varepsilon \to \chi_0 \quad \textit{in} \quad L^p(\Omega),\ \forall p \in [1,\infty),$$

where χ_ε is the characteristic function of I_ε.

Proof. From the Lewy-Stampacchia inequalities we have

$$f \le A_\varepsilon u_\varepsilon \le f + (A_\varepsilon \psi_\varepsilon - f)^+ \ \text{ a.e. in } \ \Omega.$$

Hence, there exists a function $q_\varepsilon \in L^\infty(\Omega)$, such that,

$$A_\varepsilon u_\varepsilon - f = q_\varepsilon (A_\varepsilon \psi_\varepsilon - f)^+ \ \text{ a.e. in } \ \Omega, \tag{4.6}$$

and

$$0 \le q_\varepsilon \le \chi_\varepsilon \le 1 \ \text{ a.e. in } \ \Omega. \tag{4.7}$$

Then for a subsequence (still denoted by ε), one has

$$q_\varepsilon \to q \ \text{ and } \ \chi_\varepsilon \rightharpoonup \chi_* \ \text{ in } \ L^\infty(\Omega) - \text{weak*} \tag{4.8}$$

for functions q, $\chi_* \in L^\infty(\Omega)$. The inequalities (4.7) imply

$$0 \le q \le \chi_* \le 1 \ \text{ a.e. in } \ \Omega. \tag{4.9}$$

Using (4.2), (4.3) and (4.8), we pass to the limit, as $\varepsilon \to 0$, in (4.6) and obtain

$$A_0 u_0 - f = q(A_0 \psi_0 - f)^+ \ \text{ a.e. in } \ \Omega.$$

The latter, combined with (4.5) provides

$$q(A_0\psi_0 - f)^+ = (A_0 \psi_0 - f)\chi_0 \ \text{ a.e. in } \ \Omega. \tag{4.10}$$

Note that in the region $\{A_0\psi_0 > f\}$, (4.10) and (4.4) imply that $q = \chi_0$, while in $\{A_0\psi_0 \le f\}$, $\chi_0 = 0$. Therefore, $q \ge \chi_0$ a.e. in Ω. Consequently, from (4.9) we get

$$\chi_0 \le \chi_* \ \text{ a.e. in } \ \Omega.$$

On the other hand, from (4.1) and (4.8) one has

$$0 = \int_\Omega \chi_\varepsilon (u_\varepsilon - \psi_\varepsilon) \to \int_\Omega \chi_* (u_0 - \psi_0) = 0,$$

thus $\chi_*(u_0 - \psi_0) = 0$ a.e. in Ω. Consequently, if $u_0 > \psi_0$, then $\chi_* = 0$, and since $0 \le \chi_* \le 1$, one obtains

$$\chi_0 \ge \chi_* \ \text{ a.e. in } \ \Omega.$$

Therefore, $\chi_0 = \chi_*$, and the whole sequence χ_ε converges to χ_0 as $\varepsilon \to 0$, first weakly, and since they are characteristic functions, also strongly in any $L^p(\Omega)$, for any $p \in [1, \infty)$. □

Remark 4.1. *If $\psi_0 = 0$ and the right hand side is regular enough, the condition (4.5) holds automatically, since in this particular case one has porosity of the free boundary from [9] (hence, the free boundary has Lebesgue measure zero), which provides (4.5).*

Remark 4.2. *The assumption* (4.4) *is a weaker version of the condition*

$$A_0\psi_0 - f \neq 0 \quad a.e. \ in \ \ \Omega, \quad when \ \ A_0\psi_0 \in L^1(\Omega).$$

Theorem 4.2. *Let the conditions of Theorem 3.1 and also $s > n/2$. If $\psi_\varepsilon \to \psi_0$, uniformly, $\psi_0\big|_{\partial\Omega} < 0$ and*

$$\overline{int\{u_0 = \psi_0\}} = \{u_0 = \psi_0\} = I_0,$$

then the coincidence sets $I_\varepsilon := \{u_\varepsilon = \psi_\varepsilon\}$ converge in the Hausdorff distance to I_0.

Proof. Using [13, Theorem 3.2], we obtain the uniform Hölder continuity of solutions. The uniform Hölder continuity of the obstacles then implies, as $\varepsilon \to 0$, the convergence $u_\varepsilon \to u_0$, uniformly in compact subsets of Ω. This, in turn, provides the convergence of the coincidence sets in Hausdorff distance as in [8] and [18, Theorem 6:6.5]. □

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