A FORMULA FOR CODENSITY MONADS AND DENSITY COMONADS

JIŘÍ ADÁMEK AND LURDES SOUSA

Dedicated to Bob Lowen on his seventieth birthday

Abstract. For a functor F whose codomain is a cocomplete, cowellpowered category $\mathcal{K}$ with a generator S we prove that a codensity monad exists iff for every oject s in S all natural transformations from $\mathcal{K}(X, F-)$ to $\mathcal{K}(s, F-)$ form a set. Moreover, the codensity monad has an explicit description using the above natural transformations. Concrete examples are presented, e.g., the codensity monad of the power-set functor $\mathcal{P}$ assigns to every set X the set of all nonexpanding endofunctions of $\mathcal{P}X$.

Dually, a set-valued functor F is proved to have a density comonad iff all natural transformations from X^F to 2^F form a set. Moreover, that comonad assigns to X the set of all those transformations. For preimages-preserving endofunctors F of Set we prove that F has a density comonad iff F is accessible.

1. Introduction

The important concept of density of a functor $F : \mathcal{A} \to \mathcal{K}$ means that every object of $\mathcal{K}$ is a canonical colimit of objects of the form FA. For general functors, the *density comonad* is the left Kan extension along itself:

$$C = \mathrm{Lan}_F F.$$

This endofunctor of $\mathcal{K}$ carries the structure of a comonad. We speak about the *pointwise density comonad* if C is computed by the usual colimit formula: given an object X of $\mathcal{K}$, form the diagram $D_X : F/X \to \mathcal{K}$ assigning to each $FA \xrightarrow{a} X$ the value FA, and put

$$CX = \mathrm{colim} D_X.$$

This assumes that the above, possibly large, colimit exists in $\mathcal{K}$. The density comonad is a measure of how far away F is from being dense: a functor is dense iff its pointwise codensity monad is trivial (i.e., $\mathrm{Id}_{\mathcal{K}}$). Pointwise density comonads were introduced by Appelgate and Tierney [5] where they are called standard constructions. For every left adjoint F the comonad given by the adjoint situation is the density comonad of F. For functors $F : \mathcal{A} \to \mathrm{Set}$ we prove that F has a density comonad iff for every set X there is only a set of natural transformations from X^F to 2^F. Moreover, the density comonad C is always pointwise, and is given by the formula

$$CX = Nat(X^F, 2^F).$$

We also prove that every accessible endofunctor between locally presentable categories has a density comonad, and, in case of set functors, conversely: the existence of a density

Date: 9 May 2018.

This work was partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

comonad for F implies its accessibility, assuming that F preserves preimages (which is a very mild condition). For $FX = X^n$ the density comonad is X^{n^n}. For general polynomial functors $FX = \coprod_{i\in I} X^{n_i}$ it is given by $CX = \coprod_{i\in I}\prod_{j\in I} X^{n_i^{n_j}}$, see Example 5.2.

The dual concept, introduced by Kock [8], is the *codensity monad*, i.e., the right Kan extension of F over itself:

$$T = \mathrm{Ran}_F F.$$

Linton proved in [9] that if $\mathcal{K} = \mathsf{Set}$, then F has a codensity monad iff for every set X all natural transformations from F^X to F form a set. We generalize this to $\mathcal{K}$ arbitrary as follows. Given a functor $F : \mathcal{A} \to \mathcal{K}$, denote by $F^{(X)} : \mathcal{A} \to \mathsf{Set}$ the composite $\mathcal{K}(X,-)\cdot F$ for every $X \in \mathcal{K}$. Assuming that $\mathcal{K}$ has a generator S which detects limits (see Definition 3.1), a functor F with codomain $\mathcal{K}$ has a codensity monad iff for every $X \in \mathcal{K}$ all natural transformations from $F^{(X)}$ to $F^{(s)}$, $s \in S$, form a set. And the codensity monad is then pointwise. All locally presentable categories posses a limit-detecting generator, and every monadic category over a category with a limit-detecting generator possesses one, too. In fact, in a cocomplete and cowellpowered category every generator detects limits. We also obtain a formula for the codensity monad T: we can view $\mathcal{K}$ as a concrete category over S-sorted sets. And for every object X the underlying set of TX has the following sorts:

$$Nat(F^X, F^s) \quad (s \in S).$$

Again, accessible functors always possess a pointwise codensity monad, that is, T is given by the limit formula (assigning to X the limit of the diagram $((X \xrightarrow{a} FA) \mapsto FA)$. However, in contrast to the density comonad, many non-accessible set functors possess a codensity monad too – and, as we show below, codensity monads of set-valued functors are always pointwise. Example: the power-set functor $\mathcal{P}$ has a codensity monad given by

$$TX = \text{ nonexpanding self-maps of } \mathcal{P}X.$$

The subfunctor $\mathcal{P}_0$ on all nonempty subsets is its own codensity monad. But the following modification $\overline{\mathcal{P}}$ of $\mathcal{P}$ is proven not to have a codensity monad: on objects X

$$\overline{\mathcal{P}}X = \mathcal{P}X$$

and on morphism $f : X \to Y$

$$\overline{\mathcal{P}}f(M) = \begin{cases} \mathcal{P}f(M) & \text{if } f/M \text{ is monic} \\ \emptyset & \text{else.} \end{cases}$$

For $FX = X^n$ the codensity monad is obvious: this is a right adjoint, so T is the monad induced by the adjoint situation, $TX = n \times X^n$. For general polynomial functors $FX = \coprod_{i\in I} X^{n_i}$ the codensity monad is $TX = \prod_{(X_i)} \coprod_{j\in I} \left(\coprod_{i\in I} n_i \times X_i \right)^{n_j}$ where the first product ranges over all disjoint decompositions $X = \bigcup_{i\in I} X_i$, see Example 5.7

2. Accessible functors

Throughout the paper all categories are assumed to be locally small.

Recall from [7] that a category $\mathcal{K}$ is called *locally presentable* if it is cocomplete and for some infinite regular cardinal λ it has a small subcategory $\mathcal{K}_\lambda$ of λ-presentable objects K (i.e. such that the hom-functor $\mathcal{K}(K,-)$ preserves λ-filtered colimits) whose closure under λ-filtered colimits is all of $\mathcal{K}$. And a functor is called *accessible* if it preserves, for some infinite regular cardinal λ, λ-filtered colimits. Recall further that every locally presentable category is complete and every object X has a presentation rank, i.e., the least regular cardinal λ such that X is λ-presentable. Finally, locally presentable categories are locally small, and $\mathcal{K}_\lambda$ can be chosen to represent all λ-presentable objects up to isomorphism.

Theorem 2.1. *Every accessible functor between locally presentable categories has:*

(a) a pointwise codensity monad

and

(b) a pointwise density comonad.

Proof. Given an accessible functor $F : \mathcal{A} \to \mathcal{K}$ and an object X of $\mathcal{K}$, we can clearly choose an infinite cardinal λ such that $\mathcal{K}$ and $\mathcal{A}$ are locally λ-presentable, F preserves λ-filtered colimits, and X is a λ-presentable object. The domain restriction of F to $\mathcal{A}_\lambda$ is denoted by F_λ.

(a) We are to prove that the diagram

$$B_X : X/F \to \mathcal{K},\ (X \xrightarrow{a} FA) \mapsto FA$$

has a limit in $\mathcal{K}$. Denote by $E : X/F_\lambda \hookrightarrow X/F$ the full embedding. Since $\mathcal{K}$ is complete, the small diagram $B_X \cdot E$ has a limit. Thus, it is sufficient to prove that E is final (the dual concept of cofinal, see [11]): (i) every object $X \xrightarrow{a} FA$ is the codomain of some morphism departing from an object of X/F_λ, and (ii) given a pair of such morphisms, they can be connected by a zig-zag in X/F_λ.

Indeed, given $a : X \to FA$, express A as a λ-filtered colimit of λ-presentable objects with the colimit cocone $c_i : C_i \to A$ $(i \in I)$. Then $Fc_i : FC_i \to FA$, $i \in I$, is also a colimit of a λ-filtered diagram. Since X is λ-presentable, $\mathcal{K}(X,-)$ preserves this colimit, and this implies that (i) and (ii) hold.

(b) Now we prove that the diagram

$$D_X : F/X \to \mathcal{K},\ (FA \xrightarrow{a} X) \mapsto FA$$

has a colimit in $\mathcal{K}$. Denote the colimit of the small subdiagram $F_\lambda/X \to \mathcal{K}$ by K with the colimit cocone

$$\overline{a} : FA \to K \ \text{ for all } \ a : FA \to X \text{ in } F/X,\, A \in \mathcal{A}_\lambda.$$

We extend this cocone to one for D_X as follows: Fix an object $a : FA \to X$ of F/X. Express A as a colimit $c_i : C_i \to A$ $(i \in I)$ of the canonical diagram $H_A : \mathcal{A}_\lambda/A \to \mathcal{A}$ assigning to each arrow the domain. Then $Fc_i : FC_i \to FA$ $(i \in I)$ is a colimit cocone, and all $\overline{a \cdot Fc_i} : FC_i \to K$ form a compatible cocone of the diagram $F \cdot H_A$. Hence, there exists a unique morphism

$$\overline{a} : FA \to K \text{ with } \overline{a} \cdot Fc_i = \overline{a \cdot Fc_i} \ \ (i \in I).$$

We claim that this yields a cocone of D_X. That is, given a morphism f from $(FA \xrightarrow{a} X)$ to $(FB \xrightarrow{b} X)$ in F/X, we prove $\overline{a} = \overline{b} \cdot Ff$.

$$\begin{tikzcd}
FC_i \arrow[rr, dashed, "Fg"] \arrow[d, "Fc_i"'] & & FC'_j \arrow[d, "Fc'_j"] \\
FA \arrow[rr, "Ff"] \arrow[dr, "a"'] & & FB \arrow[dl, "b"] \\
& X &
\end{tikzcd}$$

Since (Fc_i) is a colimit cocone, it is sufficient to prove

$$\overline{a} \cdot Fc_i = \overline{b} \cdot F(f \cdot c_i) \text{ for a all } i \in I.$$

Indeed, let $c'_j : C'_j \to B$ $(j \in J)$ be the canonical colimit cone of $H_B : \mathcal{A}_\lambda/B \to \mathcal{A}$. Since C_i is λ-presentable, the morphism $f \cdot c_i$ factorizes through some c'_j, $j \in J$, say

$$f \cdot c_i = c'_j \cdot g.$$

This makes g a morphism from $FC_i \xrightarrow{a \cdot Fc_i} X$ to $FC'_j \xrightarrow{b \cdot Fc'_j} X$ in F_λ/X, hence the following triangle

$$\begin{tikzcd}
FC_i \arrow[rr, "Fg"] \arrow[dr, "\overline{a \cdot Fc_i}"'] & & FC_j \arrow[dl, "\overline{b \cdot Fc'_j}"] \\
& K &
\end{tikzcd}$$

commutes. That is, we have derived the required equality:

$$\overline{a} \cdot Fc_i = \overline{b} \cdot Fc'_j \cdot Fg = \overline{b} \cdot Ff \cdot Fc_i.$$

It is now easy to verify that the above cocone is a colimit of D_X. Given another cocone $\widetilde{a} : FA \to \widetilde{K}$ for all $a : FA \to X$ in F/X, the subcocone with domain F_λ/X yields a unique morphism $r : K \to \widetilde{K}$ with

$$r \cdot \overline{a} = \widetilde{a} \text{ for all } a : FA \to X,\ A \in \mathcal{A}_\lambda.$$

It remains to observe that given $a : FA \to X$ arbitrary, we also have $r \cdot \overline{a} = \widetilde{a}$:

$$\begin{tikzcd}
& FC_i \arrow[dl, "Fc_i"'] \arrow[dr, "\overline{a \cdot Fc_i}"] & \\
FA \arrow[rr, "\overline{a}"] \arrow[dr, "\widetilde{a}"'] & & K \arrow[dl, "r"] \\
& \widetilde{K} &
\end{tikzcd}$$

Indeed, the cocone (Fc_i) is collectively epic and for each i we know that $r \cdot \overline{a \cdot Fc_i} = \widetilde{a \cdot Fc_i}$. Now $\widetilde{a \cdot Fc_i} = \widetilde{a} \cdot Fc_i$ since c_i is a morphism from $FC_i \xrightarrow{a \cdot Fc_i} X$ to $FA \xrightarrow{a} X$. We conclude $r \cdot \overline{a} \cdot Fc_i = \widetilde{a} \cdot Fc_i$ for all i, thus, $\widetilde{a} = r \cdot \overline{a}$. □

Proposition 2.2. *Let $\mathcal{K}$ be a category with a generator. Every functor $F : \mathcal{A} \to \mathcal{K}$ with a codensity monad has only a set of natural transformations $\alpha : F \to F$.*

Proof. By the universal property of $T = \mathrm{Ran}_F F$, natural self-transformations of F bijectively correspond to natural transformations from $\mathrm{Id}_{\mathcal{K}}$ to T. If $(K_i)_{i\in I}$ is a generator, we will prove that every natural transformation $\alpha : \mathrm{Id}_{\mathcal{K}} \to T$ is determined by its components α_{K_i}, $i \in I$, which proves our claim.

Let $\beta : \mathrm{Id}_{\mathcal{K}} \to T$ be a natural transformation with $\beta_{K_i} = \alpha_{K_i}$ for all i. Then for every object X we have $\beta_X = \alpha_X$. Indeed, otherwise there exists $i \in I$ and a morphism $h : K_i \to X$ with $\alpha_X \cdot h \neq \beta_X \cdot h$.

$$\begin{tikzcd} K_i \arrow[r, "\alpha_{K_i}=\beta_{K_i}"] \arrow[d, "h"'] & TK_i \arrow[d, "Th"] \\ X \arrow[r, shift left, "\alpha_X"] \arrow[r, shift right, "\beta_X"'] & TX \end{tikzcd}$$

This contradicts to the naturality squares for α and β. □

Corollary 2.3. *Let $\mathcal{K}$ be a category with a cogenerator. Every functor $F : \mathcal{A} \to \mathcal{K}$ with a density comonad has only a set of natural transformations $\alpha : F \to F$.*

Example 2.4. *A set functor without a codensity monad or a density comonad.* Recall the modified power-set functor $\overline{\mathcal{P}}$ in Introduction. By Proposition 2.2 it has no codensity monad since for every cardinal λ we have a natural transformation

$$\alpha^\lambda : \overline{\mathcal{P}} \to \overline{\mathcal{P}}.$$

It assigns to a subset M of power $|M| \geq \lambda$ itself, otherwise $\emptyset$. The naturality squares are easy to verify. Thus, $\mathrm{Nat}(\overline{\mathcal{P}}, \overline{\mathcal{P}})$ is a proper class.

3. Codensity Monad Theorem

Let S be a generator of a category $\mathcal{K}$. Then $\mathcal{K}$ can be viewed as a concrete category over S-sorted sets: the forgetful functor

$$U : \mathcal{K} \to \mathsf{Set}^S$$

has components

$$U_s = \mathcal{K}(s, -) : \mathcal{K} \to \mathsf{Set} \quad (s \in S).$$

Recall that a functor U is said to *detect limits* if for every (possibly large) diagram D in $\mathcal{K}$ for which $\lim U \cdot D$ has a limit, a limit exists in $\mathcal{K}$.

In case of the functor U above the existence of $\lim U \cdot D$ says precisely that for every $s \in S$ the diaram D has only a set of cones with domain s. This leads us to the following

Definition 3.1. A generator S of $\mathcal{K}$ is called *limit-detecting* if

(a) every (possibly large) diagram D in $\mathcal{K}$ which has only a set of cones with domains in S has a limit,

and

(b) copowers of every object of S exist.

Examples 3.2. Every generator is limit-detecing in the following categories:

(1) Every *total* category $\mathcal{K}$, i.e., such that the Yoneda embedding into $[\mathcal{K}^{op}, \mathrm{Set}]$ has a left adjoint, as introduced by Street and Walters [12]. They also proved that a total category is

cocomplete and hypercomplete, i.e., every diagram D such that for any object $K \in \mathcal{K}$ there exists only a set of cones with domain K has a limit.

Suppose D has the property in 3.1(a) above. Then given K we express it as quotient of a coproduct of objects in S:

$$e : \coprod_{i \in I} s_i \twoheadrightarrow K.$$

Every cone with domain K yields one with domain $\coprod_{i \in I} s_i$ which, since e is epic, determines the original one. Since there is only a set of cones with domain $\coprod_{i \in I} s_i$, it follows that there is only a set of cones with domain K. Thus $\lim D$ exists.

(2) Every cocomplete and cowellpowered category. Indeed, $\mathcal{K}$ is total, see [6].

(3) Every locally presentable category. This follows from (2), see [7] or [3].

(4) Categories from general topology, e.g., Top, Top_2 (Hausdorff spaces), Unif (uniform spaces), approach spaces of Lowen [10], etc. These are concrete categories over Set which are solid, thus total, see [13].

(5) Monadic categories over categories with a limit-detecting generator. Indeed, let S be a limit-detecting generator of $\mathcal{K}$. For every monad $\mathbb{T} = (T, \eta, \mu)$ the set of free algebras

$$S' = \{(Ts, \mu_s) \, ; \, s \in S\}$$

is a limit-detecting generator of $\mathcal{K}^{\mathbb{T}}$. In fact, it is clearly a generator, (a) above follows since (large) limits are created by the forgetful functor $U^{\mathbb{T}}$ of $\mathcal{K}^{\mathbb{T}}$, and (b) is clear since the left adjoint of $U^{\mathbb{T}}$ preserves copowers.

Notation 3.3. For every functor $F : \mathcal{A} \to \mathcal{K}$ and every object X of $\mathcal{K}$ we denote by $F^{(X)}$ the set-valued functor

$$F^{(X)} \equiv \mathcal{A} \xrightarrow{F} \mathcal{K} \xrightarrow{\mathcal{K}(X,-)} \mathsf{Set}$$

Thus in case $\mathcal{K} = \mathsf{Set}$ this is just the power F^X of $F : \mathcal{A} \to \mathsf{Set}$ to X. The following theorem generalizes Linton's result, see [9], that a set-valued functor F has a pointwise codensity monad iff there is only a set of natural transformations from F^X to F (for every set X):

Theorem 3.4. ***(Codensity Monad Theorem)*** *Let S be a limit-detecting generator of a category $\mathcal{K}$. For every functor F with codomain $\mathcal{K}$ the following conditions are equivalent:*

(i) *F has a codensity monad,*

(ii) *F has a pointwise codensity monad, and*

(iii) *for every pair of objects $s \in S$ and $X \in \mathcal{K}$ the collection*

$$Nat(F^{(X)}, F^{(s)})$$

of natural transformations from $F^{(X)}$ to $F^{(s)}$ is small.

Remark. We will see in the proof that the object CX assigned to $X \in \mathcal{K}$ by the codensity monad C has the S-sorted underlying set given by

$$U(CX) \cong \left(Nat(F^{(X)}, F^{(s)})\right)_{s \in S}.$$

Proof. (i) $\to$ (iii). Since $s \in S$ has all copowers, $\mathcal{K}(s, -)$ is left adjoint to $\phi_s : M \mapsto \coprod_M s$.

Let C be a codensity monad of F. We prove that the set $\mathcal{K}(s, CX)$ is isomorphic to $Nat(F^{(X)}, F^{(s)})$. Indeed, we have the following bijections:

$$\begin{array}{cl}
\mathcal{K}(s, CX) & \\ \hline
\mathcal{K}(X,-) \to \mathcal{K}(s,-)\cdot C & \text{Yoneda lemma} \\ \hline
\phi_s \cdot \mathcal{K}(X,-) \to C & \phi_s \dashv \mathcal{K}(s,-) \\ \hline
\phi_s \cdot \mathcal{K}(X,-)\cdot F \to F & \text{universal property of } C \\ \hline
\mathcal{K}(X,-)\cdot F \to \mathcal{K}(s,-)\cdot F & \phi_s \dashv \mathcal{K}(s,-) \\ \hline
F^{(X)} \to F^{(s)} &
\end{array}$$

(iii)→(ii). For every object $X \in \mathcal{K}$ it is our task to prove that the diagram $D_X : X/F \to \mathcal{K}$ given by

$$D_X(X \xrightarrow{a} FA) = FA$$

has a limit. Given $s \in S$, a cone of D_X with domain s has the following form

$$\frac{X \xrightarrow{a} FA}{s \xrightarrow{a'} FA}$$

and we obtain a natural transformation

$$\alpha : F^{(X)} \to F^{(s)}$$

assigning to every $a \in F^{(X)}A = \mathcal{K}(X, FA)$ the value $\alpha_A(a) = a' \in F^{(s)}A$. Indeed, the naturality square

$$\begin{tikzcd}
F^{(X)}A \arrow[r, "\alpha_A"] \arrow[d, "F^{(X)}f"'] & F^{(s)}A \arrow[d, "F^{(s)}f"] \\
F^{(X)}B \arrow[r, "\alpha_B"'] & F^{(s)}B
\end{tikzcd}$$

commutes for every $f : A \to B$ in $\mathcal{A}$. This follows from the morphism

$$\begin{tikzcd}
& X \arrow[dl, "a"'] \arrow[dr, "b"] & \\
FA \arrow[rr, "Ff"'] & & FB
\end{tikzcd}$$

in X/F: Our cone $(-)'$ is compatible, thus

$$Ff \cdot a' = b' = (Ff \cdot a)',$$

which proves that the above square commutes when applied to a.

Conversely, every natural transformation $\alpha : F^X \to F^{(s)}$ has the above form. We obtain a cone of evaluations at a:

$$a' = \alpha_A(a) \quad \text{for every } a : A \to FX \text{ (i.e., } a \in F^{(X)}A)$$

Indeed the above triangle commutes since the naturality square does when applied to a.

It is easy to verify that we obtain a bijection between $Nat(F^{(X)}, F^{(s)})$ and the collection of all cones of D_X with domain s. Consequently, the latter collection is small for every $s \in S$. Since S is limit-detecting, D_X has a limit in $\mathcal{K}$.

(ii)→(i). This is trivial.

Finally, the claim in the remark above

$$U_s(CX) \cong Nat(F^{(X)}, F^{(s)}) \qquad \text{for } s \in S$$

follows from the fact that $U_s = \mathcal{K}(s,-)$ preserves limits. We have seen above that D_X has a limit, say, with the following cone

$$\frac{X \xrightarrow{a} FA}{CX \xrightarrow{\widehat{a}} FA} \qquad \text{for all } a : X \to FA \text{ with } A \in \mathcal{A}.$$

Then the cone of underlying functions $U(CX) \xrightarrow{U\widehat{a}} U(FA)$ is, up to isomorphism of the domain, the cone of evaluations $ev_a : Nat(F^{(X)}, F^{(s)}) \to U_s(FA)$, $s \in S$. □

Remark 3.5. (a) Suppose $\mathcal{K}$ is *transportable*, i.e., given an object K and an isomorphism $i : M \to UK$ in Set^S there exists an object $K' \in \mathcal{K}$ such that $UK' = M$ and i carries an isomorphism $K' \xrightarrow{\cong} K$ in $\mathcal{K}$. (Up to equivalence, all categories concrete over Set^S have this property, see [1], Lemma 5.35.) Then the codensity monad C can be chosen so that the underlying set of CX has components

$$U_s(CX) = Nat(F^{(X)}, F^{(s)}) \quad s \in S.$$

(b) Moreover, the evaluation maps with sorts

$$ev_a : Nat(F^{(X)}, F^{(s)}) \to U_s(FA) \qquad (\text{for } s \in S)$$

given by

$$ev_a(\alpha) = \alpha_A(a) \quad (\text{for all } a : X \to FA)$$

carry morphisms from CX to FA. Indeed, the limit cone $(\widehat{a})$ of CX was shown to fulfil this in the above proof.

(c) To characterize the object CX of $\mathcal{K}$, we use the concept of *initial lifting*, see [1]. Given a (possibly large) collection of objects $K_i \in \mathcal{K}$, $i \in I$, and a cone $v_i : V \to UK_i$ $(i \in I)$ in Set^S, the initial lifting is an object K of $\mathcal{K}$ with $UK = V$ such that

(i) each v_i carries a morphism from K to K_i $(i \in I)$

and

(ii) given an object K' of $\mathcal{K}$, then a function $f : UK' \to UK$ carries a morphism from K' to K iff all composites $v_i \cdot f$ carry morphisms from K' to K_i $(i \in I)$.

Corollary 3.6. ***(Codensity Monad Formula)*** *Let S be a limit-detecting generator making $\mathcal{K}$ a transportable category over Set^S. If a functor $F : \mathcal{A} \to \mathcal{K}$ has a codensity monad C, then C assigns to every object X the initial lifting of the cone of evaluations*

$$ev_a : \left(Nat(F^{(X)}, F^{(s)})\right)_{s \in S} \to UFA$$

for $A \in \mathcal{A}$ and $a : X \to FA$. Here $(ev_a)_s(\alpha) = \alpha_A(a)$ for every natural transformation $\alpha : F^{(X)} \to F^{(s)}$.

Indeed, the limit cone $\widehat{a} : CX \to FA$ can (due to transportability) be chosen so that $U\widehat{a} = ev_a$ for all $a : X \to FA$ in X/F. Given an object K' and a function $f : UK' \to U(CX)$ such that each composite $ev_a \cdot f$ carries a morphism $\widetilde{a} : K' \to FA$ in $\mathcal{K}$, the fact that U is faithful implies that $(\widetilde{a})$ forms a cone of D_X. Thus there exists $\overline{f} : K' \to CX$ with $\widetilde{a} = \widehat{a} \cdot \overline{f}$ for every a in X/F. This is the desired morphism carrying f: we have $U\overline{f} = f$ because the limit cone (ev_a) is collectively monic and for each $a : X \to FA$ we have

$$ev_a \cdot U\overline{f} = U(\widehat{a} \cdot \overline{f}) = U\widetilde{a} = ev_a \cdot f.$$

Remark 3.7. The definition of C on morphisms $f : X \to Y$ of $\mathcal{K}$ is canonical: Cf is carried by the S-sorted function from $Nat(F^{(X)}, F^{(s)})$ to $Nat(F^{(Y)}, F^{(s)})$ which takes a natural transformation $\alpha : \mathcal{K}(X,-) \cdot F \to \mathcal{K}(s,-) \cdot F$ to the composite

$$\mathcal{K}(Y,-) \cdot F \xrightarrow{\mathcal{K}(f,-) \cdot F} \mathcal{K}(X,-) \cdot F \xrightarrow{\alpha} \mathcal{K}(s,-) \cdot F.$$

This follows easily from the fact that Cf is the unique morphism such that the above limit morphisms $\widehat{a}: CX \to FA$ make the following triangles

$$\begin{tikzcd} CX \arrow[r, "Cf"] \arrow[d, "\widehat{a \cdot f}"'] & CY \arrow[dl, "\widehat{a}"] \\ FA & \end{tikzcd} \qquad \text{for all } a : Y \to FA$$

commutative.

4. Density Comonads

Notation 4.1. For every functor $F : \mathcal{A} \to \mathcal{K}$ and every object X of $\mathcal{K}$ we denote by X^F the set-valued functor

$$X^F \equiv \mathcal{A}^{op} \xrightarrow{F^{op}} \mathcal{K}^{op} \xrightarrow{\mathcal{K}(-,X)} \mathsf{Set}$$

Theorem 4.2. ***Density Comonad Theorem.*** *Let S be a cogenerator of a complete and wellpowered category. For every functor F with codomain $\mathcal{K}$ the following conditions are equivalent:*

(i) *F has a density comonad,*
(ii) *F has a pointwise density comonad, and*
(iii) *for every pair of objects $s \in S$ and $X \in \mathcal{K}$ the collection*

$$Nat(X^F, s^F)$$

of natural transformations from X^F to s^F is small.

Indeed, since S detects colimits by the dual of Example 3.2(2), this is just a dualization of Theorem 3.4.

Corollary 4.3. *A set-valued functor F has a density comonad iff for every set X there is only a set of natural transformations from X^F to 2^F. Moreover, the density comonad is then given by*

$$CX = Nat(X^F, 2^F).$$

For set-valued functors preserving preimages (i.e., pullbacks of monomorphisms along arbitrary morphisms) and with "set-like" domains, we intend to prove that

$$\text{accessibility} \Leftrightarrow \text{existence of a density comonad.}$$

For that we are going to use Theorem 4.6 below. The "set-like" flavour is given by the following:

Definition 4.4. A locally λ-presentable category is called *sctritly locally λ-presentable* if for every morphism $b : B \to A$ with a λ-presentable domain there exists a commutative square

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "b"'] & A \\ A \arrow[r, "f"'] & B' \arrow[u, "b'"'] \end{tikzcd}$$

with B' also λ-presentable.

Examples 4.5. (See [2]) Let λ be an infinite regular cardinal.

(1) Set is strictly locally λ-presentable.

(2) Many-sorted sets, Set^S, are strictly locally λ-presentable iff $\operatorname{card} S < \lambda$.

(3) K-Vec, the category of vector spaces over a field K, is strictly locally λ-presentable.

(4) The category of groups and homomorphisms is not strictly locally λ-presentable.

(5) For every group G the category G-Set of sets with an action of G is strictly locally λ-presentable iff $|G| < \lambda$.

The same holds for the category $\mathsf{Set}^{\mathbb{G}^{op}}$ of presheaves on a small groupoid $\mathbb{G}$, i.e., a category with invertible morphisms: it is strictly locally λ-presentable iff $\mathbb{G}$ has less than λ morphisms.

We are going to use the following characterization of accessibility proved in [2]:

Theorem 4.6. *A functor $F : \mathcal{A} \to \mathcal{B}$ with $\mathcal{A}$ and $\mathcal{B}$ strictly locally λ-presentable is λ-accessible iff for every object $A \in \mathcal{A}$ and every strong subobject $m_0 : M_0 \to FA$ with M_0 λ-presentable in $\mathcal{B}$ there exists a strong subobject $m : M \to A$ with M λ-presentable in $\mathcal{A}$ such that m_0 factorizes through Fm:*

$$\begin{tikzcd} & FM \arrow[d, "Fm"] \\ M_0 \arrow[ur, dashed] \arrow[r, "m_0"'] & FA \end{tikzcd}$$

Examples 4.7. (1) A set functor F is λ-accessible iff for every element of FA there exists a subset $m : M \hookrightarrow A$ with $\operatorname{card} M < \lambda$ such that the element lies in $Fm[FM]$.

(2) Analogously for endofunctors of K-Vec: just say $\dim M < \lambda$ here.

(3) For S finite, an endofunctor of Set^S is finitary iff every element of FA lies in $Fm[FM]$ for some finite subset $m : M \hookrightarrow FA$.

This does not generalize for S infinite. Consider the endofunctor F of $\mathsf{Set}^{\mathbb{N}}$ given as the identity function on objects (and morphisms) having all but finitely many components empty. And F is otherwise constant with value $\mathbb{1}$, the terminal object. This functor is not finitary: it does not preserve, for $\mathbb{2} = \mathbb{1} + \mathbb{1}$, the canonical filtered colimit of all morphisms from finitely presentable objects to $\mathbb{2}$. But it satisfies the condition that every element of FA lies in $Fm[FM]$ for some finite subset $m : M \hookrightarrow FA$.

Theorem 4.8. *Let $\mathcal{A}$ be a category where epimorphisms split and such that there is a cardinal μ for which $\mathcal{A}$ is strictly locally λ-presentable and λ-presentable objects are closed under subobjects, whenever $\lambda \geq \mu$.*

Then a functor $F : \mathcal{A} \to$ Set preserving preimages has a density comonad iff it is accessible.

Proof. Since epimorphisms split, $\mathcal{A}$ has regular factorizations – indeed, locally presentable categories have (strong epi, mono)-factorizations, see [3]. In view of Theorem 2.1 we only need to prove the non-existence of a density comonad in case F is not accessible. Let us call an element $x \in FA$ *λ-accessible* if there exists a λ-presentable subobject $m : M \rightarrowtail A$ with $x \in Fm[FM]$. From the preceding theorem we know that, for all $\lambda \geq \mu$, F posesses an element that is not λ-accessible. Without loss of generality, μ is an infinite regular cardinal.

(1) Define regular cardinals λ_i ($i \in \mathrm{Ord}$) by transfinite recursion as follows:

$\lambda_0 = \mu$;

Given λ_i choose an element $x_i \in FA_i$ for some $A_i \in \mathcal{A}$ which is not λ_i-accessible and define λ_{i+1} as the least regular cardinal with A_i λ_{i+1}-presentable;

Given a limit ordinal j define λ_j as the successor cardinal of $\bigvee_{i<j} \lambda_i$.

We thus see that for every ordinal i the element x_i is λ_{i+1}-accessible but not λ_i-accessible.

(2) To prove that F does not have a density comonad, we present pairwise distinct natural transformations

$$\alpha^i : 2^F \to 2^F \ (i \in \mathrm{Ord}).$$

For every object $A \in \mathcal{A}$, a subset $M \subseteq FA$ (i.e., an element of 2^{FA}) and an element $a \in M$, we call the triple (A, M, a) *λ_i-stable* if there exists a subobject $u_a : U_a \rightarrowtail A$ in $\mathcal{A}$ with $a \in Fu_a[FU_a]$ such that for all subobjects $v : V \rightarrowtail U_a$ we have

$$\text{if } V \text{ is } \lambda_i\text{-presentable, then } M \cap F(u_a v)[FV] = \emptyset.$$

Our natural transformation α^i has the following components $\alpha^i_A : 2^{FA} \to 2^{FA}$:

$$\alpha^i_A(M) = \{a \in M\,;\, (A, M, a) \text{ is } \lambda_i\text{-stable}\}.$$

We must prove that for every morphism $h : A \to B$ the naturality square

$$\begin{tikzcd} 2^{FB} \arrow[r, "\alpha^i_B"] \arrow[d, "(Fh)^{-1}(-)"'] & 2^{FB} \arrow[d, "(Fh)^{-1}(-)"] \\ 2^{FA} \arrow[r, "\alpha^i_A"'] & 2^{FA} \end{tikzcd}$$

commutes. That is, given

$$M \subseteq FB \ \text{ and } \ \overline{M} = (Fh)^{-1}(M) \subseteq FA$$

then for all elements

$$a \in \overline{M} \ \text{ and } \ b = Fh(a) \in M$$

we need to verify that

$$(A, \overline{M}, a) \text{ is } \lambda_i\text{-stable} \iff (B, M, b) \text{ is } \lambda_i\text{-stable}.$$

(a) Let $(A,\overline{M},a)$ be λ_i-stable. For the given subobject $u_a : U_a \rightarrowtail A$ form a regular factorization of hu_a:

$$\begin{tikzcd}
U_a \arrow[r, two heads, "e"] \arrow[d, "u_a"'] & U_b \arrow[l, dashed, bend left, "w"] \arrow[d, tail, "u_b"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

We have $a' \in FU_a$ with $a = Fu_a(a')$, therefore b lies in the image of Fu_b:

$$b = Fh(a) = Fu_b(Fe(a')).$$

For every subobject $v : V \to U_b$ with V λ_i-presentable we need to prove that $M \cap F(u_b v)[FV] = \emptyset$. Choose a splitting w of e, i.e., $e \cdot w = \mathrm{id}_{U_b}$. Then for the subobject

$$wv : V \to U_a$$

we know that $\overline{M} = (Fh)^{-1}(M)$ is disjoint from the image of $F(u_a wv)$. Suppose there exists an element of $M \cap F(u_b v)[FV]$, say, $F(u_b v)(t)$ for some $t \in FV$. Put $t' = F(u_a wv)(t)$, then we derive a contradiction by showing that $t' \in \overline{M}$. Indeed

$$\begin{aligned}
Fh(t') &= F(hu_a wv)(t) \\
&= F(u_b ewv)(t) \\
&= F(u_b v)(t) \in M.
\end{aligned}$$

Thus, $t' \in (Fh)^{-1}(M) = \overline{M}$.

(b) Let (B,M,b) be λ_i-stable. Since $Fh(a) = b \in M$ we have

$$a \in (Fh)^{-1}(M) = \overline{M}.$$

Given the above subobject $u_b : U_b \to B$, we define $u_a : U_a \to A$ as the preimage under h:

$$\begin{tikzcd}
V \arrow[r, two heads, "e"] \arrow[d, tail, "v"'] & W \arrow[d, tail, "w"] \\
U_a \arrow[r, "\overline{h}"] \arrow[d, tail, "u_a"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_b \arrow[d, tail, "u_b"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

We have $b' \in FU_b$ with $b = Fu_b(b') = Fh(a)$, and since F preserves preimages, there exists $a' \in FU_a$ with $Fu_a(a') = a$.

Given a subobject $v : V \to U_a$ with V λ_i-presentable, we prove that $F(u_a v)([FV]$ is disjoint from $\overline{M}$. For that take the regular factorization of $\overline{h}v$ as in the diagram above. Since e is a split epimorphism, W is a λ_i-presentable object. Therefore, the image of $F(u_b w)$ is disjoint from M.

Assuming that we have $t \in FV$ with $F(u_a v)(t) \in \overline{M}$, we derive a contradiction by showing that for $t' = Fe(t)$ we have $F(u_b w)(t') \in M$. Indeed, since $\overline{M} = (Fh)^{-1}(M)$, we see that

$F(hu_a v)(t) \in Fh[\overline{M}] \subseteq M$ and we have

$$hu_a v = u_b \overline{h} v = u_b w e.$$

(3) We have established that each $i \in \mathrm{Ord}$ yields a natural transformation $\alpha^i : 2^F \to 2^F$. We conclude the proof by verifying for all ordinals $i \neq j$ that $\alpha^i \neq \alpha^j$. Suppose $i < j$. In (1) we have presented an element $x_i \in FA_i$ which is λ_{i+1}-accessible (because A_i is λ_{i+1}-accessible) but not λ_i-accessible. Let $M_i \subseteq FA_i$ be the set of all elements that are not λ_i-accessible. Then

$$(A_i, M_i, x_i)$$

is clearly λ_i-stable. But it is not λ_j-stable because A_i is λ_j-presentable (since λ_{i+1} is a presentability rank of A_i and $\lambda_{i+1} \leq \lambda_j$). Indeed, no subobject $u_{x_i} : U_{x_i} \to A$ has the property that $x_i \in Fu_{x_i}[FU_{x_i}]$ but $M_i \cap F(u_{x_i} v)[FV] = \emptyset$ for all λ_j-presentable subobjects $v : V \to U_{x_i}$: since A_i is λ_j-presentable, so is U_{x_i}, because λ_j-presentable objects are closed under subobjects in $\mathcal{A}$. Put $v = \mathrm{id}_{U_{x_i}}$; then $x_i \in M \cap F(u_{x_i} v)[FV]$.

Consequently, we have

$$x_i \in \alpha^i_{A_i}(M_i) \text{ but } x_i \notin \alpha^j_{A_i}(M_i).$$

$\square$

The following corollary works with set functors preserving preimages. This is a very weak assumption since all "everyday" set functors preserve them:

(1) The identity and constant functors preserve preimages.
(2) Products, coproducts, and composites of functors preserving preimages preserve them.
(3) Thus polynomial functors preserve images.
(4) The power-set functor, the filter functor and the ultrafilter functor preserve preimages.

Corollary 4.9. *A set functor preserving preimages has a density comonad iff it is accessible.*

5. Examples of set functors

Example 5.1. The density comonad of $FX = X^n$ is

$$CX = X^{n^n}.$$

More detailed: we prove that the colimit of the diagram $D_X : (-)^n/X \to \mathrm{Set}$ has the component at $a : A^n \to X$ defined as follows

$$\hat{a} : A^n \to X^{n^n},\ t \mapsto a \cdot t^n \text{ (for all } t : n \to A)$$

It is easy to see that this is a cocone.

Let $\widetilde{a} : A^n \to B$ (for all $a : A^n \to X$) be another cocone. Consider the following morphisms of $(-)^n/X$ for every $a : A^n \to X$ and every $t : n \to A$:

$$\begin{tikzcd} n^n \arrow[rr, "t^n"] \arrow[dr, "a\cdot t^n"'] & & A^n \arrow[dl, "a"] \\ & X & \end{tikzcd}$$

Thus the following triangle

$$\begin{tikzcd} n^n \arrow[rr, "t^n"] \arrow[dr, "\widetilde{a\cdot t^n}"'] & & A^n \arrow[dl, "\widetilde{a}"] \\ & B & \end{tikzcd}$$

commutes. Applied to id_n this yields

$$\widetilde{a}(t) = \widetilde{a \cdot t^n}(\mathrm{id}_n).$$

Therefore we have a factorization $f : X^{n^n} \to B$ through the colimit cocone defined by

$$f(u) = \widetilde{u}(\mathrm{id}_n).$$

Indeed $\widetilde{a} = f \cdot \hat{a}$ since for every t we have $\widetilde{a}(t) = \widetilde{a \cdot t^n}(\mathrm{id}_n) = f(a \cdot t^n) = f \cdot \hat{a}(t)$. It is easy to see that f is unique.

Example 5.2. More generally, for a polynomial functor

$$FX = \coprod_{i\in I} X^{n_i}$$

the density comonad is

$$CX = \coprod_{i\in I} \prod_{j\in I} X^{n_i^{n_j}}.$$

The colimit cocone for D_X has for $a : \coprod_{i\in I} A^{n_i} \to X$ the component $\hat{a} = \coprod_{i\in I} \hat{a}_i : \coprod_{i\in I} A^{n_i} \to CX$, where

$$\hat{a}_i : A^{n_i} \to \prod_{j\in I} X^{n_i^{n_j}} \ \text{ sends } \ t : n_i \to A \ \text{ to } \ a \cdot \coprod_{j\in I} t^{n_j} : \coprod_{j\in I} n_i^{n_j} \to X.$$

(The last map is an element of $\prod_{j\in I} X^{n_i^{n_j}}$.) The proof is completely analogous to 5.1: for every $a : \coprod_{i\in I} A^{n_i} \to X$ and $t : n_i \to A$ use the following triangle

$$\begin{tikzcd} \coprod_{j\in I} n_i^{n_j} \arrow[rr, "\coprod t^{n_j}"] \arrow[dr, "a\cdot\coprod t^{n_j}"'] & & \coprod_{j\in I} A^{n_j} \arrow[dl, "a"] \\ & X & \end{tikzcd}$$

Recall that $\mathcal{P}_0$ denotes the subfunctor of $\mathcal{P}$ with $\mathcal{P}_0 X = \mathcal{P}X - \{\emptyset\}$.

Example 5.3. The power-set functor $\mathcal{P}$ and its subfunctor $\mathcal{P}_0$ do not have a density comonad, since they are not accessible.

Proposition 5.4. *The codensity monad of $\mathcal{P}_0$ is itself.*

Proof. (1) We first prove the equality on objects X by verifying that natural transformations $\alpha : \mathcal{P}_0^X \to \mathcal{P}_0$ bijectively correspond to nonempty subsets of X as follows: we assign to α the subset

$$\alpha_X(\eta_X) \subseteq X$$

where η is the unit of $\mathcal{P}_0$. The inverse map takes a nonempty set $M \subseteq X$ to the natural transformation $\widehat{M} : \mathcal{P}_0^X \to \mathcal{P}_0$ assigning to each $u : X \to \mathcal{P}_0 A$ the value

$$\widehat{M}_A(u) = \bigcup_{x\in M} u(x).$$

(1a) The naturality squares for $\widehat{M}$ are easy to verify.

(1b) Given α, put $M = \alpha_X(\eta_X)$. We prove that for all $u : X \to \mathcal{P}_0 A$ we have

$$\alpha_A(u) = \widehat{M}_A(u).$$

We first verify this for all u such that A has a disjoint decomposition $u(x)$, $x \in X$. We then have the obvious projection $f : A \to X$ with

$$\mathcal{P}_0 f \cdot u = \eta_X.$$

Thus, the naturality square

$$\begin{tikzcd}
(\mathcal{P}_0 A)^X \arrow[r, "\alpha_A"] \arrow[d, "\mathcal{P}_0 f\cdot(-)"'] & \mathcal{P}_0 A \arrow[d, "\mathcal{P}_0 f"] \\
(\mathcal{P}_0 X)^X \arrow[r, "\alpha_X"'] & \mathcal{P}_0 X
\end{tikzcd}$$

yields

$$\mathcal{P}_0 f(\alpha_A(u)) = \alpha_X(\eta_X) = M.$$

This clearly implies $\alpha_A(u) = \bigcup\limits_{x\in M} u(x)$.

Next let $u : X \to \mathcal{P}_0 A$ be arbitrary and consider its "disjoint modification" $\overline{u} : X \to \mathcal{P}_0 \overline{A}$ where

$$\overline{A} = \bigcup_{x\in X} u(x) \times \{x\} \quad \text{and} \quad \overline{u}(x) = u(x) \times \{x\}.$$

We know already that $\alpha_{\overline{A}}(\overline{u}) = \bigcup\limits_{x\in M} \overline{u}(x)$. The obvious projection $g : \overline{A} \to A$ fulfils

$$u = \mathcal{P}_0 g \cdot \overline{u}.$$

The naturality square thus gives

$$\alpha_A(u) = \mathcal{P}_0 g(\alpha_A(\overline{u})) = \mathcal{P}_0 g\left(\bigcup_{x\in M} \overline{u}(x)\right) = \bigcup_{x\in M} g[\overline{u}(x)].$$

This concludes the proof, since $g[\overline{u}(x)] = u(x)$.

(1c) The map $M \mapsto \widehat{M}$ is inverse to $\alpha \mapsto \alpha_X(\eta_X)$. Indeed, if we start with $M \subseteq X$ and form $\alpha = \widehat{M}$, we get

$$\widehat{M}_X(\eta_X) = \bigcup_{x\in M} \eta_X(x) = M.$$

Conversely, if we start with α and put $M = \alpha_X(\eta_X)$, then $\alpha = \widehat{M}$: see (1b).

(2) The definition of the pointwise codensity monad for $\mathcal{P}_0$ on morphisms $f : X \to Y$ is as follows: a natural transformation $\alpha : \mathcal{P}_0^X \to \mathcal{P}_0$ is taken to the following composite

$$\mathcal{P}_0^Y \xrightarrow{\mathcal{P}_0^f} \mathcal{P}_0^X \xrightarrow{\alpha} \mathcal{P}_0$$

If α corresponds to $M (= \alpha_X(\eta_X))$, it is our task to verify that $\alpha \cdot \mathcal{P}_0^f$ corresponds to $\mathcal{P}_0 f(M)$. Indeed:

$$\begin{aligned}
\mathcal{P}_0 f(M) &= \alpha_Y(\eta_Y \cdot f), && \text{by naturality of } \alpha \text{ and } \eta, \\
&= \left(\alpha \cdot \mathcal{P}_0^f\right)_Y (\eta_Y).
\end{aligned}$$

□

Recall from [14] that a set functor is *indecomposable*, i.e., not a coproduct of proper subfunctors, iff it preserves the terminal objects.

Proposition 5.5. *Let F be an indecomposable set functor with a codensity monad T.*

(1) The functor $F+1$ has the codensity monad

$$\widehat{T}X = \prod_{Y \subseteq X} (TY + 1)$$

with projections π_Y. This monad assigns to a morphism $f : X \to X'$ the morphism $\widehat{T}f : \widehat{T}X \to \prod_{Z \subseteq X'} T(Z+1)$ with components

$$\widehat{T}X \xrightarrow{\pi_Y} TY + 1 \xrightarrow{Tf_Z + 1} TZ + 1 \quad \textit{for all } Z \subseteq X'$$

where $f_Z : Y \to Z$ is the restriction of f with $Y = f^{-1}[Z]$.

(2) Every copower $\coprod_M F$ has the codensity monad

$$X \mapsto (M \times TX)^{M^X}$$

assigning to a morphism f the morphism $(M \times Tf)^{M^f}$.

Proof. (1) Since F is indecomposable, so is F^X for every set X, hence,

$$\mathrm{Nat}(F^X, F+1) \simeq \mathrm{Nat}(F^X, F) + 1 = TX + 1,$$

consequently, from the natural isomorphism $[F+1]^X \simeq \coprod_{Y \subseteq X} F^Y$ we get

$$\begin{aligned} \mathrm{Nat}([F+1]^X, F+1) &\simeq \mathrm{Nat}(\textstyle\coprod_{Y \subseteq X} F^Y, F+1) \\ &\simeq \textstyle\prod_{Y \subseteq X} \mathrm{Nat}(F^Y, F+1) \\ &= \textstyle\prod_{Y \subseteq X} (TY+1) \end{aligned}$$

(2) We compute

$$\begin{aligned} \mathrm{Nat}\Big((\textstyle\coprod_M F)^X, \coprod_M F\Big) &\simeq \mathrm{Nat}(M^X \times F^X, \textstyle\coprod_M F) \\ &\simeq \textstyle\prod_{M^X} \mathrm{Nat}(F^X, \coprod_M F). \end{aligned}$$

Since F^X is indecomposable, $\mathrm{Nat}(F^X, \coprod_M F) \simeq \coprod_M \mathrm{Nat}(F^X, F) \simeq M \times TX$. This yields $(M \times TX)^{M^X}$, as claimed. □

Corollary 5.6. *The codensity monad of $\mathcal{P}$ is given by*

$$X \mapsto \prod_{Y \subseteq X} \mathcal{P}Y.$$

Indeed, $\mathcal{P} = \mathcal{P}_0 + 1$ and $\mathcal{P}_0$ is indecomposable.

Another description of the codensity monad of $\mathcal{P}$: it assigns to every set X all nonexpanding selfmaps ψ of $\mathcal{P}X$ (i.e., self-maps with $\psi Y \subseteq Y$ for all $Y \in \mathcal{P}X$).

Example 5.7. Polynomial functors.

(1) The functor $FX = X^n$ has the codensity monad

$$TY = (n \times Y)^n.$$

Indeed, F is a right adjoint yielding the monad $T = (-)^n \cdot (n \times -) = (n \times -)^n$.

(2) The polynomial functor

$$FX = \coprod_{i\in I} X^{n_i} \qquad (n_i \text{ arbitrary cardinals})$$

has the following codensity monad

$$TY = \prod_{(Y_i)} \coprod_{j\in I} \left(\coprod_{i\in I} n_i \times Y_i \right)^{n_j}$$

where the product ranges over disjoint decompositions

$$Y = \bigcup_{i\in I} Y_i$$

indexed by I. (Here Y_i is allowed to be empty.) This follows from the Codensity Monad Theorem where we compute $(FX)^Y$ as follows: a mapping from Y to $\coprod_{i\in I} X^{n_i}$ is given by specifying a decomposition (Y_i) and an I-tuple of mappings from Y_i to X^{n_i}. The latter is an element of $\prod_{i\in I} X^{n_i\times Y_i} \simeq X^{\coprod_{i\in I}(n_i\times Y_i)}$, therefore

$$F^Y \cong \coprod_{(Y_i)} \mathsf{Set}(\coprod_{i\in I} n_i \times Y_i, -).$$

We conclude, using Yoneda lemma, that

$$\begin{aligned} TY &= \mathsf{Nat}(F^Y, F) \\ &\simeq \textstyle\prod_{(Y_i)} F\,(\coprod_{i\in I} n_i \times Y_i) \\ &= \textstyle\prod_{(Y_i)} \coprod_{j\in I} (\coprod_{i\in I} n_i \times Y_i)^{n_j} \end{aligned}$$

as stated.

Open Problem 5.8. Which set functors possess a codensity monad?

References

[1] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories*, John Wiley and Sons, New York 1990. Freely available at www.math.uni-bremen.de/~dmb/acc.pdf

[2] J. Adámek, S. Milius and L. Sousa, On finitary functors and finitely presentable algebras, preprint.

[3] J. Adámek and J. Rosický: *Locally presentable and accessible categories*, Cambridge University Press, 1994.

[4] J. Adámek and H.-E.Porst, On tree coalgebras and coalgebra presentations, *Theoret. Comput. Sci.* 311 (2004), 257-283.

[5] Appelgate and Tierney, Categories with models, *Lect. N. Mathem.* 80, Springer 1969, see also Reprints in Theory and Applications of Categories, 18 (2008), 122-185.

[6] B. Day, Further criteria for totality, *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 28 (1987), 77-78.

[7] P. Gabriel and F. Ulmer, *Local Präsentierbare Kategorien*, Lect. Notes in Math. 221, Springer-Verlag, Berlin 1971.

[8] A. Kock, Continuous Yoneda representations of a small category, Aarhus University (preprint) 1966.

[9] F. E. J. Linton, An outline of functorial semantics, *Lect. N. Mathem.* 80, Springer 1969, see also Reprints in Theory and Applications of Categories, 18 (2008), 11-43.

[10] R. Lowen, *Approach spaces: the missing link in the topology-uniformity-metric trial*, Oxford Mathematical Publications, Oxford 1997.

[11] S. Mac Lane, *Categories for the Working Mathematician, 2nd ed.*, Springer-Verlag, Berlin-Heidelberg-New York 1998.

[12] R. Street, B. Walters, Yoneda structures on 2-category, *J. Algebra* 50 (1978), 350-379.

[13] W. Tholen, Note on total categories, *Bulletin of the Australian Mathematical Society* 21 (1980), 169-173.
[14] V. Trnková, Some properties of set functors, *Comment. Math. Univ. Carolinae* 10 (1969) 323-352.

Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University in Prague, Czech Republic
E-mail address: adamek@iti.cs.tu-bs.de

CMUC, University of Coimbra, 3001-501 Coimbra, Portugal & ESTGV, Polytechnic Institute of Viseu, 3504-510 Viseu, Portugal
E-mail address: sousa@estv.ipv.pt