# ON FINITARY FUNCTORS

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ABSTRACT. A simple criterion for a functor to be finitary is presented: we call $F$ finitely bounded if for all objects $X$ every finitely generated subobject of $FX$ factorizes through the $F$-image of a finitely generated subobject of $X$. This is equivalent to $F$ being finitary for all functors between 'reasonable' locally finitely presentable categories, provided that $F$ preserves monomorphisms. We also discuss the question when that last assumption can be dropped. The answer is affirmative for functors between categories such as Set, K-Vec (vector spaces), boolean algebras, and actions of any finite group either on Set or on K-Vec for fields K of characteristic 0.

All this generalizes to locally $\lambda$-presentable categories, $\lambda$-accessible functors and $\lambda$-presentable algebras. As an application we obtain an easy proof that the Hausdorff functor on the category of complete metric spaces is $\aleph_1$-accessible.

## 1. Introduction

In a number of applications of categorical algebra, *finitary functors*, i.e. functors preserving filtered colimits, play an important role. For example, the classical varieties are precisely the categories of algebras for finitary monads over $\mathsf{Set}$. How does one recognize that a functor $F$ is finitary? For endofunctors of $\mathsf{Set}$ there is a simple necessary and sufficient condition: given a set $X$, every finite subset of $FX$ factorizes through the image by $F$ of a finite subset of $X$. This condition can be formulated for general functors $F\colon \mathscr{A} \to \mathscr{B}$: given an object $X$ of $\mathscr{A}$, every finitely generated subobject of $FX$ in $\mathscr{B}$ is required to factorize through the image by $F$ of a finitely generated subobject of $X$ in $\mathscr{A}$. We call such functors *finitely bounded*. For functors between locally finitely presentable categories which preserve monomorphisms we prove

$$\text{finitary} \iff \text{finitely bounded}$$

whenever finitely generated objects are finitely presentable. (The last condition is, in fact, not only sufficient but also necessary for the above equivalence.)

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What about general functors, not necessarily preserving monomorphisms? We prove the above equivalence whenever $\mathscr{A}$ is a strictly locally finitely presentable category, see Definition 3.9. Examples of such categories are sets, vector spaces, group actions of finite groups, and $S$-sorted sets with $S$ finite. Conversely, if the above equivalence is true for all functors from $\mathscr{A}$ to $\mathsf{Set}$, we prove that a weaker form of strictness holds for $\mathscr{A}$.

All of the above results can be also formulated for locally $\lambda$-presentable categories and $\lambda$-accessible functors. We use this to provide a simple proof that the Hausdorff functor on the category of complete metric spaces is countably accessible.

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## 2. Preliminaries

In this section we present properties of finitely presentable and finitely generated objects which will be useful in the subsequent sections.

Recall that an object $A$ in a category $\mathscr{A}$ is called *finitely presentable* if its hom-functor $\mathscr{A}(A,-)$ preserves filtered colimits, and $A$ is called *finitely generated* if $\mathscr{A}(A,-)$ preserves filtered colimits of monomorphisms – more precisely, colimits of filtered diagrams $D\colon \mathscr{D} \to \mathscr{A}$ for which $Dh$ is a monomorphism in $\mathscr{A}$ for every morphism $h$ of $\mathscr{D}$.

2.1. Notation. For a category $\mathscr{A}$ we denote by

$$\mathscr{A}_{\mathsf{fp}} \qquad \text{and} \qquad \mathscr{A}_{\mathsf{fg}}$$

full subcategories of $\mathscr{A}$ representing (up to isomorphism) all finitely presentable and finitely generated objects, respectively.

Subobjects $m\colon M \rightarrowtail A$ with $M$ finitely generated are called *finitely generated subobjects.*

Recall that $\mathscr{A}$ is a *locally finitely presentable* category, shortly *lfp* category, if it is cocomplete, $\mathscr{A}_{\mathsf{fp}}$ is small, and every object is a colimit of a filtered diagram in $\mathscr{A}_{\mathsf{fp}}$.

We now recall a number of standard facts about lfp categories [5].

2.2. Remark. Let $\mathscr{A}$ be an lfp category.

(1) By [5, Proposition 1.61], $\mathscr{A}$ has (strong epi, mono)-factorizations of morphisms.

(2) By [5, Proposition 1.57], every object $A$ of $\mathscr{A}$ is the colimit of its *canonical filtered diagram*

$$D_A\colon \mathscr{A}_{\mathsf{fp}}/A \to \mathscr{A} \qquad (P \xrightarrow{p} A) \mapsto P,$$

with colimit injections given by the $p$'s.

(3) By [5, Theorem 2.26], $\mathscr{A}$ is a free completion of $\mathscr{A}_{\mathsf{fp}}$ under filtered colimits. That is, for every functor $H\colon \mathscr{A}_{\mathsf{fp}} \to \mathscr{B}$, where $\mathscr{B}$ has filtered colimits, there is an (essentially unique) extension of $H$ to a finitary functor $\bar{H}\colon \mathscr{A} \to \mathscr{B}$. Moreover, this extensions can be formed as follows: for every object $A \in \mathscr{A}$ put

$$\bar{H}A = \operatorname{colim} H \cdot D_A.$$

(4) By [5, Proposition 1.62], a colimit of a filtered diagram of monomorphisms has monomorphisms as colimit injections. Moreover, for every compatible cocone formed by monomorphisms, the unique induced morphism from the colimit is a monomorphism too.

(5) By [5, Proposition 1.69], an object $A$ is finitely generated iff it is a strong quotient of a finitely presentable object, i.e. there exists a finitely presentable object $A_0$ and a strong epimorphism $e\colon A_0 \twoheadrightarrow A$.

(6) It is easy to verify that every split quotient of a finitely presentable object is finitely presentable again.

2.3. Lemma. *Let $\mathscr{A}$ be an lfp category. A cocone of monomorphisms $c_i\colon Di \rightarrowtail C$ ($i \in I$) of a filtered diagram $D$ of monomorphisms is a colimit of $D$ iff it is a* union*; that is, iff $\mathsf{id}_C$ is the supremum of the subobjects $c_i\colon Di \rightarrowtail C$.*

Proof. The 'only if' direction is clear. For the 'if' direction suppose that $c_i\colon Di \rightarrowtail C$ have the union $C$, and let $\ell_i\colon Di \to L$ be the colimit of $D$. Then, since $c_i$ is a cocone of $D$, we get a unique morphism $m\colon L \to C$ with $m \cdot \ell_i = c_i$ for every $i$. By Remark 2.2(4), all the $\ell_i$ and $m$ are monomorphisms, hence $m$ is a subobject of $C$. Moreover, we have that $c_i \leq m$, for every $i$. Consequently, since $C$ is the union of all $c_i$, $L$ must be isomorphic to $C$ via $m$, because $id_C$ is the largest subobject of $C$. Thus, the original cocone $c_i$ is a colimit cocone. ∎

2.4. Remark. Colimits of filtered diagrams $D\colon \mathscr{D} \to \mathsf{Set}$ are precisely those cocones $c_i\colon D_i \to C$ ($i \in \mathsf{obj}\,\mathscr{D}$) of $D$ that have the following properties:

(1) $(c_i)$ is jointly surjective, i.e. $C = \bigcup c_i[D_i]$, and

(2) given $i$ and elements $x, y \in D_i$ merged by $c_i$, then they are also merged by a connecting morphism $D_i \to D_j$ of $D$.

This is easy to see: for every cocone $c_i'\colon D_i \to C'$ of $D$ define $f\colon C \to C'$ by choosing for every $x \in C$ some $y \in D_i$ with $x = c_i(y)$ and putting $f(x) = c_i'(y)$. By the two properties above, this is well defined and is unique with $f \cdot c_i = c_i'$ for all $i$.

2.5. Lemma. [Finitely presentable objects collectively reflect filtered colimits.]
*Let $\mathscr{A}$ be an lfp category and $D\colon \mathscr{D} \to \mathscr{A}$ a filtered diagram with objects $D_i$ ($i \in I$). A cocone $c_i\colon D_i \to C$ of $D$ is a colimit iff for every $A \in \mathscr{A}_{\mathsf{fp}}$ the cocone*

$$c_i \cdot (-)\colon \mathscr{A}(A, D_i) \longrightarrow \mathscr{A}(A, C)$$

*is a colimit of the diagram $\mathscr{A}(A, D-)$ in $\mathsf{Set}$.*

Explicitly, the above property of the cocone $(c_i)$ states that for every morphism $f\colon A \to C$ where $A \in \mathscr{A}_{\mathsf{fp}}$

(1) a factorization through some $c_i$ exists, and

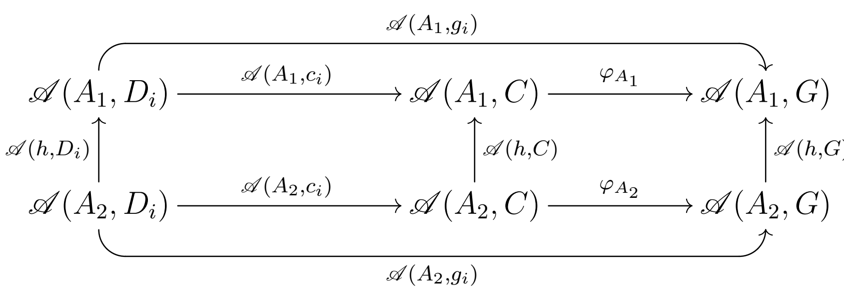
(2) given two factorizations $f = c_i \cdot q_k$ for $k = 1, 2$, then $q_1, q_2\colon A \to D_i$ are merged by a connecting morphism of $\mathscr{D}$. The proof that this describes $\operatorname{colim} \mathscr{A}(A, D-)$ follows from Remark 2.4.

Proof. If $(c_i)$ is a colimit, then since $\mathscr{A}(A, -)$ preserves filtered colimits, the cocone of all $\mathscr{A}(A, c_i) = c_i \cdot (-)$ is a colimit in $\mathsf{Set}$.

Conversely, assume that, for every $A \in \mathscr{A}_{\mathsf{fp}}$, the colimit cocone of the functor $\mathscr{A}(A, D-)$ is $\big(\mathscr{A}(A, c_i)\big)_{i \in \mathscr{D}}$. For every cocone $g_i\colon D_i \to G$ it is our task to prove that there exists a unique $g\colon C \to G$ with $g_i = g \cdot c_i$ for all $i$. We first prove uniqueness of $g$. If $g \cdot c_i = g' \cdot c_i$ for all $i$, then $\mathscr{A}(A, g) \cdot \mathscr{A}(A, c_i) = \mathscr{A}(A, g') \cdot \mathscr{A}(A, c_i)$. Since the $\mathscr{A}(A, c_i)$ are jointly surjective, we obtain $\mathscr{A}(A, g) = \mathscr{A}(A, g')$. Since this holds for all $A \in \mathscr{A}_{\mathsf{fp}}$, and $\mathscr{A}_{\mathsf{fp}}$ is a generator, we have $g = g'$.

Now $\big(\mathscr{A}(A, g_i)\big)_{i \in \mathscr{D}}$ forms a cocone of the functor $\mathscr{A}(A, -) \cdot D$. Consequently, there is a unique map $\varphi_A\colon \mathscr{A}(A, C) \to \mathscr{A}(A, G)$ with $\varphi_A \cdot \mathscr{A}(A, c_i) = \mathscr{A}(A, g_i)$ for all $i \in \mathscr{D}$.

For every morphism $h\colon A_1 \to A_2$ between objects of $\mathscr{A}_{\mathsf{fp}}$ the square on the right of the following diagram is commutative:

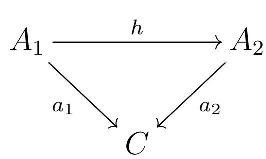



This follows from the commutativity of the left-hand square and the outside one combined with the fact that $\big(\mathscr{A}(A_2, c_i)\big)_{i \in \mathscr{D}}$, being a colimit cocone, is jointly epic.

As a consequence, the morphisms

$$A \xrightarrow{\varphi_A(a)} C \qquad \text{with } a\colon A \to C \text{ in } \mathscr{A}_{\mathsf{fp}}/C,$$

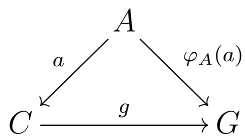
form a cocone for the canonical filtered diagram $D_C\colon \mathscr{A}_{\mathsf{fp}}/C \to \mathscr{A}$, of which $C$ is the colimit. Indeed, given a morphism $h$ in $\mathscr{A}_{\mathsf{fp}}/C$




we have

$$\varphi_{A_1}(a_1) = \varphi_{A_1}(a_2 \cdot h) = \varphi_{A_1} \cdot \mathscr{A}(h, C)(a_2) = \mathscr{A}(h, G) \cdot \varphi_{A_2}(a_2) = \varphi_{A_2}(a_2) \cdot h.$$

Thus there is a unique morphism $g\colon C \to G$ making for each $a\colon A \to C$ in $\mathscr{A}_{\mathsf{fp}}/C$ the following triangle commute:




It satisfies $g \cdot c_i = g_i$ for all $i \in \mathscr{D}$. Indeed, fix $i$; for every $A \in \mathscr{A}_{\mathsf{fp}}$ and $b\colon A \to D_i$, we have $g_i b = \mathscr{A}(A, g_i)(b) = \varphi_A \cdot \mathscr{A}(A, c_i)(b) = \varphi_A(c_i b) = g c_i b$. And the morphisms $b \in \mathscr{A}_{\mathsf{fp}}/D_i$ are jointly epimorphic, thus $g_i = g \cdot c_i$. Thus $g$ is the desired factorization morphism. ■

2.6. Lemma. [Finitely generated objects collectively reflect filtered colimits of monomorphisms.] *Let $\mathscr{A}$ be an lfp category and $D\colon \mathscr{D} \to \mathscr{A}$ a filtered diagram of monomorphisms with ojects $D_i$ $(i \in I)$. A cocone $c_i\colon D_i \to C$ of $D$ is a colimit iff for every $A \in \mathscr{A}_{\mathsf{fg}}$ the cocone*

$$c_i \cdot (-)\colon \mathscr{A}(A, D_i) \longrightarrow \mathscr{A}(A, C) \qquad (i \in I)$$

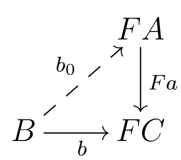
*is a colimit of the diagram $\mathscr{A}(A, D-)$ in $\mathsf{Set}$.*

Proof. If $(c_i)$ is a colimit, then since $\mathscr{A}(A, -)$ preserves filtered colimits of monomorphisms, the cocone $c_i \cdot (-)\colon \mathscr{A}(A, D_i) \to \mathscr{A}(A, C)$ is a colimit in $\mathsf{Set}$.

Conversely, if for every $A \in \mathscr{A}_{\mathsf{fg}}$, the cocone $c_i \cdot (-)\colon \mathscr{A}(A, D_i) \to \mathscr{A}(A, C)$, $i \in I$, is a colimit of the diagram $\mathscr{A}(A, D-)$, then we have for every $A \in \mathscr{A}_{\mathsf{fp}}$ that the cocone $c_i \cdot (-), i \in I$, is a colimit of the diagram $\mathscr{A}(A, D-)$. Hence by Lemma 2.5, the cocone $(c_i)$ is a colimit. ■

2.7. Corollary. *A functor $F\colon \mathscr{A} \to \mathscr{B}$ between lfp categories is finitary iff it preserves the canonical colimits: $FA = \operatorname{colim} FD_A$ for every object $A$ of $\mathscr{A}$.*

Proof. Indeed, in the notation of Lemma 2.5 we are to verify that $Fc_i\colon FD_i \to FC$ $(i \in I)$ is a colimit of $FD$. For this, taking into account that lemma and Remark 2.4, we take any $B \in \mathscr{B}_{\mathsf{fp}}$ and prove that every morphism $b\colon B \to FC$ factorizes essentially uniquely through $Fc_i$ for some $i \in \mathscr{D}$. Since $FC = \operatorname{colim} FD_C$ we have a factorization




$(A \in \mathscr{A}_{\mathsf{fp}})$

By Lemma 2.5 there is some $i \in \mathscr{D}$ and $a_0 \in \mathscr{A}(A, D_i)$ with $a = c_i \cdot a_0$ and hence $b = Fc_i \cdot (Fa_0 \cdot b_0)$. The essential uniqueness is clear. ■

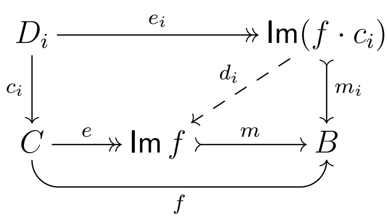
2.8. Notation. Throughout the paper, given a morphism $f\colon X \to Y$ we denote by $\mathsf{Im}\, f$ the *image of* $f$, that is, any choice of the intermediate object defined by taking the (strong epi, mono)-factorization of $f$:

$$f = (X \xrightarrow{e} \mathsf{Im}\, f \xrightarrow{m} Y).$$

We will make use of the next lemma in the proof of Proposition 3.3.

2.9. Lemma. *In an lfp category, images of filtered colimits are directed unions of images.*

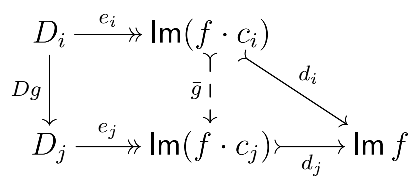
More precisely, suppose we have a filtered diagram $D\colon \mathscr{D} \to \mathscr{A}$ with objects $D_i$ $(i \in I)$ and a colimit cocone $(c_i\colon D_i \to C)_{i\in I}$. Given a morphism $f\colon C \to B$, take the factorizations of $f$ and all $f \cdot c_i$ as follows:




$$(i \in I) \qquad (2.1)$$

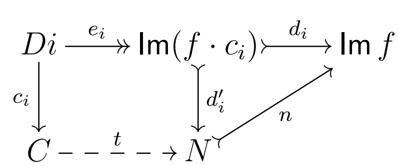
Then the subobject $m$ is the union of the subobjects $m_i$.

Proof. We have the commutative diagram (2.1), where $d_i$ is the diagonal fill-in. Since $m \cdot d_i = m_i$, we see that $d_i$ is monic. Furthermore, for every connecting morphism $Dg\colon D_i \to D_j$ we get a monomorphism $\bar{g}\colon \mathsf{Im}(f \cdot c_i) \rightarrowtail \mathsf{Im}(f \cdot c_j)$ as a diagonal fill-in in the diagram below:




Since $D$ is a filtered diagram, we see that the objects $\mathsf{Im}(f \cdot c_i)$ form a filtered diagram of monomorphisms; in fact, since $d_i$ and $d_j$ are monic there is at most one connecting morphism $\mathsf{Im}(f \cdot c_i) \to \mathsf{Im}(f \cdot c_j)$.

In order to see that $m$ is the union of the subobjects $m_i$, let $d_i'\colon \mathsf{Im}(f \cdot c_i) \rightarrowtail N$ and $n\colon N \rightarrowtail \mathsf{Im}\, f$ be monomorphisms such that $n \cdot d_i' = d_i$ for every $i \in I$.




Since $n$ is monic, the morphisms $d_i' \cdot e_i$ clearly form a cocone of $D$, and this induces a unique morphism $t\colon C \to N$ such that $t \cdot c_i = d_i' \cdot e_i$. Then $n \cdot t \cdot c_i = e \cdot c_i$; hence, $n \cdot t = e$. Since $n$ is monic, it follows that it is an isomorphism, i.e. the subobjects $\mathsf{id}_{\mathsf{Im}\, f}$ and $n$ are isomorphic. This shows that $m$ is the desired union. ∎

## 3. Finitary and Finitely Bounded Functors

In this section we introduce the notion of a finitely bounded functor on a locally presentable category, and investigate when these functors are precisely the finitary ones.

3.1. Definition. *A functor $F\colon \mathscr{A} \to \mathscr{B}$ is called* finitely bounded *provided that, given an object $A$ of $\mathscr{A}$, every finitely generated subobject of $FA$ in $\mathscr{B}$ factorizes through the $F$-image of a finitely generated subobject of $A$ in $\mathscr{A}$.*

In more detail, given a monomorphism $m_0\colon M_0 \rightarrowtail FA$ with $M_0 \in \mathscr{B}_{\mathsf{fg}}$ there exists a monomorphism $m\colon M \rightarrowtail A$ with $M \in \mathscr{A}_{\mathsf{fg}}$ and a factorization as follows:

$$\begin{array}{ccc} & & FM \\ & \nearrow & \downarrow{\scriptstyle Fm} \\ M_0 & \xrightarrow[m_0]{} & FA \end{array}$$

3.2. Example.

(1) If $\mathscr{B}$ is the category of $S$-sorted sets, then $F$ is finitely bounded iff for every object $A$ of $\mathscr{A}$ and every element $x \in FA$ there exists a finitely generated subobject $m\colon X \rightarrowtail A$ such that $x \in Fm[FX]$.

(2) Let $\mathscr{A}$ be a category with (strong epi, mono)-factorizations. An object of $\mathscr{A}$ is finitely generated iff its hom-functor is finitely bounded. Indeed, by applying (1) we see that $\mathscr{A}(A, -)$ is finitely bounded iff for every morphism $f\colon A \to B$ there exists a factorization $f = m \cdot g$, where $m\colon A' \rightarrowtail B$ is monic and $A'$ is finitely generated. This implies that $A$ is finitely generated: for $f = \mathsf{id}_A$ we see that $m$ is invertible. Conversely, if $A$ is finitely generated, then we can take the (strong epi, mono)-factorization of $f$ and use that finitely generated objects are closed under strong quotients [5].

3.3. Proposition. *Let $F$ be a functor between lfp categories preserving monomorphisms. Then $F$ is finitely bounded iff it preserves filtered colimits of monomorphisms.*

Proof. We are given lfp categories $\mathscr{A}$ and $\mathscr{B}$ and a functor $F\colon \mathscr{A} \to \mathscr{B}$ preserving monomorphisms.

(1) Let $F$ preserve filtered colimits of monomorphisms. Then, for every object $A$ we express it as a canonical filtered colimit of all $p\colon P \to A$ in $\mathscr{A}_{\mathsf{fp}}/A$ (see Remark 2.2(2)). By Lemma 2.9 applied to $f = \mathsf{id}_A$ we see that $A$ is the colimit of its subobjects $\mathsf{Im}\, p$ where $p$ ranges over $\mathscr{A}_{\mathsf{fp}}/A$. Hence, $F$ preserves this colimit,

$$FA = \operatorname*{colim}_{p \in \mathscr{A}_{\mathsf{fp}}/A} F(\mathsf{Im}\, p),$$

and it is a colimit of monomorphisms since $F$ preserves monomorphisms. Given a finitely generated subobject $m_0\colon M_0 \rightarrowtail FA$, we thus obtain some $p$ in $\mathscr{A}_{\mathsf{fp}}/A$ such that $m_0$ factorizes through the $F$-image of $\mathsf{Im}(p) \rightarrowtail A$. Hence $F$ is finitely bounded.

(2) Let $F$ be finitely bounded. Let $D\colon \mathscr{D} \to \mathscr{A}$ be a filtered diagram of monomorphisms with a colimit cocone:

$$c_i\colon D_i \rightarrowtail C \qquad (i \in I).$$

In order to prove that $Fc_i\colon FD_i \to FC$, $i \in I$, is a colimit cocone, we show that its image under $\mathscr{B}(B, -)$ is a colimit cocone for every finitely generated object $B$ in $\mathscr{B}$ (cf. Lemma 2.6). In other words, given $f\colon B \to FC$ with $B \in \mathscr{B}_{\mathsf{fg}}$ then

(a) $f$ factorizes through $Fc_i$ for some $i$ in $I$, and

(b) the factorization is unique.

We do not need to take care of (b): since every $c_i$ is monic by Remark 2.2(4), so is every $Fc_i$. In order to prove (a), factorize $f\colon B \to FC$ as a strong epimorphism $q\colon B \twoheadrightarrow M_0$ followed by a monomorphism $m_0\colon M_0 \rightarrowtail FC$. Then $M_0$ is finitely generated by Remark 2.2(5). Thus, there exists a finitely generated subobject $m\colon M \rightarrowtail C$ with $m_0 = Fm \cdot u$ for some $u\colon M_0 \to FM$. Furthermore, since $\mathscr{A}(M, -)$ preserves the colimit of $D$, $m$ factorizes as $m = c_i \cdot \overline{m}$ for some $i \in I$. Thus $F\overline{m} \cdot u \cdot q$ is the desired factorization:

$$f = m_0 \cdot q = Fm \cdot u \cdot q = Fc_i \cdot F\overline{m} \cdot u \cdot q. \qquad \blacksquare$$

In the following theorem we work with an lfp category whose finitely generated objects are finitely presentable. This holds e.g. for the categories of sets, many-sorted sets, posets, graphs, vector spaces, unary algebras on one operation and nominal sets. Further examples are the categories of commutative monoids (this is known as Redei's theorem [17], see Freyd [11] for a rather short proof), positive convex algebras (i.e. the Eilenberg-Moore algebras for the (sub-)distribution monad on sets [19]), semimodules for Noetherian semirings (see e.g. [9] for a proof). The category of finitary endofunctors of sets also has this property as we verify in Corollary 3.33.

On the other hand, the categories of groups, lattices or monoids do not have that property. A particularly simple counter-example is the slice category $\mathbb{N}/\mathsf{Set}$; equivalently, this is the category of algebras with a set of constants indexed by $\mathbb{N}$. Hence, an object $a\colon \mathbb{N} \to A$ is finitely generated iff $A$ has a finite set of generators, i.e. $A \setminus a[\mathbb{N}]$ is a finite set. It is finitely presentable iff, moreover, $A$ is presented by finitely many relations, i.e. the kernel of $a$ is a finite subset of $\mathbb{N} \times \mathbb{N}$.

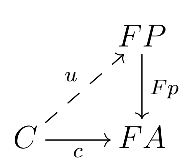
3.4. Theorem. *Let $\mathscr{A}$ be an lfp category in which every finitely generated object is finitely presentable ($\mathscr{A}_{\mathsf{fp}} = \mathscr{A}_{\mathsf{fg}}$). Then for all functors preserving monomorphisms from $\mathscr{A}$ to lfp categories we have the equivalence*

$$\textit{finitary} \iff \textit{finitely bounded.}$$

Proof. Let $F\colon \mathscr{A} \to \mathscr{B}$ be a finitely bounded functor preserving monomorphisms, where $\mathscr{B}$ is lfp. We prove that $F$ is finitary. The converse follows from Proposition 3.3.

According to Corollary 2.7 it suffices to prove that $F$ preserves the colimit of the canonical filtered diagram of every object $A$. The proof that $FD_A$ has the colimit cocone given by $Fp$ for all $p\colon P \to A$ in $\mathscr{A}_{\mathsf{fp}}/A$ uses the fact that this is a filtered diagram in the lfp category $\mathscr{B}$. By Remark 2.4, it is therefore sufficient to prove that for every object $C \in \mathscr{B}_{\mathsf{fp}}$ and every morphism $c\colon C \to FA$ we have the following two properties:

(1) $c$ factorizes through some of the colimit maps




$(P \in \mathscr{A}_{\mathsf{fp}})$,

(2) given another such factorization, $c = Fp \cdot v$, then $u$ and $v$ are merged by some connecting morphism; i.e. we have a commutative triangle

$$\begin{array}{ccc} P & \xrightarrow{\ h\ } & P' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & A & \end{array} \qquad (P, P' \in \mathscr{A}_{\mathsf{fp}})$$

with $Fh \cdot u = Fh \cdot v$.

Indeed, by applying Lemma 2.9 to $f = \mathsf{id}_A$, we see that the monomorphisms $m_p \colon \mathsf{Im}\, p \rightarrowtail A$ for $p \in \mathscr{A}_{\mathsf{fp}}/A$ form a colimit cocone of a diagram of monomorphisms. By Proposition 3.3, $F$ preserves this colimit, therefore any $c \colon C \to FA$ factorizes through some $Fm_p \colon F(\mathsf{Im}\, p) \to FA$. Observe that, since $\mathscr{A}_{\mathsf{fg}} = \mathscr{A}_{\mathsf{fp}}$, we know by Remark 2.2(5) that every $\mathsf{Im}\, p$ is finitely presentable, hence the morphisms $m_p$ are colimit injections and all $e_p \colon P \twoheadrightarrow \mathsf{Im}\, p$ are connecting morphisms of $D_A$. Consequently, (1) is clearly satisfied. Moreover, given $u, v \colon C \to FP$ with $Fp \cdot u = Fp \cdot v$, we have that $Fe_p \cdot u = Fe_p \cdot v$, since $Fm_p$ is monic, thus (2) is satisfied, too. ∎

3.5. Remark. Conversely, if every functor from $\mathscr{A}$ to an lfp category fulfils the equivalence in the above theorem, then $\mathscr{A}_{\mathsf{fp}} = \mathscr{A}_{\mathsf{fg}}$. Indeed, for every finitely generated object $A$, since $F = \mathscr{A}(A, -)$ preserves monomorphisms, we can apply Proposition 3.3 and conclude that $F$ is finitary, i.e. $A \in \mathscr{A}_{\mathsf{fp}}$.

3.6. Example. For $\mathsf{Un}$, the category of algebras with one unary operation, we present a finitely bounded endofunctor that is not finitary. Since in $\mathsf{Un}$ finitely generated algebras are finitely presentable, this shows that the condition of preservation of monomorphisms cannot be removed from Theorem 3.4.

Let $C_p$ denote the algebra on $p$ elements whose operation forms a cycle. Define $F \colon \mathsf{Un} \to \mathsf{Un}$ on objects by

$$FX = \begin{cases} C_1 + X & \text{if } \mathsf{Un}(C_p, X) = \emptyset \text{ for some prime } p, \\ C_1 & \text{else.} \end{cases}$$

Given a homomorphism $f \colon X \to Y$ with $FY = C_1 + Y$, then also $FX = C_1 + X$; indeed, in case $FX = C_1$ we would have $\mathsf{Un}(C_p, X) \neq \emptyset$ for all prime numbers $p$, and then the same would hold for $Y$, a contradiction. Thus we can put $Ff = \mathsf{id}_{C_1} + f$. Otherwise $Ff$ is the unique homomorphism to $C_1$.

(1) We now prove that $F$ is finitely bounded. Suppose we are given a finitely generated subalgebra $m_0 \colon M_0 \rightarrowtail FX$. If $FX = C_1$ then take $M = \emptyset$ and $m \colon \emptyset \rightarrowtail X$ the unique homomorphism. Otherwise we have $FX = C_1 + X$, and we take the preimages of the coproduct injections under $Ff$ to see that $m_0 = u + m$, where $u$ is the unique

homomorphism into the terminal algebra $C_1$ as shown below:

$$\begin{array}{ccc} M' & \xrightarrow{\;u\;} & C_1 \\ \downarrow & & \downarrow \\ M_0 & \xrightarrow{\;m_0\;} & C_1 + X \\ \uparrow & & \uparrow \\ M & \xrightarrow[\;m\;]{} & X \end{array}$$

Then we obtain the desired factorization of $m_0$:

$$\begin{array}{ccc} & & C_1 + M = FM \\ & \nearrow^{u+M} & \downarrow \scriptstyle{\mathsf{id}_{C_1}+m=Fm} \\ M_0 = M' + M & \xrightarrow[u+m]{} & C_1 + X = FX \end{array}$$

(2) However, $F$ is not finitary; indeed, it does not preserve the colimit of the following chain of inclusions

$$C_2 \hookrightarrow C_2 + C_3 \hookrightarrow C_2 + C_3 + C_5 \hookrightarrow \cdots$$

since every object $A$ in this chain is mapped by $F$ to $C_1 + A$ while its colimit $X = \coprod_{i \text{ prime}} C_i$ is mapped to $C_1$.

We now turn to the question for which lfp categories $\mathscr{A}$ the equivalence

$$\text{finitary} \iff \text{finitely bounded}$$

holds for *all* functors with domain $\mathscr{A}$.

In the following we call a morphism $u\colon X \to Y$ *finitary* if it factorizes through a finitely presentable object:

$$\begin{array}{ccc} & & C \;\in \mathscr{A}_{\mathsf{fp}} \\ & \nearrow^{v} & \downarrow w \\ X & \xrightarrow[u]{} & Y \end{array} \tag{3.1}$$

3.7. Example. In the category of graphs consider the following graph on $\mathbb{N}$:

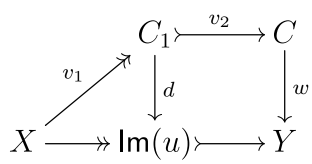
$$\circlearrowright 0 \qquad 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

The constant self-map of value 0 is finitary, but no other endomorphism on this graph is finitary.

3.8. Remark.

(1) If a morphism in an lfp category has a finitely presentable image (see Notation 2.8), then it is of course finitary.

(2) The converse, namely that every finitary morphism has a finitely presentable image, holds whenever $\mathscr{A}_{\mathsf{fp}}$ is closed under subobjects and $\mathscr{A}_{\mathsf{fp}} = \mathscr{A}_{\mathsf{fg}}$. Indeed, given a finitary morphism $u\colon X \to Y$, let $w \cdot v$ be a factorization through a finitely presentable object $C$. Take a (strong epi, mono)-factorization $v = v_2 \cdot v_1$ of $v$:




Then $C_1$ is finitely presentable and the diagonal fill-in $d$ is strongly epic thus, $\mathsf{Im}(u)$ is finitely presentable. This holds e.g. for sets, graphs, posets, vector spaces and semilattices.

3.9. Definition. *An lfp category is called*

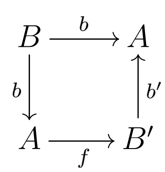
(1) semi-strictly lfp *if every object has a finitary endomorphism;*

(2) strictly lfp *if every object has, for each finitely generated subobject $m$, a finitary endomorphism $u$ fixing that subobject (i.e. $u \cdot m = m$).*

3.10. Remark.

(1) 'strictly' implies 'semi-strictly' due to $0 \in \mathscr{A}_{\mathsf{fp}}$: use the image of the unique $b\colon 0 \to A$.

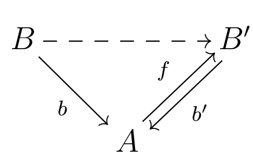
(2) An lfp category is strictly lfp iff for every morphism $b\colon B \to A$ with $B \in \mathscr{A}_{\mathsf{fp}}$ there exist morphisms $b'\colon B' \to A$ and $f\colon A \to B'$ with $B' \in \mathscr{A}_{\mathsf{fp}}$ such that the square below commutes.




Indeed, this condition is necessary: choose, for the image $m$ of $b$, a finitary $u\colon A \to A$ with $m = u \cdot m$, thus $b = u \cdot b$. We have a factorization $u = b' \cdot f$ where $b'\colon B' \to A$ has a finitely presentable domain.

The condition is also sufficient: given a square as above, the morphism $u = b' \cdot f$ is finitary and $b = u \cdot b$.

(3) An lfp category is semi-strictly lfp iff for every morphism $b\colon B \to A$ with $B \in \mathscr{A}_{\mathsf{fp}}$ there exists a factorization of $b$ through a morphism $b'\colon B' \to A$ with $B' \in \mathscr{A}_{\mathsf{fp}}$ such that $\mathscr{A}(B', A) \neq \emptyset$.



Indeed, this condition is necessary: given a finitary morphism $u\colon A \to A$ we have $u = w \cdot v$ as in (3.1). Moreover, $B' = B + C$ is finitely presentable since both $B$ and $C$ are. Put $b' = [b, w]\colon B' \to A$ and

$$f = \big(A \xrightarrow{v} C \xrightarrow{\mathsf{inr}} B + C\big),$$

where $\mathsf{inr}$ is the right-hand coproduct injection. Then $b$ factorizes through $b'$ via the left-hand coproduct injection $\mathsf{inl}\colon B \to B + C$.

The condition is also sufficient: consider $b\colon 0 \to A$ and put $a = b' \cdot f$.

(4) In every strictly lfp category we have $\mathscr{A}_{\mathsf{fg}} = \mathscr{A}_{\mathsf{fp}}$. Indeed, given $A \in \mathscr{A}_{\mathsf{fg}}$ express it as a strong quotient $b\colon B \twoheadrightarrow A$ of some $B \in \mathscr{A}_{\mathsf{fp}}$, see Remark 2.2(5). Then the equality $b = b' \cdot f \cdot b$ in (2) above implies $b' \cdot f = \mathsf{id}$. Thus, $A$ is a split quotient of a finitely presentable object $B'$, hence, $A$ is finitely presentable by Remark 2.2(6).

3.11. Examples.

(1) $\mathsf{Set}$ is strictly lfp: given $b\colon B \to A$ with $B \neq \emptyset$ factorize it as $e\colon B \twoheadrightarrow \mathsf{Im}\, b$ followed by a split monomorphism $b'\colon \mathsf{Im}\, b \to A$. Given a splitting, $f \cdot b' = \mathsf{id}$, we have $b = b' \cdot f \cdot b$. The case $B = \emptyset$ is trivial: for $A \neq \emptyset$, $b'$ may be any map from a singleton set to $A$.

(2) Vector spaces (over a given field) form a strictly lfp category. This can be seen directly quite easily, we show this in Example 3.19(2) as a consequence of Proposition 3.18.

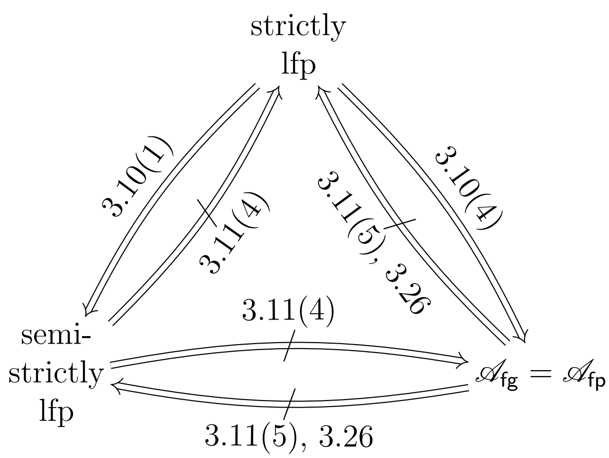
(3) For every finite group $G$ the category $G$-$\mathsf{Set}$ of sets with an action of $G$ is strictly lfp. This category is equivalent to that of presheaves on $G^{\mathrm{op}}$, see Lemma 3.20.

(4) Every lfp category with a zero object $0 \cong 1$ is semi-strictly lfp. This follows from the fact that 0 is finitely presentable and every object $A$ has the finitary endomorphism $\big(A \to 1 \cong 0 \to A\big)$. Examples include the categories of monoids and groups, which are not strictly lfp because in both cases the classes of finitely presentable and finitely generated objects differ.

A bit more generally: let an lfp category $\mathscr{A}$ have a finitely presentable terminal object from which morphisms exist to all objects outside of $\mathscr{A}_{\mathsf{fp}}$. Then it is semi-strictly lfp. For example, the category of posets is semi-strictly lfp.

(5) An example of an lfp category $\mathscr{A}$ which fulfils $\mathscr{A}_{\mathsf{fp}} = \mathscr{A}_{\mathsf{fg}}$ but is not semi-strictly lfp is the category of graphs. The subgraph of the graph of Example 3.7 on $\mathbb{N} \setminus \{0\}$ has no finitary endomorphism. Another such example is the category of nominal sets which is discussed in Example 3.26.

We will see other examples (and non-examples) below. The following figure shows the relationships between the different properties:




3.12. Theorem. *Let $\mathscr{A}$ be a strictly lfp category, and $\mathscr{B}$ an lfp category with $\mathscr{B}_{\mathsf{fg}} = \mathscr{B}_{\mathsf{fp}}$. Then for all functors from $\mathscr{A}$ to $\mathscr{B}$ we have the equivalence*

$$\textit{finitary} \iff \textit{finitely bounded.}$$

Proof. ($\Longrightarrow$) Let $F\colon \mathscr{A} \to \mathscr{B}$ be finitary. By Remark 3.10(4) we know that $\mathscr{A}_{\mathsf{fp}} = \mathscr{A}_{\mathsf{fg}}$. Given a finitely generated subobject $m\colon M \rightarrowtail FA$, write $A$ as the directed colimit of all of its finitely generated subobjects $m_i\colon A_i \rightarrowtail A$. Since $F$ is finitary, it preserves this colimit, and since $M$ is finitely generated, whence finitely presentable, we obtain some $i$ and some $f\colon M \to FA_i$ such that $Fm_i \cdot f = m$ as desired.

($\Longleftarrow$) Suppose that $F\colon \mathscr{A} \to \mathscr{B}$ is finitely bounded. We verify the two properties (1) and (2) in the proof of Theorem 3.4. In order to verify (1), let $c\colon C \to FA$ be a morphism with $C$ finitely presentable. Then we have the finitely generated subobject $\mathsf{Im}\, c \rightarrowtail FA$, and this factorizes through $Fm\colon FM \to FA$ for some finitely generated subobject $m\colon M \rightarrowtail A$ since $F$ is finitely bounded. Then $c$ factorizes through $Fm$, too, and we are done since $M$ is finitely presentable by Remark 3.10(4).

To verify (2), suppose that we have $u, v\colon C \to FB$ and $b\colon B \to A$ in $\mathscr{A}_{\mathsf{fp}}/A$ such that $Fb \cdot u = Fb \cdot v$. Now choose $f\colon A \to B'$ with $b = b' \cdot (b \cdot f)$ (see Remark 3.10(2)). Put $h = f \cdot b$ to get $b = b' \cdot h$ as required. Since $Fb \cdot u = Fb \cdot v$, we conclude $Fh \cdot u = Ff \cdot Fb \cdot u = Ff \cdot Fb \cdot u = Fh \cdot v$. ■

3.13. Corollary. *A functor between strictly lfp categories is finitary iff it is finitely bounded.*

3.14. Remark. Consequently, a set functor $F$ is finitary if and only if it is finitely bounded. The latter means precisely that every element of $FX$ is contained in $Fm[FM]$ for some finite subset $m\colon M \hookrightarrow X$.

This result was formulated already in [4], but the proof there is unfortunately incorrect.

3.15. Open Problem. Is the above implication an equivalence? That is, given an lfp category $\mathscr{A}$ such that every finitely bounded functor into lfp categories is finitary, does this imply that $\mathscr{A}$ is strictly lfp?

3.16. Theorem. *Let $\mathscr{A}$ be an lfp category such that for functors $F\colon \mathscr{A} \to \mathsf{Set}$ we have the equivalence*

$$\textit{finitary} \iff \textit{finitely bounded.}$$

*Then $\mathscr{A}$ is semi-strictly lfp and $\mathscr{A}_{\mathsf{fg}} = \mathscr{A}_{\mathsf{fp}}$.*

Proof. The second statement easily follows from Example 3.2(2). Suppose that $\mathscr{A}$ is an lfp category such that the above equivalence holds for all functors from $\mathscr{A}$ to $\mathsf{Set}$. Then the same equivalence holds for all functors $F\colon \mathscr{A} \to \mathsf{Set}^S$, for $S$ a set of sorts. To see this, denote by $C\colon \mathsf{Set}^S \to \mathsf{Set}$ the functor forming the coproduct of all sorts. It is easy to see that $C$ creates filtered colimits. Thus, a functor $F\colon \mathscr{A} \to \mathsf{Set}^S$ is finitary iff $C \cdot F\colon \mathscr{A} \to \mathsf{Set}$ is. Moreover, $F$ is finitely bounded iff $C \cdot F$ is; indeed, this follows immediately from Example 3.2(1).

We proceed to prove the semi-strictness of $\mathscr{A}$. Put $S = \mathscr{A}_{\mathsf{fp}}$. Given a morphism

$$b\colon B \to A \qquad \text{with } B \in \mathscr{A}_{\mathsf{fp}}$$

we present $b'$ and $f$ as required in Remark 3.10(2). Define a functor $F\colon \mathscr{A} \to Set^S$ on objects $Z$ of $\mathscr{A}$ by

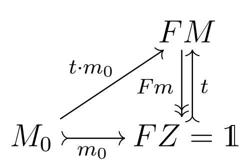
$$FZ = \begin{cases} \mathbb{1} + (\mathscr{A}(s,Z))_{s\in S} & \text{if } \mathscr{A}(A,Z) = \emptyset \\ \mathbb{1} & \text{else,} \end{cases}$$

where $\mathbb{1}$ denotes the terminal $S$-sorted set. Given a morphism $f\colon Z \to Z'$ we need to specify $Ff$ in the case where $\mathscr{A}(A,Z') = \emptyset$: this implies $\mathscr{A}(A,Z) = \emptyset$ and we put

$$Ff = \mathsf{id}_{\mathbb{1}} + (\mathscr{A}(s,f))_{s\in S}.$$

Here $\mathscr{A}(s,f)\colon \mathscr{A}(s,Z) \to \mathscr{A}(s,Z')$ is given by $u \mapsto f \cdot u$, as usual. It is easy to verify that $F$ is a well-defined functor.

(1) Let us prove that $F$ is finitely bounded. The category $\mathsf{Set}^S$ is lfp with finitely generated objects $(X)_{s\in S}$ precisely those for which the set $\coprod_{s\in S} X_s$ is finite. Let $m_0\colon M_0 \rightarrowtail FZ$ be a finitely generated subobject. We present a finitely generated subobject $m\colon M \rightarrowtail Z$ such that $m_0$ factorizes through $Fm$. This is trivial in the case where $\mathscr{A}(A,Z) \neq \emptyset$: choose any finitely generated subobject $m\colon M \rightarrowtail Z$ (e.g. the image of the unique morphism from the initial object to $Z$: cf. Remark 2.2(5)). Then $Fm$ is either $\mathsf{id}_{\mathbb{1}}$ or a split epimorphism, since $FZ = \mathbb{1}$ and in $FM$ each sort is non-empty. Thus, we have $t$ with $Fm \cdot t = \mathsf{id}$ and $m_0$ factorizes through $Fm$:



In the case where $\mathscr{A}(A,Z)=\emptyset$ we have $m_0 = m_1 + m_2$ for subobjects

$$m_1\colon M_1 \rightarrowtail \mathbb{1} \qquad \text{and} \qquad m_2\colon M_2 \rightarrowtail (\mathscr{A}(s,Z))_{s\in S}.$$

For notational convenience, assume $(M_2)_s \subseteq \mathscr{A}(s,Z)$ and $(m_2)_s$ is the inclusion map for every $s\in S$. Since $M_0$ is finitely generated, $M_2$ contains only finitely many elements $u_i\colon s_i \to Z$, $i=1,\dots,n$. Factorize $[u_1,\dots,u_n]$ as a strong epimorphism $e$ followed by a monomorphism $m$ in $\mathscr{A}$ (see Remark 2.2(1)):

$$\textstyle\coprod_{i=1}^n s_i \xrightarrow{\;e\;}\!\!\!\!\rightarrow M \overset{m}{\rightarrowtail} Z\,.$$

Then $\mathscr{A}(A,M)=\emptyset$, therefore $Fm = \mathsf{id}_{\mathbb{1}} + (\mathscr{A}(s,m))_{s\in S}$. Since every element $u_i\colon s_i\to Z$ of $M_2$ factorizes through $m$ in $\mathscr{A}$, we have

$$u_i = m\cdot u_i' \quad \text{for } u_i'\colon s_i \to M \text{ with } [u_1',\dots,u_n'] = e.$$

Let $v: M_2 \to \mathscr{A}(s,M)$ be the $S$-sorted map taking each $u_i$ to $u_i'$. Then the inclusion map $m_2\colon M_2 \to (\mathscr{A}(s,Z))_{s\in S}$ has the following form

$$m_2 = \Big(M_2 \xrightarrow{v} (\mathscr{A}(s,M))_{s\in S} \xrightarrow{(\mathscr{A}(s,m))_{s\in S}} (\mathscr{A}(s,Z))_{s\in S}\Big)\,.$$

The desired factorization of $m_0 = m_1 + m_2$ through $Fm = \mathsf{id}_{\mathbb{1}} + (\mathscr{A}(s,m))_{s\in S}$ is as follows:

$$\begin{array}{ccc} & & \mathbb{1} + (\mathscr{A}(s,M))_{s\in S} \\ & \overset{m_1+v}{\nearrow} & \big\downarrow {\scriptstyle \mathsf{id}+(\mathscr{A}(s,m))_{s\in S}} \\ M_0 = M_1 + M_2 & \underset{m_0=m_1+m_2}{\rightarrowtail} & \mathbb{1} + (\mathscr{A}(s,Z))_{s\in S} \end{array}$$

(2) We thus know that $F$ is finitary, and we will use this to prove that $\mathscr{A}$ is semi-strictly lfp. That is, as in Remark 3.10(3) we find $b'\colon B'\to A$ in $\mathscr{A}_{\mathsf{fp}}/A$ through which $b$ factorizes and which fulfils $\mathscr{A}(A,B')\neq\emptyset$. Recall from Remark 2.2(2) that $A = \operatorname{colim} D_A$. Our morphism $b$ is an object of the diagram scheme $\mathscr{A}_{\mathsf{fp}}/A$ of $D_A$. Let $D_A'$ be the full subdiagram of $D_A$ on all objects $b'$ such that $b$ factorizes through $b'$ in $\mathscr{A}$ (that is, such that a connecting morphism $b\to b'$ exists in $\mathscr{A}_{\mathsf{fp}}/A$). Then $D_A'$ is also a filtered diagram and has the same colimit, i.e. $A = \operatorname{colim} D_A'$. Since $F$ preserves this colimit and $FA = \mathbb{1}$, we get

$$\mathbb{1} \cong \operatorname{colim} FD_A'.$$

Assuming that $\mathscr{A}(A,B')=\emptyset$ for all $b'\colon B'\to A$ in $D_A'$, we obtain a contradiction: the objects of $FD_A'$ are $\mathbb{1} + (\mathscr{A}(s,B'))_{s\in S}$, and since for every $s\in S$ the functor $\mathscr{A}(s,-)$ is finitary, the colimit of all $\mathscr{A}(s,B')$ is $\mathscr{A}(s,A)$. We thus obtain an isomorphism

$$\mathbb{1} \cong \mathbb{1} + (\mathscr{A}(s,A))_{s\in S}.$$

This means $\mathscr{A}(s,A)=\emptyset$ for all $s\in S$, in particular $\mathscr{A}(B,A)=\emptyset$, in contradiction to the existence of the given morphism $b\colon B\to A$.

Therefore, there exists $b'\colon B'\to A$ in $D_A'$, i.e. $b'$ through which $b$ factorizes with $\mathscr{A}(A,B')\neq\emptyset$, as required. ∎

We now present examples of strictly lfp categories. All of them happen to be either atomic toposes or semi-simple (aka atomic) abelian categories. Recall that an object $A$ is called *simple*, or an *atom*, if it has no nontrivial subobject. That is, every subobject of $A$ is either invertible or has the initial object as a domain.

3.17. Definition. *A category is called* semi-simple *or* atomic *if every object is a coproduct of simple objects.*

3.18. Proposition. *Let a semi-simple, cocomplete category have only finitely many simple objects (up to isomorphism), all of them finitely presentable. Then it is strictly lfp.*

Proof.

(1) The given category $\mathscr{A}$ is lfp. Indeed, it is cocomplete and every finite coproduct of simple objects is finitely presentable. Moreover, every object $\coprod_{i\in I} A_i$, $A_i$ simple, is a filtered colimit of finite subcoproducts. Conversely, every finitely presentable object is, obviously, a split quotient of a finite coproduct of simple objects. Thus, for the countable set $M$ representing all these finite coproducts we see that $\mathscr{A}_{\mathsf{fp}}$ consists of split quotients of objects in $M$. Therefore $\mathscr{A}_{\mathsf{fp}}$ is essentially a set: split quotients of any object $X$ correspond bijectively to idempotent endomorphisms of $X$, and thus form a set. Hence, $\mathscr{A}$ is lfp.

(2) Let $b\colon B \to A = \coprod_{i\in I} A_i$ be a morphism with all $A_i$ simple and $B$ finitely presentable. Then $b$ factorizes through a finite subcoproduct $a_J\colon \coprod_{i\in J} A_i \to A$ ($J \subseteq I$ finite), say, $b = a_J \cdot b'$. Since $\mathscr{A}$ has essentially only a finite set of simple objects, $J$ can be chosen so that each $A_i$ is isomorphic to some $A_j$, $j \in J$. Consequently, there exists a morphism $g\colon \coprod_{i\in I\setminus J} A_i \to \coprod_{j\in J} A_j$. The following composite $u\colon A \to A$

$$A = \Big(\coprod_{j\in J} A_j + \coprod_{i\in I\setminus J} A_i\Big) \xrightarrow{[\mathsf{id},g]} \coprod_{j\in J} A_j \xrightarrow{a_J} A$$

is finitary and fulfils, since $[\mathsf{id}, g] \cdot a_J = \mathsf{id}$, the desired equation

$$u \cdot b = a_J \cdot [\mathsf{id}, g] \cdot a_J \cdot b' = a_J \cdot b' = b.$$

∎

3.19. Examples.

(1) $\mathsf{Set}^S$ is strictly lfp iff $S$ is finite. Indeed, the sufficiency is a clear consequence of Proposition 3.18. Conversely, if $S$ is infinite then the identity on the terminal object, which is its unique endomorphism, is not finitary, whence $\mathsf{Set}^S$ is not semi-strictly lfp.

(2) For every field $K$ the category $K$-$\mathsf{Vec}$ of vector spaces is strictly lfp. Indeed, the simple spaces are those of dimension 0 or 1, and every space is a coproduct of copies of $K$.

(3) We recall that a ring $R$ is called *semi-simple* if the category $R$-$\mathsf{Mod}$ of left modules is semi-simple. For example, the matrix ring $K^{(n)}$ for every field $K$ and every finite $n$ is semi-simple.

The category $R$-$\mathsf{Mod}$ is strictly lfp for every finite semi-simple ring $R$. Indeed, every simple module $A$ is a quotient of the module $R$: in case $A \neq \mathbf{0}$, choose $a \in A \setminus \{0\}$. Since $Ra$ is a submodule of $A$, we conclude

$$A = Ra \cong R/{\sim}$$

where $\sim$ is the congruence defined by $x \sim y$ iff $Rx = Ry$.

Each quotient module $R/{\sim}$ is finitely presentable. Indeed, let $a_i \colon A_i \to A$, $i \in I$, be a filtered colimit and $f \colon R/{\sim} \to A$ a homomorphism. Since $R/{\sim}$ is finite, $f$ factorizes in $\mathsf{Set}$ through $a_j$ for some $j \in J$: $f = a_j \cdot f'$. It remains to choose $j$ so that $f' \colon R/{\sim} \to A_j$ is a homomorphism. Given $r, s \in R$ we know that $rf([s]) = f([rs])$, thus $a_j$ merges $rf'([s])$ and $f'([rs])$. Our colimit is filtered, hence for the given pair we can assume, without loss of generality, that $rf'([s]) = f'([rs])$. Moreover, since $R \times R$ is finite, this assumption can be made for all pairs $(r, s)$ at once. That is, by a suitable choice of $j$ we achieve that $f'$ preserves scalar multiplication. A completely analogous argument shows that $j$ can be chosen so that, moreover, $f'$ preserves addition. Thus, it is a homomorphism.

(4) A Grothendieck topos is called *atomic*, see [8], if it is semi-simple. For example, the presheaf topos $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is atomic iff $\mathcal{C}$ is a groupoid, i.e. its morphisms are all invertible, see Sect. 7(2) in op. cit. It follows from the Proposition 3.18 that every atomic Grothendieck topos with a finite set of finitely presentable atoms (up to isomorphism) is strictly lfp.

More atomic toposes can be found in [12, Example 3.5.9]. Not all atomic Grothendieck toposes are semi-strictly lfp. See Example 3.26 below: in the category of nominal sets (aka the Schanuel topos), the set of atoms is infinite. Next we provide a class of examples of strict lfp toposes, see also Example 4.10 below.

3.20. Lemma. *The category of presheaves on a finite groupoid is strictly lfp.*

Proof. In view of Example 3.19(4) all we need proving is that for every finite groupoid $\mathbb{G}$ the category $\mathsf{Set}^{\mathbb{G}^{\mathrm{op}}}$ has, up to isomorphism, a finite set of finitely presentable atoms.

(1) Put $S = \mathsf{obj}\,\mathbb{G}$. Then the category $\mathsf{Set}^{\mathbb{G}^{\mathrm{op}}}$ can be considered as a variety of $S$-sorted unary algebras. The signature is given by the set of all morphisms of $\mathbb{G}^{\mathrm{op}}$: every morphism $f \colon X \to Y$ of $\mathbb{G}^{\mathrm{op}}$ corresponds to an operation symbol of arity $X \to Y$ (i.e. variables are of sort $X$ and results of sort $Y$). This variety is presented by the equations corresponding to the composition in $\mathbb{G}^{\mathrm{op}}$: represent $g \cdot f = h \colon X \to Y$ in $\mathbb{G}^{\mathrm{op}}$ by $g(f(x)) = h(x)$ for a variable $x$ of sort $X$. Moreover, for every object $X$, add the equation $\mathsf{id}_X(x) = x$ with $x$ of sort $X$.

For every algebra $A$ and every element $x \in A$ of sort $X$ the subalgebra which $x$ generates is denoted by $A^x$. Denote by $\sim_A$ the equivalence on the set of all elements of $A$ defined by $x \sim_A y$ iff $A^x = A^y$. If $I(A)$ is a choice class of this equivalence, then we obtain a representation of $A$ as the following coproduct:

$$A = \coprod_{x \in I(A)} A^x.$$

This follows from $\mathbb{G}$ being a groupoid: whenever $A^x \cap A^y \neq \emptyset$, then $x \sim_A y$.

Moreover, for every homomorphism $h\colon A \to B$ there exists a function $h_0\colon I(A) \to I(B)$ such that on each $A^x$, $x \in I(A)$, $h$ restricts to a homomorphism $h_0\colon A^x \to B^{h(x)}$. Indeed, define $h_0(x)$ as the representative of $\sim_B$ with $B^{h(x)} = B^{h_0(x)}$.

(2) Given $x \in A$ of sort $X$, the algebra $A^x$ is a quotient of the representable algebra $\mathbb{G}(-, X)$. Indeed, the Yoneda transformation corresponding to $x$, an element of $A^x_X$ of sort $X$, has surjective components (by the definition of $A^x$).

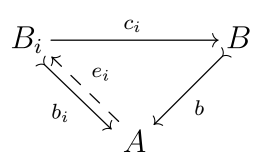
Observe that every representable algebra has only finitely many quotients. This follows from the fact that $\mathbb{G}(-, X)$ has finitely many elements, hence, finitely many equivalence relations exist on the set of all elements.

(3) We conclude that the finite set $\mathscr{B}$ of all algebras representing quotients of representable algebras $\mathbb{G}(-, X)$ consists of finitely presentable algebras. Moreover, every algebra is a coproduct of algebras from $\mathscr{B}$. ■

3.21. Remark. Recall from [5, Proposition 2.30] that *pure subobjects* $b\colon B \rightarrowtail A$ in an lfp category $\mathscr{A}$ are precisely the filtered colimits of split subobjects of $A$ in the slice category $\mathscr{A}/A$.

3.22. Proposition. *Let $\mathscr{A}$ be an lfp category in which all subobjects are pure. If $\mathscr{A}_{\mathsf{fp}} = \mathscr{A}_{\mathsf{fg}}$, then $\mathscr{A}$ is strictly lfp.*

Proof. Let $b\colon B \rightarrowtail A$ be a finitely generated subobject. Express it as a filtered colimit of split subobjects $b_i\colon B_i \rightarrowtail A$ (with $e_i \cdot b_i = \mathsf{id}_{B_i}$ for $e_i\colon A \to B_i$), $i \in I$, with the following colimit cocone in $\mathscr{A}/A$:




Then in $\mathscr{A}$ we have expressed $B$ as a filtered colimit of the objects $B_i$ with the cocone $(c_i)_{i\in I}$. It follows from our assumptions that $B$ is also finitely presentable, and therefore $\mathscr{A}(B, -)$ preserves that colimit. Hence, some $c_i$ is invertible (being both monic, due to $b_i = b \cdot c_i$, and split epic). Consequently, $B_i$ is finitely presentable. The finitary endomorphism $f = b_i \cdot e_i$ fixes the subobject $b$, as desired:

$$f \cdot b = (b_i \cdot e_i) \cdot (b_i \cdot c_i^{-1}) = b_i \cdot c_i^{-1} = b.$$ ■

3.23. Example. The following categories are strictly lfp because they satisfy all the assumptions of the above proposition. By a variety we mean an equational class of finitary (one-sorted) algebras.

(1) A variety $\mathscr{A}$ of algebras with $\mathscr{A}_{\mathsf{fp}} = \mathscr{A}_{\mathsf{fg}}$ in which every finitely generated subobject of a finitely generated object splits. By [10, Theorem 2.1] all monomorphisms are pure.

An example of such a variety are boolean algebras. Here $\mathscr{A}_{\mathsf{fg}} = \mathscr{A}_{\mathsf{fp}}$ are precisely the finite algebras. Since every epimorphism in $\mathsf{Set}_{\mathsf{fp}}$ splits, by Stone's Duality every monomorphism between finite boolean algebras splits.

(2) $R$-$\mathsf{Mod}$ for all regular, left-Noetherian rings $R$. Recall that $R$ is left-Noetherian if every left ideal $I \subseteq R$ is finitely generated; this implies that finitely generated left modules are finitely presentable [18, Example 3.8.28]. Recall further that regularity (in von Neumann's sense) means that for every $a \in R$ there exists $\bar{a} \in R$ with $a = a \cdot \bar{a} \cdot a$. For left-Noetherian rings, this condition is equivalent to $R$-$\mathsf{Mod}$ having all monomorphisms pure, see [18, Proposition 2.11.20].

Regular rings are a wider class than semi-simple rings, so in the realm of left-Noetherian rings we have a simplification of the argument of Example 3.19(3).

(3) A special case of (1), which is the 'non-abelian generalization' of (2), are varieties $\mathscr{A}$ with $\mathscr{A}_{\mathsf{fp}} = \mathscr{A}_{\mathsf{fg}}$ such that for every morphism $a \colon X \to Y$ of $\mathscr{A}_{\mathsf{fp}}$ there exists $\bar{a} \colon Y \to X$ with $a = a \cdot \bar{a} \cdot a$, See [10, Proposition 3.4].

(4) $G$-modules over a field $K$, i.e. the functor category

$$(K\text{-}\mathsf{Vec})^G,$$

for a finite group $G$ and a field of characteristic 0. (More generally: every field whose characteristic does not divide $|G|$.)

By the classical Maschke's Theorem [14, Theorem XIII.1.1] for every subobject $b \colon B \rightarrowtail A$ there exists a coproduct $A = B + C$ with $b$ as the left injection. Thus $b$ splits: consider $[\mathsf{id}_B, 0] \colon A \to B$. Hence all monomorphisms are pure.

The forgetful functor to $K$-$\mathsf{Vec}$ preserves colimits (computed object-wise). The free $G$-module $\phi n$ on $n$ generators thus has finite dimension (of the underling vector space). Indeed, $\phi 1$ has dimension $|G|$ because its underlying space is spanned by $G$, see XIII, Section 1 of [14]. Hence $\phi n = \phi 1 + \cdots + \phi 1$ has dimension $n \cdot |G|$.

It follows that every finitely generated $G$-module is finitely presentable. Indeed, it is a quotient of $\phi n$ for some $n$, thus, it is finite-dimensional. And every finite-dimensional $G$-module $A$ is finitely presentable in $(K\text{-}\mathsf{Vec})^G$. This follows easily from $A$ being finitely presentable in $K$-$\mathsf{Vec}$, since the group action $G \times A \to A$ is determined by its domain restriction to the finite set $G \times X$, where $X$ is a base of $A$.

3.24. Examples. Here we present lfp categories $\mathscr{A}$ which are not semi-strictly lfp. For that it would be sufficient to exhibit an object $A$ such that no endomorphism is finitary. However, we also provide something stronger: In each case we present a non-finitary *endofunctor* that is finitely bounded.

(1) The category $\mathsf{Un}$. In Example 3.6 we have already shown the promised endofunctor. Thus $\mathsf{Un}$ is not semi-strictly lfp. For the algebra $A = \coprod_p C_p$, where $p$ ranges over all prime numbers, there exists no finitary endomorphism.

(2) The category $\mathbb{Z}$-$\mathsf{Set}$ (of actions of the integers on sets). Since this category is equivalent to that of unary algebras with one invertible operation, the argument is as in (1).

(3) The category $\mathsf{Gra}$ of graphs and their homomorphisms is not semi-strictly lfp (see Example 3.11(5)).

Analogously to Example 3.6 define an endofunctor $F$ on $\mathsf{Gra}$ by

$$FX = \begin{cases} \mathbb{1} + X & \text{if } X \text{ contains no cycle and no infinite path} \\ \mathbb{1} & \text{else,} \end{cases}$$

where $\mathbb{1}$ is the terminal object, and $Ff = \mathsf{id}_{\mathbb{1}} + f$ if the codomain $X$ of $f$ fulfils $FX = \mathbb{1} + X$. This functor is clearly finitely bounded, but for the graph $A$ consisting of a single infinite path, it does not preserve the colimit $A = \operatorname{colim} D_A$ of Remark 2.2(2).

(4) $\mathsf{Set}^{\mathbb{N}}$. If $\mathbb{1}$ is the terminal object, then $\mathsf{Set}^{\mathbb{N}}(\mathbb{1}, B') = \emptyset$ for all finitely presentable objects $B$. We define $F$ on $\mathsf{Set}^{\mathbb{N}}$ by $FX = \mathbb{1} + X$ if $X$ has only finitely many non-empty components, and $FX = \mathbb{1}$ else.

3.25. Open Problem. Is the category $\mathsf{Pos}$ of posets strictly lfp? Is every finitely bounded endofunctor on $\mathsf{Pos}$ finitary?

We next present two examples of rather important categories for which we prove that they are not semi-strictly lfp either.

3.26. Example. Nominal sets are not semi-strictly lfp. Let us first recall the definition of the category $\mathsf{Nom}$ of nominal sets (see e.g. [16]). We fix a countably infinite set $\mathbb{A}$ of *atomic names*. Let $\mathfrak{S}_{\mathsf{f}}(\mathbb{A})$ denote the group of all finite permutations on $\mathbb{A}$ (generated by all transpositions). Consider a set $X$ with an action of this group, denoted by $\pi \cdot x$ for a finite permutation $\pi$ and $x \in X$. A subset $A \subseteq \mathbb{A}$ is called a *support* of an element $x \in X$ provided that every permutation $\pi \in \mathfrak{S}_{\mathsf{f}}(\mathbb{A})$ that fixes all elements of $A$ also fixes $x$:

$$\pi(a) = a \text{ for all } a \in A \implies \pi \cdot x = x.$$

A *nominal set* is a set with an action of the group $\mathfrak{S}_{\mathsf{f}}(\mathbb{A})$ where every element has a finite support. The category $\mathsf{Nom}$ is formed by nominal sets and *equivariant maps*, i.e. maps preserving the given group action. $\mathsf{Nom}$ is a Grothendieck topos, it is an lfp category (see e.g. Pitts [16, Remark 5.17]), and, as shown by Petrişan [15, Proposition 2.3.7], the finitely presentable nominal sets are precisely those with finitely many orbits (where an orbit of $x$ is the set of all $\pi \cdot x$).

It is a standard result that every element $x$ of a nominal set has the least support, denoted by $\mathsf{supp}(x)$. In fact, $\mathsf{supp}\colon X \to \mathcal{P}_{\mathsf{f}}(\mathbb{A})$ is itself an equivariant map, where $\mathcal{P}_{\mathsf{f}}(\mathbb{A})$ is the set of all finite subsets of $\mathbb{A}$ with the action given by $\pi \cdot Y = \{\pi(v) \mid v \in Y\}$. Consequently, any two elements of the same orbit $x_1$ and $x_2 = \pi \cdot x_1$ have a support of the same size. In addition, if $f\colon X \to Y$ is an equivariant map, it is clear that

$$\mathsf{supp}(f(x)) \subseteq \mathsf{supp}(x), \quad \text{for every } x \in X. \tag{3.2}$$

Now we present a non-finitary endofunctor on $\mathsf{Nom}$ which is finitely bounded. Consider for every natural number $n$ the nominal set $P_n = \{Y \subseteq \mathbb{A} \mid |Y| = n\}$ with the nominal structure given element-wise, as for $\mathcal{P}_{\mathsf{f}}(\mathbb{A})$ above. Clearly, $\mathsf{supp}(Y) = Y$ for every $Y \in P_n$.

For $A = \coprod_{0<n<\omega} P_n$ the existence of a finitary endomorphism leads to a contradiction. In fact, let the corresponding pair of morphisms $A \underset{g}{\overset{f}{\rightleftarrows}} X$ with $X$ orbit-finite be given. It is clear that, for every $x \in X$, $\mathsf{supp}(x) \neq \emptyset$, otherwise, by (3.2), we would have $\mathsf{supp}(g(x)) = \emptyset$, which contradicts the fact that $\mathsf{supp}(Y) = Y \neq \emptyset$ for all $Y \in A$. We show below that for every $Y \in A$, $\mathsf{supp}(f(Y)) = \mathsf{supp}(Y) = Y$, thus $X$ admits infinitely many cardinalities for $\mathsf{supp}(x)$ with $x \in X$, contradicting the orbit-finiteness of $X$.

By (3.2), it remains to prove that $\mathsf{supp}(Y) \subseteq \mathsf{supp}(f(Y))$. To see this, fix an element $v$ of $\mathsf{supp}(f(Y))$, which is already known to be nonempty. Now for any given element $w$ of $\mathsf{supp}(Y) = Y$, the equivariance of $f$ applied to the transposition $\pi$ of $v$ and $w$ implies that

$$w \in \pi \cdot \mathsf{supp}(f(Y)) = \mathsf{supp}(\pi \cdot f(Y)) = \mathsf{supp}(f(\pi \cdot Y)) = \mathsf{supp}(f(Y)).$$

This proves that $\mathsf{Nom}$ is not semi-strictly lfp.

Analogously to Example 3.6 we define an endofunctor $F$ on $\mathsf{Nom}$ by

$$FX = \begin{cases} \mathbb{1} + X & \text{if } \mathsf{Nom}(P_n, X) = \emptyset \text{ for some } n < \omega \\ \mathbb{1} & \text{else.} \end{cases}$$

For an equivariant map $f\colon X \to Y$, if $FY = \mathbb{1} + Y$, then also $FX = \mathbb{1} + X$: given $\mathsf{Nom}(P_n, Y) = \emptyset$ for some $n$, then also $\mathsf{Nom}(P_n, X) = \emptyset$ holds. In that case put $Ff = \mathsf{id}_{\mathbb{1}} + f$ and else $Ff$ is the unique equivariant map to $FY = \mathbb{1}$. A very similar argument as in Example 3.6 shows that $F$ is finitely bounded. However, $F$ is not finitary, as it does not preserve the colimit $\coprod_{n<\omega} P_n$ of the chain $P_1 \hookrightarrow P_1 + P_2 \hookrightarrow P_1 + P_2 + P_3 \hookrightarrow \cdots$.

We prove next that in the category $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ of finitary set functors (known to be lfp [5, Theorem 1.46]) finitely generated objects coincide with the finitely presentable ones, yet this category fails to be semi-strictly lfp.

3.27. Remark. Recall that a quotient of an object $F$ of $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ is represented by a natural transformation $\varepsilon\colon F \to G$ with epic components. Equivalently, $G$ is isomorphic to $F$ modulo a *congruence* $\sim$. This is a collection of equivalence relations $\sim_X$ on $FX$ ($X \in \mathsf{Set}$) such that for every function $f\colon X \to Y$ given $p_1 \sim_X p_2$ in $FX$, it follows that $Ff(p_1) \sim_Y Ff(p_2)$.

We are going to characterize finitely presentable objects of $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ as the super-finitary functors introduced in [7]:

3.28. Definition. *A set functor $F$ is called* super-finitary *if there exists a natural number $n$ such that $Fn$ is finite and for every set $X$, the maps $Ff$ for $f\colon n \to X$ are jointly surjective, i.e. they fulfil $FX = \bigcup_{f\colon n \to X} Ff[Fn]$.*

3.29. Examples.

(1) The functors $A \times \mathsf{Id}^n$ are super-finitary for all finite sets $A$ and all $n \in \mathbb{N}$.

(2) More generally, let $\Sigma$ be a finitary signature, i.e. a set of operation symbols $\sigma$ of finite arities $|\sigma|$. The corresponding *polynomial set functor*

$$H_\Sigma X = \coprod_{\sigma \in \Sigma} X^{|\sigma|}$$

is super-finitary iff the signature has only finitely many symbols. We call such signatures *super-finitary.*

(3) Every subfunctor $F$ of $\mathsf{Set}(n, -)$, $n \in \mathbb{N}$, is super-finitary. Indeed, assuming $FX \subseteq \mathsf{Set}(n, X)$ for all $X$, we are to find, for each $p \colon n \to X$ in $FX$, a member $q \colon n \to n$ of $Fn$ with $p = Ff(q)$ for some $f \colon n \to X$. That is, with $p = f \cdot q$. Choose a function $g \colon X \to n$ monic on $p[n]$. Then there exists $f \colon n \to X$ with $p = f \cdot g \cdot p$. From $p \in FX$ we deduce $Fg(p) \in Fn$, that is, $g \cdot p \in Fn$. Thus $q = g \cdot p$ is the desired element: we have $p = f \cdot q = Ff(q)$.

(4) Every quotient $\varepsilon \colon F \twoheadrightarrow G$ of a super-finitary functor $F$ is super-finitary. Indeed, given $p \in GX$, find $p' \in FX$ with $p = \varepsilon_X(p')$. There exists $q' \in Fn$ with $p' = Ff(q')$ for some $f \colon n \to X$. We conclude that $q = \varepsilon_n(q')$ fulfils $p = Gf(q)$ from the naturality of $\varepsilon$.

3.30. Lemma. *The following conditions are equivalent for every set functor $F$:*

(1) *$F$ is super-finitary*

(2) *$F$ is a quotient of the polynomial functor $H_\Sigma$ for a super-finitary signature $\Sigma$, and*

(3) *$F$ is a quotient of a functor $A \times \mathsf{Id}^n$ for $A$ finite and $n \in \mathbb{N}$.*

Proof. (3) $\Longrightarrow$ (2) is clear and for (2) $\Longrightarrow$ (1) see the Examples (2) and (4) above. To prove (1) $\Longrightarrow$ (3), let $F$ be super-finitary and put $A = Fn$ in the above definition. Apply Yoneda Lemma to $\mathsf{Id}^n \cong \mathsf{Set}(n, -)$ and use that $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ is cartesian closed:

$$\frac{Fn \xrightarrow{\cong} [\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}(\mathsf{Set}(n, -), F)}{\varepsilon \colon Fn \times \mathsf{Set}(n, -) \longrightarrow F}$$

The definition of super-finitary shows that $\varepsilon_X$ is surjective for every $X$. ∎

3.31. Proposition. *Super-finitary functors are closed in $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ under finite products, finite coproducts, subfunctors, and hence under finite limits.*

Proof.

(1) Finite products and coproducts are clear: given quotients $\varepsilon_i \colon A_i \times \mathsf{Id}^{n_i} \twoheadrightarrow F_i$, $i \in \{1, 2\}$, then $F_1 \times F_2$ is super-finitary due to the quotient

$$\varepsilon_1 \times \varepsilon_2 \colon (A_1 \times A_2) \times \mathsf{Id}^{n_1 + n_2} \to F_1 \times F_2.$$

Suppose $n_1 \geq n_2$, then we can choose a quotient $\varphi \colon A_2 \times \mathsf{Id}^{n_1} \twoheadrightarrow A_2 \times \mathsf{Id}^{n_2}$. This proves that $F_1 + F_2$ is super-finitary due to the quotient

$$\varepsilon_1 + (\varepsilon_2 \cdot \varphi) \colon (A_1 + A_2) \times \mathsf{Id}^{n_1} \cong A_1 \times \mathsf{Id}^{n_1} + A_2 \times \mathsf{Id}^{n_1} \to F_1 + F_2.$$

(2) Let $\mu\colon G \rightarrowtail F$ be a subfunctor of a super-finitary functor $F$ with a quotient $\varepsilon\colon A \times \mathsf{Id}^n \twoheadrightarrow F$. Form a pullback (object-wise in $\mathsf{Set}$) of $\varepsilon$ and $\mu$:

$$\begin{array}{ccc} H & \overset{\bar{\mu}}{\rightarrowtail} & A \times \mathsf{Id}^n \\ {\scriptstyle \bar{\varepsilon}}\downarrow & \lrcorner & \downarrow{\scriptstyle \varepsilon} \\ G & \underset{\mu}{\rightarrowtail} & F \end{array}$$

For each $a \in A$, the preimage $H_a$ of $\{a\} \times \mathsf{Id}^n \cong \mathsf{Set}(n, -)$ under $\bar{\mu}$ is super-finitary by Example (3) above. Since $A \times \mathsf{Id}^n = \coprod_{a \in A} \{a\} \times \mathsf{Id}^n$ and preimages under $\bar{\mu}$ preserve coproducts, we have $H = \coprod_{a \in A} H_a$ and so $G$ is a quotient of the super-finitary functor $H$.

■

3.32. Lemma. *Let $\mathscr{C}$ be an lfp category with finitely generated objects closed under kernel pairs and in which strong epimorphisms are regular. Then finitely presentable and finitely generated objects coincide.*

Proof. We apply Remark 2.2(5): Consider a strong epimorphism $c\colon X \twoheadrightarrow Y$ with $X$ finitely presentable. We are to show that $Y$ is finitely presentable. Let $p, q\colon K \rightrightarrows X$ be the kernel pair of $c$, then $K$ is finitely generated. Hence there is some finitely presentable object $K'$ and a strong epimorphism $e\colon K' \twoheadrightarrow K$:

$$K' \overset{e}{\twoheadrightarrow} K \underset{q}{\overset{p}{\rightrightarrows}} X \overset{c}{\twoheadrightarrow} Y$$

Since the strong epimorphism $c$ is also regular, it is the coequalizer of its kernel pair $(p, q)$; furthermore $e$ is epic, thus $c$ is also the coequalizer of $p \cdot e$ and $q \cdot e$. This means that $Y$ is a finite colimit of finitely presentable objects and thus it is finitely presentable. ■

3.33. Corollary. $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ *is not semi-strictly lfp.*

Proof. We use the subfunctors

$$\bar{\mathcal{P}} \subseteq \mathcal{P}_0 \subseteq \mathcal{P}$$

of the power-set functor $\mathcal{P}$ given by $\mathcal{P}_0 X = \mathcal{P}X \setminus \{\emptyset\}$ and $\bar{\mathcal{P}}X = \{M \in \mathcal{P}_0 X \mid M \text{ finite}\}$. Then $\bar{\mathcal{P}}$ is an object of $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ which is clearly not super-finitary. The only endomorphism of $\bar{\mathcal{P}}$ is $\mathsf{id}_{\bar{\mathcal{P}}}$. Indeed for $\mathcal{P}_0$ this has been proven in [6, Proposition 5.4]; the same proof applies to $\bar{\mathcal{P}}$. And $\mathsf{id}_{\bar{\mathcal{P}}}$ is not finitary: otherwise $\bar{\mathcal{P}}$ would be a quotient of a finitely presentable object, thus, it would be super-finitary (due to Lemma 3.30). ■

3.34. Corollary. *For a finitary set functor, as an object of* $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$*, the following conditions are equivalent:*

(1) *finitely presentable,*

(2) *finitely generated, and*

(3) *super-finitary.*

PROOF. To verify (2) $\implies$ (3), let $F$ be finitely generated. For every finite subset $A \subseteq Fn$, $n \in \mathbb{N}$, we have a subfunctor $F_{n,A} \subseteq F$ given by

$$F_{n,A}X = \bigcup_{f\colon n \to X} Ff[A].$$

Since $F$ is finitary, it is a directed union of all these subfunctors. This implies $F \cong F_{n,A}$ for some $n$ and $A$, and $F_{n,A}$ is clearly super-finitary.

For (3) $\implies$ (2), combine Lemma 3.30 and Example 3.29(1).

(1) $\iff$ (2) follows by Lemma 3.32. ∎

## 4. $\lambda$-Accessible Functors

Almost everything we have proved above generalizes to locally $\lambda$-presentable categories for every infinite regular cardinal $\lambda$. Recall that an object $A$ of a category $\mathscr{A}$ is *$\lambda$-presentable* (*$\lambda$-generated*) if its hom-functor $\mathscr{A}(A, -)$ preserves $\lambda$-filtered colimits (of monomorphisms). A category $\mathscr{A}$ is *locally $\lambda$-presentable* if it is cocomplete and has a set of $\lambda$-presentable objects whose closure under $\lambda$-filtered colimits is all of $\mathscr{A}$. Functors preserving $\lambda$-filtered colimits are called *$\lambda$-accessible*. We denote by $\mathscr{A}_{\lambda\mathsf{p}}$ and $\mathscr{A}_{\lambda\mathsf{g}}$ full subcategories representing (up to isomorphism) all $\lambda$-presentable and $\lambda$-generated objects, respectively.

All of Remark 2.2 holds for $\lambda$ in lieu of $\aleph_0$, with the same references in [5].

If $\lambda = \aleph_1$ we speak about *locally countably presentable categories*, *countably presentable objects*, etc.

4.1. EXAMPLES.

(1) Complete metric spaces. We denote by

$$\mathsf{CMS}$$

the category of complete metric spaces of diameter $\leq 1$ and non-expanding functions, i.e. functions $f\colon X \to Y$ such that for all $x, y \in X$ we have $d_Y(f(x), f(y)) \leq d_X(x, y)$. This category is locally countably presentable. The classes of countably presentable and countably generated objects coincide and these are precisely the compact spaces.

Indeed, every compact (= separable) complete metric space is countably presentable, see [2, Corollaries 2.9]. And every countably generated space $A$ in $\mathsf{CMS}$ is separable: consider the countably filtered diagram of all spaces $\bar{X} \subseteq A$ where $X$ ranges over countable subsets of $A$ and $\bar{X}$ is the closure in $A$. Since $A$ is the colimit of this diagram, $\mathsf{id}_A$ factorizes through one of the embeddings $\bar{X} \hookrightarrow A$, i.e. $A = \bar{X}$ is separable.

(2) Complete partial orders. Denote by

$$\omega\mathsf{CPO}$$

the category of $\omega$-cpos, i.e. of posets with joins of $\omega$-chains and monotone functions preserving joins of $\omega$-chains. This is also a locally countably presentable category. An

$\omega$-cpo is countably presentable (equivalently, countably generated) iff it has a countable subset which is dense w.r.t. joins of $\omega$-chains.

Following our convention in Section 3 we speak about a *$\lambda$-generated subobject* $m\colon M \rightarrowtail A$ of $A$ if $M$ is a $\lambda$-generated object of $\mathscr{A}$. This leads to a generalization of the notion of finitely bounded functors to $\lambda$-bounded ones. The latter terminology stems from Kawahara and Mori [13], where endofunctors on sets were considered. Our terminology is slightly different in that $\lambda$-generated subobjects in $\mathsf{Set}$ have cardinality less than $\lambda$, whereas subsets of cardinality less than or equal to $\lambda$ were considered in loc. cit.

4.2. Definition. *A functor $F\colon \mathscr{A} \to \mathscr{B}$ is called* $\lambda$-bounded *provided that given an object $A$ of $\mathscr{A}$, every $\lambda$-generated subobject $m_0\colon M_0 \rightarrowtail FA$ in $\mathscr{B}$ factorizes through the $F$-image of a $\lambda$-generated subobject $m\colon M \rightarrowtail A$ in $\mathscr{A}$:*

$$\begin{array}{ccc} & & FM \\ & \nearrow & \big\downarrow {\scriptstyle Fm} \\ M_0 & \xrightarrow[m_0]{} & FA \end{array}$$

4.3. Theorem. *Let $\mathscr{A}$ be a locally $\lambda$-presentable category in which every $\lambda$-generated object is $\lambda$-presentable. Then for all functors from $\mathscr{A}$ to locally $\lambda$-presentable categories preserving monomorphisms we have the equivalence*

$$\lambda\text{-}accessible \iff \lambda\text{-}bounded.$$

The proof is completely analogous to that of Theorem 3.4.

4.4. Example. The Hausdorff endofunctor $\mathscr{H}$ on $\mathsf{CMS}$ was proved to be accessible (for some $\lambda$) by van Breugel et al. [20]. Later it was shown to be even finitary [2]. However, these proofs are a bit involved. Using Theorem 4.3 we provide an easy argument why the Hausdorff functor is countably accessible. (Which, since $\mathsf{CMS}$ is not lfp but is locally countably presentable, seems to be the 'natural' property.)

Recall that for a given metric space $(X, d)$ the distance of a point $x \in X$ to a subset $M \subseteq X$ is defined by $d(x, M) = \inf_{y\in M} d(x, y)$. The *Hausdorff distance* of subsets $M, N \subseteq X$ is defined as the maximum of $\sup_{x\in M} d(x, N)$ and $\sup_{y\in N} d(y, M)$. The *Hausdorff functor* assigns to every complete metric space $X$ the space $\mathscr{H}X$ of all non-empty compact subsets of $X$ equipped with the Hausdorff metric. It is defined on non-expanding maps by taking the direct images. We now easily see that $\mathscr{H}$ is countably accessible:

(1) $\mathscr{H}$ preserves monomorphisms. Indeed, given $f\colon X \rightarrowtail Y$ monic, then $f[M] \neq f[N]$ for every pair $M, N$ of distinct elements of $\mathscr{H}X$, thus $\mathscr{H}f$ is monic, too.

(2) $\mathscr{H}$ is countably bounded. In order to see this, let $m_0\colon M_0 \hookrightarrow \mathscr{H}X$ be a subspace with $M_0$ compact, and choose a countable dense subset $S \subseteq M_0$. For every element $s \in S$ the set $m_0(s) \subseteq X$ is compact, hence, separable; choose a countable dense set $T_s \subseteq m_0(s)$. For the countable set $T = \bigcup_{s\in S} T_s$ form the closure in $X$ and denote it by $m\colon M \hookrightarrow X$. Then $M$ is countably generated, and $M_0 \subseteq \mathscr{H}m[\mathscr{H}M]$; indeed, for every $x \in M_0$ we have $m_0(x) \subseteq M$ because $M$ is closed, and this holds whenever $x \in S$ (due to $m_0(x) = \overline{T_x}$).

In the following definition a morphism is called *$\lambda$-ary* if it factorizes through a $\lambda$-presentable object.

4.5. Definition. *A locally $\lambda$-presentable category is called*

(1) semi-strictly locally $\lambda$-presentable *if every object has a $\lambda$-ary endomorphism;*

(2) strictly locally $\lambda$-presentable *if every object has, for each $\lambda$-generated subobject $m$, a finitary endomorphism $u$ fixing that subobject (i.e. $u \cdot m = m$).*

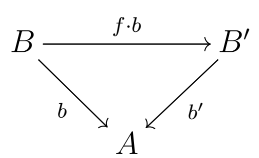
Observe that Remark 3.10 immediately generalizes to an arbitrary $\lambda$.

4.6. Examples.

(1) $\mathsf{Set}^S$ is strictly locally $\lambda$-presentable iff $\mathsf{card}\, S < \lambda$. This is analogous to Example 3.19(1).

(2) The category $\mathsf{Grp}$ of groups is semi-strictly locally $\lambda$-presentable by the same argument as in Example 3.11(4). However, $\mathsf{Grp}$ is not strictly locally $\lambda$-presentable for any infinite cardinal $\lambda$.

To see this, let $A$ be a simple group of cardinality at least $\lambda^\lambda$. (Recall that for every set $X$ of cardinality $\geq 5$ the group of even permutations on $X$ is simple.) Since $\mathsf{Grp}$ is an lfp category, there exists a non-zero homomorphism $b\colon B \to A$ with $B$ finitely presentable. Given a commutative diagram




for some $f\colon A \to B'$

we show that $B'$ is not $\lambda$-presentable. Indeed, since $b$ is non-zero, we see that so is $f\colon A \to B'$. Since $A$ is simple, $f$ is monic, hence $\mathsf{card}\, B' \geq \lambda^\lambda$. However, every $\lambda$-presentable group has cardinality at most $\lambda$. Thus, by an argument analogous to Remark 3.10(2), $\mathsf{Grp}$ is not strictly locally $\lambda$-presentable.

(3) The category $\mathsf{Nom}$ of nominal sets is strictly locally countably presentable. In order to prove this, we first verify that countably presentable objects are precisely the countable nominal sets.

(a) Let $X$ be a countably presentable nominal set. Then every countable choice of orbits of $X$ yields a countable subobject of $X$ in $\mathsf{Nom}$. Thus $X$ is a countably directed union of countable subobjects. Since $X$ is countably presentable, it follows that $X$ is isomorphic to one of these subobjects. Thus, $X$ is countable.

(b) Conversely, every countable nominal set is countably presentable since countably filtered colimits of nominal sets are formed on the level of sets (i.e. these colimits are preserved and reflected by the forgetful functor $\mathsf{Nom} \to \mathsf{Set}$).

Now let $b\colon B \to A$ be a morphism in $\mathsf{Nom}$ with $B$ countable. We have $A = \mathsf{Im}(b) + C$ for some subobject $C$ of $A$. Indeed, every nominal set is a coproduct of its orbits, and the equivariance of $b$ implies that $\mathsf{Im}(b)$ is a coproduct of some of the orbits of $A$. Furthermore, let $m\colon C_1 \rightarrowtail C$ be a subobject obtained by choosing one orbit from each isomorphism class of orbits of $C$. We obtain a surjective equivariant map $e\colon C \twoheadrightarrow C_1$ by choosing, for every orbit in $C \setminus C_1$, a concrete isomorphism to an orbit of $C_1$ and for every $x \in C_1 \subseteq C$ putting $e(x) = x$. Then we have $e \cdot m = \mathsf{id}_{C_1}$, i.e. $m$ is a split monomorphism of $\mathsf{Nom}$. In the appendix we prove that there are (up to isomorphism) only countably many single-orbit nominal sets. Hence, $C_1$ is countable, and thus so is $B' = \mathsf{Im}(b) + C_1$. Moreover, the morphisms $b' = \mathsf{id} + m\colon B' \to A$ and $f\colon \mathsf{id} + e\colon A \to B'$ clearly satisfy the desired property $b = b' \cdot f \cdot b$, see Remark 3.10(2).

4.7. Proposition. *Every semi-simple locally presentable category is strictly locally $\lambda$-presentable for some $\lambda$.*

Proof. Let $\mathscr{A}$ be a locally $\kappa$-presentable category that is semi-simple.

(1) $\mathscr{A}$ has only a set of simple objects up to isomorphism. Indeed, we have a set $\mathscr{A}_\kappa$ representing all $\kappa$-presentable objects. Given a simple object $A$, express it as a colimit of a $\kappa$-filtered diagram in $\mathscr{A}_\kappa$ with a colimit cocone $c_i\colon C_i \to A$, $i \in I$. Since $\mathscr{A}$ is locally presentable, it has (strong epi, mono)-factorizations [5, Proposition 1.61]. Then, since $A$ is simple, either it is a strong quotient of some $C_i$ or it is an initial object. Thus, every simple object is a strong quotient of a $\kappa$-presentable one. The desired statement follows since every locally presentable category is cowellpowered [5, Theorem 1.58].

(2) Let $\lambda \geq \kappa$ be a regular cardinal such that every semi-simple object is $\lambda$-presentable. Then $\mathscr{A}$ is locally $\lambda$-presentable, and the rest of the proof is completely analogous to point (2) in the proof of Proposition 3.18.

■

4.8. Corollary. *For every semi-simple ring $R$ the category $R$-$\mathsf{Mod}$ is strictly locally $\lambda$-presentable provided that $\lambda > 2^{|R\times R|}$.*

Indeed, the module $R$ has less than $\lambda$ quotient modules. As in Example 3.19(3) each quotient is $\lambda$-presentable in $R$-$\mathsf{Mod}$, and the rest is as in that example.

4.9. Corollary. *Every atomic Grothendieck topos with a set of atoms (up to isomorphism) is strictly locally $\lambda$-presentable for some $\lambda$.*

Being a Grothendieck topos, our category is locally $\lambda$-presentable for some $\lambda$. We can choose $\lambda$ to be (a) larger than the number of atoms up to isomorphism and (b) such that every atom is $\lambda$-presentable. Then our topos is strictly locally $\lambda$-presentable.

4.10. Example. The category of presheaves on a small groupoid is strictly locally $\lambda$-presentable. Indeed, the proof that there is, up to isomorphism, only a set of atomic presheaves is analogous to Lemma 3.20.

4.11. Theorem. *Let $\mathscr{A}$ be a locally $\lambda$-presentable category.*

(1) *If $\mathscr{A}$ is strictly locally $\lambda$-presentable, then for all functors from $\mathscr{A}$ to a locally $\lambda$-presentable category $\mathscr{B}$ with $\mathscr{B}_{\lambda\mathsf{p}} = \mathscr{B}_{\lambda\mathsf{g}}$ we have*

$$\lambda\text{-accessible} \iff \lambda\text{-bounded}.$$

(2) *Conversely, if this equivalence holds for all functors to* Set*, then* $\mathscr{A}$ *is semi-strictly locally* $\lambda$*-presentable and* $\mathscr{A}_{\lambda \mathsf{p}} = \mathscr{A}_{\lambda \mathsf{g}}$.

The proofs are completely analogous to those of Theorems 3.12 and 3.16.

4.12. Remark. Assume that we work in a set theory distinguishing between sets and classes (e.g. Zermelo-Fraenkel theory) or distinguishing universes, so that by 'a class' we take a member of the next higher universe of that of all small sets. Then we form a super-large category

$$\mathsf{Class}$$

of classes and class functions. It plays a central role in the paper of Aczel and Mendler [1] on terminal coalgebras. An endofunctor $F$ of Class in that paper is called *set-based* if for every class $X$ and every element $x \in FX$ there exists a subset $i\colon Y \rightarrowtail X$ such that $x$ lies in $Fi[FX]$. This corresponds to $\infty$-bounded where $\infty$ stands for 'being large'. The corresponding concept of $\infty$-accessibility is evident:

4.13. Definition. *A diagram* $D\colon \mathscr{D} \to \mathsf{Class}$*, with* $\mathscr{D}$ *not necessarily small, is called* $\infty$-filtered *if every small subcategory of* $\mathscr{D}$ *has a cocone in* $\mathscr{D}$*. An endofunctor of* Class *is called* $\infty$-accessible *if it preserves colimits of* $\infty$*-filtered diagrams.*

4.14. Proposition. *An endofunctor of* Class *is set-based iff it is* $\infty$*-accessible.*

Proof. (1) For every morphism $b\colon B \to A$ in Class with $B$ small factorizes in $\mathsf{Set}/A$ through a morphism $b'\colon B' \to A$ in $\mathsf{Set}/A$ where the factorization $f$ fulfils $b = b' \cdot (f \cdot b)$. (Shortly: Class is strictly locally $\infty$-presentable.) The proof is the same as that of Example 3.11(2).

(2) The rest is completely analogous to part (1) of the proof of Theorem 3.12 ∎

4.15. Remark. Assuming, moreover, that all proper classes are mutually bijective, it follows that *every* endofunctor on Class is $\infty$-accessible, see [3].

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# A. Details on Single-Orbit Nominal Sets

In this appendix we prove that in the category Nom of nominal sets there are (up to isomorphism) only countably many nominal sets having only one orbit. To this end we consider the nominal sets $\mathbb{A}^{\#n}$ of injective maps from $n = \{0, 1, \ldots, n-1\}$ to $\mathbb{A}$. The group action on $\mathbb{A}^{\#n}$ is component-wise, in other words, it is given by postcomposition: for $t\colon n \rightarrowtail \mathbb{A}$ and $\pi \in \mathfrak{S}_\mathsf{f}(\mathbb{A})$ (which is a bijective map $\pi\colon \mathbb{A} \to \mathbb{A}$) the group action is the composed map $\pi \cdot t\colon n \rightarrowtail \mathbb{A}$. Thus, for every $t : n \rightarrowtail \mathbb{A}$ of $\mathbb{A}^{\#n}$, $\mathsf{supp}(t) = \{t(i) \mid i < n\}$.

A.1. Lemma. *Up to isomorphism, there are only countably many single-orbit nominal sets.*

Proof. Every single-orbit nominal set $Q$ whose elements have supports of cardinality $n$ is a quotient of the (single-orbit) nominal set $\mathbb{A}^{\#n}$ (see [16, Exercise 5.1]). Indeed, if $Q = \{\pi \cdot x \mid \pi \in \mathfrak{S}_\mathsf{f}(\mathbb{A})\}$ with $\mathsf{supp}(x) = \{a_0, \ldots, a_{n-1}\}$, let $t\colon n \rightarrowtail \mathbb{A}$ be the element of $\mathbb{A}^{\#n}$ with $t(i) = a_i$ and define $q\colon \mathbb{A}^{\#n} \twoheadrightarrow Q$ as follows: for every $u \in \mathbb{A}^{\#n}$ it is clear that there is some $\pi \in \mathfrak{S}_\mathsf{f}(\mathbb{A})$ with $u = \pi \cdot t$; put $q(u) = \pi \cdot x$. This way, $q$ is well-defined (since $\mathsf{supp}(x) = \{t(i) \mid i < n\}$) and equivariant.

For every $n \in \mathbb{N}$, the quotients of $\mathbb{A}^{\#n}$ are given by equivariant equivalence relations on $\mathbb{A}^{\#n}$. We prove that we have a bijective correspondence between the set of all quotients with $|\mathsf{supp}([t]_\sim)| = n$ for all $t \in \mathbb{A}^{\#n}$ and the set of all subgroups of $\mathfrak{S}_\mathsf{f}(n)$.

(1) Given an equivariant equivalence $\sim$ on $\mathbb{A}^{\#n}$ put

$$S = \{\sigma \in \mathfrak{S}_\mathsf{f}(n) \mid \forall (t\colon n \rightarrowtail \mathbb{A})\colon t \cdot \sigma \sim t\}.$$

Note that since $\sim$ is equivariant (and composition of maps is associative), $\forall$ can equivalently be replaced by $\exists$:

$$S = \{\sigma \in \mathfrak{S}_\mathsf{f}(n) \mid \exists (t\colon n \rightarrowtail \mathbb{A})\colon t \cdot \sigma \sim t\}.$$

It is easy to verify that $S$ is a subgroup of $\mathfrak{S}_\mathsf{f}(n)$. Moreover, we have that, for every $t, u \in \mathbb{A}^{\#n}$,

$$t \sim u \quad\iff\quad u = t \cdot \sigma \quad \text{for some } \sigma \in S. \tag{A.1}$$

Indeed, "$\Longleftarrow$" is obvious. For "$\Longrightarrow$" suppose that $t \sim u$. Since $|\mathsf{supp}([t]_\sim)| = n$, we have that $\mathsf{supp}(t) = \mathsf{supp}([t]_\sim) = \mathsf{supp}([u]_\sim) = \mathsf{supp}(u)$; thus, there is some $\sigma \in \mathfrak{S}_\mathsf{f}(n)$ such that $u = t \cdot \sigma$. Consequently, $t \sim t \cdot \sigma$, showing that $\sigma \in S$.

(2) For every subgroup $S$ of $\mathfrak{S}_\mathsf{f}(n)$, it is clear that the relation $\sim$ defined by (A.1) is an equivariant equivalence. We show that, moreover, $|\mathsf{supp}([t]_\sim)| = n$ for every $t \in \mathbb{A}^{\#n}$. We have $|\mathsf{supp}([t]_\sim)| \leq n$ because the canonical quotient map $[-]_\sim$ is equivariant. In order to see that $|\mathsf{supp}([t]_\sim)|$ is not smaller than $n$, assume $a \in \mathsf{supp}(t) \setminus \mathsf{supp}([t]_\sim)$ and take any element $b \notin \mathsf{supp}(t)$. Then $(a\,b) \cdot [t]_\sim = [t]_\sim$, i.e. there is some $\sigma \in \mathfrak{S}_\mathsf{f}(n)$ with $(a\,b) \cdot t \cdot \sigma = t$, which is a contradiction to $b \notin \mathsf{supp}(t) = \mathsf{supp}(t \cdot \sigma) = \{t(i) \mid i < n\}$.

(3) It remains to show that, given two subgroups $S$ and $S'$ which determine the same equivariant equivalence relations $\sim$ via (A.1), then $S = S'$. Indeed, given $\sigma \in S$, we have

$t = (t \cdot \sigma) \cdot \sigma^{-1}$ and therefore $t \cdot \sigma \sim t$ for every $t \in \mathbb{A}^{\# n}$. By (A.1) applied to $S'$, this implies that $t = t \cdot \sigma \cdot \sigma'$ for some $\sigma' \in S'$. Since $t$ is monic, we obtain $\sigma \cdot \sigma' = \mathsf{id}_n$, i.e. $\sigma = (\sigma')^{-1} \in S'$. This proves $S \subseteq S'$, and the reverse inclusion holds by symmetry. ∎

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