Decompositions of linear spaces induced by n-linear maps

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ABSTRACT. Let $\mathbb{V}$ be an arbitrary linear space and $f : \mathbb{V}\times\ldots\times\mathbb{V} \to \mathbb{V}$ an n-linear map. It is proved that, for each choice of a basis $\mathcal{B}$ of $\mathbb{V}$, the n-linear map f induces a (nontrivial) decomposition $\mathbb{V} = \oplus V_j$ as a direct sum of linear subspaces of $\mathbb{V}$, with respect to $\mathcal{B}$. It is shown that this decomposition is f-orthogonal in the sense that $f(\mathbb{V},\ldots,V_j,\ldots,V_k,\ldots,\mathbb{V}) = 0$ when $j \neq k$, and in such a way that any V_j is strongly f-invariant, meaning that $f(\mathbb{V},\ldots,V_j,\ldots,\mathbb{V}) \subset V_j$. A sufficient condition for two different decompositions of $\mathbb{V}$ induced by an n-linear map f, with respect to two different bases of $\mathbb{V}$, being isomorphic is deduced. The f-simplicity – an analogue of the usual simplicity in the framework of n-linear maps – of any linear subspace V_j of a certain decomposition induced by f is characterized. Finally, an application to the structure theory of arbitrary n-ary algebras is provided. This work is a close generalization the results obtained by A. J. Calderón (2018) [1].

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1 Introduction

The main idea of this paper is to present an n-ary ($n > 2$) generalization of the results achieved by the first author on the decomposition of linear spaces induced by bilinear maps on a linear space [1].

In the mentioned paper, given a linear space $\mathbb{V}$ of arbitrary dimension and a bilinear map f on $\mathbb{V}$, Calderón introduced the notions of f-orthogonal, f-invariant and strongly f-invariant subspaces, as well as the notion of f-simplicity, which are just the usual notions of orthogonality, invariance and simplicity, but now defined with respect to f. Then, for a fixed basis of $\mathbb{V}$, he developed connection techniques allowing to obtain a first nontrivial decomposition of $\mathbb{V}$ as the direct sum of f-orthogonal vector subspaces. In order to improve the obtained decomposition he introduced an adequate equivalence relation on the above family of linear subspaces, leading to the first main result: a non-trivial decomposition of $\mathbb{V}$ as an f-orthogonal direct sum of strongly f-invariant linear subspaces, with respect to a fixed basis. After that, observing that different choices of the bases of $\mathbb{V}$ may lead to different decompositions, he studied sufficient conditions to assure induced isomorphic decompositions of $\mathbb{V}$ with respect to different bases of $\mathbb{V}$. Another important result gives necessary and sufficient conditions for the f-simplicity of the linear subspaces in the second decomposition of $\mathbb{V}$. The author ends the paper providing an application of the previous results to the structure theory of arbitrary algebras.

At this point, a parenthesis is due to underline the considerable amount of recent works where the above mentioned and similar connection techniques are applied as a tool to obtain interesting results in the frameworks of several types of algebras. Without being exhaustive, these techniques were used, for instance, along with the notions of multiplicative basis and quasi-multiplicative basis not only related with algebras (see Caledrón and Navarro, [2, 3]), but also with some n-ary generalizations (see, *e.g.*, the works of Calderón, Barreiro, Kaygorodov and Sánchez in [4, 5, 6]). Further, connection techniques were also applied in the context of graded Lie algebras (see Calderón (2014) [7]) and to obtain structural results on graded Leibniz triple systems (see Cao and Chen (2016) [8]).

The present work follows an approach that uses, as close as possible, generalized n-ary versions of the techniques applied in [1], obtaining generalized results which are similar to those of Calderón.

The paper is organized as follows. In Section 2 we present the necessary basic notions related with n-linear maps and develop all connection techniques needed to obtain the main results. As a consequence, we get that each choice of a basis $\mathcal{B}$ of $\mathbb{V}$ rises a first nontrivial decomposition of $\mathbb{V}$, induced by f, as an f-orthogonal direct sum of linear subspaces with respect to $\mathcal{B}$. This decomposition is then enhanced by the introduction of an adequate equivalence relation on the above family of linear subspaces, leading to our first main result: $\mathbb{V}$ decomposes as a nontrivial f-orthogonal direct sum of strongly f-invariant linear subspaces, with respect to a fixed basis.

In Section 3 the relation among the previous decompositions of $\mathbb{V}$ given by different choices of its bases is discussed. Concretely, after defining the notion of orbit associated to an n-linear map f, it is shown that if two bases, $\mathcal{B}$ and $\mathcal{B}'$ of $\mathbb{V}$ belong to the same orbit under an action of a certain subgroup of $\mathrm{GL}(\mathbb{V})$ on the set of all bases of $\mathbb{V}$, then they induce isomorphic decompositions of $\mathbb{V}$.

In Section 4 we generalize the concept of i-division basis to the case of n-ary algebras. After that, we obtain a characterization of the f-simplicity of the components of the main decomposition obtained in Section 2. That is, we prove that any of the linear subspaces in the decomposition of $\mathbb{V}$ in f-orthogonal, strongly f-invariant linear subspaces of $\mathbb{V}$ is f-simple if and only if its annihilator is zero and it admits an i-division basis.

Finally, in Section 5 an application of the previous results to the the structure theory of arbitrary n-ary algebras is included.

2 Development of the techniques. First decomposition theorem

We begin by noting that throughout the paper all of the linear spaces $\mathbb{V}$ considered are of arbitrary dimension and over an arbitrary base field $\mathbb{F}$. Hereinafter, $\mathbb{V}$ is a linear space and $f : \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{V}$ an n-linear map on $\mathbb{V}$, $n \geq 2$. We start recalling some notions concerning $\mathbb{V}$ and f.

Definition 2.1. Two linear subspaces V_1 and V_2 of $\mathbb{V}$ are called *f-orthogonal* if

$$f(\mathbb{V}, \ldots, V_1^{(i)}, \ldots, V_2^{(j)}, \ldots, \mathbb{V}) = 0,$$

for all $i, j \in \{1, \ldots, n\}$, $i \neq j$, where the notations $V_1^{(i)}$ and $V_2^{(j)}$ mean that V_1 and V_2 occupy the i-th and j-th entries of f, respectively.

It is also said that a decomposition of $\mathbb{V}$ as a direct sum of linear subspaces

$$\mathbb{V} = \bigoplus_{j \in J} V_j$$

is *f-orthogonal* if V_j and V_k are f-orthogonal for any $j, k \in J$, with $j \neq k$.

Definition 2.2. A linear subspace W of $\mathbb{V}$ is called *f-invariant* if

$$f(W, \ldots, W) \subset W.$$

The linear space W is called *strongly f-invariant* if

$$f(\mathbb{V}, \ldots, W^{(i)}, \ldots, \mathbb{V}) \subset W,$$

for all $i \in \{1, \ldots, n\}$. The linear space $\mathbb{V}$ will be called *f-simple* if

$$f(\mathbb{V}, \ldots, \mathbb{V}) \neq 0$$

and its only strongly f-invariant subspaces are $\{0\}$ and $\mathbb{V}$.

Definition 2.3. *The annihilator of f is defined as the set*

$$\mathrm{Ann}(f) = \{v \in \mathbb{V} : f(\mathbb{V}, \ldots, v^{(i)}, \ldots, \mathbb{V}) = 0, \text{ for all } i \in \{1, \ldots, n\}\}.$$

Let us fix a basis $\mathcal{B} = \{e_i\}_{i \in I}$ of $\mathbb{V}$. For each $e_i \in \mathcal{B}$, we introduce a symbol $\overline{e}_i \notin \mathcal{B}$ and the following set

$$\overline{\mathcal{B}} := \{\overline{e}_i : e_i \in \mathcal{B}\}.$$

We will also write $\overline{(\overline{e}_i)} := e_i \in \mathcal{B}$, $\mathbb{V}^* := \mathbb{V} \setminus \{0\}$ and $\mathcal{P}(\mathbb{V}^*)$ the power set of $\mathbb{V}^*$.

We define the n-linear mapping

$$F : \mathcal{P}(\mathbb{V}^*) \times \big((\mathcal{B} \dot{\cup} \overline{\mathcal{B}}) \times \cdots \times (\mathcal{B} \dot{\cup} \overline{\mathcal{B}})\big) \to \mathcal{P}(\mathbb{V}^*) \tag{1}$$

as

(i) $F(\emptyset, \big(\mathcal{B} \dot{\cup} \overline{\mathcal{B}}, \ldots, \mathcal{B} \dot{\cup} \overline{\mathcal{B}}\big)) = \emptyset.$

(ii) For any $\emptyset \neq U \in \mathcal{P}(\mathbb{V}^*)$ and $\xi_i \in \mathcal{B}$, $i = 1, \ldots, n-1$,

$$F(U, (\xi_1, \ldots, \xi_{n-1})) = \left(\bigcup_{\substack{k \in \{1,\ldots,n\} \\ \sigma \in \mathbb{S}_{n-1}}} \{ f(\xi_{\sigma(1)}, \ldots, u^{(k)}, \ldots, \xi_{\sigma(n-1)}) : u \in U \} \right) \backslash \{0\}.$$

(iii) For any $\emptyset \neq U \in \mathcal{P}(\mathbb{V}^*)$ and $\overline{\xi}_i \in \overline{\mathcal{B}}$, $i = 1, \ldots, n-1$,

$$F(U, \big(\overline{\xi}_1, \ldots, \overline{\xi}_{n-1}\big)) = \left(\bigcup_{\substack{k \in \{1,\ldots,n\} \\ \sigma \in \mathbb{S}_{n-1}}} \{ u \in \mathbb{V} : f(\xi_{\sigma(1)}, \ldots, u^{(k)}, \ldots, \xi_{\sigma(n-1)}) \in U \} \right) \backslash \{0\}.$$

(iv) $F(U, (\xi_1, \ldots, \xi_{n-1})) = \emptyset$, if there are $i, j \in \{1, \ldots, n-1\}$, $i \neq j$, such that $\xi_i \in \mathcal{B}$, $\xi_j \in \overline{\mathcal{B}}$.

Remark 2.4. *It is clear that*

$$F(U, \big(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(n-1)}\big)) = F(U, (\xi_1, \ldots, \xi_{n-1})),$$

and

$$F(U, \big(\overline{\xi}_{\sigma(1)}, \ldots, \overline{\xi}_{\sigma(n-1)}\big)) = F(U, \big(\overline{\xi}_1, \ldots, \overline{\xi}_{n-1}\big)),$$

for all $\xi_1, \ldots, \xi_{n-1} \in \mathcal{B}$, $\sigma \in \mathbb{S}_{n-1}$.

Lemma 2.5. *Concerning the mapping F previously defined, we have*

1. *For any* $v \in \mathbb{V}^*$ *and* $\xi_1, \ldots, \xi_{n-1} \in \mathcal{B}$,
   $w \in F(\{v\}, (\xi_1, \ldots, \xi_{n-1}))$ *if and only if* $v \in F(\{w\}, \big(\overline{\xi}_1, \ldots, \overline{\xi}_{n-1}\big))$.

2. *For any* $U \in \mathcal{P}(\mathbb{V}^*)$ *and* $\xi_1, \ldots, \xi_{n-1} \in \mathcal{B}\dot{\cup}\overline{\mathcal{B}}$,
   $v \in F(U, (\xi_1, \ldots, \xi_{n-1}))$ *if and only if* $F(\{v\}, \big(\overline{\xi}_1, \ldots, \overline{\xi}_{n-1}\big)) \cap U \neq \emptyset$.

Proof. 1. Let us start admitting that $w \in F(\{v\}, (\xi_1, \ldots, \xi_{n-1}))$, being $v \in \mathbb{V}^*$ and $\xi_1, \ldots, \xi_{n-1} \in \mathcal{B}$. This means that

$$w = f(\xi_{\sigma(1)}, \ldots, v^{(k)}, \ldots, \xi_{\sigma(n-1)}),$$

for some $k \in \{1, \ldots, n-1\}$ and $\sigma \in \mathbb{S}_{n-1}$, and thus

$$v \in F(\{w\}, (\overline{\xi}_{\sigma(1)}, \ldots, \overline{\xi}_{\sigma(n-1)})).$$

According to the previous remark, we have:

$$v \in F(\{w\}, (\overline{\xi}_1, \ldots, \overline{\xi}_{n-1})).$$

The reciprocal result can be proved analogously.

2. Suppose that $U \in \mathcal{P}(\mathbb{V}^*)$ and $\xi_i \in \mathcal{B}\dot{\cup}\overline{\mathcal{B}}$, $i = 1, \ldots, n-1$. Let us first admit that $v \in F(U, (\xi_1, \ldots, \xi_{n-1}))$. Then $v \in F(\{w\}, (\xi_1, \ldots, \xi_{n-1}))$ for some $w \in U$. By item 1., this is equivalent to $w \in F(\{v\}, (\overline{\xi}_1, \ldots, \overline{\xi}_{n-1}))$ and thus

$$w \in F(\{v\}, (\overline{\xi}_1, \ldots, \overline{\xi}_{n-1})) \cap U \neq \emptyset.$$

The reciprocal assertion can be proved in a similar way. □

Definition 2.6. Let $e_i, e_j \in \mathcal{B}$. We say that e_i is *connected* to e_j if either,

(i) $e_i = e_j$ or

(ii) there exists an ordered list $(X_1, X_2, \ldots, X_m)$, where $X_i := (a_{i1}, \ldots, a_{in-1})$ such that $a_{ik} \in \mathcal{B}\dot{\cup}\overline{\mathcal{B}}$, $i \in \{1, \ldots, m\}$, $k \in \{1, \ldots, n-1\}$, satisfying:

1. $F(\{e_i\}, X_1) \neq \emptyset$,
   $F(F(\{e_i\}, X_1), X_2) \neq \emptyset$,
   $\vdots$
   $F(\ldots(F(F(\{e_i\}, X_1), X_2), \ldots, X_{m-1}) \neq \emptyset$.
2. $e_j \in F(F(\ldots(F(F(\{e_i\}, X_1), X_2), \ldots, X_{m-1}), X_m)$.

In this case we say that $(X_1, X_2, \ldots, X_m)$ is a *connection* from e_i to e_j.

Along with the notation $X = (\xi_1, \ldots, \xi_{n-1})$ for any given tuple of elements of $\mathcal{B}\dot{\cup}\overline{\mathcal{B}}$, we will also write $\overline{X} := (\overline{\xi}_1, \ldots, \overline{\xi}_{n-1})$.

Lemma 2.7. *Let $(X_1, X_2, \ldots, X_{m-1}, X_m)$ be any connection from e_i to e_j, where e_i and e_j are arbitrary elements in $\mathcal{B}$, with $e_i \neq e_j$. Then the ordered list $(\overline{X}_m, \overline{X}_{m-1}, \ldots, \overline{X}_2, \overline{X}_1)$ is a connection from e_j to e_i.*

Proof. The proof will be done by induction on m. In the case $m = 1$ we have that $e_j \in F(\{e_i\}, X_1) = F(\{e_i\}, (a_{11}, \ldots, a_{1n-1}))$ implying that

$$e_i \in F(\{e_j\}, (\overline{a}_{11}, \ldots, \overline{a}_{1n-1})) = F(\{e_j\}, \overline{X}_1),$$

by 1. of Lemma 2.5. Thus $(\overline{X}_1)$ is a connection from e_j to e_i.

Admit now that the assertion holds for any connection with $m \geq 1$ elements, and let us show this assertion also holds for any connection

$$(X_1, X_2, \ldots, X_m, X_{m+1})$$

with $m + 1$ ($(n-1)$-tuples) elements. So, consider a connection $(X_1, X_2, \ldots, X_m, X_{m+1})$ from e_i to e_j. Let us begin by setting

$$U := F(F(\ldots(F(F(\{e_i\}, X_1), X_2), \ldots, X_{m-1}), X_m).$$

Applying 2. of Definition 2.6 we have that $e_j \in F(U, X_{m+1})$. Then, by 2. of Lemma 2.5, $F(\{e_j\}, \overline{X}_{m+1}) \cap U \neq \emptyset$. Admit that

$$x \in F(\{e_j\}, \overline{X}_{m+1}) \cap U \neq \emptyset. \tag{2}$$

Since $x \in U$ we have that $(X_1, X_2, \ldots, X_{m-1}, X_m)$ is a connection from e_i to x with m elements. Henceforth $(\overline{X}_m, \overline{X}_{m-1}, \ldots, \overline{X}_2, \overline{X}_1)$ connects x to e_i. From here, and by equation (2), we obtain

$$e_i \in F(F(\ldots(F(F(\{e_j\}, \overline{X}_{m+1}), \overline{X}_m), \ldots, \overline{X}_2), \overline{X}_1),$$

which means that

$$(\overline{X}_{m+1}, \overline{X}_m, \ldots, \overline{X}_2, \overline{X}_1)$$

connects e_j to e_i. $\square$

Proposition 2.8. *The relation $\sim$ in $\mathcal{B}$, defined by $e_i \sim e_j$ if and only if e_i is connected to e_j, is an equivalence relation.*

Proof. The relation $\sim$ is clearly reflexive (see (i) of Definition 2.6) and symmetric (see Lemma 2.7). Hence let us verify its transitivity.

Admit that $e_i, e_j, e_k \in \mathcal{B}$ are pairwise different such that $e_i \sim e_j$ and $e_j \sim e_k$ (the cases in which two among those elements are equal are trivial). Then there are connections $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_p)$ from e_i to e_j and from e_j to e_k, respectively. Therefore, $(X_1, \ldots, X_m, Y_1, \ldots, Y_p)$ is a connection from e_i to e_k showing the transitivity of $\sim$, and the result is proved. $\square$

Henceforth, by the above defined equivalence relation, we introduce the quotient set

$$\mathcal{B}/\sim := \{[e_i] : e_i \in \mathcal{B}\},$$

where $[e_i]$ stands for the set of elements in $\mathcal{B}$ which are connected to e_i.

For each $[e_i] \in \mathcal{B}/\sim$ we may introduce the linear subspace

$$V_{[e_i]} := \bigoplus_{e_j \in [e_i]} \mathbb{F}e_j,$$

allowing us to write

$$\mathbb{V} = \bigoplus_{[e_i] \in \mathcal{B}/\sim} V_{[e_i]}. \tag{3}$$

Next we show that this is a decomposition of $\mathbb{V}$ in pairwise f-orthogonal subspaces.

Lemma 2.9. *For any $[e_i], [e_j] \in \mathcal{B}/\sim$ with $[e_i] \neq [e_j]$, we have that*

$$f(\mathbb{V}, \ldots, V_{[e_i]}^{(k_1)}, \ldots, V_{[e_j]}^{(k_2)}, \ldots, \mathbb{V}) = 0, \tag{4}$$

for all $k_1, k_2 \in \{1, \ldots, n\} suchthat$ $k_1 \neq k_2$.

Proof. In order to prove equation (4) it is sufficient to show that

$$f(\xi_{\sigma(1)}, \ldots, V_{[e_i]}^{(k_1)}, \ldots, V_{[e_j]}^{(k_2)}, \ldots, \xi_{\sigma(n-2)}) = 0,$$

for any permutation $\sigma \in \mathbb{S}_{n-2}$, $\xi_1, \ldots, \xi_{n-2} \in \mathcal{B}$. Admit the opposite assertion. Then there are $e_k \in [e_i]$, $e_p \in [e_j]$ and $v \in \mathbb{V}^*$ such that

$$v = f(\xi_{\sigma(1)}, \ldots, e_k^{(k_1)}, \ldots, e_p^{(k_2)}, \ldots, \xi_{\sigma(n-2)}), \tag{5}$$

for some $\sigma \in \mathbb{S}_{n-2}$. By definition of F, from equation (5) we may deduce two facts:

$$\text{(i) } v \in F(\{e_k\}, (e_p, \xi_1, \ldots, \xi_{n-2})),$$
$$\text{(ii) } v \in F(\{e_p\}, (e_k, \xi_1, \ldots, \xi_{n-2})).$$

From (ii) and 1. of Lemma 2.5, we have

$$\text{(iii) } e_p \in F(\{v\}, (\overline{e}_k, \overline{\xi}_1, \ldots, \overline{\xi}_{n-2}))$$

From (i) and (iii), we observe that (X_1, X_2), where

$$X_1 = (e_p, \xi_1, \ldots, \xi_{n-2}) \text{ and } X_2 = \left(\overline{e}_k, \overline{\xi}_1, \ldots, \overline{\xi}_{n-2}\right),$$

is a connection from e_k to e_p. Thus, $[e_i] = [e_k] = [e_p] = [e_j]$, causing a contradiction. $\square$

As a consequence of Lemma 2.9 and equation (3) we have:

Proposition 2.10. *Given* $\mathbb{V}$ *and* f *as initially defined,* $\mathbb{V}$ *decomposes as the* f*-orthogonal direct sum of linear subspaces*

$$\mathbb{V} = \bigoplus_{[e_i]\in\mathcal{B}/\sim} V_{[e_i]}.$$

The family of linear subspaces of $\mathbb{V}$ formed by all of the $V_{[e_i]}$, $[e_i] \in \mathcal{B}/\sim$, which gives rise to the decomposition in Proposition 2.10, is not good enough for our purposes. So we need to introduce a new equivalence relation on this family, as follows.

We begin by observing that the above mentioned decomposition of $\mathbb{V}$ allows us to consider, for each $V_{[e_i]}$, the projection map

$$\Pi_{V_{[e_i]}} : \mathbb{V} \to V_{[e_i]}.$$

Also, let us consider these family of nonzero linear subspaces of $\mathbb{V}$,

$$\mathcal{F} := \{V_{[e_i]} : [e_i] \in \mathcal{B}/\sim\}.$$

Definition 2.11. We will say that $V_{[e_i]} \approx V_{[e_j]}$ if and only if either $V_{[e_i]} = V_{[e_j]}$ or there exists a subset

$$\{[\xi_1], [\xi_2], \dots, [\xi_m]\} \subset \mathcal{B}/\sim,$$

such that

(i) $[\xi_1] = [e_i]$ and $[\xi_m] = [e_j]$.

(ii)

$$\sum_{1\leq k_1<k_2\leq n} \Big(\Pi_{V_{[\xi_1]}}(f(\mathbb{V},\dots,V^{(k_1)}_{[\xi_2]},\dots,V^{(k_2)}_{[\xi_2]},\dots,\mathbb{V})) + \Pi_{V_{[\xi_2]}}(f(\mathbb{V},\dots,V^{(k_1)}_{[\xi_1]},\dots,V^{(k_2)}_{[\xi_1]},\dots,\mathbb{V}))\Big) \neq 0.$$

$$\sum_{1\leq k_1<k_2\leq n} \Big(\Pi_{V_{[\xi_2]}}(f(\mathbb{V},\dots,V^{(k_1)}_{[\xi_3]},\dots,V^{(k_2)}_{[\xi_3]},\dots,\mathbb{V})) + \Pi_{V_{[\xi_3]}}(f(\mathbb{V},\dots,V^{(k_1)}_{[\xi_2]},\dots,V^{(k_2)}_{[\xi_2]},\dots,\mathbb{V}))\Big) \neq 0.$$

$$\vdots$$

$$\sum_{1\leq k_1<k_2\leq n} \Big(\Pi_{V_{[\xi_{m-1}]}}(f(\mathbb{V},\dots,V^{(k_1)}_{[\xi_m]},\dots,V^{(k_2)}_{[\xi_m]},\dots,\mathbb{V})) + \Pi_{V_{[\xi_m]}}(f(\mathbb{V},\dots,V^{(k_1)}_{[\xi_{m-1}]},\dots,V^{(k_2)}_{[\xi_{m-1}]},\dots,\mathbb{V}))\Big) \neq 0.$$

Clearly $\approx$ is an equivalence relation on $\mathcal{F}$ and so we can introduce the quotient set

$$\mathcal{F}/\approx := \{[V_{[e_i]}] : V_{[e_i]} \in \mathcal{F}\}.$$

For each $[V_{[e_i]}] \in \mathcal{F}/\approx$, we denote by $\overbrace{V_{[e_i]}}$ the linear subspace of $\mathbb{V}$

$$\overbrace{V_{[e_i]}} := \bigoplus_{V_{[e_j]} \in [V_{[e_i]}]} V_{[e_j]}.$$

By equation (3) and the definition of $\approx$, we clearly have

$$\mathbb{V} = \bigoplus_{[V_{[e_i]}] \in \mathcal{F}/\approx} \overbrace{V_{[e_i]}}. \tag{6}$$

Also, we can assert by Lemma 2.9 that

$$f(\mathbb{V}, \dots, \overbrace{V_{[e_i]}}^{(k_1)}, \dots, \overbrace{V_{[e_j]}}^{(k_2)}, \dots, \mathbb{V}) = 0$$

when $[V_{[e_i]}] \neq [V_{[e_j]}]$ in $\mathcal{F}/\approx$, for all $k_1, k_2 \in \{1, \dots, n\}$ such that $k_1 \neq k_2$.

Proposition 2.12. *For any $[V_{[e_i]}] \in \mathcal{F}/\approx$, $\overbrace{V_{[e_i]}}$ is a strongly f-invariant linear subspace of $\mathbb{V}$.*

Proof. We begin by proving that

$$f(\mathbb{V}, \dots, \overbrace{V_{[e_i]}}^{(k_1)}, \dots, \overbrace{V_{[e_i]}}^{(k_2)}, \dots, \mathbb{V}) \subset \overbrace{V_{[e_i]}}. \tag{7}$$

Indeed, in case some $0 \neq w \in f(\mathbb{V}, \dots, \overbrace{V_{[e_i]}}^{(k_1)}, \dots, \overbrace{V_{[e_i]}}^{(k_2)}, \dots, \mathbb{V})$, decomposition (6) allows us to write

$$w = w_1 + w_2 + \cdots + w_m$$

for some $0 \neq w_j \in \overbrace{V_{[\xi_j]}}$ for $j = 1, \dots, m$ and $\xi_j \in \mathcal{B}$. Observe now that Lemma 2.9 gives us that there exist nonzero $x, y \in V_{[e_k]}$ with $V_{[e_k]} \subset \overbrace{V_{[e_i]}}$ and $z_1, \dots z_{n-2} \in \mathbb{V}$, such that

$$0 \neq w = f(z_1, \dots, x^{(k_1)}, \dots, y^{(k_2)}, \dots, z_{n-2}) \tag{8}$$

Let us consider $0 \neq w_j \in \overbrace{V_{[\xi_j]}}$, being so $w_j \in V_{[e_r]}$ for some $V_{[e_r]} \subset \overbrace{V_{[\xi_j]}}$. By equation (8) we have

$$\Pi_{V_{[e_r]}}(f(z_1, \ldots, x^{(k_1)}, \ldots, y^{(k_2)}, \ldots, z_{n-2})) = w_j \neq 0.$$

That is

$$\Pi_{V_{[e_r]}}(f(\mathbb{V}, \ldots, V_{[e_k]}^{(k_j)}, \ldots, V_{[e_k]}^{(k_2)}, \ldots, \mathbb{V})) \neq 0$$

and we get that the set $\{[e_k], [e_r]\}$ gives us $V_{[e_k]} \approx V_{[e_r]}$. Hence

$$V_{[e_i]} \approx V_{[e_k]} \approx V_{[e_r]} \approx V_{[\xi_j]}$$

and we conclude $V_{[\xi_j]} \subset \overbrace{V_{[e_i]}}$. From here $w_j \in \overbrace{V_{[e_i]}}$ and so $w \in \overbrace{V_{[e_i]}}$. Consequently, the inclusion (7) holds, as desired.

Finally, by decomposition (6), Lemma 2.9 and equation (7), we have the following inclusion

$$\sum_{k=1}^{n} f(\mathbb{V}, \ldots, \overbrace{V_{[e_i]}}^{(k)}, \ldots, \mathbb{V}) \subset \overbrace{V_{[e_i]}},$$

and thus $f(\mathbb{V}, \ldots, \overbrace{V_{[e_i]}}^{(k)}, \ldots, \mathbb{V}) \subset \overbrace{V_{[e_i]}}$ for all $k \in \{1, \ldots, n\}$. $\square$

Theorem 2.13. *Let $\mathbb{V}$ be a linear space equipped with an n-linear map $f : \mathbb{V} \times \ldots \times \mathbb{V} \to \mathbb{V}$. For any basis $\mathcal{B} = \{e_i : i \in I\}$ of $\mathbb{V}$ we have that $\mathbb{V}$ decomposes as the f-orthogonal direct sum of strongly f-invariant linear subspaces*

$$\mathbb{V} = \bigoplus_{[V_{[e_i]}] \in \mathcal{F}/\approx} \overbrace{V_{[e_i]}}.$$

Proof. Consider the decomposition, as direct sum of linear subspaces

$$\mathbb{V} = \bigoplus_{[V_{[e_i]}] \in \mathcal{F}/\approx} \overbrace{V_{[e_i]}},$$

given by equation (6). Now Lemma 2.9 shows that this decomposition is f-orthogonal and Proposition 2.12 that all of the linear subspaces $\overbrace{V_{[e_i]}}$ are strongly f-invariant. $\square$

3 On the relation among the decompositions given by different choices of bases

Observe that the decomposition of $\mathbb{V}$ as an f-orthogonal direct sum of strongly f-invariant linear subspaces given by Theorem 2.13 is related with the initial choice of the basis. Indeed, as it was exemplified in [1], for $n = 2$, two different bases of $\mathbb{V}$ may lead to two different of those decompositions of $\mathbb{V}$. The same happens in the n-ary case, with $n > 2$, as shown in the following example.

Let $\mathbb{V}$ be the $\mathbb{R}$-linear space $\mathbb{V} := \mathbb{R}^4$ equipped with the n-linear map $f : \mathbb{R}^4 \times \cdots \times \mathbb{R}^4 \to \mathbb{R}^4$ defined as

$$f(\overline{x}_1, \ldots, \overline{x}_n) = (x_{11}x_{21}, x_{11}x_{21}, 0, 0),$$

where

$$\overline{x}_i = (x_{i1}, \ldots, x_{i4})$$

for each $i \in \{1, \ldots, n\}$.

Let us consider the following two bases of $\mathbb{R}^4$:

$$\mathcal{B} := \{e_1, \ldots, e_4\},$$

that is, the canonical basis, and

$$\mathcal{B}' := \{(1, 0, 1, 0), (1, 0, -1, 0), e_2, e_4\}.$$

Then it is possible to observe that the decomposition of $\mathbb{V} = \mathbb{R}^4$, given in Theorem 2.13 with respect to the basis $\mathcal{B}$ is given by

$$\mathbb{R}^4 = (\mathbb{R}e_1 \oplus \mathbb{R}e_2) \bigoplus (\mathbb{R}e_3) \bigoplus (\mathbb{R}e_4).$$

However, the same kind of decomposition with respect to $\mathcal{B}'$ is given by

$$\mathbb{R}^4 = (\mathbb{R}(1, 0, 1, 0) \oplus \mathbb{R}(1, 0, -1, 0) \oplus \mathbb{R}e_2) \bigoplus (\mathbb{R}e_4).$$

Thus, it will be an interesting task to find a sufficient condition for two different decompositions of a linear space $\mathbb{V}$, induced by an n-linear map f and with respect to two different bases of $\mathbb{V}$, being isomorphic. The following notion will help us in this purpose.

Definition 3.1. Let $\mathbb{V}$ be a linear space equipped with an n-linear map $f : \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{V}$ and consider

$$\Gamma := \ \mathbb{V} = \bigoplus_{i \in I} V_i \text{ and } \Gamma' := \ \mathbb{V} = \bigoplus_{j \in J} W_j$$

two decompositions of $\mathbb{V}$ as an f-orthogonal direct sum of strongly f-invariant linear subspaces. It is said that Γ and Γ' are *isomorphic* if there exists a linear isomorphism $g : \mathbb{V} \to \mathbb{V}$ satisfying

$$f(g(v_1), \ldots g(v_n)) = g(f(v_1, \ldots, v_n))$$

for any $v_1, \ldots, v_n \in \mathbb{V}$, and a bijection $\sigma : I \to J$ such that

$$g(V_i) = W_{\sigma(i)}$$

for any $i \in I$.

Lemma 3.2. *Let $\mathbb{V}$ be a linear space equipped with an n-linear map $f : \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{V}$ and consider $\mathcal{B} = \{e_i : i \in I\}$ a fixed basis of $\mathbb{V}$. Let also $g : \mathbb{V} \to \mathbb{V}$ be a linear isomorphism satisfying*

$$f\left(g\left(\xi_1\right), \ldots, g\left(\xi_n\right)\right) = g\left(f\left(\xi_1, \ldots, \xi_n\right)\right)$$

for any $\xi_i \in \mathcal{B}$. Then for any $U \in \mathcal{P}(\mathbb{V}^*)$ and $\xi_k \in \mathcal{B}$, with $k \in I$, the following assertions hold:

(i) $g\left(F\left(U, \left(\xi_1, \ldots, \xi_{n-1}\right)\right)\right) = F\left(g(U), \left(g\left(\xi_1\right), \ldots, g\left(\xi_{n-1}\right)\right)\right)$,

(ii) $g\left(F\left(U, \left(\overline{\xi}_1, \ldots, \overline{\xi}_{n-1}\right)\right)\right) = F\left(g(U), \left(\overline{g\left(\xi_1\right)}, \ldots, \overline{g\left(\xi_{n-1}\right)}\right)\right)$,

where F is the mapping defined by equation (1).

Proof. (i) We have

$$g\left(F\left(U, \left(\xi_1, \ldots, \xi_{n-1}\right)\right)\right) = \left(\bigcup_{\substack{k \in \{1, \ldots, n\} \\ \sigma \in \mathbb{S}_{n-1}}} \left\{ g\left(f(\xi_{\sigma(1)}, \ldots, u^{(k)}, \ldots, \xi_{\sigma(n-1)})\right) : u \in U \right\} \right) \backslash \{0\}$$

$$= \left(\bigcup_{\substack{k \in \{1,\dots,n\} \\ \sigma \in \mathbb{S}_{n-1}}} \left\{ f\left(g\left(\xi_{\sigma(1)}\right), \dots, g(u)^{(k)}, \dots, g\left(\xi_{\sigma(n-1)}\right)\right) : u \in U \right\} \right) \setminus \{0\}$$

$$= F\left(g(U), \left(g\left(\xi_1\right), \dots, g\left(\xi_{n-1}\right)\right)\right).$$

(ii) In this case we have

$$g\left(F\left(U, \left(\overline{\xi}_1, \dots, \overline{\xi}_{n-1}\right)\right)\right)$$

$$= \left(\bigcup_{\substack{k \in \{1,\dots,n\} \\ \sigma \in \mathbb{S}_{n-1}}} \left\{ u \in \mathbb{V} : f\left(\xi_{\sigma(1)}, \dots, (g^{-1}(u))^{(k)}, \dots, \xi_{\sigma(n-1)}\right) \in U \right\} \right) \setminus \{0\}$$

$$= \left(\bigcup_{\substack{k \in \{1,\dots,n\} \\ \sigma \in \mathbb{S}_{n-1}}} \left\{ u \in \mathbb{V} : f\left(g\left(\xi_{\sigma(1)}\right), \dots, u^{(k)}, \dots, g\left(\xi_{\sigma(n-1)}\right)\right) \in g(U) \right\} \right) \setminus \{0\}$$

$$= F\left(g(U), \left(\overline{g\left(\xi_1\right)}, \dots, \overline{g\left(\xi_{n-1}\right)}\right)\right).$$

Observe that in both cases we took into account Remark 2.4. □

Proposition 3.3. *Let $\mathbb{V}$ be a linear space equipped with an n-linear map $f : \mathbb{V} \times \dots \times \mathbb{V} \to \mathbb{V}$ and consider $\mathcal{B} = \{e_i : i \in I\}$ a fixed basis of $\mathbb{V}$. Further, admit that $g : \mathbb{V} \to \mathbb{V}$ is a linear isomorphism satisfying*

$$f\left(g\left(\xi_1\right), \dots, g\left(\xi_n\right)\right) = g\left(f\left(\xi_1, \dots, \xi_n\right)\right)$$

for any $\xi_i \in \mathcal{B}$. Then the decompositions

$$\Gamma := \mathbb{V} = \bigoplus_{[V_{[e_i]}] \in \mathcal{F}/\approx} \widetilde{V_{[e_i]}} \text{ and } \Gamma' := \mathbb{V} = \bigoplus_{[V_{[g(e_i)]}] \in \mathcal{F}'/\approx} \widetilde{V_{[g(e_i)]}},$$

corresponding to the choices of $\mathcal{B}$ and $\mathcal{B}' := \{g(e_i) : i \in I\}$ respectively in Theorem 2.13, are isomorphic.

Proof. Firstly, let us observe that, according to the previous result, we may state that if e_i is connected to some e_j, for some $i, j \in I$, $e_i, e_j \in \mathcal{B}$ through a connection $(X_1, X_2, \ldots, X_m)$, where $X_i = (a_{i1}, \ldots, a_{in-1})$ such that $a_{ik} \in \mathcal{B}\dot{\cup}\overline{\mathcal{B}}$, $i \in \{1, \ldots, m\}$, $k \in \{1, \ldots, n-1\}$, then $g(e_i)$ is connected to $g(e_j)$ through the connection $(g(X_1), g(X_2), \ldots, g(X_n))$, where $g(X_i) := (g(a_{i1}), \ldots, g(a_{in-1}))$ and $g(a_{ik}) \in \mathcal{B}' \cup \overline{\mathcal{B}'}$, (where $g(\bar{e}_k) := \overline{g(e_k)}$). Thus, it is possible to conclude that

$$g(V_{[e_i]}) = V_{[g(e_i)]}$$

for any $[e_i] \in \mathcal{B}/\sim$. Further, it is also clear that the mapping μ such that

$$\mu(V_{[e_i]}) = V_{[g(e_i)]}$$

defines a bijection between the families $\mathcal{F} := \{V_{[e_i]} : [e_i] \in \mathcal{B}/\sim\}$ and $\mathcal{F}' := \{V_{[g(e_i)]} : [g(e_i)] \in \mathcal{B}'/\sim\}$.

Now, from Lemma 3.2 we have

$$g\left(\Pi_{V_{[e_i]}}(f(\mathbb{V}, \ldots, V_{[e_j]}^{(k_1)}, \ldots, V_{[e_j]}^{(k_2)}, \ldots, \mathbb{V})\right) = \Pi_{V_{[g(e_i)]}}\left(f(\mathbb{V}, \ldots, V_{[g(e_j)]}^{(k_1)}, \ldots, V_{[g(e_j)]}^{(k_2)}, \ldots, \mathbb{V})\right)$$

for $i, j \in I$ and $k_1, k_2 \in \{1, \ldots, n\}$, with $k_1 < k_2$. This allows to deduce that

$$g(\widetilde{V_{[e_i]}}) = \widetilde{V_{[g(e_i)]}} \tag{9}$$

for any $[V_{[e_i]}] \in \mathcal{F}/\approx$, which induces a second bijection, σ, now between the families $\mathcal{F}/\approx$ and $\mathcal{F}'/\approx$ given by

$$\sigma([V_{[e_i]}]) = [V_{[g(e_i)]}]. \tag{10}$$

From equations (9) and (10) we conclude that the decompositions Γ and Γ' are isomorphic. $\square$

Being f an n-linear map on $\mathbb{V}$, the following set

$$\mathrm{O}_f(\mathbb{V}) = \{g \in \mathrm{GL}(\mathbb{V}) : f(g(v_1), \ldots, g(v_n)) = g(f(v_1, \ldots, v_n)) \text{ for any } v_1, \ldots, v_n \in \mathbb{V}\},$$

(where $\mathrm{GL}(\mathbb{V})$ denotes the group of all linear isomorphisms of $\mathbb{V}$), is known as the *orbit* of $\mathbb{V}$ (associated to f). We have that $\mathrm{O}_f(\mathbb{V})$ is a subgroup of $\mathrm{GL}(\mathbb{V})$. If we also denote by $\mathfrak{B}$ the set of all bases of $\mathbb{V}$ we get the action

$$\begin{aligned} \mathrm{O}_f(\mathbb{V}) \times \mathfrak{B} &\to \mathfrak{B} \\ (g, \{e_i\}_{i \in I}) &\mapsto \{g(e_i)\}_{i \in I}. \end{aligned} \tag{11}$$

The previous result states that if two bases $\mathcal{B}$ and $\mathcal{B}'$ of $\mathbb{V}$ belong to the same orbit under the action (11), then they induce two isomorphic decompositions of $\mathbb{V}$. Finally, this can be stated as follows.

Corollary 3.4. *Let $\mathbb{V}$ be a linear space equipped with an n-linear map $f : \mathbb{V}\times\cdots\times\mathbb{V} \to \mathbb{V}$ and fix two bases $\mathcal{B} = \{e_i : i \in I\}$ and $\mathcal{B}' = \{u_i : i \in I\}$ of $\mathbb{V}$. Suppose there exists a bijection $\mu : I \to I$ such that the linear isomorphism $g : \mathbb{V} \to \mathbb{V}$ determined by $g(e_i) := u_{\mu(i)}$ for any $i \in I$, satisfies*

$$f\left(g(v_1), \ldots, u_{\mu(i)}^{(k_1)}, \ldots, u_{\mu(j)}^{(k_2)}, \ldots, g(v_{n-2})\right) = g(f(v_1, \ldots, e_i^{(k_1)}, \ldots, e_j^{(k_2)}, \ldots, v_{n-2}))$$

for any $i, j \in I$, $k_1, k_2 \in \{1, \ldots, n\}$, with $k_1 < k_2$. Then the decompositions

$$\Gamma := \mathbb{V} = \bigoplus_{[V_{[e_i]}] \in \mathcal{F}/\approx} \widehat{V_{[e_i]}} \text{ and } \Gamma' := \mathbb{V} = \bigoplus_{[V_{[u_i]}] \in \mathcal{F}'/\approx} \widehat{V_{[u_i]}},$$

corresponding to the choices of $\mathcal{B}$ and $\mathcal{B}'$, respectively, in Theorem 2.13, are isomorphic.

4 A characterization of the f-simplicity of the components

In this section we intend to establish a characterization theorem on the f-simplicity of the linear subspaces $\widehat{V_{[e_i]}}$, which appear in the decomposition of $\mathbb{V}$ given in Theorem 2.13.

Let us begin by recalling several concepts from the theory of algebras.

Let $\mathbb{A}$ be an algebra equipped with an n-ary multiplication $[., \ldots, .] : \mathbb{A} \times \cdots \times \mathbb{A} \to \mathbb{A}$ and $\mathcal{B}$ a basis of $\mathbb{A}$. The basis $\mathcal{B}$ is said to be an *i-division basis* if for any $e_i \in \mathcal{B}$ and $b_1, \ldots, b_{n-1} \in \mathbb{A}$ such that

$$[b_1, \ldots, e_i^{(k)}, \ldots, b_{n-1}] = w \neq 0$$

for some $w \in \mathbb{A}$ and $k \in \{1, \ldots, n\}$ we have that $e_i, b_1, \ldots, b_{n-1} \in \mathcal{I}(w)$, where $\mathcal{I}(w)$ denotes the *ideal of $\mathbb{A}$ generated by w*.

The above notion can be generalized to the case of a linear space $\mathbb{V}$ equipped with an n-linear map $f : \mathbb{V}\times\cdots\times\mathbb{V} \to \mathbb{V}$. We refer to the minimal strongly f-invariant subspace of $\mathbb{V}$ that contains v as the *strongly f-invariant subspace of* $\mathbb{V}$ *generated by* v, and will be denoted by $\mathcal{I}(v)$. Observe that the sum of two strongly f-invariant subspaces of $\mathbb{V}$ is also a strongly f-invariant subspace, and that the whole $\mathbb{V}$ is a trivial strongly f-invariant subspace.

Definition 4.1. Let $\mathbb{V}$ be a linear space, $\mathcal{B} = \{e_i : i \in I\}$ a fixed basis of $\mathbb{V}$ and $f : \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{V}$ an n-linear map. It is said that $\mathcal{B}$ is an *i-division basis* of $\mathbb{V}$ respect to f, if for any $e_i \in \mathcal{B}$ and $b_1, \ldots, b_{n-1} \in \mathbb{V}$ such that

$$f\left(b_1, \ldots, e_i^{(k)}, \ldots, b_{n-1}\right) = w \neq 0$$

for some $k \in \{1, \ldots, n\}$ we have that $e_i, b_1, \ldots, b_{n-1} \in \mathcal{I}(w)$, where $\mathcal{I}(w)$ denotes the strongly f-invariant subspace of $\mathbb{V}$ generated by w.

Let us return to the decomposition of the linear space $\mathbb{V}$, given an n-linear map $f : \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{V}$ and fixed a basis $\mathcal{B}$,

$$\mathbb{V} = \bigoplus_{[V_{[e_i]}] \in \mathcal{F}/\approx} \widehat{V_{[e_i]}}$$

as deduced by Theorem 2.13. For any $\widehat{V_{[e_i]}}$ we can restrict f to the n-linear map

$$f' : \widehat{V_{[e_i]}} \times \cdots \times \widehat{V_{[e_i]}} \to \widehat{V_{[e_i]}}$$

and consider on $\widehat{V_{[e_i]}}$ the basis $\mathcal{B}' := \mathcal{B} \cap \widehat{V_{[e_i]}}$. Then we can assert:

Theorem 4.2. *The linear space $\widehat{V_{[e_i]}}$ is f'-simple if and only if* $\mathrm{Ann}(f') = 0$ *and $\mathcal{B}'$ is an i-division basis of $\widehat{V_{[e_i]}}$ with respect to f'.*

Proof. Suppose that $\widehat{V_{[e_i]}}$ is f'-simple. Observe firstly that $\mathrm{Ann}(f')$ is a strongly f'-invariant subspace of $\widehat{V_{[e_i]}}$, and thus $\mathrm{Ann}(f') = 0$. Additionally, if we consider some $e_j \in \mathcal{B}'$ and $b_1, \ldots, b_{n-1} \in \widehat{V_{[e_i]}}$ such that

$$f'\left(b_1, \ldots, e_j^{(k)}, \ldots, b_{n-1}\right) = w \neq 0$$

for some $k \in \{1, \ldots, n\}$, since $\widehat{V_{[e_i]}}$ is f'-simple, we have

$$\mathcal{I}(w) = \widehat{V_{[e_i]}}$$

and so $e_j, b_1, \dots, b_{n-1} \in \mathcal{I}(w)$. Thus, the basis $\mathcal{B}'$ is an i-division basis of $\widehat{V_{[e_i]}}$ with respect to f'.

Conversely, let us suppose that $\mathrm{Ann}(f') = 0$ and that the set $\mathcal{B}'$ is an i-division basis of $\widehat{V_{[e_i]}}$ with respect to f'. Consider any nonzero strongly f'-invariant linear subspace W of $\widehat{V_{[e_i]}}$ and take some nonzero $w \in W$. Since $\mathrm{Ann}(f') = 0$, there are nonzero elements

$$\xi_1, \dots, \xi_{n-1} \in \mathcal{B}'$$

such that

$$0 \neq f\left(\xi_1, \dots, w^{(j)}, \dots, \xi_{n-1}\right) \in W$$

for some $j \in \{1, \dots, n\}$. Since $\mathcal{B}'$ is an i-division basis, we get

$$\xi_k \in \mathcal{I}(w) \subset W, \tag{12}$$

for all $k \in \{1, \dots, n-1\}$.

In order to deduce the f'-simplicity of $\widehat{V_{[e_i]}}$, we must arrive to $W = \widehat{V_{[e_i]}}$, which can be proved following two main steps:
(I) Show that $V_{[\xi_1]}, \dots, V_{[\xi_{n-1}]} \subset W$;
(II) Recalling that

$$\widehat{V_{[\xi_k]}} := \bigoplus_{V_{[\nu_j]} \in [V_{[\xi_k]}]} V_{[\nu_j]},$$

show that, for each k all $V_{[\nu_j]} \approx V_{[\xi_k]}$ also satisfy $V_{[\nu_j]} \subset W$.

(I) In order to prove that $V_{[\xi_k]} \subset W$ for each $k \in \{1, \dots, n-1\}$, we have to show that for any $\nu_j \in [\xi_k]$ such that $\nu_j \neq \xi_k$, we must conclude that $\nu_j \in W$. It is clear that ξ_k is connected to any $\nu_j \in [\xi_k]$, and thus there is a connection $(X_1, X_2, \dots, X_m)$, where $X_i = (a_{i1}, \dots, a_{in-1})$ such that $a_{il} \in \mathcal{B}\dot{\cup}\overline{\mathcal{B}}$, $i \in \{1, \dots, m\}$, $l \in \{1, \dots, n-1\}$, from ξ_k to ν_j.

Recall that we are dealing with an f-orthogonal and strongly f-invariant decomposition of $\mathbb{V}$ (by Theorem 2.13). Thus, we may claim that the elements a_{il} satisfy

$$a_{il} \in \mathcal{B}'\dot{\cup}\overline{\mathcal{B}'}, \tag{13}$$

and that the whole connection process from ξ_k to ν_j can be deduced in $\widehat{V_{[e_i]}}$.

According to Definition 2.6, we have that

$$F(\{\xi_k\}, X_1) = F\left(\{\xi_k\}, (a_{11}, \dots, a_{1n-1})\right) \neq \emptyset.$$

Given an arbitrary $x \in F(\{\xi_k\}, X_1)$, $x \neq 0$, there are two cases to discuss.

First case: $a_{1l} \in \mathcal{B}'$, $l = 1, \ldots, n-1$ and so

$$0 \neq x = f\left(a_{11}, \ldots, \xi_k^{(r)}, \ldots, a_{1n-1}\right),$$

for some $r \in \{1, \ldots, n\}$.
Second case: $a_{1l} \in \overline{\mathcal{B}'}$, $l = 1, \ldots, n-1$ and so $x \in \widehat{V_{[e_i]}}$ is such that:

$$f\left(\overline{a}_{11}, \ldots, x^{(r)}, \ldots, \overline{a}_{1n-1}\right) = \xi_k,$$

for some $r \in \{1, \ldots, n\}$.

Consider the first case. As a consequence of the inclusion (12), we obtain $x \in W$.

Consider now the second case. By the i-division property of the basis $\mathcal{B}'$ and due to inclusion (12) we conclude that $x \in \mathcal{I}(\xi_k) \subset W$.

So, in both cases we have shown that

$$F(\{\xi_k\}, X_1) \subset W. \tag{14}$$

By the connection definition, we have

$$F(F(\{\xi_k\}, X_1), X_2) \neq \emptyset,$$

where $F(\{\xi_k\}, a_1) \subset W$ as seen in (14).

Given an arbitrary $t \in F(F(\{\xi_k\}, X_1), X_2)$, as before, we have two cases to distinguish. In the first one $a_{2l} \in \mathcal{B}'$, $l = 1, \ldots, n-1$ and so there exists $z \in F(\{\xi_k\}, X_1)$ such that

$$0 \neq t = f\left(a_{21}, \ldots, z^{(r')}, \ldots, a_{2n-1}\right),$$

for some $r' \in \{1, \ldots, n\}$.
In the second one $a_{2l} \in \overline{\mathcal{B}'}$, and then there exists $z \in F(\{\xi_k\}, X_1)$ such that $0 \neq f(\overline{a}_{21}, \ldots, t^{(r')}, \ldots, \overline{a}_{2n-1}) = z$.

In the first case the inclusion (14) shows that $t \in W$. In the second case the i-division property of $\mathcal{B}'$ gives us that $t \in \mathcal{I}(z) \subset W$.

In both cases, we have

$$F(F(\{\xi_k\}, X_1), X_2) \subset W.$$

Iterating this argument on the connection (13), we obtain that

$$\nu_j \in F(F(\dots(F(F(\{\xi_k\}, X_1), X_2), \dots, X_{m-1}), X_m) \subset W$$

and so we can assert that

$$V_{[\xi_k]} \subset W. \tag{15}$$

(II) To finish the proof, we must show that all $V_{[\nu_j]}$ such that $V_{[\nu_j]} \approx V_{[\xi_k]}$ verifies $V_{[\nu_j]} \subset W$.

Under the above assumption, there exists a subset

$$\{[\xi_k], [\nu_2], \dots, [\nu_j]\} \subset \mathcal{B}/\sim \tag{16}$$

satisfying the conditions in Definition 2.11. From here,

$$\sum_{1 \leq i < i' \leq n} \Big(\Pi_{V_{[\xi_k]}}(f(\mathbb{V}, \dots, V_{[\nu_2]}^{(i)}, \dots, V_{[\nu_2]}^{(i')}, \dots, \mathbb{V})) + \Pi_{V_{[\nu_2]}}(f(\mathbb{V}, \dots, V_{[\xi_k]}^{(i)}, \dots, V_{[\xi_k]}^{(i')}, \dots, \mathbb{V}))\Big) \neq 0.$$

Therefore, there are $i, i' \in \{1, \dots, n\}$ with $i < i'$, such that

$$\Pi_{V_{[\nu_2]}}(f(\mathbb{V}, \dots, V_{[\xi_k]}^{(i)}, \dots, V_{[\xi_k]}^{(i')}, \dots, \mathbb{V})) \neq 0$$

or

$$\Pi_{V_{[\xi_k]}}(f(\mathbb{V}, \dots, V_{[\nu_2]}^{(i)}, \dots, V_{[\nu_2]}^{(i')}, \dots, \mathbb{V})) \neq 0.$$

Consider the first case, in which

$$\Pi_{V_{[\nu_2]}}(f(\mathbb{V}, \dots, V_{[\xi_k]}^{(i)}, \dots, V_{[\xi_k]}^{(i')}, \dots, \mathbb{V})) \neq 0.$$

Then there exist $e'_k, e''_k \in [\xi_k]$ and $b_1, \dots, b_{n-2} \in \mathbb{V}$ such that

$$0 \neq f(b_1, \dots, {e'_k}^{(i)}, \dots, {e''_k}^{(i')}, \dots, b_{n-2}) = x_2 + c$$

where $0 \neq x_2 \in V_{[\nu_2]}$ and $c \in \bigoplus\limits_{[\nu_j] \neq [\nu_2]} V_{[\nu_j]}$.

Since $\mathrm{Ann}(f') = 0$, and taking into account Lemma 2.9, there exist $e'_{21}, \dots, e'_{2n-1} \in [\nu_2]$ such that

$$0 \neq f(e'_{21}, \dots, {x_2}^{(r)}, \dots, e'_{2n-1}) = q$$

for some $r \in \{1,\dots,n\}$. By Lemma 2.9 and (15) we have that

$$0 \neq f(e'_{21},\dots,f(b_1,\dots,{e'_k}^{(i)},\dots,{e''_k}^{(i')},\dots,b_{n-2})^{(r)},\dots,e'_{2n-1}) =$$

$$f(e'_{21},\dots,(x_2+c)^{(r)},\dots,e'_{2n-1}) = f(e'_{21},\dots,{x_2}^{(r)},\dots,e'_{2n-1}) = q \in W.$$

From here, by the i-division property of $\mathcal{B}'$ we conclude that

$$e'_{21},\dots,e'_{2n-1} \in \mathcal{I}(q) \subset W.$$

Concerning the second case, recall that we have

$$\Pi_{V_{[\xi_k]}}(f(\mathbb{V},\dots,V^{(i)}_{[\nu_2]},\dots,V^{(i')}_{[\nu_2]},\dots,\mathbb{V})) \neq 0.$$

Similarly to the first case, there exist $e'_2, e''_2 \in [\nu_2]$ and $b_1,\dots,b_{n-2} \in \mathbb{V}$ such that

$$0 \neq f(b_1,\dots,{e'_2}^{(i)},\dots,{e''_2}^{(i')},\dots,b_{n-2}) = x_k + d$$

where $0 \neq x_k \in V_{[\xi_k]}$ and $d \in \bigoplus\limits_{[\nu_j]\neq[\xi_k]} V_{[\nu_j]}$. Again, since $\mathrm{Ann}(f') = 0$, there exist $e'_{k1},\dots,e'_{kn-1} \in [\xi_k]$ such that

$$0 \neq f(e'_{k1},\dots,{x_k}^{(r)},\dots,e'_{kn-1}) = s$$

for some $r \in \{1,\dots,n\}$.

By Lemma 2.9 and inclusion (15) we have that

$$0 \neq f(e'_{k1},\dots,f(b_1,\dots,{e'_2}^{(i)},\dots,{e''_2}^{(i')},\dots,b_{n-2})^{(r)},\dots,e'_{kn-1}) =$$

$$f(e'_{k1},\dots,(x_k+d)^{(r)},\dots,e'_{kn-1}) = f(e'_{k1},\dots,{x_k}^{(r)},\dots,e'_{kn-1}) = s \in W.$$

Applying the i-division property of $\mathcal{B}'$ this leads to

$$f(b_1,\dots,{e'_2}^{(i)},\dots,{e''_2}^{(i')},\dots,b_{n-2}) \in \mathcal{I}(s) \subset W.$$

A second application of the i-division property of $\mathcal{B}'$ allows us to write $e'_2, e''_2 \in W$.

At this point, we have shown in both cases that there are elements in $[\nu_2]$ belonging to W. Hence by using the same previous argument as done with ξ_k, (see inclusions (12) and (15)), we get that

$$V_{[\nu_2]} \subset W.$$

It is clear that this reasoning can be repeated for all other elements of the set (16). Henceforth

$$V_{[\nu_j]} \subset W$$

and consequently, since

$$\widehat{V_{[e_i]}} = \widehat{V_{[\xi_k]}} := \bigoplus_{V_{[e_j]} \in [V_{[\xi_k]}]} V_{[e_j]}$$

we proved that

$$\widehat{V_{[e_i]}} = W,$$

that is $\widehat{V_{[e_i]}}$ is f-simple. □

Remark 4.3. The above result can be restated as follows.

The linear space $\widehat{V_{[e_i]}}$ *is* f'*-simple if and only if* $\mathrm{Ann}(f') = 0$ *and every non-zero element in* $\widehat{V_{[e_i]}}$ *is an* i*-division element with respect to* f'*.*

5 Application to the structure theory of arbitrary n-ary algebras

In this section we will apply the results obtained in the previous sections to the structure theory of arbitrary n-ary algebras.

We will denote by $\mathfrak{A}$ an arbitrary n-ary algebra in the sense that there are no restrictions on the dimension of the algebra nor on the base field $\mathbb{F}$, and that no specific identity on the product (n-Lie (Filippov) [9], n-ary Jordan [10], n-ary Malcev [11], etc.) is supposed. That is, $\mathfrak{A}$ is just a linear space over $\mathbb{F}$ endowed with a n-linear map

$$[\cdot, \ldots, \cdot] : \mathfrak{A} \times \ldots \times \mathfrak{A} \to \mathfrak{A}$$

$$(x_1,\dots,x_n)\mapsto [x_1,\dots,x_n]$$

called *the product* of $\mathfrak{A}$.

We recall that given an n-ary algebra $(\mathfrak{A},[\cdot,\dots,\cdot])$, a *subalgebra* of $\mathfrak{A}$ is a linear subspace $\mathfrak{B}$ closed for the product. That is, such that $[\mathfrak{B},\dots,\mathfrak{B}]\subset\mathfrak{B}$. A linear subspace $\mathfrak{I}$ of $\mathfrak{A}$ is called an *ideal* of $\mathfrak{A}$ if $[\mathfrak{A},\dots,\mathfrak{I}^{(r)},\dots,\mathfrak{A}]\subset\mathfrak{I}$, for all $r\in\{1,\dots,n\}$. An n-ary algebra $\mathfrak{A}$ is said to be *simple* if its product is nonzero and its only ideals are $\{0\}$ and $\mathfrak{A}$. We finally recall that the *annihilator* of the algebra $(\mathfrak{A},[.,\dots,.])$ is defined as the linear subspace

$$\mathrm{Ann}(\mathfrak{A})=\{x\in\mathfrak{A}:[\mathfrak{A},\dots,x^{(k)},\dots,\mathfrak{A}]=0,\ \text{for all } k\in\{1,\dots,n\}\ \}.$$

If we fix any basis $\mathcal{B}=\{e_i\}_{i\in I}$ of $\mathfrak{A}$, and denote the product $[.,\dots,.]$ of $\mathfrak{A}$ as f, Theorem 2.13 applies to get that $\mathfrak{A}$ decomposes as the f-orthogonal direct sum of strongly f-invariant linear subspaces

$$\mathfrak{A}=\bigoplus_{[\mathfrak{A}_{[e_i]}]\in\mathcal{F}/\approx}\widehat{\mathfrak{A}_{[e_i]}}.$$

Now observe that the f-orthogonality of the linear subspaces means that, when $[\mathfrak{A}_{[e_i]}]\neq[\mathfrak{A}_{[e_j]}]$, we have

$$[\mathfrak{A},\dots,\widehat{\mathfrak{A}_{[e_i]}}^{(k_1)},\dots,\widehat{\mathfrak{A}_{[e_j]}}^{(k_2)},\dots,\mathfrak{A}]=0,$$

for all $k_1,k_2\in\{1,\dots,n\}$, $k_1\neq k_2$, and that the strongly f-invariance of a linear subspace $\widehat{\mathfrak{A}_{[e_i]}}$ means that $\widehat{\mathfrak{A}_{[e_i]}}$ is actually an ideal of $\mathfrak{A}$. From here, we can state:

Theorem 5.1. *Let $(\mathfrak{A},[\cdot,\dots,\cdot])$ be an arbitrary algebra. Then for any basis $\mathcal{B}=\{e_i:i\in I\}$ of $\mathfrak{A}$ one has the decomposition*

$$\mathfrak{A}=\bigoplus_{[\mathfrak{A}_{[e_i]}]\in\mathcal{F}/\approx}\widehat{\mathfrak{A}_{[e_i]}},$$

being any $\widehat{\mathfrak{A}_{[e_i]}}$ an ideal of $\mathfrak{A}$. Furthermore, any pair of different ideals in this decomposition is f-orthogonal.

In the same context, if we restrict the product $[\cdot, \ldots, \cdot]$ of $\mathfrak{A}$ to any ideal $\widehat{\mathfrak{A}_{[e_i]}}$, we get the algebra $(\widehat{\mathfrak{A}_{[e_i]}}, [\cdot, \ldots, \cdot])$. Now, by observing that the f'-simplicity of $(\widehat{\mathfrak{A}_{[e_i]}}, [\cdot, \ldots, \cdot])$ is equivalent to the simplicity of $(\widehat{\mathfrak{A}_{[e_i]}}, [\cdot, \ldots, \cdot])$ as an algebra, and that $\mathrm{Ann}(f') = \mathrm{Ann}(\widehat{\mathfrak{A}_{[e_i]}})$, Theorem 4.2 allows us to assert the following.

Theorem 5.2. *The ideal* $(\widehat{\mathfrak{A}_{[e_i]}}, [\cdot, \ldots, \cdot])$ *is simple if and only if* $\mathrm{Ann}(\widehat{\mathfrak{A}_{[e_i]}}) = 0$ *and* $\mathcal{B}' := \mathcal{B} \cap \widehat{\mathfrak{A}_{[e_i]}}$ *is an* i*-division basis of* $\widehat{\mathfrak{A}_{[e_i]}}$*.*

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