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INVOLUTORY LR SYMMETRY BIJECTION ON HIVES

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Abstract: Littlewood–Richardson (LR) coefficients $c^{\lambda}_{\mu\nu}$ may be evaluated by means of several combinatorial models. These include not only the original one based on the LR rule for enumerating LR tableaux of skew shape λ/μ and weight ν, but also one based on the enumeration of LR hives with boundary edge labels λ, μ and ν. Unfortunately, neither of these reveal in any obvious way the well-known symmetry property $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$. Here we introduce a map $\sigma^{(n)}$ on LR hives that interchanges contributions to $c^{\lambda}_{\mu\nu}$ and $c^{\lambda}_{\nu\mu}$ for any partitions λ, μ, ν of lengths no greater than n, and then prove that it is a bijection, thereby making manifest the required symmetry property. The map $\sigma^{(n)}$ involves repeated path removals from a given LR hive with boundary edge labels (λ, μ, ν) that give rise to a sequence of hives whose left-hand boundary edge labels define a partner LR hive with boundary edge labels (λ, ν, μ). A new feature of our hive model is its realisation in terms of edge labels and rhombus gradients, with the latter playing a key role in defining the action of path removal operators in a manner designed to preserve the required hive conditions. A consideration of the detailed properties of the path removal procedures also leads to a wholly combinatorial self-contained hive-based proof that $\sigma^{(n)}$ is an involution.

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1. Introduction and statement of results

Let n be a fixed positive integer and let $x = (x_1, x_2, \ldots, x_n)$ be a sequence of indeterminates. Then, for each partition λ of length $\ell(\lambda) \leq n$ and weight $|\lambda|$, there exists a Schur function $s_\lambda(x)$ which is a homogeneous symmetric polynomial in the x_k of total degree $|\lambda|$. These Schur functions $s_\lambda(x)$ for all such λ form a linear basis of the ring Λ_n of symmetric polynomials in the components of x. It follows that

$$s_\mu(x)\ s_\nu(x) = \sum_{\lambda} c^{\lambda}_{\mu\nu}\ s_\lambda(x)\,, \tag{1.1}$$

where the coefficients $c^{\lambda}_{\mu\nu}$ are known as Littlewood–Richardson (LR) coefficients. These coefficients are independent of n. They are non-negative integers that may be evaluated by means of the Littlewood–Richardson rule [LR34] as the number of Littlewood–Richardson tableaux of skew shape λ/μ and of weight ν, where the parts of ν specify the numbers of its entries k for $k = 1, 2, \ldots, n$,

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with the entries satisfying certain semistandardness and lattice permutation conditions.

Alternatively, $c^{\lambda}_{\mu\nu}$ is the number of Littlewood–Richardson n-hives with boundary edge labels specified by the ordered triple (λ, μ, ν) [KT99, Buc00], where each of the three partitions has n parts through the inclusion if necessary of trailing zeros. Further details of the hive model may be found in Section 2. Put briefly, an LR n-hive is a labelling of the vertices of an equilateral triangular graph of side length n subdivided by its edges into n^2 elementary triangles of side length 1, as illustrated in the case $n = 4$ in 2.1 on the left, with the vertex labels of those pairs of elementary triangles sharing a common edge satisfying rhombus conditions, see 2.1 on the right.

Here we find it convenient to use an edge representation of an LR hive [KTT06], whereby each edge is labelled by the label of the vertex at its rightmost end minus the label of the vertex at its leftmost end. In this setting the boundary edge labels are the parts of the relevant partitions λ, μ and ν. What we call the gradient of a rhombus formed from two elementary triangles sharing a common edge is the difference between the sum of the vertex labels at the two ends of this edge and the sum of the remaining two vertex labels. When expressed in terms of edge labels, the gradient of a rhombus is the difference between the labellings of either pair of two opposite edges (see 2.8).

Although logically distinct, the tableau and hive models may be thought of as being equivalent thanks to the existence of a bijection between LR tableaux and LR hives described by Fulton in the Appendix to [Buc00], see also [KT99, PV05]. Within these two models, the set of LR tableaux of shape λ/μ and weight ν is denoted by $\mathcal{LR}(\lambda/\mu, \nu)$, and the corresponding set of n-hives is denoted by $\mathcal{H}^{(n)}(\lambda, \mu, \nu)$ for any fixed $n \geq \ell(\lambda)$. We then have

$$c^{\lambda}_{\mu\nu} = \#\{T \in \mathcal{L}R(\lambda/\mu, \nu)\} = \#\{H \in \mathcal{H}^{(n)}(\lambda, \mu, \nu)\}\,. \tag{1.2}$$

Unfortunately, although the definition 1.1 makes it immediately clear that $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$, the same cannot be said of either of the combinatorial formulae in 1.2. Within a variety of equivalent combinatorial models, people have succeeded in defining bijective maps between objects with parameters (λ, μ, ν) and those with parameters (λ, ν, μ), which we may call *Littlewood–Richardson commutativity bijections*, and in some cases showed their involutive nature (see e.g. [BSS96, HK06b, PV10, DK05, DK08]). However, this has not yet been done for the map originally defined in a tableaux setting by the third author in [Aze99, Aze00] and described as ρ_3 in [PV10], nor has its coincidence with other known LR commutativity bijections been fully established. Our aim here is to define such a bijection $\sigma^{(n)}\colon \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$ such that $\sigma^{(n)}\colon \mathcal{H}^{(n)}(\lambda,\mu,\nu) \ni H \mapsto K \in \mathcal{H}^{(n)}(\lambda,\nu,\mu)$, in the arena of hives, that corresponds to the map ρ_3 in the arena of tableaux, and show that it is involutive, independently of the existing involutiveness results on other LR commutativity bijections. The issue of coincidence, an approach to whose proof has been proposed in [Aze08], will be deferred to another publication.

The present paper is organised as follows. In Section 2 we recall the notion of hives, putting emphasis on their *edge representation* and *rhombus gradients* which we actually rely upon, and in Section 3 three *path removal operators* on hives are introduced. These correspond to the deletion operators on tableaux first introduced in [Aze99, Aze00]. In Section 4 they are shown to preserve the hive properties.

In Section 5, we give the precise algorithmic definition of our LR commutativity map $\sigma^{(n)}$. The procedure allowing us to define $\sigma^{(n)}$ as a map taking any LR n-hive $H \in \mathcal{H}^{(n)}$ to some *partner* LR n-hive $K \in \mathcal{H}^{(n)}$ involves a succession of pairs, $(H^{(r)}, K^{(n-r)})$, for $r = n, n-1, \ldots, 0$, where $H^{(r)}$ is an r-hive and $K^{(n-r)}$ is what we call an r-truncated n-hive. For the initial $r = n$ pair one sets $H^{(n)} = H$ with $K^{(0)}$ an empty n-truncated n-hive, and constructs the final $r = 0$ pair with $K = K^{(n)}$ and $H^{(0)}$ the empty 0-hive. The passage from one pair to the next is effected by performing a sequence of path removals from $H^{(r)}$ to give $H^{(r-1)}$ and using the data on the location of the initial and final edges on each path that is removed to build $K^{(n-r+1)}$ from $K^{(n-r)}$. By this means one evacuates H and builds K. All this is illustrated in 5.3. The main result in this section is then the proof of Theorem 5.4 which states that for $H \in \mathcal{H}^{(n)}(\lambda,\mu,\nu)$ and $K = \sigma^{(n)}H$ we have $K \in \mathcal{H}^{(n)}(\lambda,\nu,\mu)$.

In Section 6 we introduce a path addition operator on hives and define a map $\overline{\sigma}^{(n)}$, and show that for $K \in \mathcal{H}^{(n)}(\lambda,\nu,\mu)$ and $H = \overline{\sigma}^{(n)}K$ we have $H \in \mathcal{H}^{(n)}(\lambda,\mu,\nu)$. This section culminates with the proof of Theorem 6.9 stating that the maps $\sigma^{(n)}$ and $\overline{\sigma}^{(n)}$ are mutually inverse bijections.

The next Section 7 is concerned with the involutory property of $\sigma^{(n)}$. A new type of path removal operator ψ_n enables us to generate from any given n-hive H a new hive $\widehat{H} = \psi_n H$, only marginally different from H. However, this difference is enough to show that $\sigma^{(n)}$ is an involution, by showing first that $K = \sigma^{(n)}H$ and $\widehat{K} = \sigma^{(n)}\widehat{H}$ are related by the action of one of our original path removal operators ϕ_n, and then exploiting this in an inductive proof of the involutory property along the lines of an approach first proposed in a tableaux setting in [Aze00]. The heart of the matter is the somewhat intricate proof that $\widehat{K} = \phi_n\, K$, where ϕ_n is one of our original path removal operators. This result emerges as a special case of the key Lemma 7.3 whose proof involves amongst other key ingredients the notion of a critical rhombus and three subsidiary lemmas whose proofs are deferred to Section 9, Appendix.

In Section 8 we offer some brief concluding remarks.

2. The hive model

It is by now well known that *hives*, as first introduced by Knutson and Tao [KT99], with properties described in more detail by Buch [Buc00], offer an alternative way to determine Littlewood–Richardson coefficients. As we have said, this comes about as a result of the existence of a bijection described by Fulton in the Appendix to [Buc00] between the set of LR n-hives $\mathcal{H}^{(n)}(\lambda,\mu,\nu)$ with boundary specified by a triple (λ,μ,ν) of partitions with $\ell(\lambda),\ell(\mu),\ell(\nu)\le n$ and the set of LR-tableaux $\mathcal{LR}^{(n)}(\lambda/\mu,\nu)$ of skew shape λ/μ and weight ν.

In its *vertex representation* an integer n-hive is a labelling of the vertices of a planar, equilateral triangular graph of side length n with integers a_{ij}, for $0\le i\le j\le n$, as illustrated below on the left in the case $n=4$, satisfying the *rhombus inequalities* indicated on the right, which are to be applied to each *elementary rhombus* formed from the union of any pair of *elementary triangles* having a common edge whatever their orientation.

$$
\begin{array}{ccccccccc}
&&&& a_{04} &&&& \\
&&& / && \backslash &&& \\
&&& a_{03} & \cdot & a_{14} &&& \\
&& / & \backslash & / & \backslash && & \\
&& a_{02} & \cdot\; a_{13} & \cdot & a_{24} &&& \\
& / & \backslash & / & \backslash & / & \backslash && \\
& a_{01} & \cdot\; a_{12} & \cdot & a_{23} & \cdot\; a_{34} &&& \\
/ & \backslash & / & \backslash & / & \backslash & / & \backslash & \\
a_{00} & \cdot\; a_{11} & \cdot & a_{22} & \cdot & a_{33} & \cdot\; a_{44} &&
\end{array}
\qquad
\begin{array}{ccc}
 & a - d & \\
/ & \backslash & / \\
c - b & &
\end{array}
\qquad
\begin{array}{ccc}
 & c & \\
/ & & \backslash \\
a & - & b \\
\backslash & & / \\
 & d &
\end{array}
\qquad
\begin{array}{ccc}
c - b & & \\
\backslash & / & \backslash \\
 & a - d &
\end{array}
\qquad a+b \ge c+d
\tag{2.1}
$$

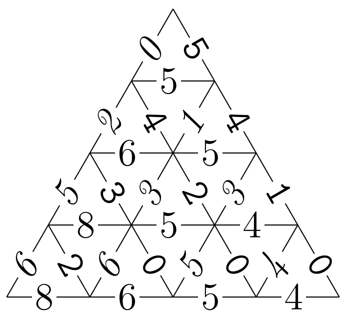
Such an integer n-hive is an LR n-hive and belongs to $\mathcal{H}^{(n)}(\lambda,\mu,\nu)$ if and only if for $k=1,2,\ldots,n$:

$$
a_{00}=0\,;\quad a_{0k}=\sum_{j=1}^{k}\mu_j\,;\quad a_{kn}=a_{0n}+\sum_{i=1}^{k}\nu_k\,;\quad a_{kk}=\sum_{i=1}^{k}\lambda_i\,. \tag{2.2}
$$

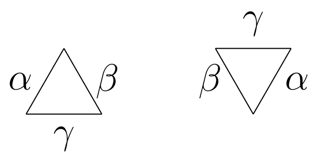
An LR hive may equally well be specified by mean of its *edge representation* as introduced by [KTT06] and used in [KTT09, CJM11], whereby each edge between neighbouring vertices labelled a and b is labelled $b - a$ if the vertex labelled b is to the right of that labelled a.

Example 2.1. *In the case* $n = 4$, $\lambda = (8, 6, 5, 4)$, $\mu = (6, 5, 2, 0)$ *and* $\nu = (5, 4, 1, 0)$, *a typical LR hive takes the following forms when expressed on the left in terms of vertex labels and on the right in terms of edge labels:*

$$\begin{array}{ccccccccc} &&&& 13 &&&& \\ &&& 13 && 18 &&& \\ && 11 && 17 && 22 && \\ & 6 && 14 && 19 && 23 & \\ 0 && 8 && 14 && 19 && 23 \end{array} \tag{2.3}$$

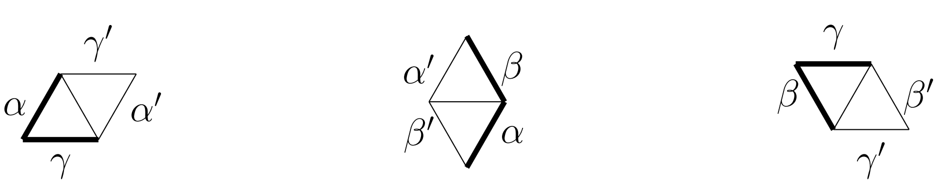


As a matter of convention we sometimes refer to any edge parallel to the left, right or lower boundary as being an α-edge, β-edge or γ-edge, respectively. It is to be noted that the sequences of α, β and γ boundary edge labels constitute the partitions μ, ν and λ, respectively. In terms of edge labels the LR hive conditions are equivalent to the following requirements: all edge labels are non-negative integers, while for each elementary triangle we have the *triangle conditions*



$$\alpha + \beta = \gamma \tag{2.4}$$

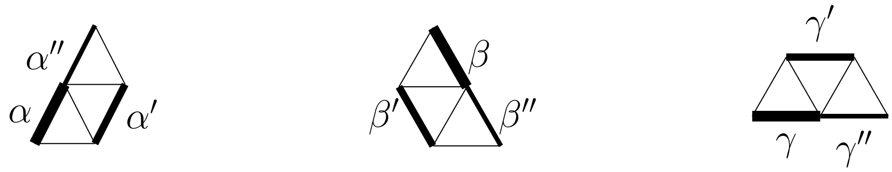
and for each elementary rhombus we have the *rhombus conditions*



$$\alpha - \alpha' = \gamma - \gamma' \geq 0 \qquad \alpha - \alpha' = \beta - \beta' \geq 0 \qquad \beta - \beta' = \gamma - \gamma' \geq 0 \tag{2.5}$$

where the equalities are a consequence of the triangle condition and as an *aide mémoire*, for each pair of parallel edges that with the larger edge label has been drawn thicker than the other.

The rhombus inequalities can be encapsulated in the form of the *betweenness conditions* specified below:



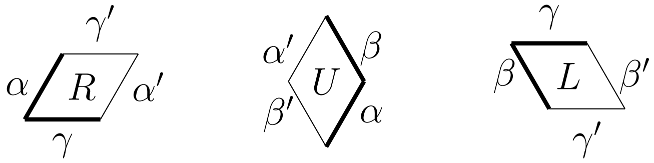
$$\alpha \geq \alpha' \geq \alpha'' \qquad \beta \geq \beta' \geq \beta'' \qquad \gamma \geq \gamma' \geq \gamma'' \tag{2.6}$$

The implication of these betweenness conditions is that if we separate the edges into those that are α-edges, β-edges and γ-edges they can be seen to form three interlocking Gelfand-Tsetlin patterns [GT50]. In our Example 2.1 these take the form:

$$\begin{array}{ccccccc} &&&0&&& \\ &&2&&1&& \\ &5&&3&&3& \\ 6&&6&&5&&4 \end{array} \qquad \begin{array}{ccccccc} &&&5&&& \\ &&4&&4&& \\ &3&&2&&1& \\ 2&&0&&0&&0 \end{array} \qquad \begin{array}{ccccccc} &&&5&&& \\ &&6&&5&& \\ &8&&5&&4& \\ 8&&6&&5&&4 \end{array} \tag{2.7}$$

By interlocking, we mean that when superposed, as on the right in 2.3, the edge labels of each elementary triangle must satisfy the triangle condition 2.4.

Within a hive there are three types of elementary rhombi: right-leaning, upright and left-leaning, as displayed in 2.1 and 2.5. We often omit the interior edge and display them in the form:



(2.8)

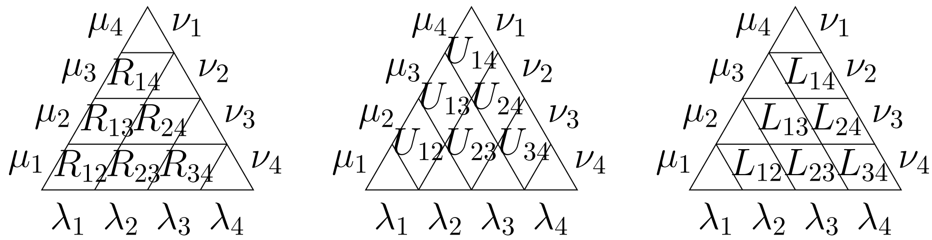
where the parameters R, U and L introduced in these diagrams are not edge labels. They are referred to as the *gradients* of the corresponding right-leaning, upright and left-leaning rhombi, respectively. Each gradient is defined to be the difference between parallel edge labels in the relevant rhombus, or, more precisely, for each pair of parallel edges, one thick and one thin in the above diagrams, the gradient is equal to the thick edge label minus the thin edge label, so that we have

$$R = \alpha - \alpha' = \gamma - \gamma' , \quad U = \alpha - \alpha' = \beta - \beta' , \quad \text{and} \quad L = \beta - \beta' = \gamma - \gamma' . \tag{2.9}$$

The hive rhombus inequalities then just take the form

$$R \geq 0\,, \qquad U \geq 0 \quad \text{and} \quad L \geq 0\,. \tag{2.10}$$

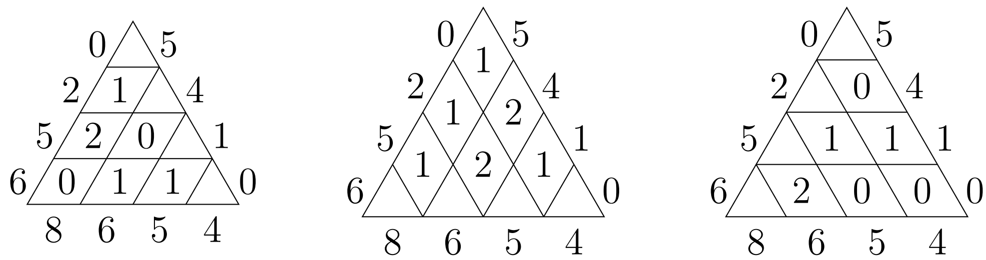
All this gives rise to a third way of specifying hives, namely *the gradient representation*, which involves labelling its boundary edges and giving the gradients of one or other of its three sets of right-leaning, upright or left-leaning elementary rhombi. This is illustrated in the case $n = 4$ by



(2.11)

and exemplified in the case of our running example by:

Example 2.2. *Rhombus gradient labellings of 2.1*



Of all these labelling schemes for LR hives the one that provides the simplest connection with LR-tableaux is that offered by specifying boundary edge labels, λ, μ and ν, together with the upright rhombus gradients U_{ij} with $1 \leq i < j \leq n$. These labels are themselves constrained by the triangle conditions applied to the elementary triangles at the base of the hive which take the form:

$$\lambda_k = (\mu_k + \sum_{i=1}^{k-1} U_{ik}) + (\nu_k - \sum_{j=k+1}^{n} U_{kj}) \quad \text{for} \quad k = 1, 2, \ldots, n\,. \tag{2.12}$$

In particular we have

$$\lambda_n = \mu_n + \nu_n + \sum_{i=1}^{n-1} U_{in}. \tag{2.13}$$

so that $\lambda_n = 0$ if and only if $\mu_n = \nu_n = 0$ and $U_{in} = 0$ for all $i < n$.

3. Path removal operators

It should be recalled first that in any given hive a *diagonal* consists of all the triangles in a strip parallel to the right hand boundary of the hive lying between an edge on the base and a corresponding edge on the left hand boundary. Our convention is that the pth diagonal is the one bounded by the edges λ_p and μ_p. Then by a *path* we mean a connected set of edges extending from an edge on the base to an edge on either the left or right hand boundary. Such paths are generally zig-zag in nature and proceed either up a diagonal or horizontally leftwards from one diagonal to another. They consist of pairs of edges taken from a sequence of neighbouring triangles always including their common edge. However, it is not the case that any set of edges having such a configuration is called a path. A precise definition will be made in association with *path removal operators*. Here we introduce three kinds of such operators, χ_r, ϕ_r and ω_r, and other kinds will be added later. Then, thanks to the bijection between LR tableaux and LR hives, the actions of the deletion operators on tableaux introduced by the third author [Aze99, Aze00] are transformed into the action of χ_r, ϕ_r and ω_r on H as described in the following definition:

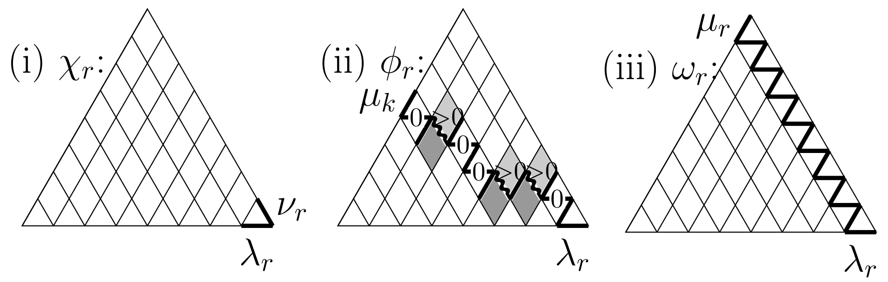
Definition 3.1. *For any given $H \in \mathcal{H}^{(r)}(\lambda, \mu, \nu)$ with $r = \ell(\lambda)$ we may define three path removal operators χ_r, ϕ_r and ω_r whose action on H is to reduce or increase edge labels by 1 along a path starting from the edge labelled λ_r on the base of the hive and specified as follows:*

(i) χ_r*: if $\nu_r > 0$ then the path consists of the edges labelled λ_r and ν_r, with both edge labels decreased by 1;*

(ii) ϕ_r*: if $\lambda_r - \mu_r - \nu_r > 0$ so that $U_{ir} > 0$ for some $i < r$ then the path proceeds up the rth diagonal from the edge labelled λ_r through upright rhombi of gradient 0 until it encounters an upright rhombus of positive gradient, at which point it moves horizontally to the left into the $(r-1)$th diagonal and proceeds up this diagonal or to the left as before, and so on until it terminates on the left-hand boundary at the top of the kth diagonal, that is at the edge labelled μ_k for some k such that $1 \leq k < r$, with all path α- and γ-edge labels being decreased by 1 and all path β-edge labels increased by 1;*

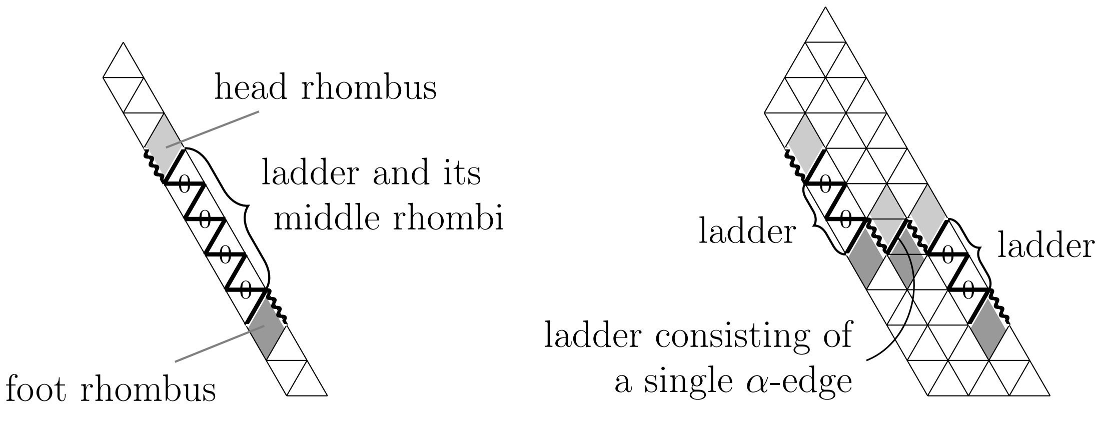
(iii) ω_r*: if $\mu_r > 0$ then the path proceeds directly up the rth diagonal until it terminates on the left-hand boundary at level r, that is at the edge labelled μ_r, with all path edge labels decreased by 1. Such a type (iii)* *path may be thought of as a special case of a type (ii) path in which the terminating level* $k = r$.

The three types of path are illustrated below, where we have used full lines and wavy lines to distinguish those edges whose labels are decreased and increased, respectively, by 1 under the relevant path removal operation. The action of χ_r and ω_r is to decrease all path edge labels by 1, whereas the action of ϕ_r is to decrease the label of each α- or γ-edge on the path by 1 and to increase that of each β-edge on the path by 1. In particular, under this action the edge label λ_r and one or other of ν_r, μ_k (with $k < r$) or μ_r are each reduced by 1 to $\lambda_r - 1$ and $\nu_r - 1$, $\mu_k - 1$ or $\mu_r - 1$, respectively, while the only changes of upright rhombus gradients are those of -1 and $+1$ immediately above and below the path in each diagonal, with the corresponding rhombi shaded light and dark grey, respectively.



(3.1)

While the structure of the paths is rather simple in cases (i) and (iii), the structure of the path in case (ii) is more complicated and consists of a sequence of *ladders* in each diagonal from the rth to the kth. Each ladder consists of a continuous zig-zag of α- or γ-edges (shown above as solid lines) passing through a sequence of upright rhombi of gradient 0 that extends up the diagonal from an edge that is either a γ-edge on the base of the hive or the α-edge at the top of an upright rhombus that we call the *foot rhombus* (shaded dark grey), to an α-edge that is either on the left-hand boundary or at the bottom of an upright rhombus that we call the *head rhombus* (shaded light grey). The upright rhombi of gradient 0 through which the ladder extends are called its *middle rhombi*. A ladder may consist of a single α-edge (possibly accompanied by a γ-edge on the base of the hive), lacking any middle rhombi. The passage between one diagonal and the next is by way of a β-edge common to both a head and a foot rhombus (shown above and in the diagrams below as a wavy line). If such an edge is the lth from the top of the diagonal then its *level* is said to be l. For example, in the right-hand diagram the passage from the rightmost diagonal to the next one on its left takes place at level 6.



(3.2)

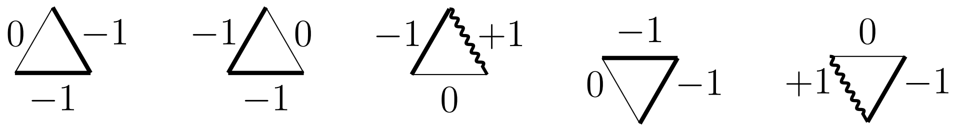
4. Preservation of the hive conditions

Before using these path removals we first establish that the action of each of the path removal operators preserves the hive conditions, as follows:

Lemma 4.1. *Let H be a hive in $\mathcal{H}^{(r)}(\lambda,\mu,\nu)$ with $r=\ell(\lambda)$. Then the path removal operators are such that if we set $\widetilde{H}=\chi_r\,H$, $\phi_r\,H$ or $\omega_r\,H$ with $\nu_r>0$, $\lambda_r-\mu_r-\nu_r>0$ and $\mu_r>0$, respectively, then in each case $\widetilde{H}$ is an LR hive.*

Proof: First note that if one confirms all triangle conditions and all rhombus inequalties for $\widetilde{H}$, and if it can be shown that the two edges initially labelled μ_r and ν_r still have non-negative labels in $\widetilde{H}$, then all α- and β-edges have non-negative labels by the betweenness of edge labels, and then all γ-edges also have non-negative labels by the triangle conditions. However, the edge label μ_r is only changed under the action of ω_r. In this case $\mu_r>0$ by hypothesis, so that its new value μ_r-1 in $\widehat{H}$ is non-negative. Similarly, the edge label ν_r is only changed under the action of χ_r. In this case $\nu_r>0$ by hypothesis, so that its new value ν_r-1 in $\widehat{H}$ is again non-negative. It therefore remains only to prove the validity of the triangle and rhombus conditions.

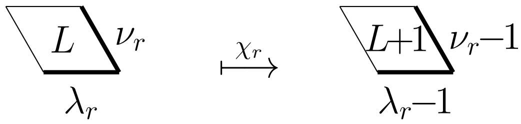
It is easy to see that the triangle conditions 2.4 are preserved under any of the three path removal procedures mapping a hive H to $\widetilde{H}$ by examining the changes to the edge labels of elementary triangles as illustrated by:



(4.1)

Turning next to rhombi, we shall show that all their gradients remain non-negative under the maps from H to $\widetilde{H}$. For upright rhombi this is clear since the gradients remain fixed except in the case of head and foot rhombi, for which, as we have seen, the gradient decreases and increases by 1, respectively. However, the gradient of each head rhombus is necessarily positive in H and must therefore remain non-negative in $\widetilde{H}$. Thus all upright rhombus gradients of $\widetilde{H}$ are non-negative, as required.

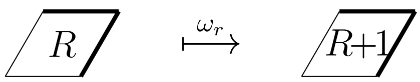
In the case of a type (i) hive path removal the action of χ_r affects only one rhombus and does so as shown below:



(4.2)

Clearly, the gradient of this rhombus is increased. It follows that all rhombus gradients remain non-negative under the action of χ_r.

Similarly, for a hive path removal of type (iii) as illustrated on the right of 3.1 the only rhombi whose gradients change are those undergoing the map:



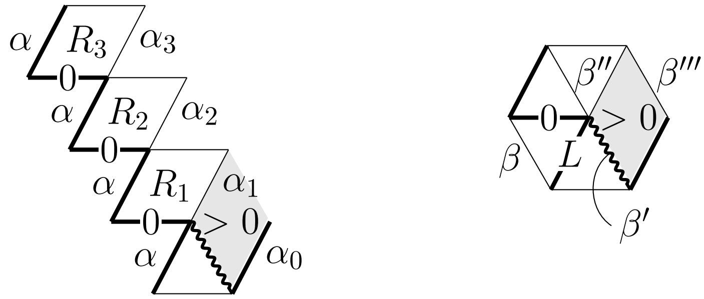
(4.3)

Thus all rhombus gradients remain non-negative under the action of ω_r.

The situation is more complicated for type (ii) hive path removals under the action of ϕ_r. However, the only right-leaning or left-leaning rhombi that undergo a reduction in gradient under a type (ii) hive path removal are those subject to the following transformations:

(4.4)

To preserve the validity of the corresponding hive condition it is therefore necessary to show that on the left the initial gradient $R = \alpha - \alpha'$ is positive, and that on the right the initial gradient $L = \beta - \beta'$ is also positive. It can be seen from the type (ii) diagram of 3.1 that the only cases that arise are those of the following types:



(4.5)

In the left-hand diagram the parallel edge labels α on the left of the diagram are identical, since the gradients of the intervening upright rhombi are all zero as indicated by the 0's appearing on horizontal edges. On the right of the diagram the fact that the relevant upright rhombi have non-negative gradients implies that $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$, while the positivity of the gradient of the upright rhombus shaded grey ensures from the betweenness conditions that $\alpha \geq \alpha_0 > \alpha_1$. Hence $\alpha \geq \alpha_0 > \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$ so that $R_k = \alpha - \alpha_k > 0$ for all $k \geq 1$, as required for the right-leaning rhombus condition to be maintained after the path removal.

In the hexagonal diagram on the right the gradients of the two upright rhombi specified in the diagram as 0 and > 0 ensure that $\beta = \beta''$ and $\beta''' > \beta'$. In addition the gradient $\beta'' - \beta'''$ of the upper left-leaning rhombus must be non-negative. Hence $\beta = \beta'' \geq \beta''' > \beta'$. It follows that $L = \beta - \beta' > 0$ so that $L - 1 \geq 0$, as required for the left-leaning rhombus condition to be maintained under the action of ϕ_r.

Thus it is confirmed that all rhombus gradients remain non-negative under the action of χ_r, ϕ_r and ω_r. This completes the proof of Lemma 4.1. ∎

This Lemma allows us to produce from an LR hive $H \in \mathcal{H}^{(r)}(\lambda, \mu, \nu)$ a new LR hive $\widetilde{H} \in \mathcal{H}^{(r-1)}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu})$. To this end it is convenient to make the following definition:

Definition 4.2. *For any given hive $H \in \mathcal{H}^{(r)}(\lambda, \mu, \nu)$ with $\ell(\lambda) \leq r$ the full r-hive path removal operator θ_r is defined by*

$$\theta_r = \kappa_r \; \omega_r^{\mu_r} \; \phi_r^{\lambda_r - \mu_r - \nu_r} \; \chi_r^{\nu_r} \, , \tag{4.6}$$

where κ_r is an operator whose action is to restrict any LR r-hive H with an empty rth diagonal to an LR $(r-1)$-hive consisting of the leftmost $(r-1)$ diagonals of H. Here an empty rth diagonal is one in which all the edge labels within and on its boundary satisfy the triangle conditions, with the top and *bottom edge labels both* 0*, and with all upright rhombus gradients also* 0*. By virtue of 2.13, this is the case if and only if the bottom edge label is* 0*. This implies that the lowest right-hand boundary edge label is also* 0*.*

With this definition we have

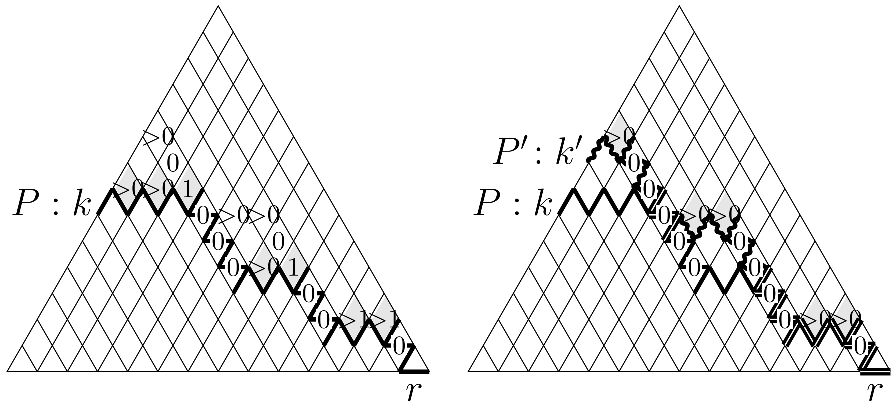
Theorem 4.3. *For a hive $H \in \mathcal{H}^{(r)}(\lambda,\mu,\nu)$ with $\ell(\lambda) \leq r$ let $\theta_r H = \widetilde{H}$. Then we have $\widetilde{H} \in \mathcal{H}^{(r-1)}(\widetilde{\lambda},\widetilde{\mu},\widetilde{\nu})$ with $\widetilde{\lambda} = (\lambda_1,\ldots,\lambda_{r-1})$, $\widetilde{\mu} = (\mu_1 - V_{1r},\ldots,\mu_{r-1} - V_{r-1,r})$ and $\widetilde{\nu} = (\nu_1,\ldots,\nu_{r-1})$, where V_{kr} is the number of type (ii) hive path removals from H that extend from the boundary edge initially labelled λ_r to that initially labelled μ_k for $1 \leq k < r$.*

Proof: If $\ell(\lambda) < r$ then $\lambda_r = \mu_r = \nu_r = 0$ and $U_{ir} = 0$ for $i = 1,2,\ldots,r-1$. Thus $\theta_r = \kappa_r$ and there are no path removals, so that $V_{kr} = 0$ for all $k = 1,2,\ldots,r-1$ and the effect of the action of κ_r is simply to remove from each partition λ, μ and ν a trailing 0. This implies that $\widetilde{\lambda} = \lambda$, $\widetilde{\mu} = \mu$, $\widetilde{\nu} = \nu$ and $\theta_r H = \widetilde{H} \in \mathcal{H}^{(r-1)}(\widetilde{\lambda},\widetilde{\mu},\widetilde{\nu})$, as required. For $\ell(\lambda) = r$ the required result is an easy consequence of the iterated action of χ_r, ϕ_r and ω_r, followed by that of κ_r. First the edge label ν_r is reduced to 0 by the action of $\chi_r^{\nu_r}$. At the same time the edge label λ_r is reduced to $\lambda_r - \nu_r$. It is then reduced to μ_r under the action of $\phi_r^{\lambda_r-\mu_r-\nu_r}$, and finally to 0 under the action of $\omega_r^{\mu_r}$, under which the edge label μ_r is also reduced to 0. Meanwhile, under the action of $\phi_r^{\lambda_r-\mu_r-\nu_r}$ the upright rhombus gradients U_{ir} are reduced one by one to 0, since $\lambda_r - \mu_r - \nu_r = U_{r-1,r} + \cdots + U_{2r} + U_{1r}$ by virtue of 2.12. It follows that the rth diagonal of the hive is now empty and is then finally removed through the action of κ_r, which includes the removal of the trailing zeros from the boundary edge label partitions. The parameters V_{kr} give the number of type (ii) hive removal paths that reach the left-hand boundary edge initially labelled μ_k, thereby reducing this label to $\mu_k - V_{kr}$ for $k = 1,2,\ldots,r-1$, as required to complete the proof. ■

Two further observations are of use in what follows.

Lemma 4.4. *In the action of θ_r on $H \in \mathcal{H}^{(r)}(\lambda,\mu,\nu)$ with $\ell(\lambda) = r$ if a hive path removal of type (ii) or (iii) follows a path P and reaches the left-hand boundary at level k, then the next such removal follows a path P' lying weakly above the path P in each diagonal they have in common. In particular P' reaches the left-hand bounday at level k' with $k' \geq k$.*

Proof: In the case where both P and P' are of type (ii) this can be seen by consideration of the following hive path removal diagrams in which the two successive removal paths P and P' are illustrated. In the left-hand diagram just the path P is shown with full line edges, and with each head rhombus shaded as usual, while in the right-hand diagram the path P' has been added, using double line and wavy line edges where it does and does not, respectively, coincide with P, now with just each head rhombus of the path P' being shaded.



(4.7)

Each of the head rhombi of P necessarily has a positive gradient before the path removal, but by way of an example two of them are *critical* in that they have gradient precisely 1, which must then be reduced to 0 by the P path removal. This means that the next path removal P', as illustrated in the diagram on the right, follows the first path P until it meets the first such critical rhombus. Since this now has gradient 0 the path P' must pass up the diagonal through this critical rhombus until it again meets an upright rhombus of positive gradient. It then proceeds in the usual way, where it may, as in this example, meet and then follow the path P until it once again meets a critical rhombus, and so on. It is clear that in this way the path P' remains weakly above P at all stages, and that if P and P' meet the left-hand boundary at levels k and k', respectively, then $k' \geq k$.

In the case where P is of type (ii) but P' is of type (iii) then P is as shown on the left with $k < r$, and P' proceeds directly up the rth diagonal, the only diagonal they have in common, and terminates at level $k' = r > k$. On the other hand if P is of type (iii) so that $k = r$, then the same must be true of the next successive path removal P', so that P and P' coincide and $k' = r = k$.

Thus in all cases P' lies weakly above P in each diagonal they have in common, and they terminate at levels k' and k, respectively, with $k' \geq k$. ■

Corollary 4.5. *Under the action of θ_r on $H \in \mathcal{H}^{(r)}(\lambda,\mu,\nu)$ with $\ell(\lambda) = r$ let V_{kr} be the number of type (ii) hive path removals reaching the left-hand* *boundary at level k for $1 \leq k < r$. Then for each such k*

$$\mu_k \geq \mu_k - V_{kr} \geq \mu_{k+1}. \tag{4.8}$$

Proof: The first inequality is immediate, since $V_{kr} \geq 0$. Now let $\widetilde{H} = \phi_r^{N_{kr}} \chi_r^{\nu_r} H$, where $N_{kr} = V_{1r} + \cdots + V_{kr}$. In view of 4.6 and Lemma 4.4 this is the intermediate hive obtained while applying θ_r to H, immediately after all those path removals produced by the action of ϕ_r that reach the left-hand boundary at or below level k. At this stage the left-hand boundary edge label μ_{k+1} remains unchanged, while the label μ_k has been reduced to $\mu_k - V_{kr}$. The hive conditions on $\widetilde{H}$ then imply the second inequality. ■

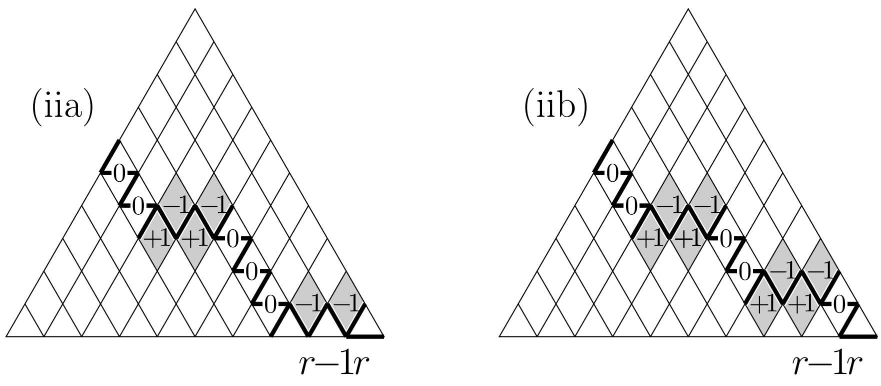
Our second observation is the following:

Lemma 4.6. *Under the action of θ_r followed by θ_{r-1} on $H \in \mathcal{H}^{(r)}(\lambda, \mu, \nu)$ with $\ell(\lambda) = r$, let N_{kr} and $N_{k-1,r-1}$ be the number of type (ii) hive path removals occurring in the action of θ_r and θ_{r-1} that reach the left-hand boundary at or below levels k and $k-1$, respectively. Then*

$$N_{k-1,r-1} \geq N_{kr} - U_{r-1,r}, \tag{4.9}$$

where $U_{r-1,r}$ is the upright rhombus gradient at the foot of the rth diagonal of H.

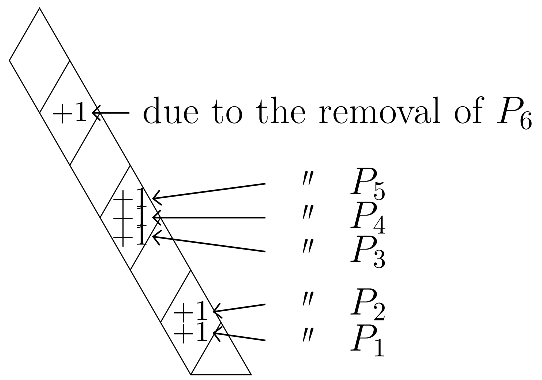
Proof: It should be noted that under the action of θ_r on H the type (ii) hive path removals may take one or other of the following two forms (iia) and (iib):



(4.10)

To be precise, the gradient $U_{r-1,r}$ of H forces exactly the first $U_{r-1,r}$ of the type (ii) paths removed by θ_r to take the form (iia). If the value of k is such that $N_{kr} \leq U_{r-1,r}$, then we immediately have $N_{k-1,r-1} \geq 0 \geq N_{kr} - U_{r-1,r}$ as required. In order to prove the required inequality for all remaining values of k, let $P_1, P_2, \ldots, P_c$ be all type (iib) paths removed by the action of θ_r on H, and $Q_1, Q_2, \ldots, Q_d$ all type (ii) paths, whether of the form (iia) or (iib), removed by the action of θ_{r-1} on $\theta_r H$, both numbered in the order of removals, and claim that (a) $c \leq d$, and (b) Q_i lies strictly below P_i for each $1 \leq i \leq c$. Once this claim has been shown, one can argue for each k with $N_{kr} > U_{r-1,r}$ that by Lemma 4.4, among the type (ii) paths removed by θ_r, those terminating at levels $\leq k$ are the first $U_{r-1,r}$ type (iia) paths and the following $N_{kr} - U_{r-1,r}$ type (iib) paths, and that by the claim each Q_i with $1 \leq i \leq N_{kr} - U_{r-1,r}$ terminates at a level strictly below the terminating level of P_i, and hence strictly below level k. The required inequality then follows.

In order to prove the claim, note that the removal of each P_i leaves $+1$, meaning an increase by 1, in the gradient of its foot rhombus in each diagonal it enters. Moreover, by Lemma 4.4, each P_j with $j \geq i$ lies weakly above P_i, and its removal decreases an upright rhombus gradient only for the head rhombus in each diagonal, lying strictly above the foot rhombus of P_i in that diagonal. Hence the $+1$ obtained by the removal of P_i is not negated by the removal of any P_j with $j > i$, but rather the effect of these $+1$'s accumulates in each diagonal, until all type (ii) paths are removed by θ_r. The situation remains unaltered by the removal of any necessary type (iii) paths under the action of θ_r and any necessary type (i) paths under the action of θ_{r-1}. See the diagram 4.11 for a typical illustration of the situation immediately before the type (ii) path removals by θ_{r-1} start. For instance, the third upright rhombus from the bottom containing three $+1$'s has gradient equal to its value before the removal of P_3 plus 3 as a result of removing P_3, P_4 and P_5, while the next upright rhombus above it, containing no $+1$, maintains the same value of its gradient as it had immediately before the removal of P_6.



(4.11)

Now entering the phase of type (ii) path removals by θ_{r-1}, first note that the $(r-1)$th diagonal at this point embraces a total of at least c upright rhombus gradients due to the above-mentioned accumulating nature of $+1$. Hence θ_{r-1}

must remove at least that many type (ii) paths, fulfilling $c \leq d$, which was part (a) of our claim.

Now consider how the path Q_1 proceeds. The $+1$'s left by the removal of P_1, due to the positions of its foot rhombi in consecutive diagonals each of which is located either precisely to the west of the previous one or further up the diagonal (see 4.10 on the right), create an inpenetrable barrier for the path Q_1, starting from the bottom of the $(r-1)$th diagonal, to climbing any of the ladders of P_1 in diagonals over which it extends. Thus Q_1 stays entirely below P_1 in an edge-disjoint manner. The removal of Q_1 decreases the gradient of the head rhombus of each of its ladders, which is located weakly below the foot rhombus of P_1 in that diagonal. Hence the removal of Q_1 leaves intact the $+1$ left by the removal P_2 in each diagonal, or by any P_i with $i \geq 2$, even in the extreme case where the head rhombus of Q_1, the foot rhombi of P_1 and P_2 all coincide, since only the $+1$ left by the removal of P_1 is annihilated.

So upon removal of the next path Q_2, again the $+1$'s left by the removal of P_2, due to their placement, serve as an inpenetrable barrier for Q_2, which starts again from the bottom of the $(r-1)$th diagonal, to climb any of the ladders of P_2, confining Q_2 to the region strictly below P_2. The decrease of the upright rhombus gradients by the removal of Q_2 occurs weakly below the foot rhombus of P_2 in each diagonal, and hence again keeps the effect of $+1$ left by the removal of P_3 or by any P_i with $i \geq 3$, even if the head rhombus of Q_2, the foot rhombi of P_2 and P_3 all coincide.

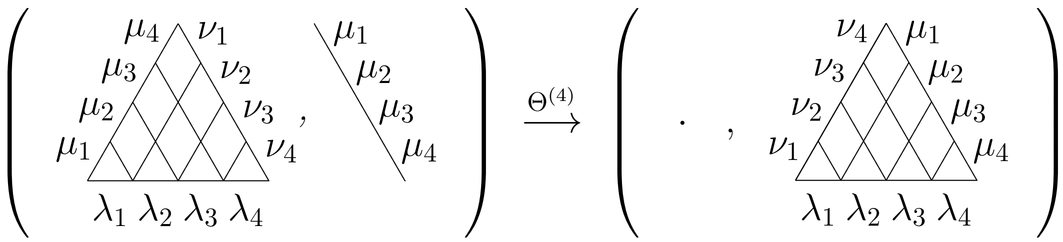
Proceeding in this manner, one concludes that Q_i lies strictly below P_i for all $1 \leq i \leq c$ as claimed in part (b), which shows the required inequality as anticipated.

■

5. Path removal map $\sigma^{(n)}$ from $\mathcal{H}^{(n)}(\lambda,\mu,\nu)$ to $\mathcal{H}^{(n)}(\lambda,\nu,\mu)$

Armed with our path removal procedures we are able to exploit them to construct from any hive $H \in \mathcal{H}^{(n)}(\lambda,\mu,\nu)$ its partner hive $K \in \mathcal{H}^{(n)}(\lambda,\nu,\mu)$. To do so it is only necessary to evacuate the initial hive H by performing a sequence of path removals that render it empty and to build the final hive K from the data on the location of the first and last edges on each path that is removed. In doing so one constructs a sequence of pairs $(H^{(r)}, K^{(n-r)})$ for each $r = n, n-1, \ldots, 0$ where $H^{(r)}$ is an r-hive and $K^{(n-r)}$ is an r-*truncated* n-*hive* consisting of the rightmost $n-r$ diagonals of some n-hive. These pairs are such that $H^{(n)} = H$ and $H^{(0)}$ is the empty hive, signified here by a single point, while $K^{(n)} = K$ and $K^{(0)}$ is an empty n-truncated n-hive, signified by a single boundary line consisting of β-edges with labels $\mu_1, \mu_2, \ldots, \mu_r$.

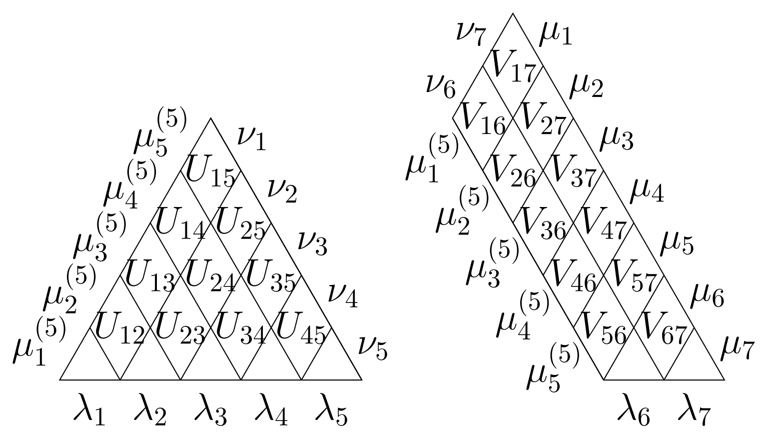
Example 5.1. *In the case $n = 4$ the map we are seeking is of the following type from $(H^{(4)}, K^{(0)})$ to $(H^{(0)}, K^{(4)})$:*



Definition 5.2. *Given any LR hive $H \in \mathcal{H}^{(n)}(\lambda, \mu, \nu)$, let $H^{(n)} = H$ and let $K^{(0)}$ be the n-truncated n-hive with edge labels μ. Then let $\Theta^{(n)} := \Theta_1 \cdots \Theta_{n-1}\Theta_n$ denote the operation which transforms the pair $(H^{(n)}, K^{(0)})$ to the pair $\Theta^{(n)}(H^{(n)}, K^{(0)}) := (H^{(0)}, K^{(n)})$ through the action of a succession of n operators that produce pairs $(H^{(r)}, K^{(n-r)})$, with $r = n-1, n-2, \ldots, 0$, respectively, as indicated by*

$$(H^{(n)}, K^{(0)}) \xmapsto{\Theta_n} (H^{(n-1)}, K^{(1)}) \xmapsto{\Theta_{n-1}} \cdots \xmapsto{\Theta_2} (H^{(1)}, K^{(n-1)}) \xmapsto{\Theta_1} (H^{(0)}, K^{(n)}). \tag{5.1}$$

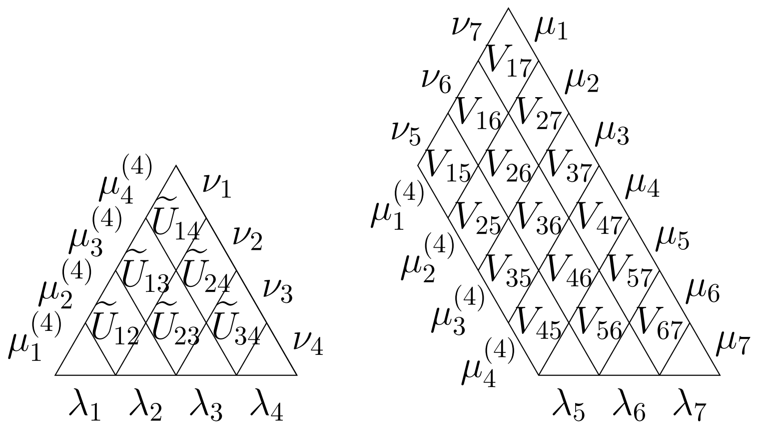
The boundary edge labels of the r-hive $H^{(r)}$ are $(\lambda_1, \ldots, \lambda_r)$, $(\mu_1^{(r)}, \ldots, \mu_r^{(r)})$ and $(\nu_1, \ldots, \nu_r)$, while those of the r-truncated n-hive $K^{(n-r)}$ with which it is paired are $(\lambda_{r+1}, \ldots, \lambda_n)$, $(\mu_1^{(r)}, \ldots, \mu_r^{(r)})$, $(\nu_{r+1}, \ldots \nu_n)$ and $(\mu_1, \ldots, \mu_n)$, as exemplified in the case $n = 7$ and $r = 5$ by



(5.2)

The operator Θ_r maps the pair $(H^{(r)}, K^{(n-r)})$ to $(H^{(r-1)}, K^{(n-r+1)})$ where $H^{(r-1)} = \theta_r H^{(r)}$ and the action of θ_r serves to define V_{kr} as in Theorem 4.3. In parallel *with this, $K^{(n-r+1)}$ is obtained from $K^{(n-r)}$, by adding to its left-hand boundary an rth diagonal of upright rhombi having gradients V_{kr}, with boundary edge labels ν_r and λ_r at its top and bottom, respectively.*

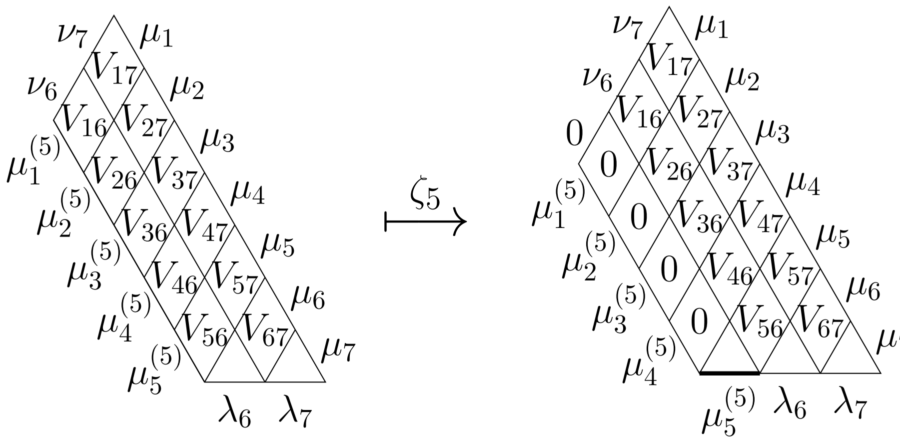
Given that the right-hand boundary edge labels of the rth diagonal of $K^{(n-r+1)}$ are $(\mu_1^{(r)},\ldots,\mu_r^{(r)})$, it follows from the fact that $V_{kr}=\mu_k^{(r)}-\mu_k^{(r-1)}$ that the left-hand boundary edge labels of $K^{(n-r+1)}$ are $(\mu_1^{(r-1)},\ldots,\mu_{r-1}^{(r-1)})$. For example Θ_5 maps the pair $(H^{(5)},K^{(2)})$ displayed in 5.2 to the pair $(H^{(4)},K^{(3)})$ given by



(5.3)

where, as a result of the type (ii) path removals, the upright rhombus gradients U_{ij} of $H^{(5)}$ have been replaced by $\widetilde{U}_{ij}$ in $H^{(4)}$.

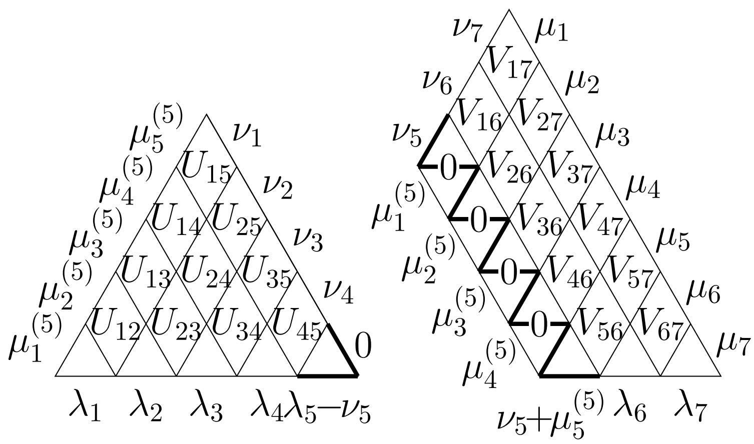
Following this lengthy definition, it is convenient to exemplify each phase of the action of Θ_r on the pair $(H^{(r)},K^{(n-r)})$. Before doing this it is helpful to introduce an operator ζ_r whose action on a truncated hive $K^{(n-r)}$ is to add to the left-hand boundary of $K^{(n-r)}$ an rth diagonal with upright rhombus gradients all 0, upper boundary edge label 0 and lower boundary edge label $\mu_r^{(r)}$. Then what might be called Phase 0 of the action of Θ_r is to act on $K^{(n-r)}$ with ζ_r, as illustrated by the following where it will be seen that the left-hand boundary edge labels automatically become $(\mu_1^{(r)},\ldots,\mu_{r-1}^{(r)})$:



(5.4)

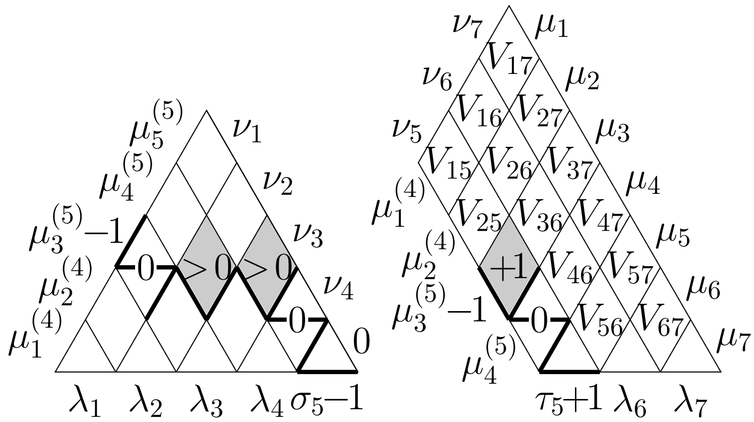
Here it might be noted that the label $\mu_5^{(5)}$ added to the the leftmost lower boundary edge automatically preserves the triangle condition.

Phase 1 then arises if $\nu_r > 0$ in which case θ_r involves ν_r type (i) hive path removals from $H^{(r)}$ and the same number of hive path additions to $\zeta_r K^{(n-r)}$ as illustrated by



(5.5)

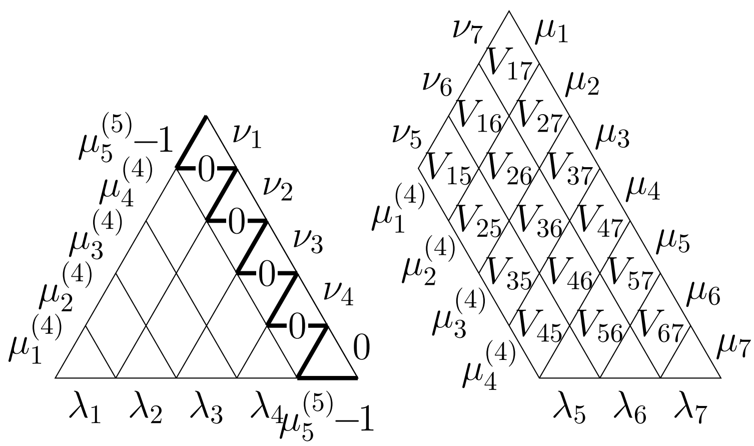
Phase 2 involves a sequence of $\lambda_r - \nu_r - \mu_r$ type (ii) hive path removals from $H^{(r)}$ and the same number of hive path additions to $\zeta_r K^{(n-r)}$, of which one such removal and addition is illustrated by



(5.6)

where $\sigma_5 = \lambda_5 - \nu_5 - V_{15} - V_{25}$ and $\tau_5 = \mu_5^{(5)} + \nu_5 + V_{15} + V_{25}$. In this example it has been assumed that the hive path removal illustrated on the left is the first that terminates at the 3rd edge on the left-hand boundary, thereby reducing the edge label $\mu_3^{(5)}$ by 1. A further $V_{35} - 1$ such path removals reduce this edge label to $\mu_3^{(4)}$ whilst increasing the shaded rhombus label on the right to V_{35}. In Phase 2 this process continues until the upright rhombus gradients in the rightmost diagonal on the left are all 0, and those in leftmost diagonal on the right are V_{k5} for $k = 1, 2, \ldots, r-1 = 4$.

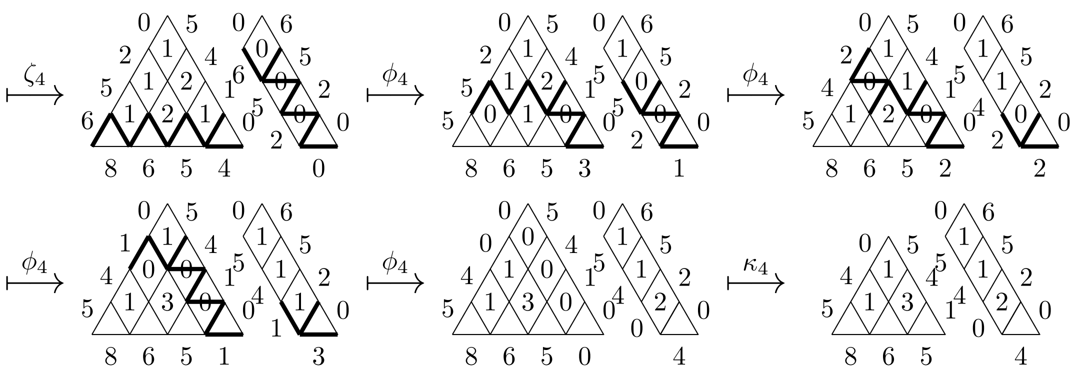
Phase 3 then involves a succession of $\mu_r^{(r)}$ type (iii) hive path removals from $H^{(r)}$. However, no corresponding hive path additions to $\zeta_r K^{(n-r)}$ are required because the addition of $\mu_r^{(r)}$ to the leftmost lower boundary edge label has already taken place in Phase 0. The first step of Phase 3 is illustrated in our example by:



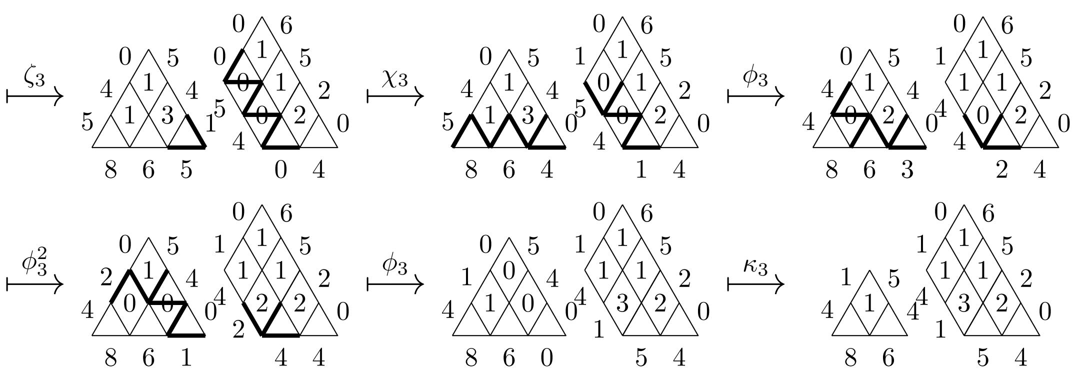
(5.7)

The repetition of this a total of $\mu_5^{(5)}$ times and the removal of the resulting redundant 5th diagonal on the left by means of the action of κ_5 then yields 5.3 as required.

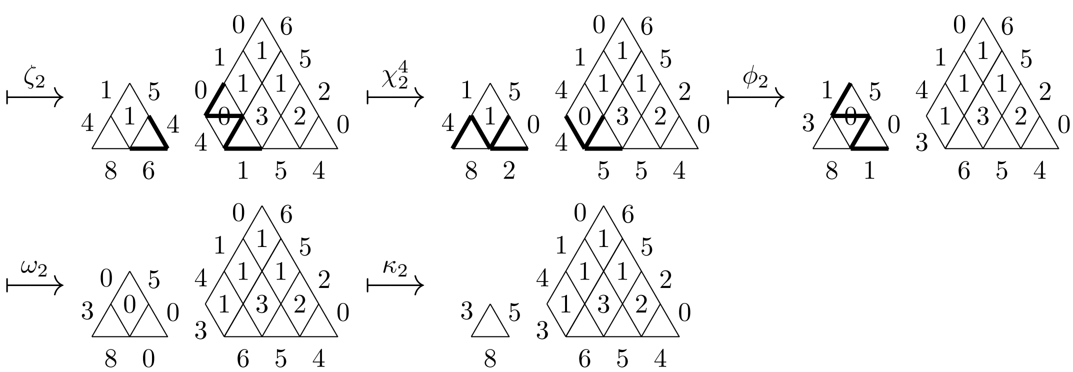
Example 5.3. *An exemplification of the map from $(H^{(4)}, K^{(0)})$ to $(H^{(0)}, K^{(4)})$ is provided by the following: $(H^{(4)}, K^{(0)}) \xmapsto{\Theta_4} (H^{(3)}, K^{(1)})$:*



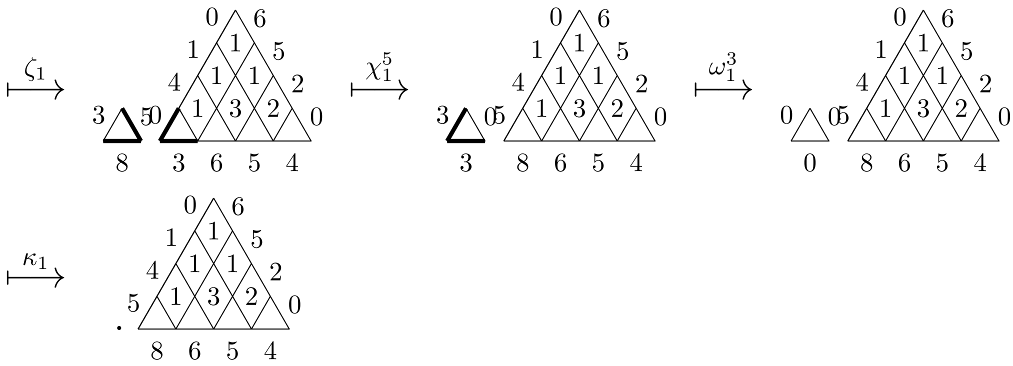
$(H^{(3)}, K^{(1)}) \xmapsto{\Theta_3} (H^{(2)}, K^{(2)})$:



$(H^{(2)}, K^{(2)}) \xmapsto{\Theta_2} (H^{(1)}, K^{(3)})$:



$(H^{(1)}, K^{(3)}) \overset{\Theta_1}{\longmapsto} (H^{(0)}, K^{(4)})$:

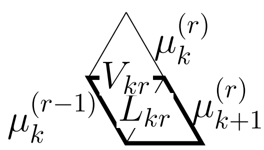


We are now in a position to state and prove

Theorem 5.4. *Let n be a positive integer and let λ, μ and ν be partitions such that $\ell(\lambda) \leq n$ and $\mu, \nu \subseteq \lambda$ with $|\lambda| = |\mu| + |\nu|$. For each LR hive $H \in \mathcal{H}^{(n)}(\lambda, \mu, \nu)$ let $H^{(n)} = H$ and let $K^{(0)}$ be an n-truncated n-hive with edge labels μ. If we let $\Theta^{(n)}(H^{(n)}, K^{(0)}) = (H^{(0)}, K^{(n)})$ as in Definition 5.2, then $H^{(0)} = \theta_1\theta_2 \cdots \theta_n H$ is an empty hive and $K = K^{(n)}$ is an LR hive $K \in \mathcal{H}^{(n)}(\lambda, \nu, \mu)$. In such a case we write $K = \sigma^{(n)} H$.*

Proof: First it should be recognised from 4.6 that the passage from $H^{(r)}$ to $H^{(r-1)} = \theta_r H^{(r)}$ involves the action of κ_r that eliminates an empty rth diagonal. Repeating this for $r = n, \ldots, 2, 1$ ensures that $H^{(0)} = \theta_1\theta_2 \cdots \theta_n H$ is the empty hive, as required. In order to determine the properties of K we adopt the same plan as described at the beginning of the proof of Lemma 4.1. From the iteration scheme of Definition 5.2 and the illustration of the action of Θ_r in mapping 5.2 to 5.3 it is clear that $K = K^{(n)}$ will have boundary edge labels λ, ν and μ, of which, in particular, ν_n and μ_n are non-negative. In addition it can be seen immediately that each Phase of the action of Θ_r preserves the triangle condition at every stage.

As far as the gradients of elementary rhombi are concerned, all the upright rhombus gradients V_{kr} are non-negative as they count the number of certain type (ii) hive path removals. As can be seen from the following diagram

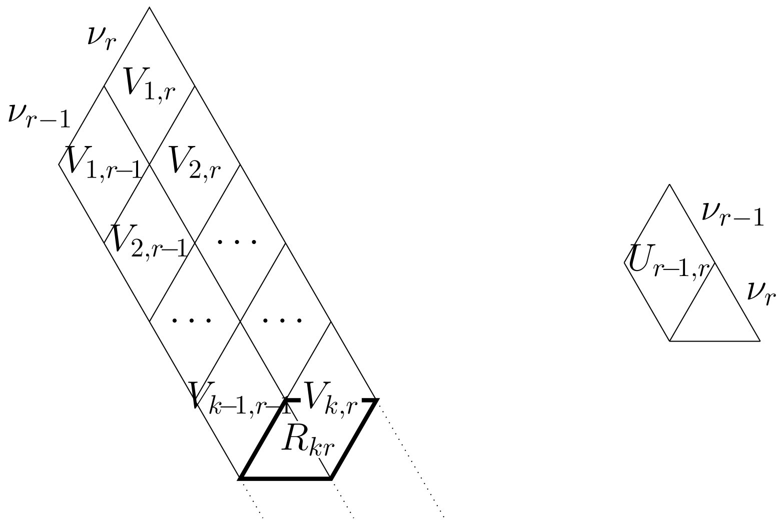


(5.8)

the left-leaning rhombus gradients L_{kr} are also non-negative since

$$L_{kr} = \mu_k^{(r-1)} - \mu_{k+1}^{(r)} = \mu_k^{(r)} - V_{kr} - \mu_{k+1}^{(r)} \geq 0, \tag{5.9}$$

where the final step is a consequence of Corollary 4.5. Similarly, as can be seen from the following pair of diagrams



(5.10)

the right-leaning rhombus gradients R_{kr} are also non-negative since

$$R_{kr} = (\nu_{r-1} + \sum_{i=1}^{k-1} V_{i,r-1}) - (\nu_r + \sum_{i=1}^{k} V_{ir}) \geq U_{r-1,r} + N_{k-1,r-1} - N_{kr} \ \geq \ 0\,, \tag{5.11}$$

where use has been made first of the hive condition $\nu_{r-1} - U_{r-1,r} \geq \nu_r$ that applies to the sub-diagram of $H^{(r)}$ that appears on the right and then of Lemma 4.6.

Thus all elementary rhombus gradients of K are non-negative, and together with the triangle conditions and the nonnegativity of ν_n and μ_n mentioned earlier, this completes the proof that K is an LR hive. The boundary edge labels then ensure that $K \in \mathcal{H}^{(n)}(\lambda, \nu, \mu)$. ∎

6. Creation of a hive by path additions and a proof of bijectivity

Having used a path removal procedure to provide a map from any $H \in \mathcal{H}^{(n)}(\lambda, \mu, \nu)$ to some $K \in \mathcal{H}^{(n)}(\lambda, \nu, \mu)$ we now wish to point out that a path addition procedure may be used to provide a map from any $K \in \mathcal{H}^{(n)}(\lambda, \nu, \mu)$ to some $H \in \mathcal{H}^{(n)}(\lambda, \mu, \nu)$. The aim is to show that these two maps are mutually inverse to one another, thereby proving that each is a bijection.

Our approach is to move successively from an $(r-1)$-hive $H^{(r-1)}$ to an r-hive $H^{(r)}$ under a procedure dictated by the rth diagonal of K. In doing so it is necessary to exploit first a new operator $\overline{\kappa}_r$ whose action is to add to $H^{(r-1)}$ an empty rth diagonal consisting of a sequence of upright rhombi all of gradient 0, with its upper and lower boundary edge labels 0, and with its remaining new edges given the unique labels that preserve the triangle conditions. At this point it will be appropriate to verify that if H' is any LR $(r-1)$-hive, then $H'' = \overline{\kappa}_r H'$ is also an LR r-hive. It is only necessary to confirm that all the new left- and right-leaning rhombi in H'' have gradients ≥ 0. The new right-leaning rhombi lie across the border of the $(r-1)$th and the rth diagonals Their right-hand edges have label 0 by construction, and their left-hand edges have non-negative labels as part of the LR hive H', so their gradients are ≥ 0. The new left-leaning rhombi sit in the rth diagonal. If the edge labels on the right-hand boundary of H' are $(\nu_1, \ldots, \nu_{r-1})$, then these constitute a partition since H' is a hive and by construction those on the right-hand boundary of H'' are $(\nu_1, \ldots, \nu_{r-1}, \nu_r)$ with $\nu_r = 0$. The kth left-leaning rhombus from the top therefore has left-hand edge label ν_k and right-hand edge label ν_{k+1}, so its gradient is $\nu_k - \nu_{k+1}$ which is ≥ 0 for all $k = 1, \ldots, r-1$, thereby confirming that H'' is a hive.

Then we require the following:

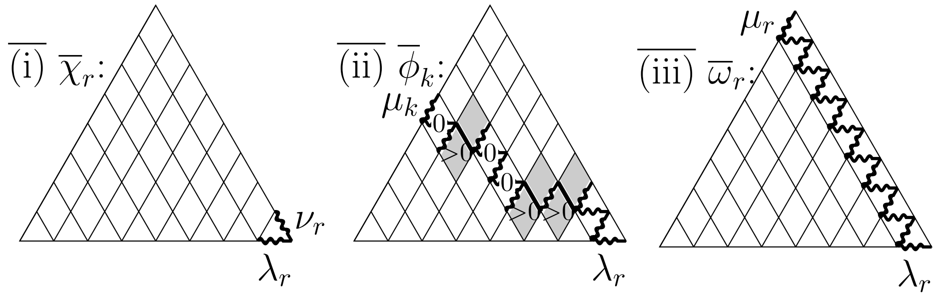
Definition 6.1. *For any given $H \in \mathcal{H}^{(r)}(\lambda, \mu, \nu)$ with $\ell(\lambda) \leq r$ we define three path addition operators $\overline{\chi}_r$, $\overline{\phi}_k$ and $\overline{\omega}_r$ whose action on H is to increase or reduce edge labels by 1 along paths specified as follows:*

$\overline{\text{(i)}}$ $\overline{\chi}_r$*: the path consists of the boundary edges labelled ν_r and λ_r, with each of these labels being increased by 1;*

$\overline{\text{(ii)}}$ $\overline{\phi}_k$*: for any $k < r$ the path proceeds down the kth diagonal from the edge labelled μ_k through upright rhombi of gradient 0 until it encounters an upright rhombus of positive gradient, at which point it moves horizontally to the right into the $(k+1)$th diagonal and proceeds down this diagonal or to the right as before, and so on until it either meets the base of the hive and then moves to the right or meets the right hand boundary and then moves down the rth diagonal regardless of its upright rhombus gradients until, in both cases, it terminates at the edge labelled λ_r, with all path α- and γ-edge labels being increased by 1 and all path β-edge labels decreased by 1 ;*

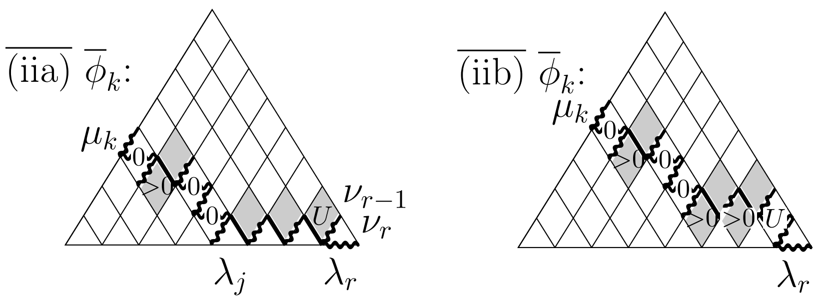
$\overline{\text{(iii)}}$ $\overline{\omega}_r$: *the path proceeds directly down the rth diagonal until it terminates at the base at the edge labelled λ_r, with all path edge labels increased by 1.*

These three types of path addition are illustrated below. In each case every α- or γ-edge label is increased by 1 and every solid β-edge label is decreased by 1. In the case of $\overline{\chi}_r$ and $\overline{\omega}_r$ the path additions are confined to the rightmost rth diagonal. On the other hand the path addition route ascribed to the action of $\overline{\phi}_k$ with $1 \le k < r$ consists of a sequence of ladders through upright rhombi of gradient 0 in each diagonal from the kth to the rth, with the passage from each diagonal to the next taking place through a solid β-edge.



(6.1)

It might be noted here that there are two distinct manners in which type $\overline{\text{(ii)}}$ paths may terminate. They are illustrated below, with the figures $\overline{\text{(iia)}}$ and $\overline{\text{(iib)}}$ applying to cases in which the path addition meets the base hive boundary first and the right-hand hive boundary first, respectively.



(6.2)

Remark 6.2. *Just as $\overline{\kappa}_r$ is the inverse of κ_r, whose action is specified in Definition 4.2, so the path addition operators $\overline{\chi}_r$ and $\overline{\omega}_r$ are the inverses of χ_r and ω_r introduced in Definition 3.1 with their action exemplified in the diagrams of 3.1. Moreover, if the action of ϕ_r on an r-hive H removes a path P terminating at level k, then applying $\overline{\phi}_k$ to $\phi_r H$ recovers H, since the foot rhombus of each ladder of P, left with positive gradient after the removal of P, and the middle rhombi of each ladder of P, left with gradient 0, direct the* *action of* $\overline{\phi}_k$ *so as to trace* P *backward, restoring each edge label and upright rhombus gradient to their original values in* H*. On the other hand, the opposite cancellation* $\phi_r(\overline{\phi}_k H') = H'$ *may not hold in general since, in the definition of the operator* $\overline{\phi}_k$*, any upright rhombus gradient test for the added path* P' *to descend the* r*th diagonal has been omitted in order to ensure that* P' *extends to the foot of this diagonal. Hence the path* P *removed by the action of* ϕ_r *may encounter an upright rhombus of gradient* > 0 *in the* r*th diagonal below the entry point of* P'*, causing* P *to leave the* r*th diagonal earlier than expected. This will occur in the case of the example illustrated in* $\overline{\text{(iib)}}$ *if* U *is positive. However, our use of the operators* $\overline{\phi}_k$ *is only through the operator* $\overline{\theta}_r$ *defined in Theorem 6.3, in which case* $\phi_r(\overline{\phi}_k H') = H'$ *also holds: see Lemmas 6.5 and 6.8.*

We then claim the validity of the following:

Theorem 6.3. *Let* n *be a positive integer and let* λ*,* μ *and* ν *be partitions such that* $\ell(\lambda) \leq n$ *and* $\mu, \nu \subseteq \lambda$ *with* $|\lambda| = |\mu| + |\nu|$*. For each LR hive* $K \in \mathcal{H}^{(n)}(\lambda, \nu, \mu)$ *with upright rhombus gradients* V_{ij} *for* $1 \leq i < j \leq n$ *let*

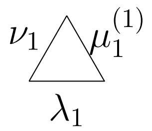
$$\overline{\theta}_r = \overline{\chi}_r^{\,\nu_r}\ \overline{\phi}_1^{\,V_{1,r}}\ \overline{\phi}_2^{\,V_{2r}}\ \cdots\ \overline{\phi}_{r-1}^{\,V_{r-1,r}}\ \overline{\omega}_r^{\,\mu_r^{(r)}}\ \overline{\kappa}_r\,, \tag{6.3}$$

where $\mu_r^{(r)} = \mu_r - V_{r,n} - V_{r,n-1} - \cdots - V_{r,r+1}$*, and let*

$$H^{(r)}(K) = \overline{\theta}_r\ \cdots\ \overline{\theta}_2\,\overline{\theta}_1\ H^{(0)}\,, \tag{6.4}$$

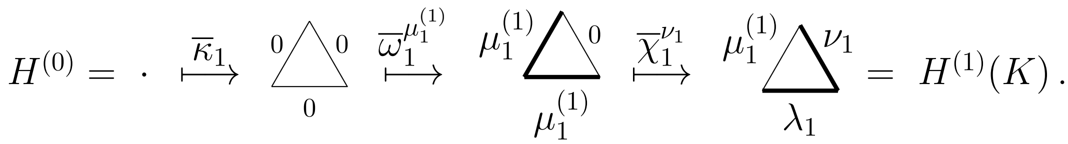
for $r = 1, 2, \ldots, n$ *with* $H^{(0)}$ *being an empty hive. Then* $H(K) := H^{(n)}(K) \in \mathcal{H}^{(n)}(\lambda, \mu, \nu)$ *and we write* $H(K) = \overline{\sigma}^{(n)}\, K$*.*

Proof: It is convenient to set $\lambda^{(r)} = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\nu^{(r)} = (\nu_1, \nu_2, \ldots, \nu_r)$ and to remind ourselves of the notation already used in connection with K whereby $\mu^{(r)} = (\mu_1^{(r)}, \mu_2^{(r)}, \ldots, \mu_r^{(r)})$ with $\mu_k^{(r)} = \mu_k - V_{k,n} - V_{k,n-1} - \cdots - V_{k,r+1}$ for $k = 1, 2, \ldots, r$. This allows us to define $K_{(r)} \in \mathcal{H}^{(r)}(\lambda^{(r)}, \nu^{(r)}, \mu^{(r)})$ to be the subhive of $K \in \mathcal{H}^{(n)}(\lambda, \nu, \mu)$ consisting of its leftmost r diagonals, for $r = 1, 2, \ldots, n$. Thus $K_{(r)}$ is essentially the complement of the truncated hive $K^{(n-r)}$ in $K = K^{(n)}$. We then claim first that $H^{(r)}(K)$ is a triangular array of side length r with boundary edge labels $\lambda^{(r)}$, $\mu^{(r)}$ and $\nu^{(r)}$ for $r = 1, 2, \ldots, n$. This may be proved by induction. In the case $r = 1$ we have

$K_{(1)} =$ 

, and $H^{(1)}(K) = \overline{\chi}_1^{\,\nu_1}\ \overline{\omega}_1^{\,\mu_1^{(1)}}\ \overline{\kappa}_1\ H^{(0)}$ so that the map from

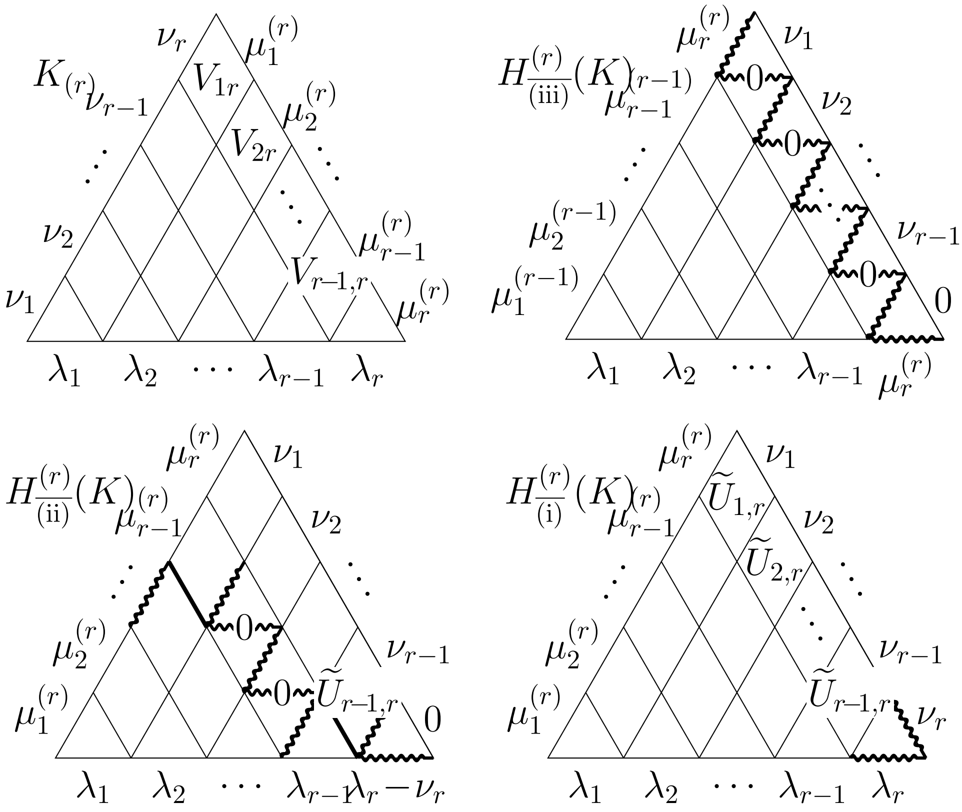
$H^{(0)}$ to $H^{(1)}(K)$ proceeds as shown below:



(6.5)

where in the final step use has been made of the fact that $\mu_1^{(1)} + \nu_1 = \lambda_1$, as implied by the hive condition on K. This demonstrates that $H^{(1)}(K)$ has edge labels (λ_1), $(\mu_1^{(1)})$ and (ν_1) as required.

By the induction hypothesis $H^{(r-1)}(K)$ is a triangular array of side length $r-1$ with boundary edge labels $\lambda^{(r-1)}$, $\mu^{(r-1)}$ and $\nu^{(r-1)}$. The passage from $H^{(r-1)}(K)$ to $H^{(r)}(K) = \overline{\theta}_r\, H^{(r-1)}(K)$, as determined by the rth diagonal of K, is then illustrated by:



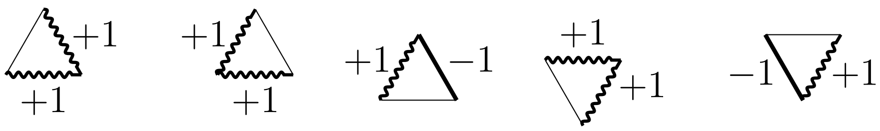
(6.6)

The hive conditions on $K_{(r)}$ imply that

$$\mu_k^{(r)} = \mu_k^{(r-1)} + V_{kr} \quad \text{for } k = 1, 2, \ldots, r-1 \text{ and } \quad \sum_{k=1}^{r-1} V_{kr} = \lambda_r - \nu_r - \mu_r^{(r)}\,. \tag{6.7}$$

In the above display 6.6 $H^{(r)}_{\text{(iii)}}(K)$ has been formed by adding to $H^{(r-1)}(K)$ an rth diagonal of upright rhombi all of gradient 0 and then applying all $\mu_r^{(r)}$ type $\overline{\text{(iii)}}$ path addition operators. Then $H^{(r)}_{\text{(ii)}}(K)$ is obtained by applying V_{kr} type $\overline{\text{(ii)}}$ path addition operators successively in the order $k = r-1, \ldots, 2, 1$. Just one type $\overline{\text{(ii)}}$ path addition has been shown for illustrative purposes. For each k the V_{kr} added paths increase the kth left hand boundary edge label from $\mu_k^{(r-1)}$ to $\mu_k^{(r-1)} + V_{kr} = \mu_k^{(r)}$, where use has been made of the first identity in 6.7. Moreover, each of these path additions extends as far as the foot of the rth diagonal, adding precisely 1 both to the rth lower boundary edge and to one or other of the upright rhombus gradients in this diagonal. It follows that on completing this type $\overline{\text{(ii)}}$ action the rth lower boundary edge becomes $\mu_r^{(r)} + V_{1r} + V_{2r} + \cdots + V_{r-1,r} = \lambda_r - \nu_r$, as shown, where use has been made of the second identity of 6.7. Finally, the application of all ν_r type $\overline{\text{(i)}}$ path addition operators adds ν_r to the two edges meeting at the lower right-hand corner of $H^{(r)}_{\text{(ii)}}(K)$ thereby yielding $H^{(r)}_{\text{(i)}}(K)$ with boundary edge labels as shown in the last diagram of 6.6. It can be seen from this $H^{(r)}(K) = H^{(r)}_{\text{(i)}}(K)$ has boundary edge labels specified by $\lambda^{(r)}$, $\mu^{(r)}$ and $\nu^{(r)}$, as required.

It remains to show that $H^{(r)}(K)$ satisfies all necessary hive conditions and is thus $\in \mathcal{H}^{(r)}(\lambda^{(r)}, \mu^{(r)}, \nu^{(r)})$, for which again we use the same method as in the proof of Lemma 4.1. The non-negativity of $\mu_r^{(r)}$ and $\nu_r^{(r)} = \nu_r$ follows immediately from that of all edge labels in K. As far as elementary triangles are concerned the path additions give rise to the following possibilities:



(6.8)

It is clear that the triangle conditions are preserved in every case, and that it is only β-edge labels that may be reduced in value. If we can confirm the rhombus gradient conditions, then these labels remain non-negative since they must then all be $\geq \nu_r \geq 0$.

It is helpful to proceed by way of an analogue of Lemma 4.4.

Lemma 6.4. *During the action of $\overline{\theta}_r$ on $H^{(r-1)}(K)$ let a hive path addition of type $\overline{\text{(ii)}}$ follow a path P starting from the left-hand boundary at level $k < r$, then the next such path addition, starting from the left-hand boundary at level*

$k' \leq k$ by the definition 6.3 of $\overline{\theta}_r$, follows a path P' lying weakly below the path P in each diagonal from the kth to the rth.

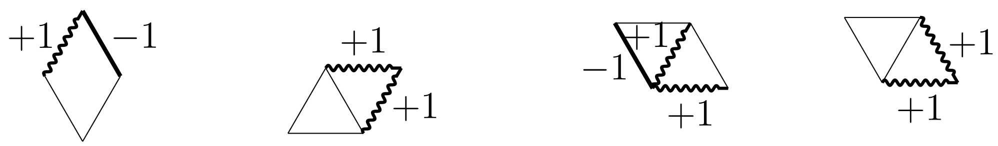
Proof: The argument is similar to the one used in the proof of Lemma 4.4 except for the direction in which the paths proceed and the exchanged roles of head and foot rhombi in guiding the paths, and so we omit the details. To derive the conclusion of Lemma 6.4, it is sufficient to apply this argument, diagonal by diagonal, until each added path meets either the bottom or the right-hand boundary, since afterwards the definition directs the path to just proceed in a zig-zag manner along that boundary. ■

Before analysing other rhombus gradients, let us settle the issue, just mentioned above, that was raised in the Remark 6.2, namely that of the upright rhombus gradients in the rth diagonal below the point of entry of each type $\overline{\text{(iib)}}$ path, since we will need it more than once.

Lemma 6.5. *Consider an application of $\overline{\phi}_k$ to an r-hive H' as occurs in the course of the action of $\overline{\theta}_r$. If the added path is of type $\overline{\text{(iib)}}$, then all the upright rhombi in the rth diagonal of H' through which P' descends necessarily have gradient* 0.

Proof: Focusing on the transformation of the rth diagonal during the action of $\overline{\theta}_r$, the gradients of the upright rhombi are initially 0 when created by $\overline{\kappa}_r$ and remain intact through actions of $\overline{\varpi}_r$. Then, by the definition of $\overline{\theta}_r$, operators $\overline{\phi}_k$ are applied in the weakly decreasing order of the starting level k, and Lemma 6.4 ensures that each path is added weakly below its predecessor, accompanied by an increase of an upright rhombus gradient in the rth diagonal only immediately above its first α-edge in the rth diagonal. Hence at the time of each type $\overline{\text{(iib)}}$ path addition, the upright rhombi below its first α-edge in the rth diagonal retain gradients 0, since all previous increments have occured above it. ■

Returning to the proof of Theorem 6.3, the only path addition configurations that gives rise to a reduction in a rhombus gradient are those shown below:

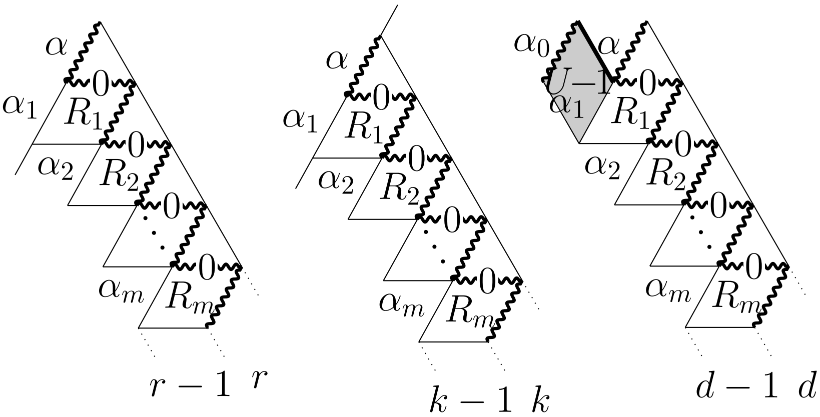


(6.9)

$U \longrightarrow U-1 \qquad R \longrightarrow R-1 \qquad L \longrightarrow L-1 \qquad L \longrightarrow L-1$

The leftmost configuration only arises in a situation where the transition is from an upright rhombus gradient $U > 0$ to $U - 1$, as can be seen from 6.1. Thus all upright rhombus gradients remain non-negative after all possible path additions.

The second configuration in 6.9 always appears as part of a ladder of one of the three types with some $m \geq 1$:



(6.10)

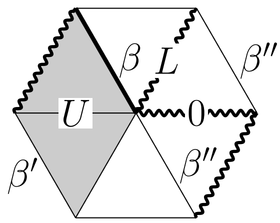
In all three cases the hive conditions on $H^{(r-1)}(K)$ imply that $\alpha_m \geq \cdots \geq \alpha_2 \geq \alpha_1$. Moreover, each addition path ladder passes through upright rhombi of gradient 0, which due to Lemma 6.5 is true even if $d = r$ in the third diagram. This implies that in each case $R_m = \alpha_m - \alpha$. Then in the first case on the left, for which $\alpha_1 = \mu_{r-1}^{(r-1)}$, under the addition of $\mu_r^{(r)}$ paths of the type shown the edge label α increases from 0 to its maximum value $\mu_r^{(r)}$. It follows that $R_m \geq \alpha_1 - \alpha \geq \mu_{r-1}^{(r-1)} - \mu_r^{(r)} \geq 0$, where the last step is a consequence of the hive conditions on K. Thus all right-leaning rhombi in this situation remain of non-negative gradient.

Similarly in the next case, for which $\alpha_1 = \mu_{k-1}^{(r-1)}$, under the addition of $V_{kr} = \mu_k^{(r)} - \mu_k^{(r-1)}$ paths of the type shown the edge label α increases from $\mu_k^{(r-1)}$ to its maximum value $\mu_k^{(r)}$. Hence $R_m = \alpha_m - \alpha \geq \alpha_1 - \alpha \geq \mu_{k-1}^{(r-1)} - \mu_k^{(r)} \geq 0$ where once again the last step is a consequence of the hive conditions on K. Hence, once again all right-leaning rhombi in this situation remain of non-negative gradient.

In the third case, the labelling is taken to be that immediately after any one of the actions of some $\overline{\phi}_k$. The fact that the path addition has moved from the $(d-1)$th to the dth diagonal implies that the shaded upright rhombus had an initial gradient $U = \alpha_1 - (\alpha_0 - 1) > 0$. with an initial hive condition $\alpha_0 - 1 \geq$

$\alpha - 1$. It follows that after the path addition $R_m = \alpha_m - \alpha \geq \alpha_1 - \alpha_0 \geq 0$, as required to show that all right-leaning rhombi remain of non-negative gradient.

Returning to 6.9 it is necessary to consider the reduction of gradients of left-leaning rhombi. The third configuration in 6.9 appears at the top of a ladder either $\underline{(1)}$ as in the third diagram of 6.10 with $m \geq 1$, or (2) at the end of a type $\overline{\text{(iia)}}$ path as in 6.2. For case (1) consider the following diagram with the edge and gradient labels specified *before* the path addition:



(6.11)

In this situation, by hypothesis, the shaded upright rhombus has gradient $U = \beta - \beta' > 0$ and the white upright rhombus has gradient 0, which is again true even if the white rhombus lies in the rth diagonal, due to Lemma 6.5. Advantage has been taken of the zero gradient of the white upright rhombus to equate the pair of edge labels labelled β''. The hive conditions before the path addition also imply that $\beta' \geq \beta''$ from which it follows that $L = \beta - \beta'' > 0$. After the path addition the rhombus gradient L is reduced to $L - 1$, which remains ≥ 0.

The only remaining left-leaning rhombi whose gradients may be reduced under path additions are those lying at the bottom right-hand corner as exemplified below, namely the third configuration in 6.9 in case (2) and the fourth configuration in 6.9 that applies in the case of each type $\overline{\text{(i)}}$ path addition.

(6.12)

On the left we have $L = \beta = \nu_{r-1} - U$ and on the right $L = \beta - \nu_r = \nu_{r-1} - \nu_r - U$. Without knowing whether L remains ≥ 0 after the path addition, nor some of the edge labels, we can continue applying path addition operators as prescribed by K, since their action is well-defined on any triangular array of edge labels satisfying the triangle conditions as well as the non-negativity of all upright rhombus gradients, and this action produces another such array. In doing so, we can still use Lemma 6.4 since both its statement and its proof only refer to the upright rhombus gradients, from which it follows that, once a type $\overline{\text{(iia)}}$ path addition occurs, all remaining type $\overline{\text{(ii)}}$ path additions are of type $\overline{\text{(iia)}}$. Then U increases steadily from 0 to, say, $\widetilde{U}_{r-1,r}$ under all path additions of the type $\overline{\text{(iia)}}$, with no further changes under path additions of type $\overline{\text{(i)}}$. Therefore all we require for L to remain non-negative is that the final value $\widetilde{U}_{r-1,r} \leq \nu_{r-1} - \nu_r$.

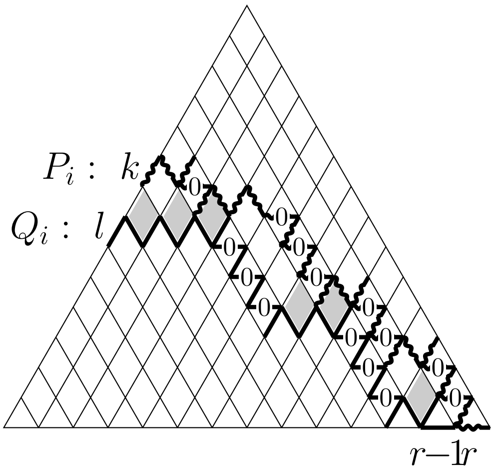
In the case $r = 1$ there is no left-leaning rhombus, while for $r = 2$ we have $H^{(2)}(K) = \overline{\theta}_2\, H^{(1)}(K) = \overline{\chi}_2^{\nu_2} \overline{\phi}_1^{V_{12}} \overline{\omega}_2^{\mu_2^{(2)}}\, \overline{\kappa}_2\, H^{(1)}(K)$ from which it can be seen that $\widetilde{U}_{12} = V_{12} \leq \nu_1 - \nu_2$, as required, where the final step is a consequence of the hive conditions on K. To then prove that $\widetilde{U}_{r-1,r} \leq \nu_{r-1} - \nu_r$ for $r \geq 3$ we first make the following observation regarding the sequential action of $\overline{\theta}_{r-1}$ and $\overline{\theta}_r$ on $H^{(r-2)}(K)$ that yields $H^{(r)}(K) = \overline{\theta}_r\, \overline{\theta}_{r-1}\, H^{(r-2)}(K)$.

Lemma 6.6. *For $r \geq 3$ let P_i for $i = 1, 2, \ldots, \lambda_r$ and Q_i for $i = 1, 2 \ldots, \lambda_{r-1}$ be the paths added by the operations $\overline{\theta}_{ri}$ and $\overline{\theta}_{r-1,i}$, respectively, lying in the ith positions counted from left to right in the following expansions of $\overline{\theta}_{r-1}$ and $\overline{\theta}_r$:*

$$\begin{gathered}
\overline{\theta}_{r-1} = \overbrace{\overline{\chi}_{r-1} \;\cdots\; \overline{\chi}_{r-1}}^{\nu_{r-1}} \overbrace{\overline{\phi}_1 \cdots \overline{\phi}_1}^{V_{1,r-1}} \;\cdots\; \overbrace{\overline{\phi}_{r-2} \cdots \overline{\phi}_{r-2}}^{V_{r-2,r-1}} \overbrace{\overline{\omega}_{r-1} \cdots \overline{\omega}_{r-1}}^{\mu_{r-1}^{(r-1)}} \;\overline{\kappa}_{r-1} \\
\overline{\theta}_r = \overbrace{\overline{\chi}_r \cdots \overline{\chi}_r}^{\nu_r} \overbrace{\overline{\phi}_1 \cdots \overline{\phi}_1}^{V_{1r}} \overbrace{\overline{\phi}_2 \cdots \overline{\phi}_2}^{V_{2r}} \;\cdots\; \overbrace{\overline{\phi}_{r-1} \cdots \overline{\phi}_{r-1}}^{V_{r-1,r}} \overbrace{\overline{\omega}_r \cdots \overline{\omega}_r}^{\mu_r^{(r)}} \;\overline{\kappa}_r \\
i = 1, \cdots, \nu_r, \;\cdots\;, \nu_{r-1}, \underbrace{\nu_{r-1}{+}1, \;\cdots\;, \nu_r {+} \sum_{j=1}^{r-1} V_{jr}}
\end{gathered} \tag{6.13}$$

Then for each i above the final brace, that is such that $\nu_{r-1} < i \leq \nu_r + \sum_{j=1}^{r-1} V_{jr}$, the paths P_i and Q_i are both of type $\overline{\text{(ii)}}$ and the path P_i lies strictly above Q_i.

Proof: Here the vertical alignment is designed to reflect not only that $\lambda_r \leq \lambda_{r-1}$ and $\nu_r \leq \nu_{r-1}$, but also that $\nu_r + \sum_{j=1}^{r-1} V_{j,r} \leq \nu_{r-1} + \sum_{j=1}^{r-2} V_{j,r-1}$, with the latter a consequence of the hive condition $R_{r-1,r} \geq 0$ in K. It follows that the paths P_i and Q_i are both of type $\overline{\text{(ii)}}$ if and only if $\nu_{r-1} + 1 \leq i \leq \nu_r + \sum_{j=1}^{r-1} V_{jr}$, as illustrated above in 6.13. For fixed i in this range, let P_i and Q_i start on the left hand boundary at levels k and l, respectively, so that $\overline{\theta}_{r,i} = \overline{\phi}_k$ and $\overline{\theta}_{r-1,i} = \overline{\phi}_l$. However, $\nu_r + \sum_{j=1}^{k} V_{j,r} \leq \nu_{r-1} + \sum_{j=1}^{k-1} V_{j,r-1}$, by virtue of the non-negativity of the right-leaning rhombus gradient R_{kr} in K. This implies that the list of operators $\overline{\phi}_k$ in the expansion of $\overline{\theta}_r$ extends no further to the right than the rightmost position of $\overline{\phi}_{k-1}$ in the expansion of $\overline{\theta}_{r-1}$. It follows that $l \leq k-1 < k$, so that the path Q_i passes from the lth diagonal to the $(r-1)$th diagonal leaving an upright rhombus of positive gradient immediately above it in each diagonal from the $(l+1)$th to the $(r-1)$th, necessarily including the kth, as illustrated below.



(6.14)

To show that P_i lies strictly above Q_i it only remains to show that the positivity condition on all the shaded upright rhombus gradients associated with the addition of Q_i remains valid up until the subsequent addition of P_i. This can be accomplished as follows. We consider first the case $i = m$ where $m = \nu_r + \sum_{j=1}^{r-1} V_{jr}$, corresponding to the first type $\overline{\text{(ii)}}$ path addition, and then proceed in the order of decreasing indices, following the argument very similar to the one given in the proof of Lemma 4.6 regarding the accumulation of $+1$'s creating inpenetrable barriers, with the roles of head and foot rhombi exchanged (namely in the current case the accumulation occurs in the head rhombi), so we omit further details. ■

We are now in a position to prove the following:

Lemma 6.7. *For $r \geq 3$ let the action of $\overline{\theta}_r$ on $H^{(r-1)}(K) = \overline{\theta}_{r-1}\, H^{(r-2)}$ yield $H^{(r)}$ with the bottommost upright rhombus in the rth diagonal having gradient $\widetilde{U}_{r-1,r}$. Then $\widetilde{U}_{r-1,r} \leq \nu_{r-1} - \nu_r$.*

Proof: For $r \geq 3$ we can exploit Lemma 6.6. Given any pair of addition paths P_i and Q_i with Q_i necessarily extending as far as the $(r-1)$th diagonal, the fact that P_i lies strictly above Q_i means that P_i enters the rth diagonal above its lowest upright rhombus, and therefore makes no contribution to $\widetilde{U}_{r-1,r}$. The only possible contributions to $\widetilde{U}_{r-1,r}$ are those that might arise from the type $\overline{\text{(ii)}}$ path additions P_i that are not paired with a corresponding type $\overline{\text{(ii)}}$ path addition Q_i. It then follows immediately from the vertical alignment of the expansions of $\overline{\theta}_r$ and $\overline{\theta}_{r-1}$ in 6.13 that $\widetilde{U}_{r-1,r} \leq \nu_{r-1} - \nu_r$, as required. ■

Returning yet again to the proof of Theorem 6.3, we now know that $\widetilde{U}_{r-1,r} \leq \nu_{r-1} - \nu_r$ for all $r \geq 2$, as required to prove that all hive conditions are satisfied by $H^{(r)}(K) = \overline{\theta}_r\, H^{(r-1)}(K)$ under the induction hypothesis that $H^{(r-1)}(K) \in \mathcal{H}^{(r)}(\lambda^{(r-1)}, \mu^{(r-1)}, \nu^{(r-1)})$. Since we have already established that $H^{(r)}(K)$ has the appropriate boundary edge labels, including the non-negativity of the topmost left-hand and the bottommost right-hand boundary edge labels, and also that it satisfies all triangle conditions, it follows that $H^{(r)}(K) \in \mathcal{H}^{(r)}(\lambda^{(r)}, \mu^{(r)}, \nu^{(r)})$.

This completes the induction argument, and applying this result in the case $r = n$ we conclude that $H(K) := H^{(n)}(K) \in \mathcal{H}^{(n)}(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}) = \mathcal{H}^{(n)}(\lambda, \mu, \nu)$, thereby proving the validity of Theorem 6.3. ■

With Lemma 6.5 at hand, we can now fill the piece that was missing in Remark 6.2 in saying that the type $\overline{\text{(ii)}}$ path additions, in the context in which we use them, and the type (ii) path removals are mutually inverse operators.

Lemma 6.8. *Consider an application of $\overline{\phi}_k$ to an r-hive H' as occurs in the course of the action of $\overline{\theta}_r$. If the operator ϕ_r is applied to such $\overline{\phi}_k H'$, then this recovers H'.*

Proof: The middle rhombi of all ladders of the path, say P', added by the action of $\overline{\phi}_k$ on H' are left with gradient 0, including the ones in the rth diagonal if P' is of type $\overline{\text{(iib)}}$, due to Lemma 6.5, and the head rhombi of all ladders of P' with positive gradients. Hence the path removed by the action of ϕ_r on $\overline{\phi}_k H'$ traces P' backwards guided by those rhombi, cancelling the effects of the addition of P' on edge labels and recovering H'. ■

The relationship between our path removal and path addition operations allows us to establish the bijective nature of the maps we have encountered in Theorems 5.4 and 6.3 with their domains extended as in the following:

Theorem 6.9. *For fixed positive integer n, let $\mathcal{H}^{(n)}$ be the union of $\mathcal{H}^{(n)}(\lambda,\mu,\nu)$ for all partitions λ, μ and ν such that $\ell(\lambda)\le n$, and with $\mu,\nu\subseteq\lambda$ and $|\lambda|=|\mu|+|\nu|$. Let $\sigma^{(n)}:\mathcal{H}^{(n)}\to\mathcal{H}^{(n)}$ be such that for each $H\in\mathcal{H}^{(n)}(\lambda,\mu,\nu)$ we have $\sigma^{(n)}:H\mapsto K\in\mathcal{H}^{(n)}(\lambda,\nu,\mu)$ with $K=\sigma^{(n)}H$, as defined in Theorem 5.4. Similarly, let $\overline{\sigma}^{(n)}:\mathcal{H}^{(n)}\to\mathcal{H}^{(n)}$ be such that for each $K\in\mathcal{H}^{(n)}(\lambda,\nu,\mu)$ we have $\overline{\sigma}^{(n)}:K\mapsto H(K)\in\mathcal{H}^{(n)}(\lambda,\mu,\nu)$ with $H(K)=\overline{\sigma}^{(n)}K$, as defined in Theorem 6.3. Then the maps $\sigma^{(n)}$ and $\overline{\sigma}^{(n)}$ are mutually inverse bijections.*

Proof: It follows from Theorems 5.4 and 6.3 that

$$H(K)=\overline{\theta}_n\cdots\overline{\theta}_2\overline{\theta}_1\,H^{(0)}=\overline{\theta}_n\cdots\overline{\theta}_2\overline{\theta}_1\,\theta_1\theta_2\cdots\theta_n\,H\,, \tag{6.15}$$

with

$$\overline{\theta}_r=\overline{\chi}_r^{\nu_r}\;\overline{\phi}_1^{V_{1r}}\overline{\phi}_2^{V_{2r}}\cdots\overline{\phi}_{r-1}^{V_{r-1,r}}\;\overline{\omega}_r^{\mu_r^{(r)}}\;\overline{\kappa}_r \tag{6.16}$$

and

$$\begin{aligned}\theta_r&=\kappa_r\;\omega_r^{\mu_r^{(r)}}\;\phi_r^{\lambda_r-\mu_r^{(r)}-\nu_r}\;\chi_r^{\nu_r}=\kappa_r\;\omega_r^{\mu_r^{(r)}}\;\phi_r^{V_{1r}+V_{2r}+\cdots+V_{r-1,r}}\;\chi_r^{\nu_r}\\&=\kappa_r\;\omega_r^{\mu_r^{(r)}}\;\phi_r^{V_{r-1,r}}\cdots\phi_r^{V_{2r}}\phi_r^{V_{1r}}\;\chi_r^{\nu_r}\,,\end{aligned} \tag{6.17}$$

where the exponents ν_r, $\mu_r^{(r)}$ and V_{kr} for $k=1,2,\ldots,r-1$ are all taken from K, and use has been made of the hive conditions on K that ensure that $\lambda_r-\mu_r^{(r)}-\nu_r=V_{1r}+\cdots+V_{r-1,r}$. The final form of θ_r reflects the fact that its action on $H^{(r)}$ to produce $H^{(r-1)}$ involves ν_r type (i) path removals, followed successively by $V_{1r},V_{2r},\ldots,V_{r-1,r}$ type (ii) path removals terminating at levels $1,2,\ldots,r-1$, respectively, and then $\mu_r^{(r)}$ type (iii) path removals. As noted in the Remark 6.2, not only are $\overline{\kappa}_r$, $\overline{\omega}_r$ and $\overline{\chi}_r$ the mutual inverses of κ_r, ω_r and χ_r, respectively, but also if the action of ϕ_r removes a path terminating at level k, then applying $\overline{\phi_k}$ restores that path; that is to say their actions mutually cancel. Since successive type (ii) paths removed by $\phi_r^{V_{kr}}$ are weakly above one another and successive type $\overline{\text{(ii)}}$ paths added by $\overline{\phi}_k^{V_{kr}}$ are weakly below one another, the operator $\overline{\phi}_k^{V_{kr}}\phi_r^{V_{kr}}$ involves V_{kr} nested pairs of operators $\overline{\phi}_k\phi_r$ whose actions cancel. This is true for $k=r-1,\ldots,2,1$ as well as for the pairs $\overline{\kappa}_r\,\kappa_r$, $\overline{\omega}_r\,\omega_r$ and $\overline{\chi}_r\,\chi_r$. It follows that

$$\overline{\theta}_r\theta_rH^{(r)}=\overline{\chi}_r^{\nu_r}\overline{\phi}_1^{V_{1r}}\cdots\overline{\phi}_{r-1}^{V_{r-1,r}}\overline{\omega}_r^{\mu_r^{(r)}}\overline{\kappa}_r\,\kappa_r\,\omega_r^{\mu_r^{(r)}}\phi_r^{V_{r-1,r}}\cdots\phi_r^{V_{1r}}\chi_r^{\nu_r}H^{(r)}=H^{(r)}. \tag{6.18}$$

Since this occurs for each r, we have $\overline{\theta}_n \cdots \overline{\theta_1} H^{(0)} = \overline{\theta}_n \cdots \overline{\theta_1}\theta_1 \cdots \theta_n H = H$, that is to say $H(K) = H$. From this we see that for any n-hive H we have $\overline{\sigma}^{(n)}\,\sigma^{(n)} H = H$.

Similarly, if one starts with $K \in \mathcal{H}^{(n)}(\lambda,\nu,\mu)$ and creates $H(K) = \overline{\sigma}^{(n)} K \in \mathcal{H}^{(n)}(\lambda,\mu,\nu)$, through a sequence of path additions determined by K, then the action of $\sigma^{(n)}$ on $H(K)$ consists of reversing the order of the path additions and applying their inverses in the form of corresponding path removals. More precisely, to deal with the cancellation of $\theta_r\overline{\theta}_r$ in

$$\theta_r\overline{\theta}_r\, H' = \kappa_r\,\omega_r^{\mu_r^{(r)}}\,\phi_r^{V_{r-1,r}} \cdots \phi_r^{V_{1r}}\chi_r^{\nu_r}\,\overline{\chi}_r^{\nu_r}\overline{\phi}_1^{V_{1r}} \cdots \overline{\phi}_{r-1}^{V_{r-1,r}}\overline{\omega}_r^{\mu_r^{(r)}}\overline{\kappa}_r\, H' \tag{6.19}$$

where $H' = H^{(r-1)}(K)$, there is first an easy cancellation of $\chi_r^{\nu_r}\,\overline{\chi}_r^{\nu_r}$, after which we apply 6.8 to cancel pairs $\phi_r\,\overline{\phi}_k$ one by one, V_{kr} times for $k = 1, 2, \ldots, r-1$. This amount to cancelling all type (ii) path removal and type $\overline{\text{(ii)}}$ path addition operators, and finally there are two more easy cancellations of $\omega_r^{\mu_r^{(r)}}\overline{\omega}_r^{\mu_r^{(r)}}$ and $\kappa_r\,\overline{\kappa}_r$. In this process the cancellation of $\phi_r\,\overline{\phi}_k$ implies that the path generated by this particular action of ϕ_r terminates at level k. Hence each exponent V_{kr} of $\overline{\phi}_k$ in the expression for $\overline{\theta}_r$, originally taken from K, is also equal to the number of those type (ii) paths terminating at level k, removed during the action of θ_r as part of $\sigma^{(n)}$ applied to $H(K)$. By this means one necessarily arrives back at $K \in \mathcal{H}^{(n)}(\lambda,\nu,\mu)$ as a record of the boundary edges of the sequence of path removals. That is to say, this time, if $H(K) = \overline{\sigma}^{(n)}K$ then $K = \sigma^{(n)}\big(H(K)\big)$ so that $\sigma^{(n)}\overline{\sigma}^{(n)}K = K$ for all $K \in \mathcal{H}^{(n)}$.

It follows that $\sigma^{(n)}$ and $\overline{\sigma}^{(n)}$ are mutually inverse maps and that both are bijective. ■

7. Hive based proof of the involutive property

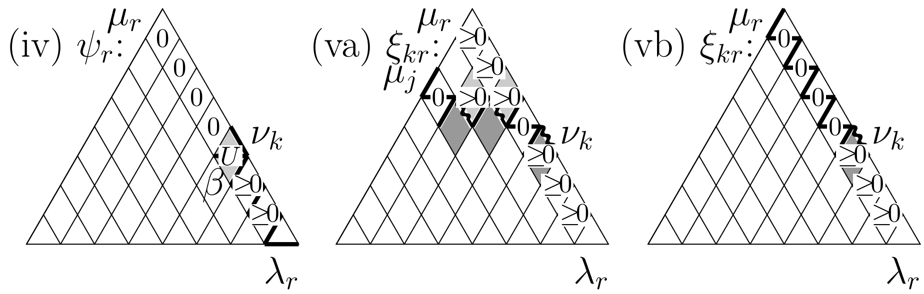
Our next task is to prove that the map $\sigma^{(n)}$ is an involution. To do this we proceed by way of a sequence of Lemmas, in connection with which we need to introduce two new types, (iv) and (v), of path removal operations on hives.

Definition 7.1. *Given any hive $H \in \mathcal{H}^{(r)}(\lambda,\mu,\nu)$ we define path removal operators ψ_r and ξ_{kr} whose action on H is to decrease or increase edge labels by 1 along paths as follows:*

- (iv) *ψ_r: provided that $U_{ir} > 0$ for some $i < r$ and $k = \min\{i \,|\, U_{ir} > 0\}$ the path proceeds downwards along the rth diagonal from the edge labelled ν_k along a zig-zag route to the edge labelled λ_r with all path edge labels being decreased by 1.*

(v) ξ_{kr}: *the path proceeds from the edge labelled* ν_k *along the route that would be followed by either a type (ii) or a type (iii) path from level* k *in diagonal* r *to the left-hand boundary at level* $j \le r$ *with all* α- *and* γ-*edge labels on the path decreased by* 1*, and all* β-*edge labels on the path increased by* 1*.*

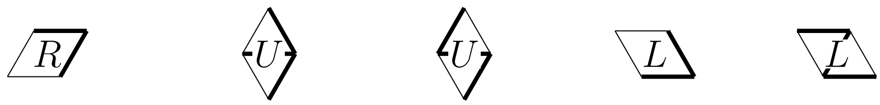
Such paths are illustrated below, where certain upright rhombus labels have been indicated as being > 0, 0 or ≥ 0 immediately before path removal. In case (iv) we have $U_{kr} = U$ with $U > 0$, while case (v) has been exemplified in two cases depending upon whether or not there exists $U_{ir} > 0$ for some $i < k$.



(7.1)

In all three cases the specified changes of ± 1 in edge labels ensure that the hive triangle conditions are satisfied, while in the case of path removals generated by ξ_{kr} the hive rhombus conditions are also satisfied since the paths of type (va) and (vb) follow, respectively, the routes determined by the type (ii) and type (iii) rules of Definition 3.1. In the case (va) the fact that $U_{ir} > 0$ for some $i < k$ is sufficient to ensure that initially all path α- and γ-edge labels, including μ_j, are positive and therefore remain non-negative after the path removal. To ensure this in the case (vb) it is necessary and sufficient that $\mu_r > 0$, and this will always be found to be the case in what follows.

In case (iv) prior to the path removal generated by ψ_r the condition $U_{kr} = U > 0$ ensures that $\nu_k > 0$ and $\lambda_r > 0$. The initial hive conditions then ensure that all α- and γ-edges on the path have labels $\ge \mu_r + U > 0$ and $\ge \lambda_r > 0$, respectively, so that they also remain ≥ 0 after the path removal. That the rhombus hive conditions are preserved can be seen from the following display of all the types of rhombi whose edge labels are affected by the path removal, in the second case of which we necessarily have $U > 0$.



$$R \mapsto R{+}1 \qquad U \mapsto U{-}1 \qquad U \mapsto U \qquad L \mapsto L{+}1 \qquad L \mapsto L \tag{7.2}$$

Thus each of the path removals of Definition 7.1 preserves the hive conditions.

As an immediate consequence of this we have:

Lemma 7.2. *For $i = 1, 2, \ldots, n$ let $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the single entry 1 in the ith position. Let $H \in \mathcal{H}^{(n)}(\lambda, \mu, \nu)$ with $\ell(\lambda) = n$ be such that $U_{in} > 0$ for some $i < n$. Then $\psi_n H \in \mathcal{H}^{(n)}(\lambda - \epsilon_n, \mu, \nu - \epsilon_k)$ where $k = \min\{i \,|\, U_{in} > 0\}$.*

We are now in a position to state what turns out to be a crucial lemma *en route* to Lemma 7.9 and our involution Theorem 7.10.

Lemma 7.3. *Let $H \in \mathcal{H}^{(n)}(\lambda, \mu, \nu)$ with $\ell(\lambda) = n$ be such that $U_{in} > 0$ for some $i < n$, with $k = \min\{\, i \mid U_{in} > 0 \,\}$, and let $\widehat{H} = \psi_n H$. Setting $H^{(n)} = H$ and $\widehat{H}^{(n)} = \widehat{H}$, let the action of $\sigma^{(n)}$ yield $K = K^{(n)}$ and $\widehat{K} = \widehat{K}^{(n)}$ by way of the chains*

$$(H^{(n)}, K^{(0)}) \xmapsto{\Theta_n} (H^{(n-1)}, K^{(1)}) \xmapsto{\Theta_{n-1}} (H^{(n-2)}, K^{(2)}) \xmapsto{\Theta_{n-2}} \cdots \xmapsto{\Theta_1} (H^{(0)}, K^{(n)})$$

and

$$(\widehat{H}^{(n)}, \widehat{K}^{(0)}) \xmapsto{\Theta_n} (\widehat{H}^{(n-1)}, \widehat{K}^{(1)}) \xmapsto{\Theta_{n-1}} (\widehat{H}^{(n-2)}, \widehat{K}^{(2)}) \xmapsto{\Theta_{n-2}} \cdots \xmapsto{\Theta_1} (\widehat{H}^{(0)}, \widehat{K}^{(n)})\,.$$

Then

$$\phi_n\, K^{(r)} = \widehat{K}^{(r)} \qquad \text{for } r = 1, 2, \ldots, n\,, \tag{7.3}$$

where the action of ϕ_n on the truncated hive $K^{(r)}$ is exactly the same as its action would be on a hive except that it terminates on reaching the lower left hand boundary of $K^{(r)}$.

In particular, we have $\phi_n K = \widehat{K}$. *That is to say we have* $\sigma^{(n)}\psi_n H = \phi_n \sigma^{(n)} H$, *namely the following diagram commutes:*

$$\begin{tikzcd} H \arrow[r, "\sigma^{(n)}"] \arrow[d, "\psi_n"'] & K \arrow[d, "\phi_n"] \\ \widehat{H} \arrow[r, "\sigma^{(n)}"'] & \widehat{K} \end{tikzcd} \tag{7.4}$$

Proof: The action of Θ_r involves applying θ_r to $H^{(r)}$ and $\widehat{H}^{(r)}$ to create $H^{(r-1)}$ and $\widehat{H}^{(r-1)}$, while recording information on the relevant path removals in $K^{(n-r+1)}$ and $\widehat{K}^{(n-r+1)}$, respectively, for each $r = n, n-1, \ldots, 1$. We divide this sequence of actions according to the following four regions of the values of r, namely $r = n$, $n > r > k$ (vacuous if $k = n-1$), $r = k$ and $r < k$ (vacuous if $k = 1$). Lemmas 7.4, 7.5 and 7.7 deal with the cases $r = n$, $n > r > k$ and $r = k$ respectively, and the remaining case $r < k$ follows easily. First we state the three Lemmas.

Lemma 7.4. *Let* n *be a positive integer,* H *an* n*-hive such that* $U_{in} > 0$ *for some* $i < n$, *with* $k = \min\{\, i \mid U_{in} > 0 \,\}$, *and* $\widehat{H} = \psi_n H$.

Then, the removals by the actions of θ_n *on* H *and* $\widehat{H}$ *involve*

$$\begin{cases} \text{\underline{type (i) paths:} the same number of them from both } H \text{ and } \widehat{H}, \\ \text{\underline{type (ii) paths:} one more of them from } H \text{ than from } \widehat{H}\text{; more precisely:} \\ \quad \text{the same type (ii) paths } P_1 = \widehat{P}_1, \ldots, P_{c-1} = \widehat{P}_{c-1} \text{ from both } H \text{ and } \widehat{H}, \text{ and} \\ \quad \text{one extra type (ii) path } P_c \text{ from } H \text{ entering the } (n-1)\text{th diagonal at level } k, \\ \text{\underline{type (iii) paths:} the same number of them from both } H \text{ and } \widehat{H}. \end{cases}$$

Moreover, the $(n-1)$*-hive* $\theta_n H$ *differs from* $\theta_n \widehat{H}$ *by the removal of one type (v) path, which is actually the* $(n-1)$*-hive part of* P_c, *so that we have* $\theta_n H = \xi_{k,n-1}(\theta_n \widehat{H})$.

Lemma 7.5. *Let* k *and* r *be integers satisfying* $1 \le k < r$. *Let* H *and* $\widehat{H}$ *be* r*-hives such that* $H = \xi_{kr} \widehat{H}$, *and* D *the path removed by the action of* ξ_{kr} *on*

$\widehat{H}$. We call D the path of difference. Let j denote the level at which D ends on the left-hand boundary of the r-hive.

Then, the removals by the actions of θ_r on H and $\widehat{H}$ involve

$$\begin{cases} \underline{\text{type (i) paths:}} \text{ the same number of them from both } H \text{ and } \widehat{H}, \text{ and} \\ \underline{\text{type (ii) and (iii) paths:}} \text{ the same number of them (counted together) from both } H \text{ and } \widehat{H}. \text{ More precisely, let } P_1, \ldots, P_m \text{ and } \widehat{P}_1, \ldots, \widehat{P}_m \text{ be such paths in the order of removal, respectively. Then, for some } 1 \le c \le m, \text{ the following hold.} \\ \quad P_1 = \widehat{P}_1, \ \ldots, \ P_{c-1} = \widehat{P}_{c-1}. \\ \quad \widehat{P}_c \text{ ends at level } j, \text{ while } P_c \text{ ends at some level } j' < j, \text{ on the left-hand boundary.} \\ \quad \text{For each } a > c, \ P_a \text{ and } \widehat{P}_a \text{ both end at the same level on the left-hand boundary.} \end{cases}$$

Moreover, the $(r-1)$-hive $\theta_r\, H$ differs from $\theta_r\, \widehat{H}$ by the removal of a type (v) path, say D', starting at level $k' = k$ on the right-hand boundary, that is to say $\theta_r\, H = \xi_{k,r-1}(\theta_r\, \widehat{H})$, and the new path of difference D' terminates on the left-hand boundary at some level $j' < j$.

Remark 7.6. *The paths P_a and $\widehat{P}_a$ with $a < c$, as well as P_c, are type (ii) paths. $\widehat{P}_c$ is a type (iii) path if $j = r$ (in which case, so are all P_a and $\widehat{P}_a$ with $a > c$), while it is a type (ii) path if $j < r$ (in which case, for $a > c$ the paths P_a and $\widehat{P}_a$ are, in general, type (ii) paths for $c < a \le d$ and type (iii) paths for $d < a \le m$, for some d such that $c < d \le m$).*

Lemma 7.7. *Let k be a positive integer. Let H and $\widehat{H}$ be k-hives such that $H = \xi_{kk}\, \widehat{H}$, D the path of difference between H and $\widehat{H}$, and j the level at which D ends on the left-hand boundary of the k-hive.*

Then, the removals by the action of θ_k on H and $\widehat{H}$ involve

$$\begin{cases} \underline{\text{type (i) paths:}} \text{ one more of them from } H \text{ than from } \widehat{H}, \text{ and} \\ \underline{\text{type (ii) and (iii) paths:}} \text{ one more of them from } \widehat{H} \text{ than from } H; \text{ more precisely:} \\ \quad \text{one extra path } \widehat{P}_0 \text{ from } \widehat{H}, \text{ ending at level } j, \text{ from } \widehat{H}, \\ \quad \text{then the same paths } P_1 = \widehat{P}_1, \ \ldots, \ P_m = \widehat{P}_m \text{ from both } H \text{ and } \widehat{H}. \end{cases}$$

Moreover, the $(k-1)$-hives $\theta_k\, H$ and $\theta_k\, \widehat{H}$ are identical. There is no longer any path of difference.

Remark 7.8. *$\widehat{P}_0$ is a type (iii) path if $j = k$ (in which case, so are all P_a and $\widehat{P}_a$ with $a \geq 1$), while it is a type (ii) path if $j < k$ (in which case, for $a > 0$ the paths P_a and $\widehat{P}_a$ are, in general, type (ii) paths for $0 < a \leq d$ and type (iii) paths for $d < a \leq m$, for some d such that $0 < d \leq m$).*

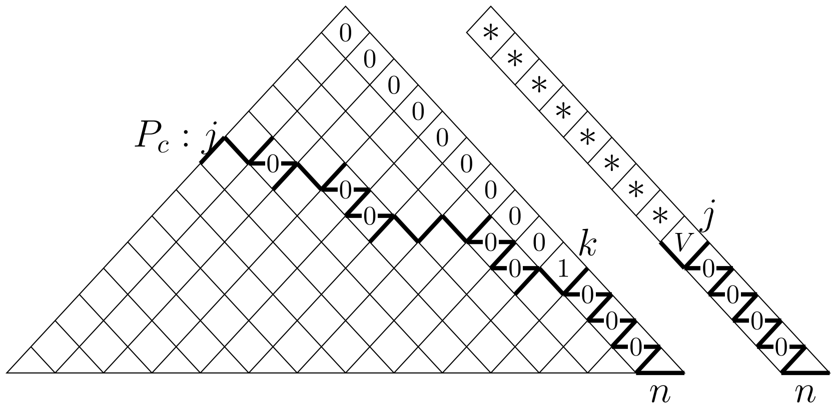
We defer the proofs of these three highly technical lemmas to Section 9, Appendix. Assuming their validity for all values of n, r and k, the proof of Lemma 7.3 can be built upon them as follows:

Proof of Lemma 7.3: Let $H = H^{(n)}$ and $\psi_n\, H = \widehat{H} = \widehat{H}^{(n)}$ be as in Lemma 7.3. Then Lemma 7.4 shows that $\theta_n\, H = H^{(n-1)}$ and $\theta_n\, \widehat{H} = \widehat{H}^{(n-1)}$ are related by $H^{(n-1)} = \xi_{k,n-1}\, \widehat{H}^{(n-1)}$, where k is the smallest value for which $U_{kn} > 0$. For this value of k, Lemma 7.5 can be applied successively to $H^{(r)}$ and $\widehat{H}^{(r)}$ for $r = n-1, n-2, \ldots, k+1$, showing in each case that $\theta_r\, H^{(r)} = H^{(r-1)}$ and $\theta_r\, \widehat{H}^{(r)} = \widehat{H}^{(r-1)}$ are related by $H^{(r-1)} = \xi_{k,r-1}\, \widehat{H}^{(r-1)}$, thereby maintaining at each stage the value of k as the starting level of the path of difference. The final case $r = k+1$ yields the relationship $H^{(k)} = \xi_{kk}\, \widehat{H}^{(k)}$. Then Lemma 7.7 shows that $\theta_k\, H^{(k)} = H^{(k-1)}$ and $\theta_k\, \widehat{H}^{(k)} = \widehat{H}^{(k-1)}$ coincide, and from then on applications of $\theta_{k-1}, \ldots, \theta_1$ produce identical hives $H^{(k-2)} = \widehat{H}^{(k-2)}$, $\ldots$, $H^{(0)} = \widehat{H}^{(0)}$.

For later use, let $D^{(r)}$, $n-1 \geq r \geq k$, denote the type (v) path removed by ξ_{kr} from $\widehat{H}^{(r)}$ to give $H^{(r)}$, and j_r its terminating level on the left-hand boundary.

Now turn attention to how their partner hives K and $\widehat{K}$ are related. Since the bottom and left-hand boundary edge labels of H and $\widehat{H}$ are the parts of λ, μ and $\lambda - \epsilon_n, \mu$, respectively, they are also, by construction, the bottom and right-hand boundary edge labels of K and $\widehat{K}$. Hence those of $\widehat{K}$ coincide with those of $\phi_n\, K$, and so, in order to show $\phi_n\, K^{(n-r+1)} = \widehat{K}^{(n-r+1)}$, it is sufficient to show that the upright rhombus gradients of $\widehat{K}^{(n-r+1)}$ coincide with those of $\phi_n\, K^{(n-r+1)}$. We shall do this inductively with $r = n, n-1, \ldots, 1$.

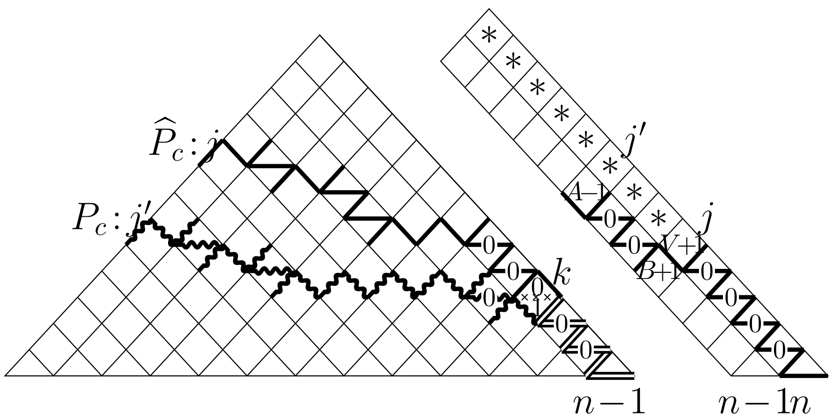
By Lemma 7.4, the terminating level j_{n-1} of the path $D^{(n-1)}$ is equal to the terminating level of the extra and final type (ii) path P_c removed from H. Since the gradients V_{in} of $K^{(1)}$ and $\widehat{K}^{(1)}$ are, by definition, the number of type (ii) path removals terminating at level i through applications of θ_n to H and $\widehat{H}$, respectively, the only difference between them resides in $V_{j_{n-1},n}$ whose value in $K^{(1)}$ is greater than that in $\widehat{K}^{(1)}$ by 1, and $V_{in} = 0$ for all $i > j_{n-1}$ in both $K^{(1)}$ and $\widehat{K}^{(1)}$. Hence, if one applies ϕ_n to $K^{(1)}$, then the removed path climbs the nth diagonal up to level j_{n-1}, where it exits the nth diagonal, decreasing $V_{j_{n-1},n}$ by 1. Thus the upright rhombus gradients of $\widehat{K}^{(1)}$ coincide with those of $\phi_n\, K^{(1)}$, and we have $\phi_n\, K^{(1)} = \widehat{K}^{(1)}$. The last type (ii) path P_c removed under the action of θ_n on $H^{(n)}$ and the path removed under the action of ϕ_n on $K^{(1)}$ are exemplified in 7.5 below, where for typographical simplicity we have represented j_{n-1} and $V_{j_{n-1},n}$ by j and V, respectively.



(7.5)

Next, assume that $k < n-1$. Lemma 7.5 applied in the case $r = n-1$ to $H^{(n-1)}$ and $\widehat{H}^{(n-1)}$ with $D = D^{(n-1)}$ and $j = j_{n-1}$, implies that there exists c such that $\widehat{H}^{(n-1)}$ affords one extra path removal of $\widehat{P}_c$ ending at level j_{n-1}, $H^{(n-1)}$ affords one extra path removal of P_c ending at level $j' = j_{n-2} < j_{n-1}$, and for each $a \neq c$ the paths P_a and $\widehat{P}_a$ end at the same level.

Hence the only difference of upright rhombus gradients in the $(n-1)$th diagonal in $K^{(2)}$ and $\widehat{K}^{(2)}$ is that, if we put $V_{j_{n-2},n-1} = A \geq 1$, $V_{j_{n-1},n-1} = B \geq 0$ in $K^{(2)}$, then $V_{j_{n-2},n-1} = A-1$, $V_{j_{n-1},n-1} = B+1$ in $\widehat{K}^{(2)}$, where $V_{j_{n-1},n-1}$ materialises only if $j_{n-1} < n-1$. Moreover, by Lemma 4.4, $P_a = \widehat{P}_a$ with $a < c$ all end at levels $\leq j_{n-2}$, being weakly lower than P_c, and P_a and $\widehat{P}_a$ with $a > c$, ending at the same level for each such a, all end at levels $\geq j_{n-1}$, being weakly higher than $\widehat{P}_c$. Hence $V_{x,n-1} = 0$ for all $j_{n-2} < x < j_{n-1}$ in both K and $\widehat{K}$. Thus, if we extend the path removal by ϕ_n from $K^{(1)}$ into the $(n-1)$th diagonal, the path enters the diagonal at level $j = j_{n-1}$, accompanied if $j_{n-1} < n-1$ by an increment of $V_{j_{n-1},n-1}$ from B to $B+1$, climbs and exits the diagonal at level $j' = j_{n-2}$, decreasing $V_{j_{n-2},n-1}$ from A to $A-1$. Hence the agreement of $\phi_n K$ and $\widehat{K}$ extends down to the $(n-1)$th diagonal, giving $\phi_n K^{(2)} = \widehat{K}^{(2)}$. All this is illustrated in 7.6 below.



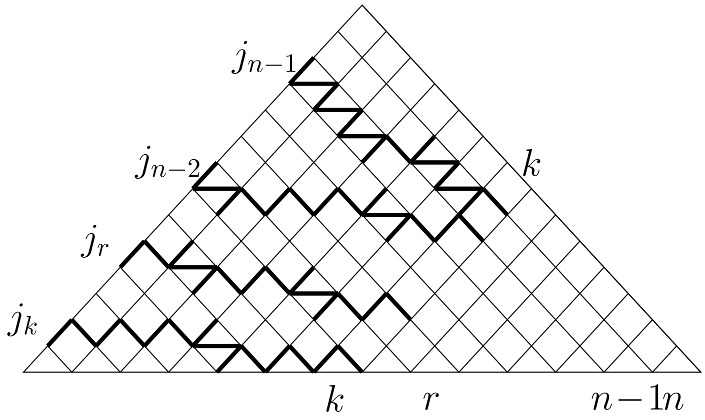
(7.6)

The same argument can then be repeated down to the $(k+1)$th diagonal, letting the path removed by ϕ_n move between diagonals at levels $j_{n-2}, \ldots, j_{k+1}$ and exit the $(k+1)$th diagonal at level j_k, and extending the agreement of $\phi_n K$ and $\widehat{K}$ down to the $(k+1)$th diagonal: $\phi_n K^{(n-k)} = \widehat{K}^{(n-k)}$.

Now, by Lemma 7.7 applied to $H^{(k)}$, $\widehat{H}^{(k)}$ and $D = D^{(k)}$ ending at level $j = j_k$, $H^{(k)}$ affords one extra type (ii) or (iii) path removal, ending at level j_k, giving a difference in the values of V_{xk} only with $x = j_k$, taking a value in $K^{(n-k+1)}$ greater than that in $\widehat{K}^{(n-k+1)}$ by 1. Moreover, we have $V_{xk} = 0$ in K for all $x < j_k$ since the extra path is the first type (ii) or (iii) path removed from $H^{(k)}$. Hence the path removed by ϕ_n from K, entering the kth diagonal at level j_k and increasing $V_{j_k,k}$ by 1 if $j_k < k$, climbs the kth diagonal to the top and terminates with its arrival on the left-hand boundary of the n-hive at level k without changing any other V_{xk}. Hence we have $\phi_n K^{(n-k+1)} = \widehat{K}^{(n-k+1)}$.

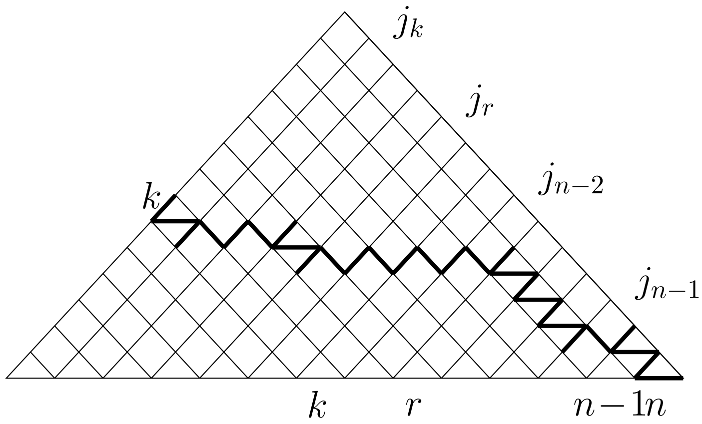
Since the path removals from $H^{(k-1)}$ and $\widehat{H}^{(k-1)}$, $\ldots$, $H^{(1)}$ and $\widehat{H}^{(1)}$ all coincide, the upright rhombus gradients of K and $\widehat{K}$ in their remaining $k-1$ diagonals also coincide. Hence we have $\phi_n K^{(r)} = \widehat{K}^{(r)}$ for all $r > n-k+1$ also, in particular $\phi_n K = \widehat{K}$. ∎

We offer the following diagram as an illustration of a succession of difference paths $D^{(r)}$ starting at level k and their end points j_r for $r = n-1, n-2, \ldots, k$:



(7.7)

The corresponding illustration of the path removal action of ϕ_n on K to give $\widehat{K}$ takes the form:



(7.8)

We may now exploit the final part of Lemma 7.3, namely the commutativity of 7.4, in the proof of the following:

Lemma 7.9. *For any n and any LR n-hive H, we have*

$$\theta_n\,(\sigma^{(n)})^2\,H = (\sigma^{(n-1)})^2\,\theta_n\,H. \tag{7.9}$$

Proof: Our goal can be expressed as the commutativity of the outer rectangle in the following diagram:

$$\begin{tikzcd} H \arrow[r, "\sigma^{(n)}"] \arrow[d, "\theta_n"'] & K \arrow[r, "\sigma^{(n)}"] \arrow[d, "\eta_n"'] & L \arrow[d, "\theta_n"'] \\ H^{(n-1)} \arrow[r, "\sigma^{(n-1)}"] & K_{(n-1)} \arrow[r, "\sigma^{(n-1)}"] & L^{(n-1)} \end{tikzcd} \tag{7.10}$$

Here H is any given LR n-hive, say in $\mathcal{H}^{(n)}(\lambda,\mu,\nu)$, $K = \sigma^{(n)}H$ and $L = \sigma^{(n)}K = (\sigma^{(n)})^2H$ so that by construction $K \in \mathcal{H}^{(n)}(\lambda,\nu,\mu)$ and $L \in \mathcal{H}^{(n)}(\lambda,\mu,\nu)$. On the lower side of the rectangle we have used the notation $H^{(n-1)} = \theta_n\, H$ and $L^{(n-1)} = \theta_n\, L$ as in Section 5 and, moreover, we have inserted a central vertical arrow representing the action of an operator which we denote by η_n taking any n-hive, in this case K, to its $(n-1)$-hive part $K_{(n-1)}$, for which we are following the notation used in the proof of Theorem 6.3.

Then the proof of the commutativity of the left-hand rectangle is straightforward and can be seen as follows. For each $1 \le j < k \le n$, the upright rhombus gradient V_{jk} of $\sigma^{(n)}H = K$ is, by definition, the number of level-j-terminating type (ii) paths removed by θ_k during the process

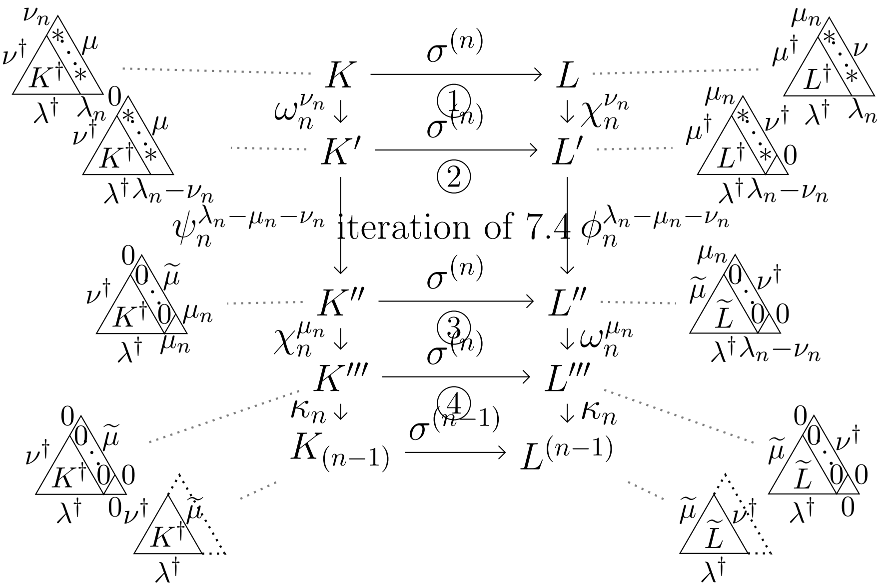
$$H = H^{(n)} \xmapsto{\theta_n} \underbrace{H^{(n-1)} \xmapsto{\theta_{n-1}} H^{(n-2)} \xmapsto{\theta_{n-2}} \cdots \xmapsto{\theta_2} H^{(1)} \xmapsto{\theta_1} H^{(0)}}_{(*)}. \tag{7.11}$$

To determine the upright rhombus gradients of $\sigma^{(n-1)}H^{(n-1)}$ the corresponding process is exactly what is marked with $(*)$ in 7.11, with removals of exactly the same paths, and so the upright rhombus gradients of $\sigma^{(n-1)}H^{(n-1)}$ are nothing but those V_{jk} in the $(n-1)$-hive part of K, namely $K_{(n-1)}$. Moreover, as explained in Definition 5.2, the left-hand boundary edges of $H^{(n-1)}$ and the lower left boundary edges of $K^{(1)}$ share the same labels, and the latter labels remain in K in those positions giving the right-hand boundary edge labels of $K_{(n-1)}$. Thus $K_{(n-1)}$ coincides with $\sigma^{(n-1)}H^{(n-1)}$ both in its boundary edge labels and upright rhombus gradients, and hence in its entirety. Hence we will be finished as soon as the right-hand rectangle is also shown to be commutative.

We turn now to the commutativity of the right-hand rectangle in 7.10. Corresponding to the definition of θ_n in the form $\kappa_n\, \omega_n^{\mu_n}\, \phi_n^{\lambda_n-\mu_n-\nu_n}\, \chi_n^{\nu_n}$ appropriate to its action on any hive in $\mathcal{H}^{(n)}(\lambda,\mu,\nu)$, it is convenient, as we shall see in the next paragraphs, to express the operator η_n in the form $\kappa_n\, \chi_n^{\mu_n}\, \psi_n^{\lambda_n-\mu_n-\nu_n}\, \omega_n^{\nu_n}$

appropriate to its action on any hive in $\mathcal{H}^{(n)}(\lambda,\nu,\mu)$. Indeed, the type (iii) action of $\omega_n^{\nu_n}$ on K reduces by ν_n each of the edge labels along a zig-zag down the nth diagonal with the top and bottom edge labels reduced from ν_n and λ_n to 0 and $\lambda_n-\nu_n$, respectively; the type (iv) action of $\psi_n^{\lambda_n-\mu_n-\nu_n}$ then reduces to 0 all upright rhombus gradients in the nth diagonal, as well as reducing the bottom edge label from $\lambda_n-\nu_n$ to μ_n whilst reducing the right hand boundary edges from $\mu=\mu^{(n)}$ to $(\mu^{(n-1)},\mu_n)$; the type (i) action of $\chi_n^{\mu_n}$ reduces the two boundary edge labels of the triangle at the foot of the nth diagonal from μ_n to 0, allowing finally the action of κ_n to remove the now empty nth diagonal, altogether fulfilling the action of η_n on K to give $K_{(n-1)}$.

This enables subdividing the right-hand rectangle in 7.10 as in 7.12 below, in which K',K'',K''' (resp. L',L'',L''') are defined to be the results of successively applying the operators represented by the vertical arrows to K (resp. L), and $K^\dagger$, $L^\dagger$ and $\widetilde{L}$ are shorthand notations for $K_{(n-1)}$, $L_{(n-1)}$ and $L^{(n-1)}$ respectively:



(7.12)

For the rectangle marked with ① in 7.12, the coincidence between the upright rhombus gradients of K and K' implies that all type (ii) path removals coincide between the actions of $\sigma^{(n)}$ on K and K', resulting in the coincidence between the upright rhombus gradients of $\sigma^{(n)}(K)$ and $\sigma^{(n)}(K')$. So the difference of $\sigma^{(n)}(K)$ and $\sigma^{(n)}(K')$ resides only in their boundary edge labels. However, since $K\in\mathcal{H}^{(n)}(\lambda,\nu,\mu)$ and $K'\in\mathcal{H}^{(n)}(\lambda-\nu_n\epsilon_n,\nu-\nu_n\epsilon_n,\mu)$ then $\sigma^{(n)}K\in\mathcal{H}^{(n)}(\lambda,\mu,\nu)$ and $\sigma^{(n)}K'\in\mathcal{H}^{(n)}(\lambda-\nu_n\epsilon_n,\mu,\nu-\nu_n\epsilon_n)$, showing that $\sigma^{(n)}K'=\chi_n^{\nu_n}(\sigma^{(n)}K)=L'$, as required to confirm the commutativity of the rectangle marked ①. A similar argument applies to the rectangle marked ③, and crucially, the rectangle ② is also commutative by virtue of the final part of Lemma 7.3, namely the commutativity of 7.4, applied $\lambda_n - \mu_n - \nu_n$ times. Note that both K''' and L''' have empty nth diagonals. Finally, since in the action of $\sigma^{(n)}$ on K''' the θ_n part simply removes the empty nth diagonal and produces an empty nth diagonal of L''', it is essentially an action of $\sigma^{(n-1)}$ on $\kappa_n K''' = K_{(n-1)}$, to produce the $(n-1)$-hive part of L'''; in other words we have $\sigma^{(n-1)} K_{(n-1)} = \kappa_n L''' = L_{(n-1)}$, thereby confirming the commutativity of rectangle marked 4.

Thus we have seen that both rectangles in 7.10 are commutative, and hence we have the equality of operators $\theta_n (\sigma^{(n)})^2 = (\sigma^{(n-1)})^2 \theta_n$. ■

As a consequence of this we have

Theorem 7.10. *For all $n \in \mathbb{N}$ and all n-hives H we have*

$$(\sigma^{(n)})^2\, H = H\,. \tag{7.13}$$

Proof: We proceed by induction with respect to n. First it should be noted that in the case $n = 1$ we have

$$\begin{array}{ccccc} \mu_1 \triangle \nu_1 & \xrightarrow{\sigma^{(1)}} & \nu_1 \triangle \mu_1 & \xrightarrow{\sigma^{(1)}} & \mu_1 \triangle \nu_1 \\ \lambda_1 & & \lambda_1 & & \lambda_1 \end{array} \tag{7.14}$$

so that $(\sigma^{(1)})^2\, H = H$ for all 1-hives H.

Next assume that $n \geq 2$ and that, by the induction hypothesis, the effect of applying $(\sigma^{(n-1)})^2$ to any $(n-1)$-hive amounts to applying the identity map to that hive. By Lemma 7.9 we have, for any n-hive H, the equality $\theta_n(\sigma^{(n)})^2 H = (\sigma^{(n-1)})^2 \theta_n H$, and by the induction hypothesis the right-hand side is equal to $\theta_n H$. This means that the two n-hives $(\sigma^{(n)})^2 H$ and H are mapped to the same $(n-1)$-hive, say $\widetilde{H}$, by θ_n. The remaining question is whether one can derive the equality $(\sigma^{(n)})^2 H = H$ from this information.

For this, it is crucial to note that both $(\sigma^{(n)})^2 H$ and H have the same boundary edge labels, say λ, μ and ν, by virtue of the definition of $\sigma^{(n)}$. Now set $L = (\sigma^{(n)})^2 H$, and consider the action of Θ_n on $(H, K^{(0)})$ and $(L, K^{(0)})$ where $K^{(0)}$ is the unique n-truncated n-hive with edge labels μ (see Definition 5.2 and the preceding paragraphs). The result of the action can expressed as $(\theta_n H, K_H^{(1)})$ and $(\theta_n L, K_L^{(1)})$, where $\theta_n H = \theta_n L = \widetilde{H}$ as we have seen, and by construction both $K_H^{(1)}$ and $K_L^{(1)}$ are $(n-1)$-truncated n-hives consisting of a single diagonal having lower and upper edge labels λ_n and ν_n, outer right-hand edge labels μ, as determined by $K^{(0)}$, and the inner left-hand boundary edge labels, say $\widetilde{\mu}$, as determined by the left-hand boundary edge labels of $\widetilde{H}$. These boundary edge labels are sufficient to determine an $(n-1)$-truncated n-hive completely. It follows that $K_H^{(1)} = K_L^{(1)}$, so that both components of $\Theta_n(H, K^{(0)})$ and $\Theta_n(L, K^{(0)})$ coincide.

We know that H and L can be recovered from $\Theta_n(H, K^{(0)})$ and $\Theta_n(L, K^{(0)})$, namely from $(\theta_n H, K_H^{(1)})$ and $(\theta_n L, K_L^{(1)})$ through applications of the path addition operator $\overline{\theta}_n$ to $\theta_n H$ and $\theta_n L$, making use of the $(n-1)$-truncated n-hives $K_H^{(1)}$ and $K_L^{(1)}$, respectively. Hence the equality $\Theta_n(H, K^{(0)}) = \Theta_n(L, K^{(0)})$ implies that $H = L$. That is to say $H = L = (\sigma^{(n)})^2 H$, thereby completing the induction argument and ensuring the validity of 7.13 for all n-hives H. ■

8. Concluding remarks

We have given a direct combinatorial proof of the bijective and involutive nature of a procedure first introduced by Azenhas [Aze99, Aze00] as a means of establishing combinatorially the symmetry of Littlewood–Richardson coefficients within the context of a tableaux based model. The model was based on the use of Littlewood–Richardson *hives*, on which we defined a commutativity operator denoted by $\sigma^{(n)}$. It transforms a given LR hive $H \in \mathcal{H}^{(n)}(\lambda, \mu, \nu)$ to a new LR hive $K \in \mathcal{H}^{(n)}(\lambda, \nu, \mu)$ by the application of what we called *path removals* from H, working from right to left, with each path starting from the base of the hive, and recording within K the level reached by each path, as exemplified in Example 5.3.

The choice of a hive as opposed to a tableaux model was made in part for pedagogical reasons and the wish to expose the power and flexibility of hives, complete with alternative vertex, edge or rhombus gradient presentations, to a wider readership. Alternative proofs of bijectivity and involutivity can be constructed purely within a tableaux model setting, and this has been done in a lengthy arXiv paper [AKT16] which sets the two models alongside one another, and illustrates the way that the interplay between the two types of model has benefitted both approaches.

9. Appendix

We now supply the proofs of the three technical Lemmas 7.4, 7.5 and 7.7 used in the proof of Lemma 7.3.

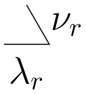
Proof of Lemma 7.4: The last statement of the Lemma compares the $(n-1)$-hives $\theta_n\, H$ and $\theta_n\, \widehat{H}$. With that in mind, let $\swarrow$ denote the $(n-1)$-hive region, and start by noting that $H|_{\swarrow} = \widehat{H}|_{\swarrow}$ since $\widehat{H}$ is obtained from H through the action of ψ_n which just removes a type (iv) path extending down the nth diagonal of H from level k to its base, causing no change in the $(n-1)$-hive region.

Since $k < n$, the lowermost right-hand boundary edge labels of H and $\widehat{H}$ are equal, and so are the number of type (i) paths removed from them. Let H_0 and $\widehat{H}_0$ denote the result of all type (i) path removals. These leave $H_0|_{\swarrow} = \widehat{H}_0|_{\swarrow}$.

Now the difference in upright rhombus gradients of H_0 and $\widehat{H}_0$ resides in the values of U_{kn} only: that of H_0 being greater than that of $\widehat{H}_0$ by 1. Hence the type (ii) path removals from H_0 and $\widehat{H}_0$ proceed in the same manner until all the gradients U_{xn} with $x > k$ have been reduced to 0 and the gradients U_{kn} of H_0 and $\widehat{H}_0$ have been reduced to 1 and 0, respectively, by removals of paths $P_1 = \widehat{P}_1, \ldots, P_{c-1} = \widehat{P}_{c-1}$, say. Let H_{c-1} and $\widehat{H}_{c-1}$ be the resulting hives. We still have $H_{c-1}|_{\swarrow} = \widehat{H}_{c-1}|_{\swarrow}$. Then there are no more type (ii) paths to remove $\widehat{H}_{c-1}$, but there is one more such path, P_c, to remove from H_{c-1}. This enters the $(n-1)$th diagonal at level k and reaches the left-hand boundary at level $j \leq n-1$ as exemplified by the **solid** path shown on the left in 7.5 above. Its removal yields $H_c = \phi_n\, H_{c-1}$, and we have $H_c|_{\swarrow} = \xi_{k,n-1}(\widehat{H}_{c-1}|_{\swarrow})$, where $\xi_{k,n-1}$ removes $P_c|_{\swarrow}$, which is of type (va) or (vb) depending on whether $j < n-1$ or $j = n-1$. Since the labels of the topmost left-hand boundary edges of H and $\widehat{H}$ are equal and unaffected by all the above, there remain the same number of type (iii) path removals from H_c and $\widehat{H}_{c-1}$, which again do not affect the $(n-1)$-hive region. Discarding the now empty nth diagonal under the action of κ_n leaves the results $\theta_n\, H$ and $\theta_n\, \widehat{H}$ that are still related by $\theta_n\, H = \xi_{k,n-1}(\theta_n\, \widehat{H})$, in which $\xi_{k,n-1}$ removes $P_c|_{\swarrow}$ reaching the left-hand boundary at level $j \leq n-1$. ■

We now proceed to the proof of Lemma 7.5. We employ some sublemmas, and even definitions, in its proof.

Proof of Lemma 7.5: By hypothesis, the r-hives $H = \xi_{kr}\, \widehat{H}$ and $\widehat{H}$ differ only by way of a path of difference D entering the rth diagonal at level $k < r$ and exiting on the left-hand boundary at level $j < r$. Let $H \in \mathcal{H}^{(r)}(\lambda, \mu, \nu)$, so that $\widehat{H} \in \mathcal{H}^{(r)}(\lambda, \mu - \epsilon_j, \nu + \epsilon_k)$. Since $k < r$, both H and $\widehat{H}$ have edge labels



at their bottom right corner, so that, in both cases, the number of type (i) path removals is ν_r, and the number of type (ii) and (iii) path removals put together is $\lambda_r - \nu_r$. Set $m = \lambda_r - \nu_r$, and let $H_0 = \chi_r^{\nu_r} H$, $\widehat{H}_0 = \chi_r^{\nu_r} \widehat{H}$. Type (i) path removals do not change any upright rhombus gradients, so we have $H_0 = \xi_{kr} \widehat{H}_0$, in which ξ_{kr} removes D. We denote the paths generated by the type (ii) and (iii) removals from H_0 and $\widehat{H}_0$, respectively, by $P_1, \ldots, P_m$ and $\widehat{P}_1, \ldots, \widehat{P}_m$. For each $1 \le a \le m$, let H_a and $\widehat{H}_a$ denote the result of removals of $P_1, \ldots, P_a$ from H_0 and $\widehat{P}_1, \ldots, \widehat{P}_a$ from $\widehat{H}_0$, respectively.

Now take any one of the paths $\widehat{P}_1, \ldots, \widehat{P}_m$, say $\widehat{P}_a$, and consider how it may intersect D in the sense of having an edge in common. Due to the form taken by a type (ii) or type (iii) path, $\widehat{P}_a$, its coincidence with D, if there is any, necessarily starts at the northwest edge of the foot rhombus of a ladder of D in some diagonal, with $\widehat{P}_a$ entering the foot rhombus by way of its southeast edge and crossing to its northwest edge along the connecting γ-edge.

Recall that, thanks to Lemma 4.4, the paths $\widehat{P}_1, \ldots, \widehat{P}_m$ lie weakly above one another. The sequence $\widehat{P}_1, \ldots, \widehat{P}_m$ can then be divided into sections $\widehat{P}_1, \ldots, \widehat{P}_{c_1-1}$; $\widehat{P}_{c_1}, \ldots, \widehat{P}_{c_2-1}$; $\widehat{P}_{c_2}, \ldots, \widehat{P}_{c_3-1}$; ......; $\widehat{P}_{c_N}, \ldots, \widehat{P}_m$, with some indices $1 \le c_1 < c_2 < c_3 < \cdots < c_N \le m$ (we do have $N \ge 1$, see the next paragraph for its reason), in such a way that the paths $\widehat{P}_1, \ldots, \widehat{P}_{c_1-1}$ do not intersect D at all, each of the paths $\widehat{P}_{c_1}, \ldots, \widehat{P}_{c_2-1}$ first intersects D in the p_1th diagonal, each of the paths $\widehat{P}_{c_2}, \ldots, \widehat{P}_{c_3-1}$ first intersects D in the p_2th diagonal, and so on, with $1 \le p_1 < p_2 < \cdots < p_N \le r$.

We first show that $N \ge 1$.

Lemma 9.1. *In the situation of Lemma 7.5, at least one of the paths $\widehat{P}_1, \ldots, \widehat{P}_m$ intersects D.*

Proof: Recalling that the path of difference D starts at level k, denote the gradient U_{kr} of $\widehat{H}$ by $X \ge 0$; then that of H is $X + 1$. The behaviour of the paths $P_1, \ldots$ and $\widehat{P}_1, \ldots$ in the rth diagonal, including the levels at which they leave the rth diagonal, are the same until, say in H_{c-1} and $\widehat{H}_{c-1}$, all the gradients U_{xr} with $x > k$ are reduced to 0 and moreover the gradient $U_{kr} = X + 1$ of H is reduced to 1 in H_{c-1} and that of $\widehat{H}$ to 0 in $\widehat{H}_{c-1}$. At this point we say that this upright rhombus is *critical*. Since all upright rhombus gradients in the rth diagonal are to be reduced to 0 through type (ii) path removals in the course of the action of θ_r, there is at least one more type (ii) path removal, namely that of P_c, involved in the application of θ_r to H. We saw above that the number of type (ii) and (iii) path removals under the action of θ_r is the same for H and $\widehat{H}$, so that there is also at least one more path removal, namely that of $\widehat{P}_c$, from $\widehat{H}$, which may be of type (ii) or (iii), whose path necessarily passes through the critical upright rhombus of gradient 0 and intersects D at its lowest edge in the rth diagonal. ■

Remark 9.2. *Since the rth diagonal is the rightmost diagonal, the c in the proof of Lemma 9.1 is c_N in the notation introduced above its statement. Also we have $p_N = r$.*

Now we introduce some terminology for use in the inductive proof of Lemma 7.5.

Definition 9.3. *Let Ω be a trapezoidal region in the shape of a hive having the following boundaries: the left and the lower right boundaries consisting of α-edges, the upper right boundary consisting of β-edges, and the bottom boundary consisting of γ-edges. Such a region will be called admissible. The lower right boundary may degenerate to a point, in which case the shape becomes triangular. The left boundary is called the terminating boundary, and the remaining three, or two in the case of a triangular region, combined together is called the starting boundary.*

Let e be an edge on the starting boundary of Ω. A prepath in Ω with starting edge e is a sequence of edges $e_0, e_1, e_2, \ldots, e_l$ in Ω such that: (a) $e_0 = e$, (b) e_l is on the terminating boundary, (c) if e_{i-1} is either a β- or γ-edge then e_i is the α-edge sharing an upward-pointing elementary triangle with e_{i-1}, (d) if e_{i-1} is an α-edge not on the terminating boundary, then e_i is either the (d1) γ- or (d2) β-edge sharing a downward-pointing elementary triangle with e_{i-1}.

If P and Q are nonempty prepaths in Ω, we say that P is strictly above Q, or equivalently Q is strictly below P, if either (1) they share at least one diagonal and in each such diagonal the edges of P lie above those of Q, or (2) the diagonals over which P extends are strictly above those over which Q extends.

Let E_Ω be a labelling of all edges of Ω with integers satisfying the triangle conditions and the nonnegativity of upright rhombus gradients. Such an edge labelling will be called admissible. We denote by $H|_\Omega$ the restriction of the edge *labelling of a hive H to Ω, which is always admissible. A prepath $(e_i)_{i=0}^{l}$ in Ω is said to be a path in E_Ω if, for any i such that e_{i-1} satisfies the condition (d) above, the option (d1) or (d2) is taken according to whether the upright rhombus having e_{i-1} as its southeast edge has gradient $= 0$ or > 0. Note that the shape of Ω is such that, whenever the situtation (d) occurs, the above-mentioned upright rhombus is contained in Ω. For each edge e on the starting boundary of Ω, there is a unique path in E_Ω with starting edge e. Restrictions of type (ii), (iii) or (v) paths in H to Ω are examples of paths in $H|_\Omega$.*

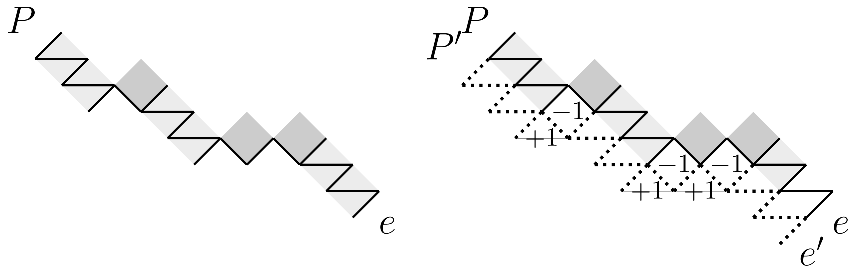
To remove a path P from E_Ω (from $H|_\Omega$ in our typical use) is to create a new edge labelling of Ω out of E_Ω by decreasing the label of each α- and γ-edge in P by 1, and increasing the label of each β-edge in P by 1 (and keeping all remaining edge labels). The resulting edge labelling will be denoted by $\phi_e\, E_\Omega$ where e is the starting edge of P. It is easy to see that applying ϕ_e preserves admissibility.

The following lemma encapsulates an easy argument used repeatedly below.

Lemma 9.4. *Let Ω be an admissible region, and E_Ω an admissible labelling of the edges of Ω. Let P, P' be paths in E_Ω with starting edges e, e' respectively. Assume that P' lies strictly below P. Then P is also a path in $\phi_{e'}\, E_\Omega$, P' is also a path in $\phi_e\, E_\Omega$, and $\phi_e\, \phi_{e'}\, E_\Omega = \phi_{e'}\, \phi_e\, E_\Omega$ holds.*

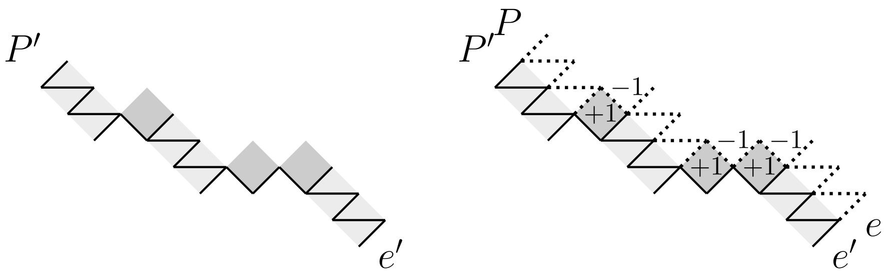
Remark 9.5. *In our usage of this lemma, E_Ω is the restriction of a hive H to Ω, and $\phi_e(H|_\Omega)$, $\phi_{e'}(H|_\Omega)$, $\phi_e\, \phi_{e'}(H|_\Omega)$ and $\phi_{e'}\, \phi_e(H|_\Omega)$ are all known to be restrictions of hives to Ω.*

Proof: Recall that a path starting from e must follow P (see below on the left) so long as the middle rhombi of each ladder of P (shaded light grey) have gradients 0 and the head rhombus of each ladder of P (shaded grey) has gradient > 0. These shaded rhombi have been called the guiding rhombi for P. Since P' lies strictly below P, as exemplified below on the right by dotted edges in the case where P' is closest to P, the removal of P', whose impact on upright rhombus gradients are shown by $+1$ and -1 below on the right, does not affect the gradient of any of the guiding rhombi for P. Hence P is also a path in $\phi_{e'}\, E_\Omega$.



(9.1)

On the other hand, first look at the guiding rhombi for P' (see below on the left). Since P lies strictly above P', the gradient of a guiding rhombus for P' can change upon removal of P only if it is a head rhombus of P', and at the same time a foot rhombus of P. In such a case, the removal of P' increases its gradient, only strengthening its positivity. Hence P' is also a path in $\phi_e\, E_\Omega$.



(9.2)

Thus ϕ_e changes the edge labels of the same set of edges whether it is applied to E_Ω or $\phi_{e'}\, E_\Omega$, and the same is true for $\phi_{e'}$ whether it is applied to E_Ω or $\phi_e\, E_\Omega$. Hence we have $\phi_e\, \phi_{e'}\, E_\Omega = \phi_{e'}\, \phi_e\, E_\Omega$. ■

We shall now return to the proof of Lemma 7.5 by induction on the number N occurring in our sequences $c_1, \ldots, c_N$ and $p_1, \ldots, p_N$, which we call the *number of critical rhombi*, whose implication will be clarified below. We start with the initial step of the induction.

Lemma 9.6. *In the situation of Lemma 7.5, if $N = 1$, namely if exactly one critical rhombus emerges during the removals of $P_1, \ldots, P_m$ from H and $\widehat{P}_1, \ldots, \widehat{P}_m$ from $\widehat{H}$, then the conclusions of Lemma 7.5 hold.*

Proof: For simplicity, set $c = c_1 = c_N$ and $p = p_1 = p_N = r$.

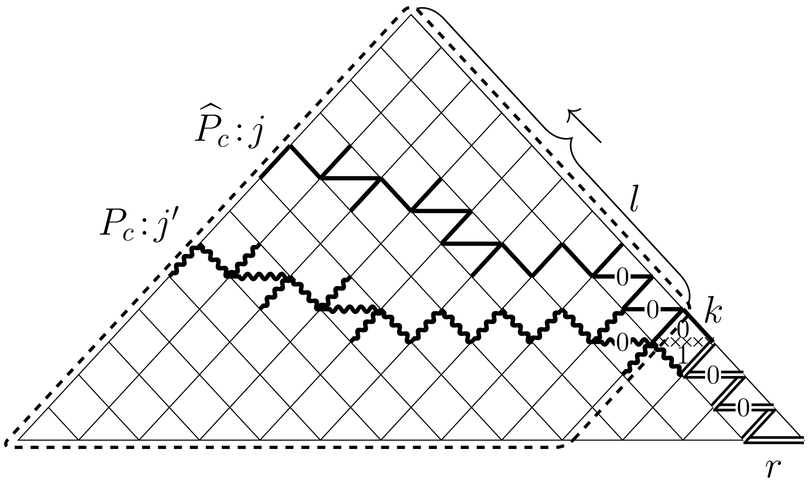
By the definition of c_1, none of the paths $\widehat{P}_1, \ldots, \widehat{P}_{c-1}$ intersect D, and since they start below D they all pass strictly below D. Then one can first apply Lemma 9.4 to the action of ξ_{kr} and ϕ_r on the r-hive $\widehat{H}_0$, namely by taking Ω to be the whole r-hive region, e to be the right-hand boundary edge of level k

and e' to be the rightmost bottom edge. Since $\xi_{kr}\widehat{H}_0 = H_0$ and $\phi_r\widehat{H}_0 = \widehat{H}_1$, Lemma 9.4 shows not only the commutativity $\xi_{kr}\phi_r\widehat{H}_0 = \phi_r\xi_{kr}\widehat{H}_0$, that is to say $\xi_{kr}\widehat{H}_1 = \phi_r H_0 = H_1$, but also that the operator ξ_{kr} removes the same path, D, from $\widehat{H}_0$ and $\widehat{H}_1$, and that the operator ϕ_r removes the same path from $\widehat{H}_0$ and H_0, namely $\widehat{P}_1 = P_1$. Then one can iterate to have $\widehat{P}_a = P_a$ for all $a \leq c-1$, and $\xi_{kr}\widehat{H}_{c-1} = H_{c-1}$ in which ξ_{kr} still removes D. In the picture 9.3 below, the path D is shown by a sequence of **solid** — edges.

The assumptions $c = c_1$ and $p_1 = r$ imply that the next path $\widehat{P}_c$, to be removed from $\widehat{H}_{c-1}$, climbs the rth diagonal following the **double** ═ edges and intersects D after traversing its foot rhombus at level k by way of the γ-edge crossing it, marked with ××××× in 9.3, to its northwest edge. This implies that $\widehat{H}_{c-1}$ has $U_{kr} = 0$. After this, $\widehat{P}_c$ follows the — path D and reaches the left-hand boundary at level j by virtue of the uniqueness of the path in $\widehat{H}_{c-1}|_{\nwarrow}$ with a given starting edge, where $\nwarrow$ denotes the region above the line passing through the northwest edge of the aforementioned upright rhombus. In 9.3, the region is enclosed by a dashed trapezium with rounded corners.

Then $H_{c-1} = \xi_{kr}\,\widehat{H}_{c-1}$ has $U_{kr} = 0+1 > 0$. Hence the path P_c, to be removed from H_{c-1}, follows the ═ edges but enters the $(r-1)$th diagonal at level k as exemplified by the ***wavy*** ∿∿∿ edges in 9.3. Let D' denote this ∿∿∿ path of the $(r-1)$-hive, starting from the right-hand boundary edge at level k. In H_{c-1}, the foot rhombi of the ladders of D have positive gradients, being greater than those in $\widehat{H}_{c-1}$ by 1, serving as an inpenetrable barrier to climbing the ladders of D. So P_c, and accordingly D', stay strictly below D and end on the left-hand boundary at some level $j' < j$.

The gradient $U_{kr} = 1$ of H_{c-1} is the smallest value to block the path P_c from climing the ladder of D, and in $\widehat{H}_{c-1}$ its value $U_{kr} = 0$ allows the path $\widehat{P}_c$ into the ladder, by a slim difference of 1. The rhombus with gradient U_{kr} thus produces a bifurcation of the ═ path into the ∿∿∿ and — paths followed by P_c and $\widehat{P}_c$, respectively, and so is said to be *critical* for the removals of P_c and $\widehat{P}_c$. Thereafter, due to their removals, this rhombus has gradient 0 in both hives and is said to be *post-critical*. Also the difference along $D|_{\nwarrow}$ has been resolved, while new differences have been introduced along the ∿∿∿ path D'.



(9.3)

Since $U_{kr} = 0$ in both H_c and $\widehat{H}_c$, as well as all U_{xr} with $x > k$, both P_{c+1} and $\widehat{P}_{c+1}$ reach the northwest edge of the post-critical rhombus without changing the gradients of that rhombus. So the situation persists and all P_a and $\widehat{P}_a$ with $a > c$ come to the northwest edge of the post-critical rhombus. By Lemma 4.4, the paths $\widehat{P}_a|_{\nwarrow}$ with $a > c$ run weakly above $\widehat{P}_c|_{\nwarrow} = D|_{\nwarrow}$, and so strictly above $P_c|_{\nwarrow} = D'|_{\nwarrow}$. Hence, by applying Lemma 9.4 to $\widehat{H}_c|_{\nwarrow}$ and its paths $\widehat{P}_{c+1}|_{\nwarrow}$ and $D'|_{\nwarrow}$, then to $\widehat{H}_{c+1}$ and its paths $\widehat{P}_{c+2}|_{\nwarrow}$ and $D'|_{\nwarrow}$, and so on, we have $P_a|_{\nwarrow} = \widehat{P}_a|_{\nwarrow}$ (so that $P_a = \widehat{P}_a$) for all $a > c$, and that $H_m|_{\nwarrow}$ and $\widehat{H}_m|_{\nwarrow}$ are related by the removal of $D'|_{\nwarrow}$. The difference in the label of the starting edge of D', namely the southwest edge of the post-critical rhombus which is the only edge of D' not included in $D'|_{\nwarrow}$, is also maintained through the removals of P_a and $\widehat{P}_a$ with $a > c$ since none of them contain that edge. Discarding the empty rth diagonal in the end, we see that $\theta_r\, H = \kappa_r\, H_m$ is related to $\theta_r\, \widehat{H} = \kappa_r\, \widehat{H}_m$ by $\xi_{k,r-1}$ which removes the $\rightsquigarrow$ path D'. ∎

Continuing with the proof of Lemma 7.5, we come to the heart of the matter, namely the inductive step on the number N of critical rhombi.

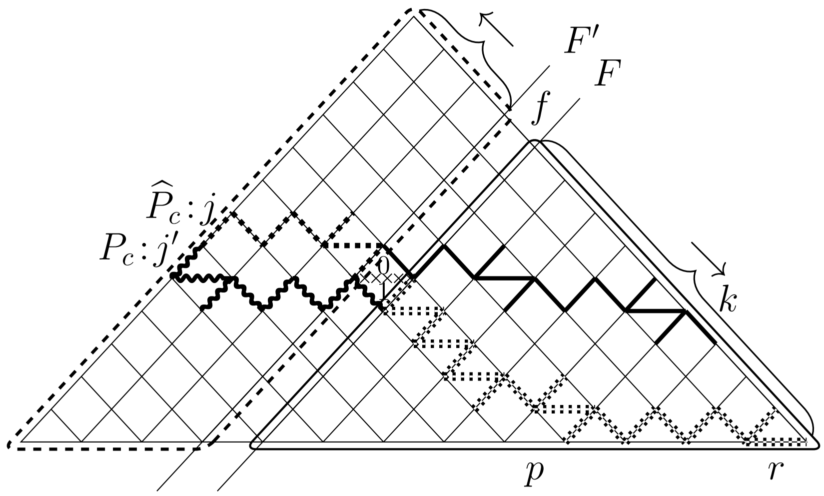
Lemma 9.7. *In the situation of Lemma 7.5, assume that $N > 1$, so that the removals of $P_1, \ldots, P_m$ and $\widehat{P}_1, \ldots, \widehat{P}_m$ involve encounters with at least two critical rhombi. Assuming, under the inductive hypothesis that Lemma 7.5 has been proved for all cases with the number of critical rhombi strictly less than N, then the conclusions of Lemma 7.5 hold for the present case involving N critical rhombi.*

Proof: Let $c = c_1$ and $p = p_1$ for simplicity, and let f denote the level of the foot rhombus of the ladder of D in the pth diagonal. The **solid** — edges in the diagram 9.4 below show the part of D up to entering the pth diagonal, and the **dotted** ····· edges show the remaining part of D. (The distinction is made since, as we shall see below, the difference of edge labels along the — part persists after the removal of P_c and $\widehat{P}_c$, but those along the ····· part resolves by the removal of P_c and $\widehat{P}_c$.) By a repeated application of Lemma 9.4 to the whole r-hive, the paths $P_1, \dots, P_{c-1}$ coincide with $\widehat{P}_1, \dots, \widehat{P}_{c-1}$ respectively, and hence all run strictly below D, and we have $H_{c-1} = \xi_{kr}\, \widehat{H}_{c-1}$ in which ξ_{kr} still removes D.

Now, by assumption, the path $\widehat{P}_c$, to be removed from $\widehat{H}_{c-1}$, runs below D up to the $(p+1)$th diagonal, but in the pth diagonal it approaches and intersects D, after crossing its foot rhombus (whose gradient U_{fp} must have been 0), in the manner discussed in the second paragraph of the proof of Lemma 7.5. Having intersected the path D in a common edge, the uniqueness of a path with a given starting edge implies that thereafter it must follow D to its end at level j. In 9.4, the part of $\widehat{P}_c$ up to this foot rhombus is shown with **dotted double** ⁞⁞⁞⁞⁞⁞ edges, and the γ-edge crossing this rhombus with a line of **crosses** ×××××, and the portion coincident with D is shown with **dotted** ····· edges. On the other hand, the path P_c, to be removed from H_{c-1}, initially coincident with $\widehat{P}_c$ along the ⁞⁞⁞⁞⁞⁞ edges, finds the same gradient U_{fp} to be 1 instead of 0, with the difference arising from D, and the path P_c therefore passes leftwards below this rhombus, decreasing its gradient to 0, and proceeds along what is exemplified by the ***wavy*** ⁓⁓ edges in 9.4, to end on the left-hand boundary at some level $j' < j$, for the same reason as before regarding an inpenetrable barrier below D.

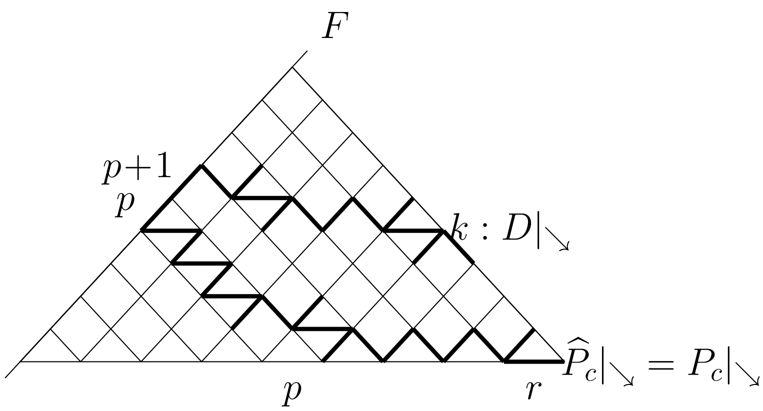
Thus, the upright rhombus carrying the gradient U_{fp}, marked in 9.4 by placing the symbols 0 above 1 as before, causes a bifurcation, and hence is critical for the removals of P_c and $\widehat{P}_c$. After this pair of removals, it carries a common gradient 0, and is post-critical.

Now consider H_c and $\widehat{H}_c$. Let F and F' denote the lines passing the southeast and northwest edges, respectively, of the above-mentioned upright rhombus, and let $\searrow$ and $\nwarrow$ denote the regions weakly below the line F (the part enclosed by a triangle with rounded corners in 9.4) and weakly above the line F' (the part enclosed by a dashed triangle with rounded corners in the same diagram), respectively. Even though $H_c|_{\searrow}$ and $\widehat{H}_c|_{\searrow}$ are $(r-f)$-hives in themselves, we designate their diagonals, edge levels and gradients using the measurement parameters specified within their parent r-hives H_c and $\widehat{H}_c$. Bearing this in mind, it is important to note that $D|_\searrow$ starts at level k and reaches the line F at level $p+1$, as part of the migration pattern of a type (v) path from the $(p+1)$th to the pth diagonal.



(9.4)

Restricting attention to the region $\searrow$ weakly below F in the prospect of using the induction hypothesis to apply the present Lemma 7.5 to $H_c|_\searrow$ and $\widehat{H}_c|_\searrow$, we start with the following picture where we represent both of the two non-intersecting paths $D|_\searrow$ and $P_c|_\searrow = \widehat{P}_c|_\searrow$ by means of solid edges —, which reach the left-hand boundary F of this region at levels $p+1$ and p, respectively:

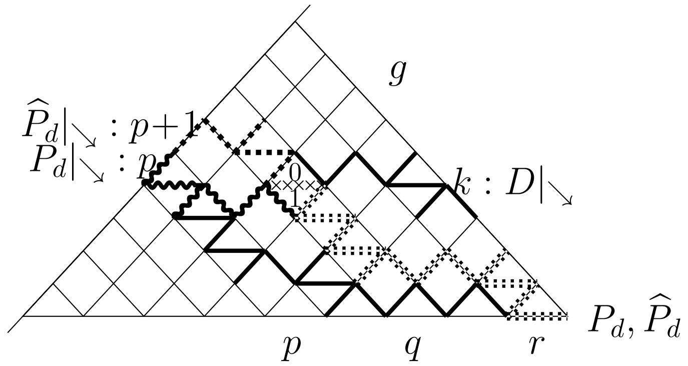


(9.5)

Maintaining our specification of diagonals, edges and rhombus gradients as dictated by our original hives, as well as our path numbering, we consider the application of θ_r to $H_c|_\searrow$ and $\widehat{H}_c|_\searrow$ with $H_c|_\searrow = \xi_{kr}\,\widehat{H}_c|_\searrow$, and path of difference $D|_{\searrow}$ starting at level k and ending at level $p+1$. Taking into account the fact that the lowermost right-hand edge label of both $H_c|_{\searrow}$ and $\widehat{H}_c|_{\searrow}$ is 0, there are no type (i) path removals. Having already removed c pairs of type (ii) paths from H and $\widehat{H}$, and accordingly from $H|_{\searrow}$ and $\widehat{H}|_{\searrow}$, the paths consecutively removed from $H_c|_{\searrow}$ and $\widehat{H}_c|_{\searrow}$ by the action of θ_r are $P_{c+1}|_{\searrow}, \ldots, P_m|_{\searrow}$ and $\widehat{P}_{c+1}|_{\searrow}, \ldots, \widehat{P}_m|_{\searrow}$, respectively. Recalling our original notation $c = c_1 < c_2 < \cdots < c_N$ and $p = p_1 < p_2 < \cdots < p_N$ with our assumption $N \geq 2$, note that the paths $\widehat{P}_a$ with $a < c_2$ do not intersect any ladder of D below the line F. Hence the number of critical rhombi encountered during the actions of θ_r on $H_c|_{\searrow}$ and $\widehat{H}_c|_{\searrow}$ is $N-1$, which allows the use of the present Lemma to $H_c|_{\searrow}$ and $\widehat{H}_c|_{\searrow}$ by the induction hypothesis.

Thus, among the $m-c$ pairs of paths being removed, for some d with $c < d \leq m$ we have: $P_a|_{\searrow} = \widehat{P}_a|_{\searrow}$ for $c < a < d$, while $P_d|_{\searrow}$ ends at level $j = p+1$ on the left-hand boundary F, and $\widehat{P}_d|_{\searrow}$ ends at some level $j' < p+1$ on F. However, $\widehat{P}_c|_{\searrow}$ ends as shown above in 9.5 at level p on F. Since $\widehat{P}_a|_{\searrow}$ lies weakly above $\widehat{P}_c|_{\searrow}$ for all $a > c$, including $a = d$, it follows that $j' = p$. Hence all paths $P_a|_{\searrow} = \widehat{P}_a|_{\searrow}$ for $c < a < d$ and $\widehat{P}_d|_{\searrow}$ end on F at level p. The inductive application of Lemma 7.5 also tells us that for each $a > d$, $P_a|_{\searrow}$ and $\widehat{P}_a|_{\searrow}$ end at the same level, say j_a, on F, with $j_a \geq p+1$ since $P_a|_{\searrow}$ lies weakly above $P_d|_{\searrow}$.

Just as we identified c with c_1 and p with p_1, we shall write d and q for c_2 and p_2, respectively, in terms of which we have the following illustration of the manner in which the paths $P_d|_{\searrow}$ and $\widehat{P}_d|_{\searrow}$ are squeezed (weakly) between the two solid lines —— representing $P_c|_{\searrow}$ and $D|_{\searrow}$:

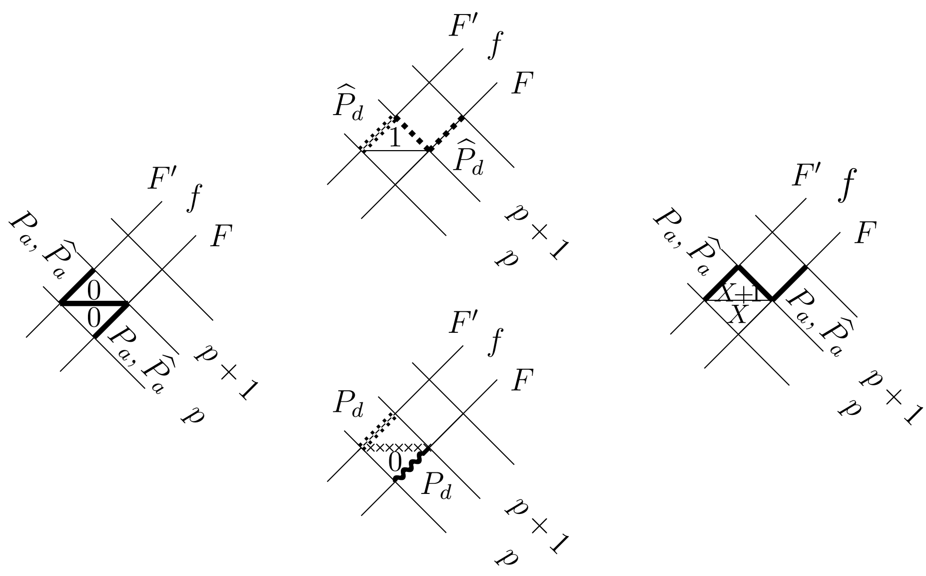


(9.6)

Here the paths $P_d|_{\searrow}$ and $\widehat{P}_d|_{\searrow}$, initially represented by a double dotted line, bifurcate in diagonal q at some level g with the critical upright rhombus gradient U_{gq} equal to 1 in $H_{d-1}|_{\searrow}$ and 0 in $\widehat{H}_{d-1}|_{\searrow}$, with the difference due to $D|_{\searrow}$. As a result, $P_d|_{\searrow}$ passes leftwards along the wavy edge path to meet F at level p, while $\widehat{P}_d|_{\searrow}$ crosses the critical rhombus and intersects $D|_{\searrow}$, thereafter following the dotted solid edge portion of $D|_{\searrow}$ to meet F at level $p+1$.

The final conclusion from the application of Lemma 7.5 to $H_c|_{\searrow}$ and $\widehat{H}_c|_{\searrow}$ is that we have $\theta_r\, H_c|_{\searrow} = \xi_{kr}(\theta_r\, \widehat{H}_c|_{\searrow})$, with a path of difference E' that will eventually be identified with $D'|_{\searrow}$. For the moment we just point out that the portion of the path of difference from the $p_2 = q$th diagonal to the $p_1 = p$th diagonal is that portion of the path $\widehat{P}_d|_{\searrow}$, with $d = c_2$, that is represented in 9.6 by means of edges.

We next consider the continuation of each of the paths $P_a|_{\searrow}$ and $\widehat{P}_a|_{\searrow}$, namely P_a and $\widehat{P}_a$, as they cross from F to F' in H_{a-1} and $\widehat{H}_{a-1}$, respectively, for all $a > c$. The outcome is illustrated below in 9.7:

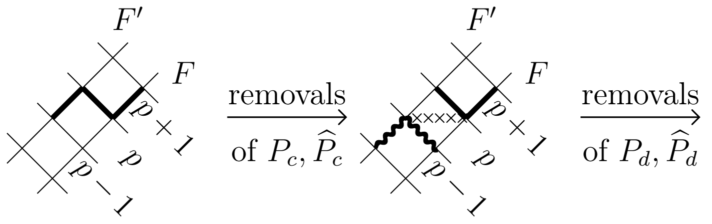


$c < a < d$ $\qquad$ $a = d$ $\qquad$ $d < a \leq m$ $\qquad$ (9.7)

For $a = c+1, \ldots, d-1$ we have $P_a|_{\searrow} = \widehat{P}_a|_{\searrow}$, meeting F at level p and both continuing across the post-critical rhombus with $U_{fp} = 0$, remaining at level p, as shown in the diagram on the left. For $a = d$ the path $P_d|_{\searrow}$ again meets F at level p and continues in the same way, as shown in the lower portion of the diagram in the middle, while the path $\widehat{P}_a|_{\searrow}$ meets F as we have seen at level $p+1$, moves to the left neighbouring β-edge since $U_{f,p+1} > 0$ (this gradient was positive in $\widehat{H}$ as evidenced by the route of D, generated by the action of ξ_{kr} on $\widehat{H}$, passing below it, and removals of $\widehat{P}_1, \ldots, \widehat{P}_{d-1}$ have not changed this gradient) and passes along the upper edges of the post-critical rhombus to level p as shown in the upper portion of the diagram in the middle. The removal of $\widehat{P}_d$ changes U_{fp} from 0 to 1, rendering it what we call post-post-critical. Finally, for each $a = d+1, \ldots, m$, both $P_a|_{\searrow}$ and $\widehat{P}_a|_{\searrow}$ meet F at the same level $j_a \geq p+1$, and each of these solid edge —— paths crosses from F to F' weakly above the post-post-critical rhombus, whose gradients U_{fp} will take values in X in H_a and $X+1$ in $\widehat{H}_a$ for some $X \geq 0$. In the diagram on the right we have illustrated this case $a > d$ in the extreme situation where the extension of $P_a|_{\searrow}$ and $\widehat{P}_a|_{\searrow}$ follows that of $P_d|_{\searrow}$, thereby each contributing 1 to X, rather than the more generic situation where it lies above that of $P_d|_{\searrow}$ and does not affect the value of X.

Concentrating on the difference of edge labels occurring in the strip flanked by F and F', first note that each of the pairs P_a and $\widehat{P}_a$ crosses this strip together without altering any differences except in the cases $a = c$ and $a = d$. The transformation of the path of difference in this strip is illustrated below in 9.8. Initially the difference occurs along the path D (represented by —— in the left-hand diagram), and this persists through removals of all P_a and $\widehat{P}_a$ with $a < c$. Then (see 9.4) the path $\widehat{P}_c$, unlike P_c, traverses the γ-edge across the critical rhombus, introducing a difference in its edge labelling (represented by ××××× in the middle diagram), and the northwest edge of that rhombus, eliminating the difference there; whereas the path P_c, unlike $\widehat{P}_c$, passes to the southwest edge of that rhombus and the α-edge to its left, introducing differences there (represented by 〰〰 in the middle diagram). Again removals of P_a and $\widehat{P}_a$ with $c < a < d$ do not change anything. Then (see 9.7) the path $\widehat{P}_d$, unlike P_d, reaches the post-critical rhombus tracing D, which eliminates the differences of the labels of the two —— edges, while $\widehat{P}_d$, unlike P_d, traverses that rhombus from its southeast edge, introducing a difference in its labelling (represented by 〰〰 in the right-hand diagram) and eliminating the difference represented by ×××××. Recalling that the post-post-critical rhombus has positive gradient in $\widehat{H}_a$ with $a \geq d$, we now see that the path of difference E' in the region weakly below F successfully extends to the region between F and F' in the manner required for a type (v) path removal under the action of ξ_{kr} on $\widehat{H}_m$ to give H_m.

Transformation of the path of difference
between H_a and $\widehat{H}_a$ in the F-F' strip



$a < c$ $\qquad$ $c \le a < d$ $\qquad$ $d \le a$ (9.8)

Finally we look at the region $\nwarrow$ weakly above the line F'. As we saw (way) above, removing the initial path of difference D from $\widehat{H}_{c-1}$ gives H_{c-1}, and in this northwestern trapezium we have $D|_{\nwarrow} = \widehat{P}_c|_{\nwarrow}$. Since, by definition, removing $\widehat{P}_c$ from $\widehat{H}_{c-1}$ gives $\widehat{H}_c$, the coincidence $D|_{\nwarrow} = \widehat{P}_c|_{\nwarrow}$ leads to the coincidence of H_{c-1} and $\widehat{H}_c$ in the northwestern trapezium. Since removing $P_c|_{\nwarrow}$ from $H_{c-1}|_{\nwarrow}$ gives $H_c|_{\nwarrow}$ by definition, it also means that $P_c|_{\nwarrow}$ is the new difference path whose removal from $\widehat{H}_c|_{\nwarrow}$ gives $H_c|_{\nwarrow}$. Note that the differences along the old path of difference $D|_{\nwarrow}$ have been resolved. Therefore, if we denote by e' the α-edge lying on F' neighbouring the northwest edge of the post-critical rhombus to its left, we have $H_c|_{\nwarrow} = \phi_{e'}\,\widehat{H}_c|_{\nwarrow}$. Now for each of $c < a \le m$, the path $\widehat{P}_a|_{\nwarrow}$, lying weakly above $\widehat{P}_c|_{\nwarrow} = D|_{\nwarrow}$, lies strictly above $P_c|_{\nwarrow}$. Hence, by applying the commutativity Lemma 9.4 repeatedly to the region, we have $\widehat{P}_a|_{\nwarrow} = P_a|_{\nwarrow}$ for all such a, and $H_m|_{\nwarrow} = \phi_{e'}\,\widehat{H}_m|_{\nwarrow}$ where in this operation $\phi_{e'}$ removes $P_c|_{\nwarrow}$.

We can now combine this with what we already have on the region $\searrow$ and the strip between F and F', and verify that the paths P_a and $\widehat{P}_a$ coincide entirely for each $a < c$, while $\widehat{P}_c$ ends at level j where D ends, whereas P_c, running strictly below $\widehat{P}_c$ in $\nwarrow$, ends at some level $j' < j$, and for each $a > c$ the paths P_a and $\widehat{P}_a$ end at the same level. Moreover, the concatenation of E', that is the path of difference arising from the induction hypothesis applied to the region $\searrow$, and the southwest edge of the post-post-critical rhombus, continues as a path in $\theta_r\,\widehat{H} = \kappa_r(\widehat{H}_m)$ to e' (note that the assumption $N \le 2$ implies

$p = p_1 < p_2 \leq r$, placing the post-post-critical rhombus in the $(r-1)$-hive region), and hence to $P_c|_{\nwarrow}$. In each section it has been verified that removing this path gives the difference between the $(r-1)$-hive parts of H_m and $\widehat{H}_m$, namely $\theta_r\, H$ and $\theta_r\, \widehat{H}$. This implies *inter alia* that E' may indeed be identified with $D'|_{\searrow}$.

By unfolding what is implied in the above inductive description, using the notation $c_1 < c_2 < \cdots < c_N$ and $p_1 < p_2 < \cdots < p_N$, we see that the final path of difference D', removed from $\theta_r\, \widehat{H}$ by $\xi_{k,r-1}$ to yield $\theta_r\, H$, is obtained by pasting the part of P_{c_1} from the southwest edge of the first critical rhombus in the p_1th diagonal to the left-hand boundary, the part of P_{c_2} from the southwest edge of the second critical rhombus in the p_2th diagonal to the southeast edge of the first critical rhombus in the p_1th diagonal, and so on, up to the part of P_{c_N} from the southwest edge of the final critical rhombus in the p_Nth $(= r\text{th})$ diagonal to the southeast edge of the $(N-1)$th critical rhombus in the p_{N-1}th diagonal.

This completes the proof of Lemma 9.7. ■

The proof of Lemma 7.5 is now complete by induction on the number N of encounters with critical rhombi, due to Lemma 9.6 which solves the case $N = 1$ and Lemma 9.7 which takes care of the inductive step. ■

To complete the proof of Lemma 7.3 we now offer a proof of Lemma 7.7.

Proof of Lemma 7.7: Upon application of θ_k, the hive H affords one extra type (i) path removal compared with $\widehat{H}$ due to the difference of $+1$ created on the lowermost right boundary edge label by applying ξ_{kk}. Let us denote the result of removing the common type (i) paths from H and $\widehat{H}$ by $H^{(0)}$ and $\widehat{H}^{(0)}$ respectively. By removing the additional type (i) path from $H^{(0)}$, and denoting the resulting hive by $H^{(1)}$, the difference of $H^{(1)}$ from $\widehat{H}^{(0)}$ is described by a path, say $\widetilde{D}$, obtained by changing the initial edge of D, namely the lowermost right boundary edge, to the rightmost bottom edge. In other words we have $H^{(1)} = \phi_k\, \widehat{H}^{(0)}$, in which operation ϕ_k removes $\widetilde{D}$. Thus the first type (ii) path removed from $\widehat{H}^{(0)}$ is $\widetilde{D}$, whose removal results in the same hive as $H^{(1)}$. Thereafter all pairs of path removals coincide, yielding $\theta_k\, H = \theta_k\, \widehat{H}$. ■

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