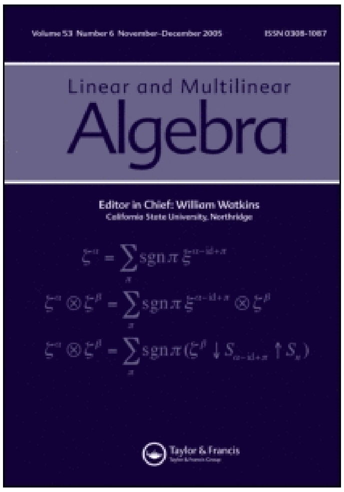

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On a conjecture about the -permanent

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On a conjecture about the μ-permanent

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Let $A = (a_{ij})$ be an n-by-n matrix. For any real μ, define the polynomial

$$P_\mu(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} \dots a_{n\sigma(n)} \, \mu^{\ell(\sigma)},$$

where $\ell(\sigma)$ is the number of inversions of the permutation σ in the symmetric group S_n. We prove that $P_\mu(A)$ is a strictly increasing function of $\mu \in [-1, 1]$, for a Hermitian positive definite nondiagonal matrix A, whose graph is a tree.

Keywords: Hermitian matrix; Permanent; Determinant; Digraph; Tree

Mathematics Subject Classifications: 15A45; 15A15; 05C50; 05C20

1. Introduction

Given an $n \times n$ matrix $A = (a_{ij})$ and a real μ we will be interested in the polynomial $P_\mu(A)$, the μ-permanent of A, defined as

$$P_\mu(A) = \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \right) \mu^{\ell(\sigma)}, \tag{1.1}$$

where $\ell(\sigma)$ is the number of inversions of the permutation σ in the symmetric group S_n of degree n, i.e.,

$$\ell(\sigma) = \#\big\{(i,j) \in \{1, \dots, n\}^2 \mid i < j \text{ and } \sigma(i) > \sigma(j)\big\}.$$

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The μ-permanent is a generalization of the determinant and of the permanent, making $\mu=-1$ and $\mu=1$, respectively. Note also that $P_0(A)=a_{11}\dots a_{nn}$.

Using the positive definiteness of $f(\sigma)=\mu^{\ell(\sigma)}$ on S_n (cf [1]), Bapat proved:

LEMMA 1.1 ([2]) *For any Hermitian positive semidefinite matrix A,*

$$P_\mu(A)\geq 0\,,\quad \textit{if}\ \mu\in[-1,1].$$

Bapat conjectured and proved for $n\leq 3$:

CONJECTURE 1.2 ([2]) *Given an $n\times n$ Hermitian positive definite nondiagonal matrix A, $P_\mu(A)$ is a strictly increasing function of $\mu\in[-1,1]$.*

This conjecture has been proved for a tridiagonal positive definite matrix in [3].

If Conjecture 1.2 is true, then

$$\det A\leq P_\mu(A)\leq \operatorname{per}A,$$

and it will give a generalization of both the classical Hadamard inequality and the permanental analogue proved by Marcus [4] more than three decades ago.

The aim of this note is to verify Conjecture 1.2 when the graph of A is a tree, in addition to the given hypothesis. In the end an illustrative example is given.

2. Weighted digraphs

A graph G consists of a finite set $\mathcal{V}$ whose members are called vertices, and a set $\mathcal{E}$ of 2-subset of $\mathcal{V}$. By a digraph or directed graph $D=(\mathcal{V},\mathcal{A})$ we mean the same finite set $\mathcal{V}$, and a subset $\mathcal{A}$ of $\mathcal{V}\times\mathcal{V}$, whose members are called arcs. Note that an arc is an ordered pair (i,j), whereas an edge of a graph is an unordered pair $\{i,j\}$. If to each arc we assign a real or a complex number, we have a weighted digraph. We write $i\sim j$, if $\{i,j\}$ is an edge of G, with $i\neq j$. For background information on graphs and digraphs, we refer the reader to [5].

A directed path from i_1 to i_r, P_{i_1,i_r}, in the digraph D is a sequence of distinct vertices $(i_1,i_2,\dots,i_{r-1},i_r)$ such that each arc $(i_1,i_2),\dots,(i_{r-1},i_r)$ is in $\mathcal{A}$. If to the path P_{i_1,i_r} we add the arc (i_r,i_1), then we have a cycle (of length r). Analogously we can define the same concepts for a graph. A tree is a connected graph without cycles.

Given an arc $e=(i,j)$ of D, $D\backslash e$ is obtained by deleting e but not the vertices i or j; on the other hand, $D\backslash i$ is obtained by deleting i and all arcs including i.

Let $A=(a_{ij})$ be an $n\times n$ matrix. The graph of A, $G(A)$, is the pair $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,\dots,n\}$ and $\{i,j\}$ is an edge if and only if $a_{ij}\neq 0$ or $a_{ji}\neq 0$. Analogously, the (weighted) digraph of $A=(a_{ij})$ is a directed graph having (i,j) as an arc if and only if $a_{ij}\neq 0$, for $i\neq j$. The matrix A can be viewed as a weighted adjacency matrix of digraph $D(A)$ on n vertices, with loops (arcs of the type (i,i)) allowed on the vertices.

3. A μ-permanental formula

A systematic and detailed account of various determinantal formulas relating the structure of the digraph and the associated matrix can be found in several papers (cf [4,6–9]). For example, the following is well known:

THEOREM 3.1 *Given an $n \times n$ matrix $A = (a_{ij})$ and $i \in \{1, \ldots, n\}$, let us assume that $\{c_1, \ldots, c_m\}$ is the set of all (directed) cycles in $D(A) = D$ containing the vertex i. Then*

$$\det A = \sum_{j=1}^{m} (-1)^{\ell(c_j)} \det A(D \setminus c_j) \prod_{e \in \mathcal{A}(c_j)} a_e. \tag{3.1}$$

Notice that if $D \setminus c_j$ is disconnected, then $\det A(D \setminus c_j)$ is a product of the determinants of the weighted adjacency matrices of each component.

We can easily generalize (3.1) to the μ-permanent defined in (1.1):

$$P_\mu(A) = \sum_{j=1}^{m} \left(\prod_{e \in \mathcal{A}(c_j)} a_e \right) P_\mu(A(D \setminus c_j))\, \mu^{\ell(c_j)}.$$

If A is Hermitian, then we have:

THEOREM 3.2 *Given an $n \times n$ Hermitian matrix $A = (a_{ij})$ and $i \in \{1, \ldots, n\}$, let us assume that $\{c_1, \ldots, c_m\}$ is the set of all cycles in $G(A) = G$ containing the vertex i. Then*

$$\begin{aligned} P_\mu(A) &= a_{ii} P_\mu(A(G \setminus i)) + \sum_{i \sim j} |a_{ij}|^2 P_\mu(A(G \setminus ij))\, \mu^{\ell(ij)} \\ &\quad + \sum_{j=1}^{m} \left(\prod_{e \in \mathcal{E}(c_j)} a_e \right) P_\mu\big(A\big(G \setminus c_j\big)\big)\, \mu^{\ell(c_j)}. \end{aligned} \tag{3.2}$$

Since a tree has no cycles, we may establish the corollary:

COROLLARY 3.3 *Given an $n \times n$ Hermitian matrix $A = (a_{ij})$ whose graph is a tree T and $i \in \{1, \ldots, n\}$, then*

$$P_\mu(A) = a_{ii} P_\mu(A(T \setminus i)) + \sum_{i \sim j} |a_{ij}|^2 P_\mu(A(T \setminus ij))\, \mu^{\ell(ij)}. \tag{3.3}$$

Notice that if $i < j$, then $\ell(ij) = 2(j - i) - 1$, and therefore $\ell(ij)$ is always odd.

4. The conjecture for trees

Given a Hermitian matrix $A = A(G)$ and a subset of indexes S let us denote throughout by A_S the complementary principal submatrix of A in the rows and columns defined by S, i.e., $A_S = A(G \setminus S)$.

In this section we prove that under the conditions of Conjecture 1.2, the derivative of $P_\mu(A)$ with respect to μ is positive, when the graph of A is a tree.

LEMMA 4.1 *If $A = (a_{ij})$ is an $n \times n$ Hermitian matrix whose graph is a tree, then*

$$\frac{d}{d\mu} P_\mu(A) = \sum_{i\sim j} \ell(ij)|a_{ij}|^2 P_\mu(A_{ij})\mu^{\ell(ij)-1}, \tag{4.1}$$

with $i<j$.

Proof We use induction on the order n. For $n=2$,

$$P_\mu(A) = a_{11}a_{22} + \mu a_{12}^2 \quad \text{and} \quad \frac{d}{d\mu} P_\mu(A) = a_{12}^2.$$

Suppose now that the result is true for matrices with order less than n. Since, from (3.3),

$$P_\mu(A) = a_{11}P_\mu(A_1) + \sum_{1\sim j} |a_{1j}|^2 P_\mu(A_{1j})\,\mu^{\ell(1j)},$$

we have

$$\begin{aligned} \frac{d}{d\mu} P_\mu(A) = a_{11}\frac{d}{d\mu} P_\mu(A_1) &+ \sum_{1\sim j} \ell(1j)|a_{1j}|^2 P_\mu(A_{1j})\,\mu^{\ell(1j)-1} \\ &+ \sum_{1\sim j} |a_{1j}|^2 \frac{d}{d\mu} P_\mu(A_{1j})\,\mu^{\ell(1j)}. \end{aligned}$$

Assume without loss of generality that if $k \sim 1$, then $k<j$, for all $j \not\sim 1$. By inductive hypothesis:

$$\begin{aligned} \frac{d}{d\mu} P_\mu(A) &= a_{11} \sum_{1<i\sim j} \ell(ij)|a_{ij}|^2 P_\mu(A_{1ij})\,\mu^{\ell(ij)-1} \\ &\quad + \sum_{1\sim k} \ell(1k)|a_{1k}|^2 P_\mu(A_{1k})\,\mu^{\ell(1k)-1} \\ &\quad + \sum_{1\sim k} |a_{1k}|^2 \sum_{k<i\sim j} \ell(ij)|a_{ij}|^2 P_\mu(A_{1kij})\mu^{\ell(ij)-1}\,\mu^{\ell(1k)} \\ &= \sum_{1<i\sim j} \ell(ij)|a_{ij}|^2 P_\mu(A_{ij})\,\mu^{\ell(ij)-1} \\ &\quad + \sum_{1\sim k} \ell(1k)|a_{1k}|^2 P_\mu(A_{1k})\,\mu^{\ell(1k)-1}, \end{aligned}$$

from (3.3). Hence we get (4.1). ■

Since the graph of a tridiagonal matrix is a path, the result of Lal [3] is obtained as a corollary.

COROLLARY 4.2 *If $A = (a_{ij})$ is an $n \times n$ Hermitian tridiagonal matrix, then*

$$\frac{d}{d\mu} P_\mu(A) = \sum_{i=1}^{n-1} |a_{i,i+1}|^2 P_\mu(A_{i,i+1}).$$

From (4.1) and Lemma 1.1 we get the main result of this note.

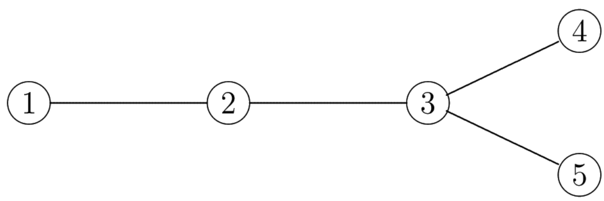
THEOREM 4.3 *Given an $n \times n$ Hermitian positive definite matrix A whose graph is a tree, $P_\mu(A)$ is a strictly increasing function of $\mu \in [-1, 1]$.*

5. An example

Consider the Hermitian matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ \overline{a}_{12} & a_{22} & a_{23} & 0 & 0 \\ 0 & \overline{a}_{23} & a_{33} & a_{34} & a_{35} \\ 0 & 0 & \overline{a}_{34} & a_{44} & 0 \\ 0 & 0 & \overline{a}_{35} & 0 & a_{55} \end{pmatrix}.$$

The graph of A is the tree



Notice that

$$\begin{aligned} P_\mu(A) = {} & a_{12}^2 a_{35}^2 a_{44}\,\mu^4 + a_{11}a_{22}a_{35}^2 a_{44}\,\mu^3 + a_{12}^2 a_{34}^2 a_{55}\,\mu^2 \\ & + \left(a_{12}^2 a_{33}a_{44}a_{55} + a_{11}a_{23}^2 a_{44}a_{55} + a_{11}a_{22}a_{34}^2 a_{55}\right)\mu + a_{11}a_{22}a_{33}a_{44}a_{55} \end{aligned}$$

Then the derivative of $P_\mu(A)$ is

$$\begin{aligned} \frac{d}{d\mu}\,P_\mu(A) = {} & 4\,a_{12}^2 a_{35}^2 a_{44}\,\mu^3 + 3\,a_{11}a_{22}a_{35}^2 a_{44}\,\mu^2 + 2\,a_{12}^2 a_{34}^2 a_{55}\,\mu \\ & + a_{12}^2 a_{33}a_{44}a_{55} + a_{11}a_{23}^2 a_{44}a_{55} + a_{11}a_{22}a_{34}^2 a_{55} \end{aligned} \tag{5.1}$$

On the other hand, by Lemma 4.1

$$\begin{aligned} \frac{d}{d\mu}\,P_\mu(A) = {} & a_{12}^2\left(a_{33}a_{44}a_{55} + a_{34}^2 a_{55}\,\mu + a_{35}^2 a_{44}\,\mu^3\right) \\ & + a_{23}^2(a_{11}a_{44}a_{55}) \\ & + a_{34}^2\left(a_{11}a_{22}a_{55} + a_{12}^2 a_{55}\,\mu\right) \\ & + 3a_{35}^2\left(a_{11}a_{22}a_{44} + a_{12}^2 a_{44}\,\mu\right)\mu^2, \end{aligned}$$

which is the equal to (5.1).

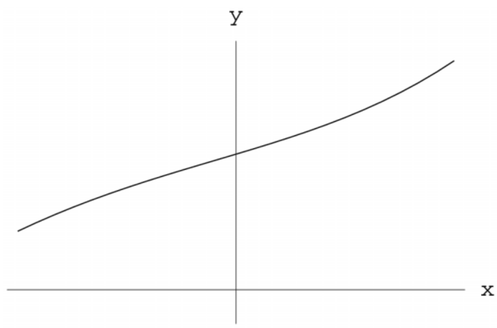
For example, consider the matrix

$$A = \begin{pmatrix} 2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1 & -1/4 & 0 & 0 \\ 0 & -1/4 & 3 & 1 & -2/3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2/3 & 0 & 1 \end{pmatrix}.$$

Then

$$P_\mu(A) = \frac{1}{9}\,\mu^4 + \frac{8}{9}\,\mu^3 + \frac{1}{4}\,\mu^2 + \frac{23}{8}\,\mu + 6.$$

whose graph, for $\mu \in [-1, 1]$, is



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