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**A deterministic bounding procedure for the global optimization of a bi-level mixed-integer problem**

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## ABSTRACT

In this paper, a deterministic bounding procedure for the global optimization of a mixed-integer bi-level programming problem is proposed. The aim has been to develop an efficient algorithm to deal with a case study in the electricity retail market. In this problem, an electricity retailer wants to define a time-of-use tariff structure to maximize profits, but he has to take into account the consumers' reaction by means of re-scheduling appliance operation to minimize costs. The problem has been formulated as a bi-level mixed-integer programming model.

The algorithm we propose uses optimal-value-function reformulations based on similar principles as the ones that have been used by other authors, which are adapted to the characteristics of this type of (pricing optimization) problems where no upper (lower) level variables appear in the lower (upper) level constraints. The overall strategy consists of generating a series of convergent upper bounds and lower bounds for the upper-level objective function until the difference between these bounds is below a given threshold. Computational results are presented as well as a comparison with a hybrid approach combining a particle swarm optimization algorithm to deal with the upper-level problem and an exact solver to tackle the lower-level problem, which we have previously developed to address a similar case study. When the lower-level model is difficult, a significant relative MIP gap is unavoidable when solving the algorithm's subproblems. Novel reformulations of those subproblems using "elastic" variables are proposed trying to obtain meaningful lower/upper bounds within an acceptable computational time.

**Keywords:** Global optimization; Bi-level optimization; Mixed-integer linear programming model; Pricing problem; Dynamic tariffs; Electricity retail market; Demand response.

## 1. Introduction

This work presents a deterministic bounding procedure for the global optimization of a certain class of mixed-integer bi-level programming problems. The algorithm generates a series of convergent upper bounds and lower bounds for the upper-level objective function until the difference between these bounds is below a given threshold.

This algorithmic approach has been motivated by a case study dealing with the setting of time-of-use (ToU) prices in the electricity retail market. In these pricing schemes, the retailer establishes time-differentiated prices, usually with higher rates during peak demand periods. This type of tariff structure is already offered in electricity retail markets with the price structure valid for long periods (e.g., one-year contract), but it is expected to become more dynamic (e.g., announced 24 or 48-hour ahead) with the evolution to smart grids, enabling to make a better use of supply availability, network assets and demand flexibility.

Bi-level (BL) programming models have been recently used to design ToU tariffs in the electricity retail market modelling the interaction between the electricity retailer and residential consumers (Meng and Zeng, 2013; Zugno et al., 2013; Alves et al., 2016; Sekizaki et al., 2016; Soares et al., 2020). A trilevel energy market model for load shifting induced by ToU pricing has been reported by Aussel et al. (2020) involving suppliers, local agents, aggregators and consumers.

A BL optimization model has a leader-follower hierarchical structure, in which the upper-level (UL) problem is associated with the leader's interests and the lower-level (LL) problem arises as a constraint of the UL problem addressing the follower's perspective. Therefore, the leader should integrate the reaction of the follower in his decision process. In the context of defining ToU electricity tariffs, the leader is the retailer, who defines the prices aiming to maximize the profit, and the follower is the consumer (or a cluster of consumers with similar consumption patterns), who reacts to the time-differentiated prices set by the retailer by redefining the operation of appliances taking into account his comfort requirements to minimize costs.

BL problems are, in general, very difficult to solve to optimality. To circumvent the inherent complexities of these problems, it is crucial to develop methodological approaches that consider the structure and features of the BL model. In the BL approaches previously developed by the authors to deal with the problem of optimizing electricity tariffs (Alves et al., 2016; Soares et al., 2020), metaheuristics (particle swarm optimization – PSO – and genetic algorithms – GA) have been used to perform the UL search for prices, calling a mixed-integer linear programming (MILP) solver to address the LL problem for each price setting. In (Alves et al. 2016), only shiftable appliances (loads for which the operation cycle cannot be interrupted once initiated) were considered in the consumer's energy management problem. The BL problem was extended in (Soares et al., 2020) to also incorporate in the LL problem interruptible appliances (loads whose operation can be interrupted as long as a given amount of energy is supplied during a specified time slot) and a thermostatic load (air conditioning system). However, the modeling of the thermostat behavior imposes a much higher computational effort, which may turn it difficult to find exact optimal LL solutions within an acceptable computational time.

The BL models in (Alves et al., 2016; Soares et al., 2020) incorporate the physical-based features related with the control and operation of the appliances, which are essential to develop a realistic representation of the consumer's problem. However, the physical modelling of the appliance operation cycles (e.g. a dishwasher or a laundry machine) or the thermostat hysteresis in the temperature control of an air conditioning system gives rise to a highly combinatorial model with many binary variables and constraints.

The models we have developed are mixed-integer nonlinear BL (MINLBL) models. The nonlinearities result from the product of UL variables by LL variables, which means that all functions in the LL problem become linear when the UL variables are instantiated and, therefore, the LL can be solved by a general MILP solver. Hybrid approaches integrating a metaheuristic to search for UL solutions and an exact solver to solve the corresponding LL problems are interesting because they enable to obtain optimal solutions to the LL for a given (feasible) instantiation of the UL variables. However, they may present difficulties to obtain the global optimal solution due to the vastness of the UL search space.

The current work aims at developing a novel approach that can find exact global solutions within a given optimality tolerance, displaying a computational efficiency higher than the hybrid approaches. This new approach is based on distinct methodological principles generally called *optimal-value-function* reformulations. Since our MINLBL model can be equivalently written as a mixed-integer linear BL (MILBL) problem, both MINLBL and MILBL approaches may be of interest for the problem of designing ToU tariffs and, in general, for other BL pricing problems that involve continuous and integer variables.

BL problems with discrete variables pose major algorithmic challenges in the development of efficient solution strategies. Concerning mixed-integer BL problems, a few exact procedures have been developed that allow for integer variables controlled by the follower (LL variables). Algorithms for nonlinear problems have received even less attention in literature, for which almost no extensive study has been performed so far.

DeNegre and Ralphs (2009) built up on the ideas of the first branch-and-bound method developed by Moore and Bard (1990) for MILBL and developed a branch-and-cut algorithm for the linear BL case in which all variables are integer and the UL constraints do not include LL variables. Xu and Wang (2014) presented a branch-and-bound algorithm for MILBL problems where all UL variables are integer, as well as the values of their functions in the LL constraints. The algorithm branches on these functions, generating multiple branches at each node. Another branch-and-cut algorithm for MILBL problems has been proposed by Caramia and Mari (2015), which works with integer UL and LL variables. Fischetti et al. (2017) also proposed a branch-and-cut for MILBL problems, which extends the valid intersection cuts for MILBL proposed in (Fischetti et al., 2016). In (Fischetti et al., 2017), new BL-specific preprocessing procedures and a general branch-and-cut exact method are developed, whose finite convergence relies on the assumption that continuous UL variables do not appear in the LL problem.

A different approach has been proposed by Zeng and An (2014) for MILBL problems, allowing continuous and integer variables in both levels. The computation scheme is based on single-level reformulations and decomposition strategies (there is a master problem and subproblems). The reformulation involves the use of Karush-Kuhn-Tucker conditions, being the master problem a MILP problem with complementarity constraints.

Concerning methods for MINLBL problems, Gümüş and Floudas (2005) proposed a deterministic global optimization method in which the LL program involves functions that are convex with respect to the continuous LL variables and multilinear with respect to the integer LL variables. Kleniati and Adjiman (2015) presented a generalization for MINLBL of the branch-and-sandwich algorithm previously proposed by the same authors for BL problems with continuous variables. The algorithm is demanding from the implementation standpoint as it requires two branch-and-bound trees.

The works of Mitsos (2010) and Lozano and Smith (2017) are representative of *optimal-value-function* reformulation approaches. Mitsos (2010) proposed a deterministic algorithm for general MINLBL problems to obtain a sequence of increasingly tighter upper and lower bounds. The algorithm finishes when the difference between these bounds is below a given tolerance ε, ensuring ε-convergence of the algorithm. Lozano and Smith (2017) proposed an exact finite algorithm based on a sample scheme for the follower's solutions. Both methods solve a relaxation of the BL problem with disjunctive constraints generated from optimal follower's responses (already known) to obtain upper bounds for the maximizing UL objective function. BL feasible solutions are used to obtain lower bounds. Lozano and Smith (2017) address the particular case in which all UL variables are integer and the functions of these variables in the LL constraints are also integer valued, profiting from this assumption to strengthen the formulation and prove convergence to an optimal global solution. Mitsos (2010) and Lozano and Smith (2017) do not prescribe tailored techniques for solving the subproblems that arise in their procedures, assuming that these subproblems are solved by appropriate algorithms available in the literature. Therefore, for the nonlinear cases, an implicit assumption is that the functions involved in the formulations of the UL and the LL problems satisfy the requirements to be solved by the mixed-integer nonlinear programming (MINLP) solvers.

The mixed-integer BL model we have to deal with has the following particular characteristics: LL constraints do not include UL variables and UL constraints do not include LL variables. These features enabled the development of an algorithm based on optimal-value-function reformulations, which is a modification of the algorithms proposed by Mitsos (2010) and Lozano and Smith (2017) adapted to this type of problems, displaying a very efficient computational performance. Moreover, the MINLBL problem we deal with has the additional particularity: the nonlinearities only arise in the objective functions and result from products of continuous variables by binary variables. This characteristic allows both (UL and LL) objective functions to be rewritten equivalently as linear functions, at the expense of additional variables and constraints. Although the size of the problem is significantly increased with this transformation, it allows the sub-problems solved within the algorithm to be tackled by a state-of-the-art MILP solver. Our algorithmic proposal has a special concern with the application in challenging real-word problems regarding convergence, implementation simplicity and computational efficiency. Being efficient and easy to implement, our algorithm can be very useful to other authors dealing with problems with the same features as ours.

The main contribution of this paper is the development of a deterministic bounding procedure (DBP) belonging to the family of optimal-value-function approaches for the global optimization of a BL dynamic pricing model in the electricity retail market (which can be applied to other problems with the characteristics mentioned above). The performance of this new approach is compared with hybrid approaches previously developed by the authors, regarding solution quality and computational effort. Moreover, further techniques are proposed for cases in which the subproblems to compute lower/upper bounds cannot be solved to optimality in a reasonable computational time, as arises frequently in practice.

The manuscript is organized as follows. Section 2 introduces the formulation and the main concepts of bilevel optimization. Section 3 presents the DBP foundations and its algorithm. Section 4 presents a BL model of the interaction between an electricity retailer and a cluster of residential consumers, similar to the one in (Alves et al., 2016), reformulated as a linear mixed-integer BL problem. The results of the DBP applied to this model are discussed in Section 5. Section 6 describes the reformulation as a MILBL problem of the extended BL model including a thermostatic load in the consumer's problem in (Soares et al., 2020). The improvements in the algorithm so that it can deal with this more challenging model are described in Section 7, whose results are presented in Section 8. The conclusions are drawn in section 9.

## 2. Bilevel optimization

A BL programming model comprises two optimization problems hierarchically related. These problems involve distinct decision makers, each one controlling a different set of decision variables. The UL decision maker is the leader, being the first to set the values of his decision variables to optimize his objective function anticipating the response of the follower. The LL decision maker is the follower, who reacts to the leader's decision by optimizing his objective function in the feasible region constrained by the instantiation of the UL variables. A general formulation of BL problems can be stated as follows. Without loss of generality we consider that the UL objective is to be maximized and the LL objective function is to be minimized, as this is the direction of optimization in our case study:

$$
\begin{aligned}
&\text{"}\max_{x\in X}\text{"}\ F(x,y)\\
&\text{s.t. } G(x,y)\le 0\\
&\qquad y\in \arg\min_{y'\in Y}\{f(x,y'): g(x,y')\le 0\}
\end{aligned}
$$

where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are compact sets with *n* being the number of UL decision variables controlled by the leader and *m* the number of LL decision variables controlled by the follower. In a mixed-integer BL problem, $X$ and $Y$ may also include integrality constraints for all or some of the $x$ and $y$ variables, respectively. $F(x, y)$ and $f(x, y)$ are the leader's and the follower's objective functions, respectively; $G(x, y)$ and $g(x, y)$ are the corresponding constraint functions.

In face of the decision variables $x \in X$ set by the leader, the *feasible* and the *rational reaction* sets of the follower are $Y(x) = \{y \in Y: g(x, y) \leq 0\}$ and $\Psi(x) = \left\{y \in Y: y \in \arg\min_{y' \in Y(x)} f(x, y')\right\}$, respectively.

The constraint set of the BL problem is $S = \{(x, y) \in X \times Y: G(x, y) \leq 0, g(x, y) \leq 0\}$. The feasible set of the BL problem, which is generally called *induced* (or *inducible*) *region*, is $IR = \{(x, y): (x, y) \in S, y \in \Psi(x)\}$.

Finding a global optimal solution to a BL optimization problem remains a great challenge due to its inherent non-convexity, since even the linear BL problem is NP-hard.

Even in the linear case with all variables being continuous and assuming non-emptiness and compactness of $S$, BL problems with UL constraints involving both UL and the LL variables (generally called connecting constraints) may have no optimal solution (Mersha and Dempe, 2006). In addition, if there are no connecting constraints but the BL (linear) problem has UL continuous variables and LL discrete variables, it also may have no optimal solution because the induced region may be not closed (thus being noncompact) (Vicente et al., 1996).

The quotation marks in the UL objective function of the general formulation of the BL problem express the ambiguity of the value $F(x, y)$ from the leader's point-of-view (who has control over $x$ only) in case of multiple optimal solutions to the LL problem. To overcome this ambiguity, the *optimistic* or the *pessimist* approach can be adopted. The optimistic approach assumes that the follower selects a solution $y \in \Psi(x)$ that maximizes $F$ among all alternative optimal solutions to his problem, thus benefiting the leader. The pessimistic approach assumes that the follower selects the worst solution for the leader among the LL optimal solutions. The optimistic approach has been the most often used, and we also consider it in this work. The optimistic approach of the BL problem is equivalent to maximize the UL objective function with respect to both $x$ and $y$, i.e., $\max_{x \in X, y \in Y} F(x, y)$.

## 3. A deterministic bounding procedure for the global optimization of a class of mixed-integer BL problems

In this section, a deterministic bounding procedure (DBP) is developed, which uses optimal-value-function reformulations to obtain increasingly tighter bounds following a structure of steps similar to (Mitsos, 2010). The algorithm we propose is devoted to mixed-integer BL problems with the following characteristics: neither UL constraints include LL variables, nor LL constraints include UL decision variables. These characteristics are present in our models to optimize ToU electricity tariffs as well as in other pricing problems. The exploitation of these features has a strong impact on the computational effort required.

### 3.1. Foundations

The optimistic formulation for the class of BL models to be addressed is

$$\begin{aligned} &\max_{x\in X, y\in Y} F(x,y) \\ &\text{s.t. } G(x)\le 0 \\ &\quad y\in \arg\min_{y'\in Y}\{f(x,y'): g(y')\le 0\} \end{aligned} \tag{P1}$$

The vectors of decision variables $x\in X$ and/or $y\in Y$ may include continuous and integer variables. Problem (P2), in which the UL objective function is optimized over the constraint set ($S$) of the BL problem, is designated as the *high point relaxation* (HPR) of the BL problem (P1):

$$\begin{aligned} &\max_{x\in X, y\in Y} F(x,y) \\ &\text{s.t. } G(x)\le 0 \\ &\quad g(y)\le 0 \end{aligned} \tag{P2}$$

In our models (which will be detailed in Sections 4 and 6), the HPR was originally a MINLP problem because $F(x,y)$ was a nonlinear function. However, since the nonlinearity resulted from products of continuous variables *x* by binary variables *y*, the objective function could be linearized using additional variables and constraints, which allows the HPR to be tackled by a general MILP solver.

The optimal solution to the HPR gives an upper bound ($UB_F$) for the optimal value of $F$ in (P1).

Let $x^u$ be a leader's solution such that $x^u\in X$, $G(x^u)\le 0$. The LL problem (P3) gives the follower's reaction set for $x^u$:

$$\begin{aligned} &\min_{y\in Y} f(x^u,y) \\ &\text{s.t. } g(y)\le 0 \end{aligned} \tag{P3}$$

If $y'$ is the unique optimal solution to (P3) for $x^u$ then $F(x^u,y')$ is a lower bound ($LB_F$) for the optimal value of $F$ in (P1). Since the UL constraints do not include LL decision variables *y*, then $(x^u,y')$ is surely a feasible solution to the BL problem (P1).

If the LL problem (P3) has alternative optimal solutions for a given $x^u$, then our algorithm searches for the alternative $y'$ that provides the best value for $F$, following an optimistic approach.

*Lemma 1*: Let $f^{u*}$ be the optimal value of $f$ in (P3) for a given $x^u\in X$, $G(x^u)\le 0$. If (P3) has alternative optimal solutions, then problem (P4) gives a $LB_F$ for the optimistic optimal $F^*$ to (P1):

$$\begin{aligned} &\max_{y\in Y} F(x^u,y) \\ &\text{s.t. } g(y)\le 0 \\ &\quad f(x^u,y)\le f^{u*} \end{aligned} \tag{P4}$$

*Proof*: Let $(x^u, y')$ be an optimal solution to (P4). Since $f(x^u, y') \leq f^{u*}$, then $(x^u, y')$ is also optimal to (P3) and provides the best $F$ among the alternative optimal solutions of (P3) due to the objective function of (P4). Thus, it provides a lower bound for $F$ in the optimistic optimal solution to (P1). •

In order to avoid numerical difficulties arising from the fact that $f^{u*}$ are floating point numbers, a small positive tolerance $\varepsilon'$ is allowed on $f^{u*}$ (i.e., $f^{u*} + \varepsilon'$) in problem (P4) solved in Step 3 of the algorithm presented below, thus ensuring feasibility of (P4). Consequently, an $\varepsilon'$-optimal solution at the LL is accepted (as in Mitsos et al. (2008); Mitsos (2010)).

Let $x^{u1}, x^{u2}, \dots, x^{uK}$ be a series of leader's solutions, $x^{uk} \in X$, $G(x^{uk}) \leq 0, k = 1, \dots, K$, and $y^1, y^2, \dots, y^K$ the corresponding follower's solutions obtained by solving (P3)+(P4) for each $x^{uk}$, $k = 1, \dots, K$.

Consider the *HPR problem with additional constraints* (P5):

$$\max_{x \in X, y \in Y} F(x, y) \tag{P5}$$

$$\text{s.t.} \quad G(x) \leq 0$$

$$g(y) \leq 0$$

$$f(x, y) \leq f\left(x, y^k\right), \quad k = 1, \dots, K$$

*Lemma 2*: An optimal solution to (P5) gives an upper bound ($UB_F$) for the optimal value of $F$ in (P1).
*Proof*: Solutions $y^1, y^2, \dots, y^K, k = 1, \dots, K$ are feasible to the LL problem of (P1) for any $x$ because the LL constraints do not depend on $x$. Therefore, for any $x'$, the value $f'^* = \min_{y \in Y} \{f(x', y) : g(y) \leq 0\}$ satisfies $f'^* \leq f\left(x', y^k\right), k = 1, \dots, K$, which means that the constraints $f(x, y) \leq f\left(x, y^k\right)$, $k = 1, \dots, K$ do not eliminate any optimal solution to the LL problem, i.e., no feasible solution of (P1) is eliminated by (P5). Thus, (P5) is a relaxation of (P1) which ensures that an $UB_F$ to (P1) is obtained. •

After obtaining a first pair of bounds for the optimal value of $F$, $[LB_F, UB_F]$, the HPR including the constraint $f(x, y) \leq f(x, y^1)$ for the first LL solution $y^1$ is solved (i.e., P5 is solved for $k = 1$) leading to a new UL solution $x^{u2}$. Lemma 3 below proves that an optimal solution to the BL problem (P1) is found if the same $x^u$ is obtained. Otherwise, the LL problem is solved for $x^{u2}$, yielding a new LL optimistic solution $y^2$ (obtained by solving P4). This solution defines another constraint valid for all $(x, y)$, $f(x, y) \leq f(x, y^2)$, which is included in (P5), and the process continues.

*Lemma 3*: Let $x^{uK}$ be the UL solution obtained by solving (P5) with $K - 1$ additional constraints ($k = 1, \dots, K - 1$) and let $y^K$ be the solution obtained by (P4) for $x^{uK}$. The constraint $f(x, y) \leq f(x, y^K)$ is added to (P5). Let $x^{u(K+1)}$ be the UL solution obtained by (P5) with $K$ additional constraints ($k = 1,\ldots,K$). If $x^{u(K+1)} = x^{uK}$ then an optimal optimistic solution to (P1) has been found.

*Proof*: By Lemma 1, $LB_F = F(x^{uK}, y^K)$ is a lower bound for the optimal optimistic $F$ of (P1). Let $(x^{u(K+1)}, y^{u(K+1)})$ be the solution obtained by solving (P5) with $k = 1,\ldots,K$. Then, $UB_F = F(x^{u(K+1)}, y^{u(K+1)})$ by Lemma 2. Since (P5) includes $f(x,y) \le f(x,y^K)$ and $x^{u(K+1)} = x^{uK}$, then $f\left(x^{uK}, y^{u(K+1)}\right) \le f(x^{uK}, y^K)$, which means that $y^{u(K+1)}$ also optimizes (P3) and (P4) for $x^{uK}$. So, $LB_F = F(x^{uK}, y^K) = F(x^{uK}, y^{u(K+1)}) = UB_F$; $(x^{uK}, y^K)$ and $(x^{uK}, y^{u(K+1)})$, which may be equal or different, are optimal solutions to (P1). •

In order to improve $UB_F$ and narrow the difference between $UB_F$ and $LB_F$, the HPR is solved iteratively by including in the problem (P5) all generated constraints $f(x,y) \le f\left(x,y^k\right), \forall\, k$ (where *k* denotes the iteration index). The $LB_F$ is updated whenever $F\left(x^{uk}, y^k\right) \ge LB_F$. In the algorithm presented below, the process stops when $UB_F - LB_F \le \varepsilon$, with $\varepsilon$ being a predefined optimality tolerance, as suggested by Mitsos (2010). If the algorithm stops with $x^{u(K+1)} \ne x^{uK}$, then the final solution is the one that provided the $LB_F$, since the solution that gives the $UB_F$ may be infeasible to (P1).

In addition, a very small ε' tolerance is added to the right-hand side of the additional constraints of (P5) to overcome numerical difficulties arising from floating point numbers, i.e., $f(x,y) \le f\left(x,y^k\right) + \varepsilon', \forall\, k$.

### 3.2. The DBP Algorithm

<u>*Parameters*</u>:

- Optimality tolerance: $\varepsilon$
- Tolerance for constraints on the LL objective function values: $\varepsilon'$

*Step 1 - Initialization* – Set initial lower and upper bounds for the UL objective function

- Lower bound of the UL objective function: $LB_F = -\infty$
- Solve the *HPR* (problem P2). Let $(x^{u0}, y^{u0})$ be the solution obtained.
  Upper bound of the UL objective function: $UB_F = F(x^{u0}, y^{u0})$
- Iteration counter: $k = 0$.

*Step 2 - Solve the LL problem for $x^{uk}$*

Solve (P3) for the leader's solution $x^{uk}$ to obtain an optimal follower's reaction.

Let $y'$ be the optimal solution to this problem. The minimum objective function value is $f^{u*} = f\left(x^{uk}, y'\right)$.

*Step 3 - Calculate the lower bound to the UL objective function corresponding to $x^{uk}$* – The following problem (P4- $\varepsilon$), corresponding to the problem (P4) above but allowing an $\varepsilon'$-tolerance for the optimal $f^{u*}$, is solved for $x^{uk}$:

$$\max_{y\in Y} F(x^{uk}, y) \qquad \text{(P4-}\varepsilon\text{)}$$
$$\text{s.t.}\quad g(y) \le 0$$
$$f(x^{uk}, y) \le f^{u*} + \varepsilon'$$

Let $(x^{uk}, y^k)$ be the solution obtained.

The lower bound for the UL objective function corresponding to $x^{uk}$ is $LB = F(x^{uk}, y^k)$.

*Step 4 - Lower bounding evaluation*

If $LB \ge LB_F$ then

- Update $LB_F = LB$
- Set the incumbent solution: $(x^*, y^*) = (x^{uk}, y^k)$, $F^* = LB_F$

*Step 5 – Update the upper bound to the UL objective function*

Update the *HPR problem with additional constraints* by including the *k-th* follower's constraint corresponding to solution $y^k$. This is problem (P5-$\varepsilon$), which corresponds to problem (P5) above, but allowing an $\varepsilon'$-tolerance for the follower's objective function value:

$$\max_{x\in X, y\in Y} F(x, y) \qquad \text{(P5-}\varepsilon\text{)}$$
$$\text{s.t.}\quad G(x) \le 0$$
$$g(y) \le 0$$
$$f(x, y) \le f(x, y^k) + \varepsilon', \quad \forall k$$

Solve (P5-$\varepsilon$). Let $(x^{u(k+1)}, y^{u(k+1)})$ be the optimal solution obtained.

$UB_F = F(x^{u(k+1)}, y^{u(k+1)})$.

*Step 6 – Stopping condition* – If the amplitude defined by the current upper-lower bounds is higher than $\varepsilon$ then the search should proceed; otherwise an $\varepsilon$–optimal solution has been obtained.

If $UB_F - LB_F > \varepsilon$ then

- $k = k + 1$
- Go to *Step 2*

Else If $x^{u(k+1)} = x^{uk}$ then the algorithm terminates with the optimal solution(s) $(x^{uk}, y^k)$ and $(x^{uk}, y^{u(k+1)})$

Else

Terminate with the $\varepsilon$–optimal solution $(x^*, y^*)$, $F^* = LB_F$ .

In our models, $f(x, y)$ was originally nonlinear for the same reason as $F(x, y)$. So, it has been also linearized to be included in the *HPR with additional constraints* (problem P5-$\varepsilon$ – Step 5 of the algorithm). Therefore, all problems solved are MILP problems.

The algorithm implementation coded in R is available at https://www.uc.pt/en/feuc/mjalves/DBP.

### 3.3. A discussion on convergence

Mitsos (2010) defines several assumptions required for the convergence of his algorithm devoted to general MINLBL problems. Under those assumptions, conditions for the $\varepsilon$ -tolerances are defined that ensure the algorithm terminates finitely. Lozano and Smith (2017) assume that all leader's variables are integer to guarantee the finite termination of their algorithm.

The DBP herein proposed for problem (P1) converges under the assumptions stated by Mitsos (2010) or Lozano and Smith (2017). In addition, the algorithm converges in a finite number of iterations for problems in which all follower's variables are integer (regardless of the leader's variables being continuous or integer) without further assumptions and considering $\varepsilon' = 0$, as we show below. The models we will deal with in the next sections fit into this case.

The algorithm converges when $UB_F$ decreases or even when it keeps the same value from one iteration to the next. We show the finiteness of the algorithm for the latter case, which also ensures convergence for the former case.

Let $(x^{u(K+1)}, y^{u(K+1)})$ be the solution obtained by solving (P5) at iteration $K$, so $UB_F = F(x^{u(K+1)}, y^{u(K+1)})$. Let us suppose that $UB_F$ did not decrease, i.e. $F(x^{uK}, y^{uK}) = F(x^{u(K+1)}, y^{u(K+1)})$, but $x^{uK} \neq x^{u(K+1)}$; so, the algorithm does not finish yet (according to Lemma 3). Due to the additional constraints $f(x,y) \leq f\left(x, y^k\right), k = 1, \dots, K$, then $f(x^{u(K+1)}, y^{u(K+1)}) \leq f\left(x^{u(K+1)}, y^k\right), k = 1, \dots, K$. Therefore, when (P3)+(P4) are solved w.r.t. to $x^{u(K+1)}$, the following situations may occur:

1) neither $y^{u(K+1)}$ nor $y^k, k = 1, \dots, K$, are yielded from (P3)+(P4) and a different solution $y^{K+1}$ is obtained;
2) $y^{u(K+1)}$ is obtained from (P3)+(P4);
3) $y^{k'}$ for some $k' \in \{1, \dots, K\}$ is obtained from (P3)+(P4), thus meaning that $f(x^{u(K+1)}, y^{u(K+1)}) = f\left(x^{u(K+1)}, y^{k'}\right)$ due to the additional constraints in (P5). So, $y^{u(K+1)}$ and $y^{k'}$ are alternative optimal solutions to the LL problem for $x^{u(K+1)}$. They also provide the same objective function value for the leader, i.e. $F(x^{u(K+1)}, y^{u(K+1)}) = F\left(x^{u(K+1)}, y^{k'}\right)$, otherwise $y^{k'}$ would have been the solution yielded by (P5).

In situations 2) and 3), the algorithm finishes at this iteration because $UB_F = LB_F$. In situation 1), if $UB_F - LB_F > \varepsilon$ then the new solution $y^{K+1}$ is included in the constraints of (P5) for the next iteration. Since all LL variables are integer, the set of all feasible $y$ is discrete and finite, which ensures the finiteness of the algorithm.

In our computational implementation we have set $\varepsilon'$ positive very small just to ensure feasibility of the subproblems solved.

## 4. A BL optimization model for electricity retail pricing problems considering Shiftable and Interruptible consumer appliances – Model SI

In this section, a BL problem for the interaction between an electricity retailer and a cluster of residential consumers with similar consumption patterns is presented. The retailer first sets the time-differentiated electricity prices aiming to maximize profit (revenue minus acquisition cost); the consumer responds to the prices communicated by the retailer by scheduling the appliance operation to minimize the electricity bill according to his preferences and comfort requirements. The consumer's problem considers two different types of controllable appliances: shiftable appliances, whose operation cycle cannot be interrupted, and interruptible loads, whose operation cycle can be interrupted provided that the required amount of energy is supplied in a predefined time slot (depending on the characteristics of the energy service, e.g. hot water). A base load not deemed for control is also considered, which has an impact on the load diagram and consequently on the consumer's decisions. In the following, this BL model – *Model SI* – is presented and briefly described. For a more detailed explanation, please see (Alves et al., 2016; Soares et al. 2020).

<u>*Notation*</u>

*Parameters*

$T$ = number of units of time the planning horizon $\overline{\overline{T}} = \{1, \dots, T\}$ is discretized into ($t \in \overline{\overline{T}}$). A typical planning horizon is one day.

$\hat{h}$ = length of the unit of time the planning horizon $\overline{\overline{T}}$ is discretized into (e.g., 15 or 5 minutes).

$I$ = number of periods of the planning horizon in which different prices apply ($i \in \{1, \dots, I\}$).

$P_i$ = periods of prices ($P_i \subset \overline{\overline{T}}$), disjoint and contiguous, which define the ToU tariff structure, $i = 1, \dots, I$.

$x^{AVG}$ = average price for the whole planning horizon $\overline{\overline{T}}$.

$\underline{x_i}$ / $\overline{x_i}$ = minimum / maximum energy price values in each period $P_i$, $i = 1, \dots, I$.

$\pi_t$ = energy acquisition prices incurred by the retailer at each unit of time $t$, $t = 1, \dots, T$.

$L$ = number of power levels defined by the retailer ($l \in \{1, \dots, L\}$), such that the consumer will pay for the peak power taken during the planning horizon.

$P_l^{Cont}$ = contracted power at level $l$ (KW), $l = 1, \dots, L$.

$e_l$ = price charged to the consumer if he takes power level $l$ (€), $l = 1, \dots, L$.

$b_t$ = power of non-controllable base load at time $t$ of the planning period (KW), corresponding to appliances that are not scheduled by the optimization model, $t = 1, \dots, T$.

For shiftable appliances:

$J$ = number of shiftable appliances ($j \in \{1, \dots, J\}$).

$d_j$ = duration of the operation cycle of shiftable appliance $j$, $j = 1, \dots, J$.

$f_{jr}$ = power requested by appliance $j$ at stage (unit of time) $r$ of its operation cycle (KW), $j = 1, \dots, J$, $r = 1, \dots, d_j$.

$\left[T_{1_j}, T_{2_j}\right]$ = comfort time slot within $\overline{\mathrm{T}}$ for appliance $j$ in which it is allowed to operate, $j = 1, \dots, J$.

For interruptible appliances:

$K$ = number of interruptible appliances ($k \in \{1, \dots, K\}$).

$Q_k$ = power requested by appliance $k$ at each unit of time (KW), $k = 1, \dots, K$

$E_k$ = total energy required by appliance $k$ (KW$\hat{h}$), $k = 1, \dots, K$.

$\left[T_{1_k}, T_{2_k}\right]$ = comfort time slot within $\overline{\mathrm{T}}$ for appliance $k$ in which it is allowed to operate, $k = 1, \dots, K$.

*Upper level decision variables*

$x_i$= energy price (€/kW$\hat{h}$) in period $P_i$, $i = 1, \dots, I$.

*Lower level decision variables*

$$w_{jrt} = \begin{cases} 1 & \text{if shiftable appliance } j \text{ is "on" at time } t \text{ and at stage } r \text{ of its operation cycle} \\ 0 & \text{otherwise} \end{cases}$$

$j = 1, \dots, J,\ r=1, \dots, d_j, t = T_{1_j}, \dots, T_{2_j}$

$$v_{kt} = \begin{cases} 1 & \text{if interruptible appliance } k \text{ is "on" at time } t \\ 0 & \text{otherwise} \end{cases} \qquad k = 1, \dots, K,\ t = T_{1_k}, \dots, T_{2_k}$$

$$u_l = \begin{cases} 1 & \text{if the consumer is charged at the power level } l \\ 0 & \text{otherwise} \end{cases} \qquad l = 1, \dots, L$$

*Lower level auxiliary variables*

$p_{jt}= \sum_{r=1}^{d_j} f_{jr} w_{jrt}$ : power required by shiftable load $j$ at time $t$, $j = 1, \dots, J$, $t = T_{1_j}, \dots, T_{2_j}$

(constraints ensure that $\sum_{r=1}^{d_j} w_{jrt} \leq 1$ so that at most one $w_{jrt} = 1$ for each pair $(j, t)$)

$q_{kt}= Q_k v_{kt}$: power required by interruptible load $k$ at time $t$, $k = 1, \dots, K$, $t = T_{1_k}, \dots, T_{2_k}$

The UL objective function is the maximization of the retailer's profit, whose components are the sale of energy to consumers (to supply the non-controllable load, the shiftable appliances and the interruptible appliances) plus the contracted power component revenue minus the cost of buying energy in the wholesale market:

$$\max_x F = \overbrace{\sum_{i=1}^{I} \sum_{t \in P_i} x_i \left( b_t + \sum_{j=1}^{J} p_{jt} + \sum_{k=1}^{K} q_{kt} \right)}^{SE=\text{sale of energy to consumers}} + \overbrace{\sum_{l=1}^{L} e_l u_l}^{CP=\text{contracted power}} \overbrace{- \sum_{t=1}^{T} \pi_t \left( b_t + \sum_{j=1}^{J} p_{jt} + \sum_{k=1}^{K} q_{kt} \right)}^{\text{cost of buying energy}}$$

The LL objective function relates to the minimization of the consumer's electricity bill and matches with the corresponding revenue terms in the UL objective function, i.e. *SE*+*CP*, the cost of the energy consumed by controllable and non-controllable (base) loads and the contracted power for the whole planning horizon.

The UL and LL objective functions have been linearized to tackle the products of the UL variables and the LL variables ($x_i p_{jt}$ and $x_i q_{kt}$). Since $x_i p_{jt} = x_i \sum_{r=1}^{d_j} f_{jr} w_{jrt}$ and $x_i q_{kt} = x_i\, Q_k v_{kt}$, then these nonlinearities arise from the product of continuous variables ($x$) by binary variables ($w$ or $v$). Introducing additional variables $y_{jrt}^1$ for the shiftable loads, the products $x_i w_{jrt}$ are equivalent to $y_{jrt}^1$ by imposing the additional constraints: $y_{jrt}^1 \leq \overline{x_i} w_{jrt}$, $y_{jrt}^1 \leq x_i$, $y_{jrt}^1 \geq x_i - (1 - w_{jrt})\overline{x_i}$. The transformation of the products $x_i v_{kt}$ for the interruptible loads is analogous by considering the additional variables $y_{kt}^2$. This linearization is the exact form of the McCormick (1976) relaxations (lower/upper envelopes) when one of the variables involved in the product is a binary variable.

The BL model with the linearized objective functions is the following:

*Linearized Model SI*

$$\max_x F = \sum_{i=1}^{I} \sum_{t \in P_i} x_i b_t + \sum_{t=1}^{T} \left( \sum_{j=1}^{J} \sum_{r=1}^{d_j} y_{jrt}^1 f_{jr} + \sum_{k=1}^{K} y_{kt}^2 Q_k \right) + \sum_{l=1}^{L} e_l u_l - \sum_{t=1}^{T} \pi_t \left( b_t + \sum_{j=1}^{J} p_{jt} + \sum_{k=1}^{K} q_{kt} \right) \quad (1)$$

s.t.

$$x_i \leq \overline{x_i}, \quad i = 1, \dots, I \quad (2)$$

$$x_i \geq \underline{x_i}, \quad i = 1, \dots, I \quad (3)$$

$$\frac{1}{T} \sum_{i=1}^{I} \overline{P_i} x_i = x^{AVG} \quad (4)$$

$$\min_{w,v,u} f = \sum_{i=1}^{I} \sum_{t \in P_i} x_i b_t + \sum_{t=1}^{T} \left( \sum_{j=1}^{J} \sum_{r=1}^{d_j} y_{jrt}^1 f_{jr} + \sum_{k=1}^{K} y_{kt}^2 Q_k \right) + \sum_{l=1}^{L} e_l u_l \quad (5)$$

s.t.

(shiftable appliances)

$$p_{jt} = \sum_{r=1}^{d_j} f_{jr} w_{jrt}, \quad j = 1, \dots, J, \ t = T_{1_j}, \dots, T_{2_j} \quad (6)$$

$$p_{jt} = 0, \quad j = 1, \dots, J, \ t < T_{1_j} \vee t > T_{2_j} \quad (7)$$

$$\sum_{r=1}^{d_j} w_{jrt} \leq 1, \quad j = 1, \dots, J, \quad t = T_{1_j}, \dots, T_{2_j} \quad (8)$$

$$w_{j(r+1)(t+1)} \geq w_{jrt}, \quad j = 1, \dots, J, \quad r = 1, \dots, (d_j - 1), \quad t = T_{1_j}, \dots, (T_{2_j} - 1) \quad (9)$$

$$\sum_{t=T_{1_j}}^{T_{2_j}} w_{jrt} = 1, \quad j = 1, \dots, J, \quad r = 1, \dots, d_j \quad (10)$$

$$\sum_{t=T_{1_j}}^{T_{2_j} - d_j + 1} w_{j1t} \geq 1, \quad j = 1, \dots, J \quad (11)$$

$$w_{jrt} \in \{0,1\}, \quad j = 1, \dots, J, \ r = 1, \dots, d_j, \ t = T_{1_j}, \dots, T_{2_j} \quad (12)$$

$$\left. \begin{array}{r} y_{jrt}^1 \leq \overline{x_i} w_{jrt} \\ y_{jrt}^1 \leq x_i \\ y_{jrt}^1 \geq x_i - (1 - w_{jrt})\overline{x_i} \end{array} \right\} i = 1, \dots, I, j = 1, \dots, J, r = 1, \dots, d_j, t = T_{1_j}, \dots, T_{2_j} \wedge t \in P_i \quad (13)$$

$$y_{jrt}^1 = 0, \quad j = 1, \dots, J, r = 1, \dots, d_j, t < T_{1_j} \vee t > T_{2_j} \quad (14)$$

(interruptible appliances)

$$q_{kt} = v_{kt} Q_k, \quad k = 1, \dots, K, \quad t = T_{1_k}, \dots, T_{2_k} \quad (15)$$

$$q_{kt} = 0, \quad k = 1, \dots, K, \quad t < T_{1_k} \vee t > T_{2_k} \quad (16)$$

$$\sum_{t=T_{1_k}}^{T_{2_k}} q_{kt} = E_k, \quad k = 1, \dots, K \quad (17)$$

$$v_{kt} \in \{0,1\}, \quad k = 1, \dots, K, t = T_{1_k}, \dots, T_{2_k} \quad (18)$$

$$\left. \begin{array}{r} y_{kt}^2 \leq \overline{x_i} v_{kt} \\ y_{kt}^2 \leq x_i \\ y_{kt}^2 \geq x_i - (1 - v_{kt})\overline{x_i} \end{array} \right\} i = 1, \dots, I, \ k = 1, \dots, K, \ t = T_{1_k}, \dots, T_{2_k} \wedge t \in P_i \quad (19)$$

$$y_{kt}^2 = 0, \quad k = 1, \dots, K, t < T_{1_k} \vee t > T_{2_k} \quad (20)$$

(power component)

$$\sum_{l=1}^{L} u_l = 1 \quad (21)$$

$$b_t + \sum_{j=1}^{J} p_{jt} + \sum_{k=1}^{K} q_{kt} \leq \sum_{l=1}^{L} P_l^{Cont} u_l, t = 1, \dots, T \quad (22)$$

$$u_l \in \{0,1\}, \quad l = 1, \dots, L \quad (23)$$

In each period $P_i$, the electricity prices are limited by minimum and maximum values, respectively $\underline{x_i}$ and $\overline{x_i}$, (UL constraints (2) and (3)). An average price $x^{AVG}$ for the whole planning horizon $\overline{\mathrm{T}}$ is also imposed to account for retail market competition (UL constraint (4)).

The LL constraints (6) – (14) model the operation of shiftable loads and the set of LL constraints (15) – (20) model the operation of interruptible appliances. The consumer should specify comfort time slots ($\left[T_{1_j}, T_{2_j}\right]$ and $\left[T_{1_k}, T_{2_k}\right]$) in which each shiftable load *j* and interruptible load *k* should operate, according to his comfort preferences and routines. Outside these time slots, the power requested to the grid by these types of appliances is always zero (constraints (7) and (16) respectively), remaining the non-controllable load only, $b_t$ (e.g. fridge, oven, etc.).

Constraints (21)-(23) define the peak power level the consumer pays for. That is, the tariff structure comprises a time-variable energy component and a volume-based power component.

## 5. Results of the DBP applied to the Model SI

This section presents results of the DBP proposed in section 3 applied to the BL model in section 4 (Model SI). A case study with a planning horizon of 24 hours is considered. First, we present the results for a discretization in units of time of quarter-hour ($\hat{h}$ = 15 minutes).

Regarding the retailer's problem, six tariff periods $P_i = \left[P_{1_i}, P_{2_i}\right]$, $i \in \{1, \ldots, 6\}$, are established for defining the electricity prices to be charged to the consumers. The minimum and maximum prices that can be charged to the consumers in each period $P_i$ and the average price over the entire planning period are displayed in Table A-1 in the Appendix. These data define an UL problem with 6 continuous variables and 13 constraints (lower/upper bounds for the prices plus the average price constraint). The electricity prices paid by the retailer in the spot market in each $t \in \overline{\mathrm{T}}$ can be seen in Table A-2 in the Appendix.

In the consumer's problem, in addition to the non-controllable base load, five controllable appliances are considered: three shiftable loads (*J* = 3) – dishwasher, laundry machine and clothes dryer – and two interruptible loads (*K* = 2) – electric vehicle and electric water heater. The data related to the base load, the operation cycles of each controllable load and the comfort time slots allowed for its operation, as well as the power levels and corresponding costs are also detailed in the Appendix (Tables A-3 – A-6). For each instantiation of the UL variables *x*, these data define a LL problem with 559 binary variables, 691 continuous variables and 2389 constraints. The tolerance parameters required by the algorithm were set as: $\varepsilon = 10^{-4}$ and $\varepsilon' = 10^{-5}$.

In all the experiments, we considered electricity prices (€/kW$\hat{h}$) with five decimal places. Therefore, the UL decision variables can be seen as integer variables by considering a $10^5$ scale factor.

The algorithm was written in R language and run in a computer with an Intel Xeon Gold 6138 CPU@3.7GHz processor. The subproblems in DBP are swiftly solved to optimality using a state-of-the-art solver (CPLEX).

The results for the Model SI are displayed in Table 1 and Figure 1. Table 1 shows the upper and lower bounds for the retailer's profit (in €) obtained in each iteration of the DBP algorithm. The DBP produced swiftly tighter bounds such that at iteration 3 the lower and upper bounds match and an optimal solution is obtained (with $F = 3.16309$). The electricity prices (in €/kWh) for each tariff period $P_i,\ i \in \{1, \ldots, 6\}$ are displayed in Figure 1: $x = (0.09960, 0.27504, 0.28360, 0.08040, 0.15400, 0.14728)$. The prices defined for the first half of the day (periods $P_1$ to $P_3$) as well as the price defined for the last period of the day ($P_6$) are equal or close to the respective maximum values $(\overline{x_i})$. The remaining prices (for periods $P_4$ and $P_5$) are equal to the respective minimum values $(\underline{x_i})$.

**Table 1.** Upper and lower bounds for the retailer's profit (in €) in each iteration of the DBP algorithm for the Model SI, discretization 15 min.

| Iteration | Lower bound | Upper bound |
|---|---|---|
| 1 | 3.11711 | 5.57038 |
| 2 | 3.11711 | 3.19822 |
| 3 | 3.16309 | 3.16309 |

We now compare the results obtained using the DBP and the hybrid approach previously developed by the authors (Soares et al., 2020). This approach consists of the combination of a PSO algorithm to perform the UL search, which calls an external solver to deal with the LL MILP problem for each setting of $x$.

Table A-9 in the Appendix compares the prices in the best solutions obtained by both approaches. In the hybrid approach, the best solution (maximum $F$) obtained over 20 runs (each with a population of 30 UL solutions and 100 iterations) has $F = 3.13996$ €, i.e. 0.73% worse than the optimal solution given by the DBP ($F = 3.16309$). The average $F$ over the 20 runs is 3.13520, which is 0.88% worse than the DBP solution. Concerning the computational effort involved, an optimal solution was obtained by the DBP in just three iterations in less than 15 seconds, whereas the best solution given by the hybrid approach (involving 3000 calls of the MILP solver per run) took 12 minutes (per run).

Figure 1 displays the electricity prices together with the power requested to the grid during the planning period $\overline{\mathrm{T}}$ for the optimal solution in the DBP and the best solution in the hybrid approach. This figure shows that the electricity prices in both approaches are very similar as well as the higher prices (within the allowed limits) are associated with the periods in which the amount of electricity required by the consumer is higher, mainly due to the charging of the electric vehicle. These results indicate that a massive deployment of electric vehicles being charged at home during the night may induce significant changes in the shape of ToU pricing schemes currently offered by electricity retailers.

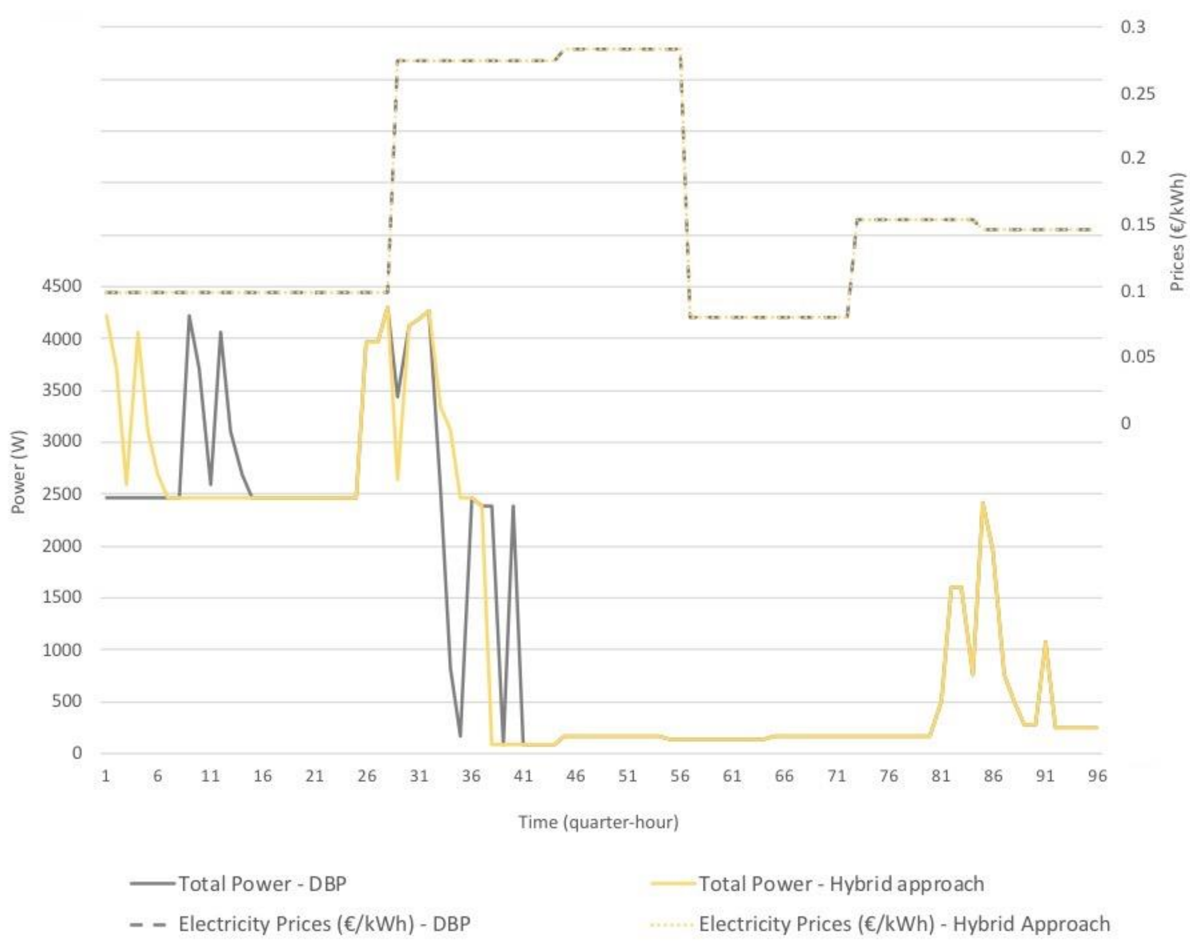


**Fig 1.** Power requested from the grid and prices imposed to the consumer in the optimal solution of DBP and in the best solution of the hybrid approach in the Model SI for the 15 min discretization.

We now present the results for a discretization in units of time of $\hat{h}$ = 5 minutes, i.e. $\overline{\mathrm{T}} = \{1, \dots, 288\}$. In this case, the follower's problem has 4617 binary variables, 5031 continuous variables and 18981 constraints. The algorithm behavior is similar to the discretization with $\hat{h}$ = 15 minutes, obtaining an optimal solution in 3 iterations with $F$ = 3.16265 but with a significant increase in the overall computational effort (around 10.5 minutes). The electricity prices (in €/kWh) are: $x = (0.0996, 0.27492, 0.28344, 0.08052, 0.15396, 0.14748)$, which are different from the solution for the discretization of $\hat{h}$ = 15 minutes just from the third decimal place onwards (*cf.* Table A-9).

For comparison purposes, five independent runs of the hybrid approach were performed, with a population of 30 UL solutions and 100 iterations. The best objective value was $F = 3.14410$ € in around 35 minutes (per run), which is 0.59% worse than the optimal solution given by the DBP. The average $F$ over the 5 runs is 3.12599, which is 1.16% worse than the DBP solution.

## 6. Including a thermostatic appliance in the consumer's problem – Model SIAC

The BL optimization model presented in section 4 (Model SI) is now extended to include a thermostatic load – an air conditioning (AC) system – in the follower's problem. This BL model – *Model SIAC* – is presented below with a brief explanation of the changes induced by the inclusion of the AC (a more complete description can be seen in (Soares et al., 2020)).

<u>*Additional notation*</u>

*Parameters*

$P_{AC}^{nom}$ = nominal power of the AC system (W)

$\theta^{max}$ = maximum allowed indoor temperature during the entire planning horizon (ºC)

$\theta^{ref}$ = reference comfort indoor temperature during the entire planning horizon (ºC)

$\theta_{Abs}^{min}$ = absolute minimum comfort indoor temperature during the entire planning horizon (ºC)

$c^{-}$= monetized cost of the discomfort (€/$\hat{h}$ per ºC below $\theta^{ref}$)

$c^{+}$= monetized cost of the discomfort (€/$\hat{h}$ per ºC above $\theta^{ref}$)

$\alpha, \beta, \gamma$ = coefficients associated with the thermal modeling of the space being conditioned

$\theta_t^{ext}$ = outdoor temperature at time *t* (ºC), $t = 1, \dots, T$

*Lower level decision variables*

$\theta_t^{min}$ = minimum indoor temperature at which the AC is set "on" at time *t* (ºC), $t = 1, \dots, T$

$s_t = \begin{cases} 1 & \text{if the AC is "on" at time } t \\ 0 & \text{otherwise} \end{cases} \quad t = 1, \dots, T$

*Lower level auxiliary variables*

$\theta_t^{in}$ = indoor temperature at time *t* (ºC), $t = 1, \dots, T$

$\Delta_t^{+}$ = indoor temperature deviation above $\theta^{ref}$ at time *t* (ºC), $t = 1, \dots, T$

$\Delta_t^{-}$ = indoor temperature deviation below $\theta^{ref}$ at time *t* (ºC), $t = 1, \dots, T$

$z_t, y_t$ = binary variables to simulate the thermostat hysteresis behavior, $t = 1, \dots, T$

In addition to the cost of the energy consumed by controllable and non-controllable loads as well as the power cost, the LL objective function also comprises the costs resulting from the monetization of the positive and negative deviations of the minimum indoor temperature $\theta_t^{min}$ at each time $t \in \overline{\mathrm{T}}$ (which is a variable controlling the thermostat operation) from the reference comfort temperature.

The consumer should specify minimum ($\theta_{Abs}^{min}$), maximum ($\theta^{max}$) and reference ($\theta^{ref}$) indoor temperatures used for setting the AC system. The LL constraints (26) – (38) below model the operation of this thermostatic load.

As in the Model SI, the UL and LL objective functions have been linearized (because of the products $\sum_{i=1}^{I} \sum_{t \in P_i} x_i (s_t P_{AC}^{nom})$ associated with the AC operation). The additional binary variables $y_t^3$ and constraints (39) are used for this purpose, which derive from products $x_i s_t$, with $i \in \{1, \dots, I\}$ and $t \in P_i$.

*Linearized Model SIAC*

$$\max_x F = \sum_{i=1}^{I} \sum_{t \in P_i} x_i b_t + \sum_{t=1}^{T} \left( \sum_{j=1}^{J} \sum_{r=1}^{d_j} y_{jrt}^1 f_{jr} + \sum_{k=1}^{K} y_{kt}^2 Q_k + y_t^3 P_{AC}^{nom} \right) + \sum_{l=1}^{L} e_l u_l - \sum_{t=1}^{T} \pi_t \left( b_t + \sum_{j=1}^{J} p_{jt} + \sum_{k=1}^{K} q_{kt} + s_t P_{AC}^{nom} \right) \quad (24)$$

s.t.

UL constraints (2) – (4)

$$\min_{w,v,u,s,\theta^{min}} f = \sum_{i=1}^{I} \sum_{t \in P_i} x_i b_t + \sum_{t=1}^{T} \left( \sum_{j=1}^{J} \sum_{r=1}^{d_j} y_{jrt}^1 f_{jr} + \sum_{k=1}^{K} y_{kt}^2 Q_k + y_t^3 P_{AC}^{nom} \right) + \sum_{l=1}^{L} e_l u_l + \sum_{t \in T} (c^+ \Delta_t^+ + c^- \Delta_t^-) \quad (25)$$

s.t.

LL constraints (6) – (14) for shiftable appliances

LL constraints (15) – (20) for interruptible appliances

(thermostatic appliance)

$$\theta_t^{in} = \alpha \theta_{t-1}^{in} + \beta \theta_{t-1}^{ext} + \gamma s_{t-1} P_{AC}^{nom}, \quad t = 1, \dots, T \quad (26)$$
$$\theta_t^{in} \geq \theta_t^{min} - M s_t, \quad t = 1, \dots, T \quad (27)$$
$$\theta_t^{in} \leq \theta_t^{min} + M z_t, \quad t = 1, \dots, T \quad (28)$$
$$\theta_t^{in} \geq \theta^{max} - M y_t, \quad t = 1, \dots, T \quad (29)$$
$$z_t + y_t - s_{t-1} + s_t \leq 2, \quad t = 1, \dots, T \quad (30)$$
$$z_t + y_t + s_{t-1} - s_t \leq 2, \quad t = 1, \dots, T \quad (31)$$
$$\theta_t^{in} \leq \theta^{max} + M(1 - s_t), \quad t = 1, \dots, T \quad (32)$$
$$\theta_t^{min} - \theta^{ref} = \Delta_t^+ - \Delta_t^-, \quad t = 1, \dots, T \quad (33)$$
$$\theta_t^{min} \geq \theta_{Abs}^{min}, \quad t = 1, \dots, T \quad (34)$$
$$\theta_t^{min} \leq \theta^{max}, \quad t = 1, \dots, T \quad (35)$$
$$s_t, z_t, y_t \in \{0,1\}, \quad t = 1, \dots, T \quad (36)$$
$$\theta_t^{min} \in \mathbb{Z}, \quad t = 1, \dots, T \quad (37)$$
$$\Delta_t^-, \Delta_t^+ \geq 0, \quad t = 1, \dots, T \quad (38)$$
$$\left. \begin{array}{r} y_t^3 \leq \overline{x_i} s_t \\ y_t^3 \leq x_i \\ y_t^3 \geq x_i - (1 - s_t)\overline{x_i} \end{array} \right\} i = 1, \dots, I, \;\; t \in P_i \quad (39)$$

(power component)

$$\sum_{l=1}^{L} u_l = 1 \quad (40)$$
$$b_t + \sum_{j=1}^{J} p_{jt} + \sum_{k=1}^{K} q_{kt} + s_t P_{AC}^{nom} \leq \sum_{l=1}^{L} P_l^{Cont} u_l, t = 1, \dots, T \quad (41)$$
$$u_l \in \{0,1\}, \quad l = 1, \dots, L \quad (42)$$

The modelling of the thermostatic load enables to capture its physical control characteristics but imposes a severe computational burden, which is associated with the introduction of a significant number of binary variables and constraints associated with the thermostat hysteresis behavior. Moreover, the linearization of the products $x_i s_t$ increases the number of continuous variables and constraints. This impairs obtaining the optimal LL solution in a practical computational time by a state-of-the-art solver (Soares et al., 2020).

## 7. Improvements in the DBP to deal with a more demanding problem

The subproblems required by the DBP revealed, in general, impossible to be solved to optimality in an acceptable timeframe for the Model SIAC presented in the previous section, having in mind the need of the retailer to announce prices short in advance, e.g., day-ahead. The solution process of most of these subproblems terminated after a specified computation budget (from five minutes until more than one hour) still with a significant MIP gap. Moreover, in some cases, the solver was not able to find a feasible integer solution to these subproblems. Therefore, new reformulations were introduced in the DBP subproblems with additional constraints. Also, the information given by the solver after a certain computation budget was considered to obtain meaningful lower/upper bounds.

In the following, the difficulties faced to solve the subproblems required by DBP are enumerated and the solution proposals adopted to circumvent these difficulties are presented. It should be noted that these difficulties have not been addressed in the literature and, in general, are not observed in benchmark problems. Therefore, it is of utmost importance to test the DBP approach in challenging conditions associated with real-world problems in which a limited timeframe for computation is generally at stake. The DBP herein proposed for this class of mixed-integer bi-level programming problems revealed a good computational performance in addressing these issues after including the proposals presented below.

*Difficulty 1:* The solver cannot find feasible integer solutions to the subproblems with additional constraints, i.e., problem (P4-$\varepsilon$) solved in Step 3 and problem (P5-$\varepsilon$) solved in Step 5 to calculate the lower bound and the upper bound to the UL objective function, respectively.

*Proposal:* “Elastic” variables are introduced, which are penalized in the objective functions of (P4-*D*) and (P5-*D*) using a big-*M*.

*1.1.* The subproblem aimed at calculating $(x^{uk}, y^k)$, which leads to a lower bound to the UL objective function (Problem P4-$\varepsilon$ solved in Step 3), is reformulated as (P4-*D*):

$$\begin{aligned} \max_{y\in Y,D}\ & F(x^{uk}, y) - M \times D \\ \text{s.t.}\ & g(y) \leq 0 \\ & f(x^{uk}, y) \leq f^{u*} + \varepsilon' + D \\ & D \geq 0 \end{aligned} \tag{P4-D}$$

If the optimal value of *D* in (P4-*D*) is different from zero, then the *F*-value obtained cannot be accepted as a lower bound because the *f*-value is not within the $\varepsilon'$-tolerance with respect to $f^{u*}$. In this case, the solution obtained in Step 2, $(x^{uk}, y')$, is kept to proceed to Step 4.

*1.2.* The *HPR with additional constraints* (Problem P5-$\varepsilon$ solved in Step 5) is reformulated as (P5-*D*):

$$\begin{aligned} \max_{x\in X, y\in Y, D_k}\ & F(x, y) - M \times \textstyle\sum_k D_k \\ \text{s.t.}\ & G(x) \leq 0 \\ & g(y) \leq 0 \\ & f(x, y) \leq f(x, y^k) + \varepsilon' + D_k, \ \ \forall k \\ & D_k \geq 0, \ \ \forall k \end{aligned} \tag{P5-D}$$

This reformulation overcomes the infeasibility issue. However, if any $D_k > 0$ then the upper bound obtained may be higher than the current $UB_F$ and turn difficult the convergence of the DBP.

*Difficulty 2:* The subproblems cannot be solved to optimality and then the resulting bounds are misleading.

*2.1.* The *HPR with additional constraints* (Problem P5-$\varepsilon$, or its reformulation P5-*D*, solved in Step 5) cannot be solved to optimality.

*Proposal:* The relative MIP gap ($gap_{HPR}$) given by the solver is used to estimate a guaranteed upper bound: $UB_F = F\left(x^{u(k+1)}, y^{u(k+1)}\right) \times (1 + gap_{HPR})$.

*2.2.* The LL problem (Problem P3 solved in Step 2) cannot be solved to optimality.

*Proposal:* In Step 2, if a solution with a relative MIP gap ($gap_{LL}$) is obtained, within the computation budget specified, the additional subproblem (P6) is considered to further improve the LL objective function value $f^{u*}$:

$$\begin{aligned} \min_{D,y}\ & D \\ \text{s.t.}\ & g(y) \leq 0 \\ & f\left(x^{uk}, y\right) \leq f^{u*}(1 - gap_{LL}) + D \\ & D \geq 0 \end{aligned} \tag{P6}$$

The solution obtained to (P6) is adopted to proceed with the algorithm in Step 3 if the corresponding *f*-value is better than $f^{u*}$.

These modifications enable the DBP to deal with challenging cases, obtaining optimal or nearly optimal solutions to the BL problem. However, due to the way the bounds are calculated, considering the elastic variables *D* and the gaps, convergence cannot be ensured. For this reason, in the implementation of the algorithm we have considered an additional stopping condition, which is defined by a maximum number of iterations without updating $UB_F$ and $LB_F$.

## 8. Results of the DBP applied to the Model SIAC

In this section, the results of the DBP with the improvements proposed in section 7 applied to the Model SIAC (in section 6) are presented for a planning horizon of 24 hours discretized in units of 15 minutes (*T*= 96 units of time) and 5 minutes (*T*= 288 units of time).

First, we present the results for the discretization in $\hat{h}$ = 15 minutes. The LL problem has 847 binary variables, 1171 continuous variables and 3627 constraints. The additional data of this BL model concerning the AC operation can be seen in Tables A-7 and A-8 in Appendix. The CPLEX models and the corresponding data are available at https://www.uc.pt/en/feuc/mjalves/DBP.

The application of the DBP, as presented in section 3, to the Model SIAC proved unable to proceed beyond the first iteration. At the first iteration, the $UB_F$ obtained in Step 1 was 7.21770 with a relative MIP gap of 4.2% for a running time limited to 5 min; the gap decreased to 3.08% after 1 hour. Solving the LL problem (Step 2) for the UL prices obtained with the *HPR* resulted into a solution with a gap of 2.41% after 5 min of computation; the gap decreased to 1.70% after 1 hour. Step 3 produced no solution and thus the UL objective function was evaluated for the solution obtained in Step 2. The problem (P5-$\varepsilon$) was then considered with an additional constraint from the previous iteration, but no solution was found within a computation budget of 5 min for the solver.

The reformulations (P4-*D*) and (P5-*D*) of the subproblems in Step 3 and Step 5, respectively, including the "elastic" variables, were thus implemented to address the *Difficulty 1* described in the previous section.

Considering a computation budget of 5 min for the solver, the reformulated subproblems became solvable (with a positive MIP gap) and the DBP found a solution after just five iterations. The $LB_F$ obtained was 4.25261 € with a consumer's cost of 7.20426 € and a MIP gap of 3%. However, the positive MIP gaps lead to misleading bounds (*Difficulty 2*). The value of $F(x^u, y^u)$ (the iteration index *k* is omitted for clarity) in a sub-optimal solution for the *HPR with additional constraints* is not a true upper bound. In our experiments the $UB_F$ taken as $F(x^u, y^u)$ oscillated over iterations and could even go below the $LB_F$. To estimate a guaranteed upper bound, the proposal in *Difficulty 2.1* was adopted, $UB_F = F(x^u, y^u) \times (1 + gap_{HPP})$, which is never lower than $LB_F$. In addition, the introduction of the constraint $F(x, y) \leq UB_F$, in both reformulated subproblems (P4-*D*) and (P5-*D*), enabled to improve the algorithm efficiency, which is in accordance with Mitsos (2010). The introduction of this constraint avoids any oscillatory behavior of the $UB_F$.

To address *Difficulty 2.2*, the additional subproblem (P6) was considered to further improve the $f^{u*}$ value obtained in (P3) with a positive gap ($gap_{LL}$).

The implementation of the proposals in section 7 allowed the DBP to quickly converge to good and reliable bounds for the UL objective function, thus being able to offer useful information to the leader/retailer helping him to make sound decisions. The respective results can be seen in Table 2.

The auxiliary stop criterion was implemented in parallel to the $\varepsilon$ threshold value to guarantee that the algorithm stops after an acceptable computational time: if both $UB_F$ and $LB_F$ remain unchanged over 10 iterations, then the DBP algorithm terminates with the incumbent solution. In this setting, the algorithm terminated after 12 iterations with the bounds $UB_F = 4.51516$ € and $LB_F = 4.31758$ €, which remained the same from iteration 3 onwards. In the final (incumbent) solution, $F = 4.31758$ € and the vector of electricity prices (in €/kWh) is $x = (0.09960, 0.27108, 0.27900, 0.08040, 0.15400, 0.15716)$. The corresponding consumer's cost is 7.21030 €, obtained with a relative MIP gap of 3.69%. Like in the Model SI (including only shiftable and interruptible loads), the highest electricity prices in the Model SIAC are also in the first half of the day, with a peak price in the period $P_3$.

We now compare the results obtained by the DBP with those obtained by the hybrid approach of Soares et al. (2020). The prices in the best solutions reached by both approaches are displayed in Table A-10 in the Appendix. The electricity prices along the whole planning horizon $\overline{\mathrm{T}}$ as well as the power requested to the grid in each $t \in \overline{\mathrm{T}}$ are displayed in Figure 2.

In the hybrid approach, the best solution has the retailer's profit in the interval $[3.86998, 4.24842]$ which compares with $[4.31758, 4.51516]$ given by the DBP. The lower bound (the worst possible retailer's profit) determined by the DBP is better than the upper bound (the best retailer's profit) found by the hybrid approach.

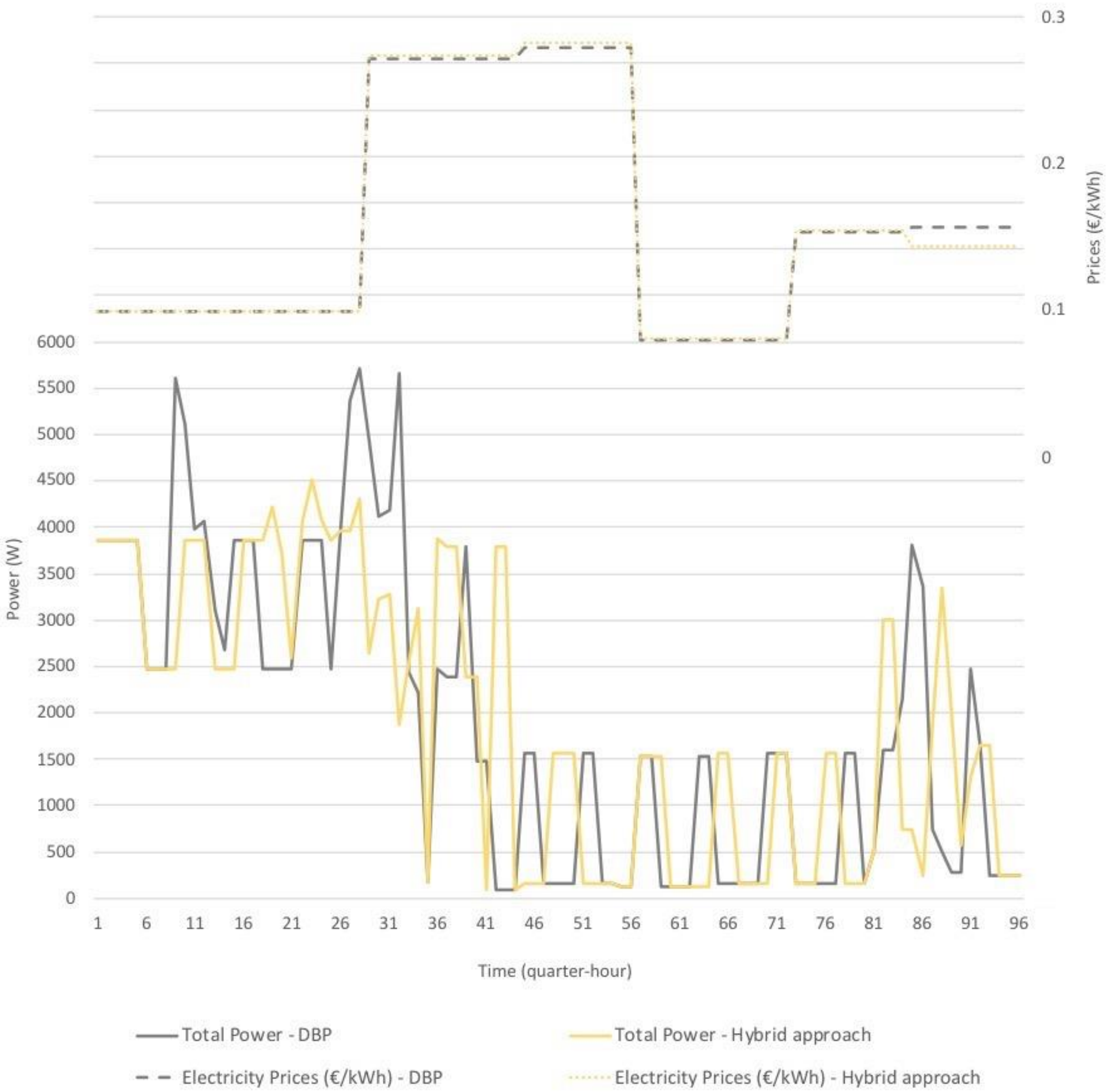


**Fig 2.** Power requested from the grid and prices imposed to the consumer in the optimal solution of DBP and in the best solution of the hybrid approach to Model SIAC for the 15 min discretization.

Table 2. Results obtained in each step of DBP in each iteration to solve the Model SIAC with a 15min discretization. The retailer's profit and the consumer's cost are in € and the relative MIP gap in % rounded to two decimal places.

| **Iteration** | **(P5-*D*)** [a] | | | | **(P3)** | | | **(P6)** | | | ***f**** | **(P4-*D*)** | | | ***LB*** | ***LB_F*** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | ***F*** | ***D*** | ***gap*** | ***UB_F*** | ***f*** | ***gap*** | ***F*** | ***D*** | ***f*** | ***F*** | | ***F*** | ***D*** | ***gap*** | | |
| 0 | 7.21770 | - | 4.20 | 7.21770 | 7.15169 | 2.38 | 4.18519 | 0.17036 | 7.15169 | 4.14934 | 7.15169 | 4.21171 | 0 | 4.78 | 4.21171 | 4.21171 |
| 1 | 4.35962 | [0] | 3.57 | 4.51516 | 7.21624 | 2.78 | 4.25774 | 0.18448 | 7.19977 | 4.22362 | 7.19977 | 4.25130 | 0.01201 | - | 4.22362 | 4.22362 |
| 2 | 4.34726 | [0,0] | 3.86 | 4.51516 | 7.23393 | 4.08 | 4.26943 | 0.27319 | 7.21192 | 4.24715 | 7.21192 | 4.31758 | 0 | 3.69 | 4.31758 | **4.31758** |
| 3 | 4.31638 | [0,0,0] | 4.61 | 4.51516 | 7.19544 | 2.88 | 4.22251 | 0.19694 | 7.18510 | 4.18392 | 7.18510 | 4.24865 | 0.00003 | - | 4.18392 | 4.31758 |
| 4 | 4.34424 | [0,0,0,0] | 3.93 | 4.51516 | 7.24744 | 3.56 | 4.29839 | 0.23166 | 7.22075 | 4.24620 | 7.22075 | 4.30570 | 0.00993 | - | 4.24620 | 4.31758 |
| 5 | 4.09413 | [0,…,0] | 10.28 | 4.51516 | 6.99685 | 4.07 | 4.01645 | 0.26786 | 6.98019 | 4.01957 | 6.98019 | 4.07412 | 0.01566 | - | 4.01957 | 4.31758 |
| 6 | 4.27569 | [0,…,0] | 5.60 | 4.51516 | 7.21012 | 2.85 | 4.23562 | 0.20574 | 7.21012 | 4.23274 | 7.21012 | 4.27906 | 0.00766 | - | 4.23274 | 4.31758 |
| 7 | 4.07345 | [0,…,0] | 10.84 | 4.51516 | 6.95674 | 3.20 | 4.00582 | 0.24626 | 6.98013 | - | 6.95674 | 4.05966 | 0.02539 | - | 4.00582 | 4.31758 |
| 8 | 3.70999 | [0,…,0] | 21.70 | 4.51516 | 6.62908 | 4.42 | 3.66111 | 0.24604 | 6.58222 | 3.61847 | 6.58222 | 3.72576 | 0.04512 | - | 3.61847 | 4.31758 |
| 9 | 4.00664 | [0,…,0] | 12.69 | 4.51516 | 6.94414 | 3.08 | 4.00540 | 0.17881 | 6.90913 | 3.93321 | 6.90913 | 4.02904 | 0.03013 | - | 3.93321 | 4.31758 |
| 10 | 4.27393 | [0,…,0] | 5.64 | 4.51516 | 7.25503 | 3.51 | 4.23056 | 0.20082 | 7.20090 | 4.20288 | 7.20090 | 4.26429 | 0 | 5.16 | 4.26429 | 4.31758 |
| 11 | 4.26184 | [0,…,0] | 5.94 | 4.51516 | 7.16764 | 3.19 | 4.23714 | 0.20694 | 7.14578 | 4.19076 | 7.14578 | 4.26184 | 0.01478 | - | 4.19076 | 4.31758 |

[a] – in iteration k = 0 problem (P2) is solved

In what concerns the computational effort, the best solution given by the DBP was obtained in less than 4 h (a total of 12 iterations) while the best solution attained by the hybrid approach (over 20 runs, each with a population of 20 UL solutions and 70 iterations, thus involving a total of 1400 MILP problems by run) took more than 6 hours (per run).

We now present a comparison of the DBP and the hybrid approach for the Model SIAC with a discretization of the planning horizon in units of time of 5 minutes (*T*= 288). The LL problem has 5481 binary variables, 6471 continuous variables and 22149 constraints.

The computational budget to solve each MILP subproblem of the DBP was set as 10 min. With this budget, no subproblem could be solved to optimality. Therefore, the bounds have been computed taken the gaps into account as described in section 7. The algorithm was stopped in the 17th iteration after 10 iterations without changes in the bounds. It took 11h20 of computation time leading to the bounds $UB_F = 4.64731$ € and $LB_F = 4.37542$ €. In the final (incumbent) solution, the UL objective value is $F^{DBP} = LB_F = 4.37542$ € and the consumer's cost is 7.32245 €, obtained in problem (P4-D) with a MIP gap of 5.8%.

Five independent runs of the hybrid approach were performed, each with a population of 20 UL solutions and 70 iterations. With a computational budget of 30 seconds to solve each MILP subproblem, each run took around 12h. At a final stage, the LL problem was solved again for the prices corresponding to the best $F$ of each run imposing the same computational budget as the DBP allowed for each subproblem (10 min). The best final solution was $F = 4.42863$ €, which is 1.22% better than $F^{DBP}$, and the consumer's cost is 7.40614 €. This solution was obtained with a MIP gap of 6.98%, which is not directly comparable with the 5.8% above since the problems that yielded the corresponding solutions are different (P4-D in the DBP and the LL problem in the hybrid approach). Although not all the five final solutions were better than $F^{DBP}$, the average $F$ is still 0.55% higher than that value. These results show that the hybrid approach can be interesting when the subproblems cannot be solved to optimality. However, due to the stochastic nature of the UL search, it requires several runs to gauge the quality of the solutions since a single run may lead to a bad solution. In this problem instance, a single run was lengthier than the DBP.

The computation time is a relevant issue in the context of announcing day-ahead dynamic ToU prices, depending on regulatory directives and the type of contracts between retailers and their clients. Therefore, the computational budget should be adjusted to the requirements imposed by the energy regulator to retailers regarding the antecedence with which those prices are broadcasted.

## 9. Conclusions

In this paper, a deterministic bounding procedure (DBP) to determine the optimal solution to a mixed-integer BL problem is proposed, which is suited to assist an electricity retailer in defining time-of-use prices to maximize profits to which consumers react to minimize costs by changing the operation of their appliances (price-based demand response). This algorithmic approach belongs to the family of methodologies based on optimal-value-function reformulations and consists of generating a series of convergent upper and lower bounds for the UL objective function. Due to the specific features of our BL pricing model (upper/lower level variables do not appear in the lower/upper level constraints), it was possible to develop a very efficient approach. When the LL problem does not impose an excessive computational burden to be solved to optimality, the DBP is able to obtain the optimal solution displaying a much better performance than a hybrid approach (a PSO algorithm to tackle the UL problem calling a solver to address the LL problem) previously developed by the authors. However, when the LL problem became more complex (in our model due to the consideration of a thermostatic appliance), the subproblems in the DBP presented several difficulties, including getting no feasible solution or yielding a sub-optimal solution with a significant MIP gap, even when the computational budget was increased. Reformulations of those subproblems using "elastic" variables and additional subproblems were proposed using the information of the MIP gaps to obtain meaningful lower/upper bounds within an acceptable computational time. The implementation of these techniques enabled the DBP to obtain good bounds for the UL objective function, outperforming the ones obtained with the hybrid approach in most cases. These key issues can only be ascertained with demanding models and, in general, are not observed in benchmark problems for which no physical analysis of results is possible. Also, the computation effort is much smaller for the DBP than for the hybrid approach. When the LL MILP problem poses excessive combinatorial difficulties, the DBP endowed with these techniques can quickly converge to good lower/upper bounds and offer reliable information to the leader to help him making sound decisions. The results showed that the hybrid approach can be useful when it is not possible to solve subproblems to optimality within a given computational budget, since it could reach solutions competitive with the ones of the DBP or even slightly better. However, this is accomplished at the expense of a very high computational time as several runs may be required to reach good quality solutions. Future work will involve the development and implementation of new subproblems and reformulations able to provide even better bounds within reasonable computation time limits whenever it is not possible to reach optimality in difficult problems.

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## References


Alves, M. J., Antunes, C. H. & Carrasqueira, P. (2016). "A Hybrid Genetic Algorithm for the Interaction of Electricity Retailers with Demand Response". Applications of Evolutionary Computation. 459-474.

Aussel, D., Brotcorne, L., Lepaul, S. & von Niederhäusern, L. (2020). "A trilevel model for best response in energy demand-side management." European Journal of Operational Research. 281(2): 299-315.

Caramia, M. & Mari, R. (2015). "Enhanced exact algorithms for discrete bilevel linear problems." Optimization Letters. 9(7): 1447-1468.

DeNegre, S. T. & Ralphs, T. K. (2009). "A Branch-and-cut Algorithm for Integer Bilevel Linear Programs". Operations Research and Cyber-Infrastructure (Springer New York). 65-78.

Fischetti, M., Ljubić, I., Monaci, M. & Sinnl, M. (2016). "Intersection Cuts for Bilevel Optimization". Integer Programming and Combinatorial Optimization 77-88.

Fischetti, M., Ljubić, I., Monaci, M. & Sinnl, M. (2017). "A New General-Purpose Algorithm for Mixed-Integer Bilevel Linear Programs." Operations Research. 65(6): 1615-1637.

Gümüş, Z. H. & Floudas, C. A. (2005). "Global optimization of mixed-integer bilevel programming problems." Computational Management Science. 2(3): 181-212.

Kleniati, P.-M. & Adjiman, C. S. (2015). "A generalization of the Branch-and-Sandwich algorithm: From continuous to mixed-integer nonlinear bilevel problems." Computers & Chemical Engineering. 72: 373-386.

Lozano, L. & Smith, J. C. (2017). "A Value-Function-Based Exact Approach for the Bilevel Mixed-Integer Programming Problem." Operations Research. 65(3): 768-786.

McCormick, G.P. (1976). “Computability of global solutions to factorable nonconvex programs: Part I - convex underestimating problems.” Mathematical Programming. 10(1): 147–175.

Meng, F. L. & Zeng, X. J. (2013). "An Optimal Real-time Pricing Algorithm for the Smart Grid: A Bi-level Programming Approach." 2013 Imperial College Computing Student Workshop (ICCSW’13) - OpenAccess Series in Informatics: 81-88.

Mitsos, A., Lemonidis, P., & Barton, P. I. (2008). "Global solution of bilevel programs with a nonconvex inner program. " Journal of Global Optimization. 42(4): 475-513.

Mitsos, A. (2010). "Global solution of nonlinear mixed-integer bilevel programs." Journal of Global Optimization. 47(4): 557-582.

Mersha, A. G., & Dempe, S. (2006). "Linear bilevel programming with upper level constraints depending on the lower level solution". Applied mathematics and computation, 180(1), 247-254.

Moore, J. T. & Bard, J. F. (1990). "The Mixed Integer Linear Bilevel Programming Problem." Operations Research. 38(5): 911-921.

Sekizaki, S., Nishizaki, I. & Hayashida, T. (2016). "Electricity retail market model with flexible price settings and elastic price-based demand responses by consumers in distribution network." International Journal of Electrical Power & Energy Systems. 81: 371-386.

Soares, I., Alves, M. J. & Antunes, C. H. (2020). "Designing time-of-use tariffs in electricity retail markets using a bi-level model – Estimating bounds when the lower level problem cannot be exactly solved." Omega. 93:102027.

Vicente, L., Savard, G., & Judice, J. (1996). "Discrete linear bilevel programming problem. " Journal of optimization theory and applications, 89(3), 597-614.

Xu, P. & Wang, L. (2014). "An exact algorithm for the bilevel mixed integer linear programming problem under three simplifying assumptions." Computers & Operations Research. 41: 309-318.

Zeng, B. & An, Y. (2014). "Solving bilevel mixed integer program by reformulations and decomposition." Optimization online: 1-34.

Zugno, M., Morales, J. M., Pinson, P. & Madsen, H. (2013). "A bilevel model for electricity retailers' participation in a demand response market environment." Energy Economics. 36: 182-197.

## APPENDIX

The data in Tables A-1 – A-8 refer to the 15 min discretization in Models SI and SIAC. The algorithm code in R, the model files to be run in CPLEX and all data for the 5 min and 15 min discretization in both models are available at https://www.uc.pt/en/feuc/mjalves/DBP.

Tables A-9 and A-10 present the best solution of the hybrid approach and the optimal solution of the DBP for the Models SI and SIAC, respectively.

**Table A-1.** Minimum ($\underline{x_i}$), maximum ($\overline{x_i}$) and average ($x^{AVG}$) of the electricity prices (in €/kWh) that can be charged to the consumers by the retailer, in each sub-period $P_i,\ i \in \{1,\dots,6\}$.

| **Periods** | | **P$_1$** | **P$_2$** | **P$_3$** | **P$_4$** | **P$_5$** | **P$_6$** | $x^{AVG}$ |
|---|---|---|---|---|---|---|---|---|
| | | [1,28] | [29,44] | [45,56] | [57,72] | [73,84] | [85,96] | |
| **Prices (€/kWh)** | $\underline{x_i}$ | 0.0440 | 0.0848 | 0.1080 | 0.0804 | 0.1540 | 0.0920 | 0.1614 |
| | $\overline{x_i}$ | 0.0996 | 0.2780 | 0.2836 | 0.2492 | 0.3240 | 0.1620 | |

**Table A-2.** Electricity spot market prices (in €/kWh) seen by the retailers.

| **Time intervals [initial time, final time]** | **Prices (€/kWh)** |
|---|---|
| [1,8] | 0.050 |
| [9,16] | 0.035 |
| [17,24] | 0.045 |
| [25,32] | 0.065 |
| [33,40] | 0.075 |
| [41,48] | 0.080 |
| [49,56] | 0.090 |
| [57,64] | 0.100 |
| [65,72] | 0.110 |
| [73,80] | 0.085 |
| [81,88] | 0.080 |
| [89,96] | 0.100 |

**Table A-3.** Power requested to the grid (in W) in each unit of time $t \in$ T (expressed in intervals of time [initial time, final time]) by (non-controllable) base load.

| **Time intervals [initial time, final time]** | **Power (W)** |
|---|---|
| [1,32] | 165 |
| [33,34] | 700 |
| [35,36] | 170 |
| [37,44] | 85 |
| [45,54] | 160 |
| [55,64] | 130 |
| [65,80] | 160 |
| [81,81] | 500 |
| [82,83] | 1600 |
| [84,85] | 750 |
| [86,86] | 250 |
| [87,87] | 450 |

| | |
|---|---|
| [88,90] | 280 |
| [91,91] | 1080 |
| [92,96] | 250 |

**Table A-4.** Comfort time slots, $T_j = \left[T_{1_j}, T_{2_j}\right]$ and $T_k = \left[T_{1_k}, T_{2_k}\right]$ for $j \in \{1, \dots, J\}$ and $k \in \{1, \dots, K\}$, allowed for the operation of each controllable appliance.

| Shiftable Loads | | | Interruptible Loads | |
|---|---|---|---|---|
| **DW** | **LM** | **CD** | **EV** | **EWH** |
| [1,32] | [28,58] | [76,96] | [1,48] | [26,34] |

**Table A-5.** Operation cycles of the controllable appliances.

| | **Appliance** | **Power required by the appliance at each stage of its operation cycle (W)** | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8-36** |
| **Shiftable Loads** | **DW** | 1750 | 1250 | 120 | 1600 | 640 | 220 | | |
| | **LM** | 1840 | 980 | 160 | 220 | 300 | 340 | 120 | |
| | **CD** | 1660 | 1720 | 300 | 220 | | | | |
| **Interruptible Loads** | **EV** | 2300 | 2300 | 2300 | 2300 | 2300 | 2300 | 2300 | 2300 |
| | **EWH** | 1500 | 1500 | 1500 | 1500 | 1500 | 1500 | | |

**Table A-6.** Power level prices (in €) charged to the consumers by the retailer.

| | **Prices (€/day)** | **Maximum Power (W)** |
|---|---|---|
| **1** | 0.2047 | 2300 |
| **2** | 0.2206 | 3450 |
| **3** | 0.2834 | 4600 |
| **4** | 0.3492 | 5750 |
| **5** | 0.4198 | 6900 |
| **6** | 0.6280 | 10350 |
| **7** | 0.8302 | 13800 |
| **8** | 1.0324 | 17250 |
| **9** | 1.2351 | 20700 |

**Table A-7.** Parameters of the thermostatic load.

| $\theta^{max}$ | $\theta^{min}_{Abs}$ | $\theta^{ref}$ | $\theta^{in}_0$ | $P^{nom}_{AC}$ | $s_0$ |
|---|---|---|---|---|---|
| 24ºC | 18ºC | 20ºC | 12ºC | 1400W | 0 |

**Table A-8.** Outdoor temperatures for a period of 24h ($T$=96).

| $t$ | $\theta^{ext}_t$ (°C) | $t$ | $\theta^{ext}_t$ (°C) | $t$ | $\theta^{ext}_t$ (°C) |
|---|---|---|---|---|---|
| 0 | 9.45 | | | | |
| 1 | 9.45 | 33 | 8.96 | 65 | 12.92 |
| 2 | 9.40 | 34 | 8.92 | 66 | 12.79 |
| 3 | 9.35 | 35 | 8.92 | 67 | 12.64 |
| 4 | 9.30 | 36 | 9.00 | 68 | 12.50 |
| 5 | 9.25 | 37 | 9.19 | 69 | 12.40 |
| 6 | 9.20 | 38 | 9.43 | 70 | 12.35 |
| 7 | 9.15 | 39 | 9.66 | 71 | 12.32 |
| 8 | 9.10 | 40 | 9.80 | 72 | 12.30 |

| | | | | | |
|---|---|---|---|---|---|
| 9 | 9.05 | 41 | 9.81 | 73 | 12.29 |
| 10 | 9.01 | 42 | 9.75 | 74 | 12.27 |
| 11 | 8.96 | 43 | 9.72 | 75 | 12.25 |
| 12 | 8.90 | 44 | 9.80 | 76 | 12.20 |
| 13 | 8.83 | 45 | 10.06 | 77 | 12.13 |
| 14 | 8.78 | 46 | 10.48 | 78 | 12.02 |
| 15 | 8.76 | 47 | 10.97 | 79 | 11.88 |
| 16 | 8.80 | 48 | 11.50 | 80 | 11.70 |
| 17 | 8.91 | 49 | 12.00 | 81 | 11.48 |
| 18 | 9.06 | 50 | 12.43 | 82 | 11.25 |
| 19 | 9.20 | 51 | 12.78 | 83 | 11.05 |
| 20 | 9.30 | 52 | 13.00 | 84 | 10.90 |
| 21 | 9.32 | 53 | 13.09 | 85 | 10.83 |
| 22 | 9.28 | 54 | 13.07 | 86 | 10.85 |
| 23 | 9.19 | 55 | 12.99 | 87 | 10.94 |
| 24 | 9.10 | 56 | 12.90 | 88 | 11.10 |
| 25 | 9.02 | 57 | 12.82 | 89 | 11.31 |
| 26 | 8.95 | 58 | 12.78 | 90 | 11.52 |
| 27 | 8.91 | 59 | 12.77 | 91 | 11.67 |
| 28 | 8.90 | 60 | 12.80 | 92 | 11.70 |
| 29 | 8.92 | 61 | 12.87 | 93 | 11.56 |
| 30 | 8.96 | 62 | 12.95 | 94 | 11.18 |
| 31 | 9.00 | 63 | 13.00 | 95 | 10.52 |
| 32 | 9.00 | 64 | 13.00 | 96 | 9.50 |

**Table A-9.** Prices (in €/kWh) obtained for the best solution of the hybrid approach and the optimal solution of the DBP for the Model SI with 15 min discretization.

| | $P_1$ [1,28] | $P_2$ [29,44] | $P_3$ [45,56] | $P_4$ [57,72] | $P_5$ [73,84] | $P_6$ [85,96] |
|---|---|---|---|---|---|---|
| **Hybrid approach** | 0.09960 | 0.27504 | 0.28360 | 0.08048 | 0.15400 | 0.14716 |
| **DBP** | 0.09960 | 0.27504 | 0.28360 | 0.08040 | 0.15400 | 0.14728 |

**Table A-10.** Prices (in €/kWh) obtained for the best solution of the hybrid approach and the optimal solution of the DBP for the Model SIAC with 15 min discretization.

| | $P_1$ [1,28] | $P_2$ [29,44] | $P_3$ [45,56] | $P_4$ [57,72] | $P_5$ [73,84] | $P_6$ [85,96] |
|---|---|---|---|---|---|---|
| **Hybrid approach** | 0.09960 | 0.27392 | 0.28284 | 0.08172 | 0.15480 | 0.14376 |
| **DBP** | 0.09960 | 0.27108 | 0.27900 | 0.08040 | 0.15400 | 0.15716 |