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On some Problems in the Theory of Orthogonal Polynomials

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Abstract

In this thesis we solve several problems in the theory of orthogonal polynomial sequences (OPS). In bellow we summarize the main contributions.

(i) Let $\mathbf{u}$ be a nonzero linear functional acting on the space of polynomials $\mathscr{P}$. Let $\mathbf{D}_{q,\omega}$ be a Hahn operator acting on the dual space of polynomials $\mathscr{P}'$. Suppose that there exist polynomials ϕ and ψ, with $\deg\phi \leq 2$ and $\deg\psi \leq 1$, so that the functional equation

$$\mathbf{D}_{q,\omega}(\phi\mathbf{u}) = \psi\mathbf{u}$$

holds, where the involved operations are defined in the distributional sense. We state necessary and sufficient conditions, involving only the coefficients of ϕ and ψ, such that $\mathbf{u}$ is regular, that is, there exists an OPS with respect to $\mathbf{u}$. In addition, the coefficients of the three-term recurrence relation (TTRR) satisfied by the corresponding monic OPS are given, as well as a distributional Rodrigues-type formula, which holds without assuming that $\mathbf{u}$ is regular.

(ii) Let M and N be fixed non-negative integer numbers and let π_N be a polynomial of degree N. Suppose that $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are two OPS such that

$$\pi_N(x)\,P_{n+m}^{(m)}(x) = \sum_{j=n-M}^{n+N} r_{n,j}Q_{j+k}^{(k)}(x) \quad (n = 0,1,\dots)\,, \tag{$*$}$$

where $r_{n,j}$ are complex numbers independent of x. It is shown that under some natural constraints, $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are semiclassical OPS. That is, there exist nonzero polynomials ϕ_1, ϕ_2, ψ_1 and ψ_2 such that the corresponding functionals $\mathbf{u}$ and $\mathbf{v}$ fulfill the functional equations

$$\mathbf{D}(\phi_1\mathbf{u}) = \psi_1\mathbf{u}, \qquad \mathbf{D}(\phi_2\mathbf{v}) = \psi_2\mathbf{v}.$$

Moreover we show that $\mathbf{u}$ and $\mathbf{v}$ are related by a rational modification in the distributional sense, meaning that $P\mathbf{u} = Q\mathbf{v}$ for some nonzero polynomials $P,Q \in \mathscr{P}$. This leads us to introduce the concept of π_N-coherent pairs with index M and order (m,k).

(iii) We extend the previous concept to the one of π_N-(q,ω)-coherent pairs with index M and order (m,k), which appears in the framework of discrete OPS by replacing in $(*)$ the ordinary derivative by the discrete Hahn's operator $D_{q,\omega}$. This leads to the (structure) relation

$$\pi_N(x)\,D_{q,\omega}^m P_{n+m}(x) = \sum_{j=n-M}^{n+N} r_{n,j}D_{q,\omega}^k Q_{j+k}(x) \quad (n = 0,1,\dots)\,.$$

Again, in this situation, it is shown that under some natural constraints, $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are semi-classical OPS (with respect to $D_{q,\omega}$) and the corresponding functionals are related by a (distributional) rational modification. Some examples of application are given, recovering in a more simple way some known results in the literature about the subject. Our results enable us to describe in a unified way all the classical OPS with respect to Jackson's operator, which appear as special or limiting cases of a four parametric family of q-polynomials.

(iv) Let us consider now that $\mathbf{u}$ is a functional on $\mathscr{P}$ satisfying the more general functional equation

$$\mathbf{D}_x(\phi\mathbf{u}) = \mathbf{S}_x(\psi\mathbf{u}) \, ,$$

where $\mathbf{D}_x$ and $\mathbf{S}_x$ are operators defined on $\mathscr{P}'$ in the usual way, in the framework of the theory of OPS on a nonuniform lattice, $x(s)$, that includes as a special case the lattice associated with the so-called Askey-Wilson operator, namely $x(s) = \frac{1}{2}q^{-s} + \frac{1}{2}q^s$. We state necessary and sufficient conditions for the regularity of $\mathbf{u}$, giving, in addition, closed formulas for the coefficients of the TTRR of the corresponding monic OPS, as well as a Rodrigues-type formula. Some examples are given to point out the power of our formulas in the framework of classical OPS on nonuniform lattices. In particular, our results enable us to derive in a simple way the coefficients of the TTRR of the Racah polynomials as well as the ones for the Askey-Wilson polynomials.

(v) Let $(P_n)_{n\geq 0}$ be a monic OPS and π a polynomial of a degree at most two such that

$$\pi(x)D_xP_n(x) = (a_nx+b_n)P_n(x)+c_nP_{n-1}(x) \quad (n = 0,1,2\ldots) \, ,$$

for some complex sequences coefficients a_n, b_n and c_n. M. E. H. Ismail posed the problem of characterizing all OPS fulfilling this structure relation, for the lattice associated with the Askey-Wilson operator. Ismail conjectured (see [26, Conjecture 24.7.8]) that the continuous $q-$Jacobi polynomials, the Al-Salam-Chihara polynomials, or special or limiting cases of them, are the only OPS fulfilling the structure relation. Using the main result obtained in (iv) we give a positive answer to Ismail's conjecture.

Resumo

Nesta dissertação resolvem-se vários problemas na âmbito da teoria das sucessões de polinómios ortogonais (SPO). As contribuições principais apresentadas são descritas em seguida.

(i) Seja $\mathbf{u}$ uma funcional linear não nula definida sobre o espaço dos polinómios, $\mathscr{P}$. Seja $\mathbf{D}_{q,\omega}$ um operador de Hahn que actua no espaço dual $\mathscr{P}'$. Suponha-se que existem polinómios ϕ e ψ, com ϕ e ψ polinómios de graus não superiores a 2 e 1, respectivamente, tais que $\mathbf{u}$ satisfaz a equação funcional

$$\mathbf{D}_{q,\omega}(\phi\mathbf{u}) = \psi\mathbf{u} \, ,$$

onde as operações são definidas no sentido usual da teoria das distribuições. Estabelecem-se condições necessárias e suficientes, envolvendo apenas os coeficientes de ϕ e ψ, tais que $\mathbf{u}$ é regular, isto é, existe uma SPO a respeito de $\mathbf{u}$. Além disso, os coeficientes da relação de recorrência a três termos (RRTT) verificada pela correspondente SPO mónica são dados de forma explícita. É também apresentada uma fórmula de tipo Rodrigues distribucional, a qual se verifica mesmo que $\mathbf{u}$ não seja regular.

(ii) Sejam M e N números inteiros não negativos fixados e π_N um polinómio de grau N. Sejam $(P_n)_{n\geq 0}$ e $(Q_n)_{n\geq 0}$ duas SPO tais que

$$\pi_N(x)\,P_{n+m}^{(m)}(x) = \sum_{j=n-M}^{n+N} r_{n,j} Q_{j+k}^{(k)}(x) \quad (n = 0,1,\ldots) \, , \qquad (*)$$

onde cada $r_{n,j}$ é um número complexo independente de x. Prova-se que, sob certas reservas naturais, $(P_n)_{n\geq 0}$ e $(Q_n)_{n\geq 0}$ são SPO semiclássicas, isto é, existem polinómios não nulos ϕ_1, ϕ_2, ψ_1 e ψ_2 tais que as correspondentes funcionais regulares $\mathbf{u}$ e $\mathbf{v}$ satisfazem as equações funcionais

$$\mathbf{D}(\phi_1\mathbf{u}) = \psi_1\mathbf{u}, \qquad \mathbf{D}(\phi_2\mathbf{v}) = \psi_2\mathbf{v}.$$

Prova-se ainda que $\mathbf{u}$ e $\mathbf{v}$ estão relacionados por uma modificação racional, no sentido distribucional, o que significa que $P\mathbf{u} = Q\mathbf{v}$ para certos polinómios $P, Q \in \mathscr{P}$. Estes factos conduzem ao conceito de pares π_N-coerentes de índice M e ordem (m,k).

(iii) O conceito anterior é estendido para o conceito de pares π_N-(q,ω)-coerentes de índice M e ordem (m,k), no contexto das SPO discretas, substituindo em $(*)$ o operador derivada usual pelo operador de Hahn $D_{q,\omega}$. Isto conduz à relação de estrutura

$$\pi_N(x)\,D_{q,\omega}^{m} P_{n+m}(x) = \sum_{j=n-M}^{n+N} r_{n,j} D_{q,\omega}^{k} Q_{j+k}(x) \quad (n = 0,1,\ldots) \, .$$

De novo, nesta situação, mostra-se que, assumindo certas condições naturais, $(P_n)_{n\geq 0}$ e $(Q_n)_{n\geq 0}$ são SPO semiclássicas (a respeito de $D_{q,\omega}$) e que as funcionais regulares associadas estão relacionadas por uma modificação racional (no sentido distribucional). São apresentados alguns exemplos de aplicação, recuperando de forma simples alguns resultados conhecidos na literatura. Os resultados apresentados permitem ainda descrever de forma unificada todas as SPO clássicas a respeito do operador de Jackson, as quais são representadas como casos especiais ou caso limite de uma família de q-polinómios envolvendo quatro parâmetros.

(iv) Seja agora $\mathbf{u}$ uma funcional linear em $\mathscr{P}$ que satisfaz a equação funcional mais geral

$$\mathbf{D}_x(\phi \mathbf{u}) = \mathbf{S}_x(\psi \mathbf{u}) \; ,$$

onde $\mathbf{D}_x$ e $\mathbf{S}_x$ são operadores definidos em $\mathscr{P}'$ da maneira usual, no contexto da teoria das SPO em redes não uniformes, $x(s)$, o que inclui como caso especial a rede associada ao chamado operador de Askey-Wilson, nomeadamente, $x(s) = \frac{1}{2}q^{-s} + \frac{1}{2}q^{s}$. Estabelecem-se condições necessárias e suficientes para a regularidade de $\mathbf{u}$. Para além disso, dão-se fórmulas fechadas para os coeficientes da RRTT da correspondente SPO mónica, bem como uma fórmula de tipo Rodrigues. São apresentados alguns exemplos que evidenciam que tais fórmulas são muito poderosas no contexto das SPO clássicas em redes não uniformes. Em particular, os resultados obtidos permitem obter de forma simples os coeficientes da RRTT para os polinómios de Racah, bem como para os polinómios de Askey-Wilson.

(v) Sejam $(P_n)_{n\geq 0}$ uma SPO mónica e π um polinómio de grau quando muito igual a dois que satisfazem a relação de estrutura

$$\pi(x) D_x P_n(x) = (a_n x + b_n) P_n(x) + c_n P_{n-1}(x) \quad (n = 0, 1, 2, \ldots) \; ,$$

onde a_n, b_n e c_n são parâmetros reais ou complexos. M. E. H. Ismail colocou o problema de caracterizar tais SPO para a rede associada ao operador de Askey-Wilson. Ismail conjecturou (veja-se [26, Conjecture 24.7.8]) que os polinómios $q-$Jacobi contínuos, os polinómios de Al-Salam Chihara, ou casos especiais ou limite destes, constituem as únicas SPO que satisfazem aquela relação de estrutura. Usando o resultado principal estabelecido em (iv), damos uma resposta positiva à conjectura de Ismail.

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Chapter 1

Introduction

The aim of this chapter is to give the outline of this thesis. This requires some basic knowledge in the theory of orthogonal polynomial sequences (OPS). So we start by giving a short introduction to orthogonal polynomials. This includes a review of some properties of Hahn's operator, as well as of operators on a nonuniform lattice, including the so-called Askey-Wilson operator. Moreover, some new properties of those operators are presented. After that, the outline of the thesis is given.

1.1 Basic results on orthogonal polynomial sequences

For the general theory of OPS (continuous and discrete) we refer the reader to the influential monographs by Szegö [60], Chihara [13], Ismail [26], Nikiforov, Suslov, and Uvarov [53], and Koekoek, Lesky, and Swarttouw [34]. As fundamental references on the so-called algebraic theory of orthogonal polynomials, we mention Maroni's works [41, 43, 46–48]. We also mention here the recent unpublished class notes [56] (where the emphasis is on the algebraic approach developed by Maroni).

The algebraic approach to orthogonal polynomials was developed by Pascal Maroni. Along this work, we will use this approach, and so we start this section by pointing out some basic facts on the algebraic theory. Most of the facts that we are going to present next may be founded on the references mentioned above.

We denote by $\mathscr{P}$ the vector space of all (complex) polynomials and by $\mathscr{P}^*$ its algebraic dual space. $\mathscr{P}$ may be endowed with the strict inductive limit topology so that

$$\mathscr{P} = \bigcup_{n\geq 0} \mathscr{P}_n \,,$$

where $\mathscr{P}_n$ is the space of all (complex) polynomials of degree at most n. With this topology, the algebraic and the topological dual spaces of $\mathscr{P}$ coincide, that is

$$\mathscr{P}^* = \mathscr{P}' \,.$$

Given a simple set of polynomials $(P_n)_{n\geq 0}$ (meaning that each $P_n \in \mathscr{P}_n$ and $\deg P_n = n$ for each $n = 0,1,\ldots$), the corresponding dual basis is a sequence of linear functionals $\mathbf{a}_n : \mathscr{P} \to \mathbb{C}$ such that

$$\langle \mathbf{a}_n, P_m \rangle := \delta_{n,m} \quad (n,m = 0,1,\ldots) ,$$

where $\delta_{n,m}$ denotes the Kronecker's symbol. As usual, $\langle \cdot,\cdot \rangle$ means the duality bracket, so that $\langle \mathbf{u}, p\rangle$ is the action of the functional $\mathbf{u}$ over the polynomial p. In addition, any functional $\mathbf{u} \in \mathscr{P}'$ can be written in the sense of the weak topology in $\mathscr{P}'$ as

$$\mathbf{u} = \sum_{k=0}^{\infty} \langle \mathbf{u}, P_k \rangle \, \mathbf{a}_k .$$

Definition 1.1 *A simple set of polynomials $(P_n)_{n\geq 0}$ is said to be an orthogonal polynomial sequence (OPS) with respect to a functional $\mathbf{u} \in \mathscr{P}^*$ if there exists a sequence of nonzero complex numbers $(k_n)_{n\geq 0}$ such that*

$$\langle \mathbf{u}, P_n P_m \rangle = k_n \delta_{n,m} \quad (n,m = 0,1,2,\ldots) .$$

We also said that $\mathbf{u}$ is regular and $(P_n)_{n\geq 0}$ is the corresponding OPS.

A monic OPS is a sequence of orthogonal polynomials for which the leading coefficient of each polynomial is one. If $(P_n)_{n\geq 0}$ is a (monic) OPS with respect to $\mathbf{u} \in \mathscr{P}^*$, then the corresponding dual basis is explicitly given by

$$\mathbf{a}_n = \left\langle \mathbf{u}, P_n^2 \right\rangle^{-1} P_n \mathrm{u} \quad (n = 0,1,2,\ldots) .$$

Here the left multiplication of a functional $\mathbf{u}$ by a polynomial ϕ is defined as in the usual sense of the theory of distributions:

$$\langle \phi \mathbf{u}, p \rangle := \langle \mathbf{u}, \phi p \rangle , \quad \forall p \in \mathscr{P} .$$

The following proposition is a useful characterization of OPS.

Theorem 1.1.1 *[13]*
Let $\mathbf{u}$ be a linear functional and let $(P_n)_{n\geq 0}$ be a simple set in $\mathscr{P}$. Then the following are equivalent:

i) *$(P_n)_{n\geq 0}$ is an OPS with respect to $\mathbf{u}$;*

ii) *For each $n \in \mathbb{N}_0$ and $R \in \mathscr{P}_n \setminus \{0\}$, there exists a nonzero sequence of complex numbers $(k_n)_{n\geq 0}$ such that $\langle \mathbf{u}, RP_n \rangle = k_n \delta_{n,m}$, with $m = degR$;*

iii) *For each $n \in \mathbb{N}_0$, there exists a nonzero sequence of complex numbers $(k_n)_{n\geq 0}$ such that $\langle \mathbf{u}, x^m P_n \rangle = k_n \delta_{n,m}$, with $m = 0,1,2,\ldots,n$.*

A very interesting situation of orthogonality appears in the so-called positive-definite case.

Definition 1.2 *[13]*
A linear functional $\mathbf{u} \in \mathscr{P}'$ is positive-definite if $\langle \mathbf{u}, p \rangle > 0$ for all nonzero and non negative real polynomial p.

In the following result, known as Favard's theorem (or spectral theorem), we see that OPS are characterized by a three-term recurrence relation.

Theorem 1.1.2 *[13]*
Let $(\beta)_{n\geq 0}$ and $(\gamma_n)_{n\geq 1}$ be two arbitrary sequence of complex numbers, and let $(P_n)_{n\geq 0}$ be a sequence of (monic) polynomials defined by the following three-term recurrence relation (TTRR).

$$\begin{aligned} &P_{-1}(x) := 0, \;\; P_0(x) := 1 \\ &P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x) \;\; (n = 0,1,2,\ldots) \,. \end{aligned} \tag{1.1}$$

Then there exists a unique functional $\mathbf{u} \in \mathscr{P}'$ such that

$$\langle \mathbf{u}, 1\rangle \neq 0, \;\; \langle \mathbf{u}, P_n P_m\rangle = 0 \text{ if } n \neq m \;\; (n,m = 0,1,2,\ldots) \,.$$

Moreover, $\mathbf{u}$ is regular and $(P_n)_{n\geq 0}$ is the corresponding (monic) OPS if and only if $\gamma_n \neq 0$ $(n = 1,2,\ldots)$, while $\mathbf{u}$ is positive-definite with $(P_n)_{n\geq 0}$ as the corresponding (monic) OPS if and only if the coefficients β_n are all real and the γ_n are positive.

The coefficients β_n and γ_n of (1.1) are given by (see [13, 46])

$$\beta_n = \frac{\langle \mathbf{u}, xP_n^2\rangle}{\langle \mathbf{u}, P_n^2\rangle}, \;\; \gamma_{n+1} = \frac{\langle \mathbf{u}, P_{n+1}^2\rangle}{\langle \mathbf{u}, P_n^2\rangle} \quad (n = 0,1,2,\ldots) \,. \tag{1.2}$$

It is also known that if $(P_n)_{n\geq 0}$ is a positive-definite OPS, then there exists a positive Borel measure μ on $\mathbb{R}$, whose support is an infinite set ($supp(\mu) := \{x \in \mathbb{R} : \; \mu\left((x-\varepsilon, \, x+\varepsilon)\right) > 0, \; \forall \varepsilon > 0\}$), and with finite moments of all orders (i.e $\langle \mathbf{u}, x^n\rangle < \infty$ for $n = 0,1,2,\ldots$), such that

$$\langle \mathbf{u}, p\rangle := \int_{\mathbb{R}} p(x)d\mu(x), \;\; \forall p \in \mathscr{P} \,.$$

When an OPS $(P_n)_{n\geq 0}$ is positive-definite, we also say that it is an OPS with respect to the measure μ (where μ represents the linear functional $\mathbf{u}$). Finally we define the derivative of a functional by the following.

Definition 1.3 *[46]*
Let $\mathbf{u}$ be a linear functional. Then we define the derivative of $\mathbf{u}$ denoted $\mathbf{Du}$ by:

$$\langle \mathbf{Du}, p\rangle := -\left\langle \mathbf{u}, p'\right\rangle, \, \forall p \in \mathscr{P} \,. \tag{1.3}$$

It is easy to show that the functional $\phi\mathbf{u}$ obeys to the following Leibniz rule

$$\mathbf{D}^n(\phi\mathbf{u}) = \sum_{k=0}^{n} \binom{n}{k} \phi^{(k)}\mathbf{D}^{n-k}\mathbf{u} \quad (n = 0,1,2,\ldots) \,.$$

1.2 Hahn's operator

Definition 1.4 *[24]*
Given complex numbers q and ω, the (ordinary) Hahn's operator $D_{q,\omega} : \mathscr{P} \to \mathscr{P}$ is defined by

$$D_{q,\omega} f(x) := \frac{f(qx+\omega)-f(x)}{(q-1)x+\omega} \quad (f \in \mathscr{P}) \,. \tag{1.4}$$

The OPS related to this operator has been studied by Hahn [24]. Hereafter (when referring to $D_{q,\omega}$) we will assume that q and ω fulfill the conditions

$$|q-1|+|\omega| \neq 0 \,, \quad q \notin \left\{0, \mathrm{e}^{2ij\pi/n} \mid 1 \leq j \leq n-1 \,;\; n=2,3,\ldots\right\} . \tag{1.5}$$

The first condition in (1.5) ensures that the right-hand side of (1.4) is well defined. The second one is imposed in order to ensure the existence of OPS in Hahn's sense (this will be made clear later in the next chapter, cf. Theorem 2.2.1 therein). The (ordinary) Hahn's operator $D_{q,\omega}$ on $\mathscr{P}$ induces a (distributional) Hahn's operator on $\mathscr{P}^*$.

Definition 1.5 ,
The (distributional) Hahn's operator $\mathbf{D}_{q,\omega} : \mathscr{P}^* \to \mathscr{P}^*$ is defined by

$$\langle \mathbf{D}_{q,\omega}\mathbf{u}, f\rangle := -q^{-1}\langle \mathbf{u}, D^*_{q,\omega} f\rangle \quad (\mathbf{u} \in \mathscr{P}^* \,,\; f \in \mathscr{P}) \,, \tag{1.6}$$

where $D^*_{q,\omega} := D_{1/q,-\omega/q}$.

This definition of $\mathbf{D}_{q,\omega}$ appears in Foupouagnigni's PhD thesis [15, Definition 3.4]. A slightly different one was considered in Häcker's PhD thesis [22, (1.16)] (under the supervision of P. Lesky and reviewed for AMS by R. Askey), where the adopted definition is $\langle \mathbf{D}_{q,\omega}\mathbf{u}, f\rangle = -\langle \mathbf{u}, D_{q,\omega} f\rangle$, as it may seem more natural *a priori*, taking into account the standard definition for the continuous case (cf. (1.3)). The advantage of (1.6) is going to be pointed out in the next chapter (see Section 2.2.2 therein). Recall that the $q-$bracket is defined by

$$[\alpha]_q := \begin{cases} \dfrac{q^\alpha - 1}{q-1} \,, & \text{if} \quad q \neq 1 \\ \alpha \,, & \text{if} \quad q = 1 \end{cases} \qquad (\alpha, q \in \mathbb{C}) \,.$$

Note that for each nonnegative integer number n, we have $[0]_q := 0$ and $[n]_q \to n$ as $q \to 1$. Note also that (1.5) ensures that $[n]_q \neq 0$ for each $n = 1, 2, \ldots$.

Definition 1.6 *For $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, the dilation operator $h_a : \mathscr{P} \to \mathscr{P}$ and the translation operator $\tau_b : \mathscr{P} \to \mathscr{P}$ are defined by*

$$h_a f(x) := f(ax) \,, \quad \tau_b f(x) := f(x-b) \quad (f \in \mathscr{P}) \,. \tag{1.7}$$

Note that if $q = 1$ in (1.4) then, setting $\triangle_\omega f(x) := f(x+\omega) - f(x)$, we have

$$D_{1,\omega} = \frac{\triangle_\omega}{\omega} \,,$$

while if $q \neq 1$ then, setting $D_q := D_{q,0}$, we have

$$D_{q,\omega} = \tau_{\omega_0} D_q \tau_{-\omega_0} \,, \quad \omega_0 := \omega/(1-q) \tag{1.8}$$

(see e.g. [9, (7.1)]). Thus, if $q \neq 1$ then there is no loss of generality by assuming $\omega = 0$, a fact remarked by Hahn himself [24]. Despite this, it seems to us preferable to present the theory for general (q,ω) fulfilling (1.5), in order to emphasize that there is no significant simplification by presenting it for specific q or ω and, more interesting, there is no need to study separately the case $q = 1$ and $q \neq 1$. As a matter of fact, the general formulas appearing in Chapter 2 (see Theorem 2.2.1 therein) allow us to emphasize a complete similarity with the corresponding ones proved by Marcellán and Petronilho for the continuous case (appearing in Theorem 2.1.1 of the same chapter). Next we introduce some basic definitions and useful notations.

Definition 1.7 *Let $q \in \mathbb{C} \setminus \{0\}$ and $\omega \in \mathbb{C}$.*

(i) *The operator $L_{q,\omega} : \mathscr{P} \to \mathscr{P}$ is defined by*

$$L_{q,\omega} := h_q \circ \tau_{-\omega} \,.$$

(ii) *The operators $L^*_{q,\omega} : \mathscr{P} \to \mathscr{P}$ and $D^*_{q,\omega} : \mathscr{P} \to \mathscr{P}$ are defined by*

$$L^*_{q,\omega} := h_{1/q} \circ \tau_{\omega/q} = L_{1/q,-\omega/q} \,, \quad D^*_{q,\omega} := D_{1/q,-\omega/q} \,.$$

(iii) *The operator $\mathbf{L}_{q,\omega} : \mathscr{P}^* \to \mathscr{P}^*$ is defined by*

$$\langle \mathbf{L}_{q,\omega}\mathbf{u}, f \rangle := q^{-1} \langle \mathbf{u}, L^*_{q,\omega} f \rangle \quad (\mathbf{u} \in \mathscr{P}^* \,,\; f \in \mathscr{P}) \,.$$

(iv) *The operators $\mathbf{D}^*_{q,\omega} : \mathscr{P}^* \to \mathscr{P}^*$ and $\mathbf{L}^*_{q,\omega} : \mathscr{P}^* \to \mathscr{P}^*$ are defined by*

$$\mathbf{D}^*_{q,\omega} := \mathbf{D}_{1/q,-\omega/q} \quad \text{and} \quad \mathbf{L}^*_{q,\omega} := \mathbf{L}_{1/q,-\omega/q} \,.$$

Remark 1.2.1 *As far as we know, the definitions appearing in (i), (ii), and (iv) were given in [22], while the ones appearing in (iii) were proposed in [15].*

The linear operators $L_{q,\omega}$ and $L^*_{q,\omega}$ are explicitly by

$$L_{q,\omega} f(x) = f(qx+\omega) \,, \quad L^*_{q,\omega} f(x) = f\Big(\frac{x-\omega}{q}\Big) \qquad (f \in \mathscr{P}) \,.$$

In bellow we summarize some useful properties involving the above operators.

Proposition 1.2.1 *[15, 22, 34]*
Let $\mathbf{u} \in \mathscr{P}^*$ *and* $f, g \in \mathscr{P}$, *then we have the following properties.*

$$L^*_{q,\omega} L_{q,\omega} = L_{q,\omega} L^*_{q,\omega} = I \; ; \quad \mathbf{L}^*_{q,\omega} \mathbf{L}_{q,\omega} = \mathbf{L}_{q,\omega} \mathbf{L}^*_{q,\omega} = \mathbf{I} \; ; \tag{1.9}$$

$$L^{-1}_{q,\omega} = L^*_{q,\omega} \; ; \quad \mathbf{L}^{-1}_{q,\omega} = \mathbf{L}^*_{q,\omega} \; ; \tag{1.10}$$

$$L^n_{q,\omega} f(x) = f\big(q^n x + \omega [n]_q\big) \quad (n = 0, \pm 1, \pm 2, \ldots) \; ; \tag{1.11}$$

$$D^*_{q,\omega} D_{q,\omega} = q D_{q,\omega} D^*_{q,\omega} \; ; \; D_{q,\omega} L^*_{q,\omega} = q^{-1} L^*_{q,\omega} D_{q,\omega} \; ; \; D_{q,\omega} L_{q,\omega} = q L_{q,\omega} D_{q,\omega} \; ; \tag{1.12}$$

$$D^*_{q,\omega} L_{q,\omega} = q D_{q,\omega} \; ; \quad \mathbf{D}^*_{q,\omega} \mathbf{L}_{q\omega} = q \mathbf{D}_{q,\omega} \; ; \tag{1.13}$$

$$L_{q,\omega}(fg) = (L_{q,\omega} f)(L_{q,\omega} g) \; ; \quad \mathbf{L}_{q,\omega}(f\mathbf{u}) = L_{q,\omega} f \, \mathbf{L}_{q,\omega} \mathbf{u} \; ; \tag{1.14}$$

$$D_{q,\omega}(fg) = (D_{q,\omega} f)(L_{q,\omega} g) + f D_{q,\omega} g \tag{1.15}$$

$$\mathbf{D}_{q,\omega}(f\mathbf{u}) = D_{q,\omega} f \, \mathbf{L}_{q,\omega} \mathbf{u} + f \mathbf{D}_{q,\omega} \mathbf{u} = D_{q,\omega} f \, \mathbf{u} + L_{q,\omega} f \, \mathbf{D}_{q,\omega} \mathbf{u} \; . \tag{1.16}$$

(In (1.9), I *and* $\mathbf{I}$ *denote the identity operators in* $\mathscr{P}$ *and in* $\mathscr{P}^*$, *respectively.)*

We also point out the following analogue of Leibniz's formula.

Proposition 1.2.2 *Let* $f, g \in \mathscr{P}$ *be two polynomials, then we have*

$$D^n_{q,\omega}(fg) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q L^k_{q,\omega}\big(D^{n-k}_{q,\omega} f\big) \cdot D^k_{q,\omega} g \quad (n = 0, 1, 2, \ldots) \; , \tag{1.17}$$

for each $n = 0, 1, 2, \ldots$ *where, defining the* $q-$*factorials as* $[0]_q! := 1$ *and* $[n]_q! := [1]_q [2]_q \cdots [n]_q$ *for* $n \in \mathbb{N}$, *the* $q-$*binomial number is given by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!} \quad (n, k \in \mathbb{N}_0 \; ; \; k \leq n) \; .$$

Note that (1.17) can be easily deduced from the well known Leibniz formula for the operator D_q (see e.g. [26, Exercise 12.1] or [34, (1.15.6)]) and using the relation (1.8) between D_q and $D_{q,\omega}$. There is also a functional version of the Leibniz formula.

Proposition 1.2.3 *Let* $\mathbf{u} \in \mathscr{P}^*$ *be a linear functional and* $f \in \mathscr{P}$ *be a polynomial.*
Then we have

$$\mathbf{D}^n_{q,\omega}(f\boldsymbol{u}) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q L^{n-j}_{q,\omega} \left(D^j_{q,\omega} f\right) \mathbf{D}^{n-j}_{q,\omega} \boldsymbol{u} = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q L^j_{q,\omega} \left(D^{n-j}_{q,\omega} f\right) \mathbf{D}^j_{q,\omega} \boldsymbol{u} \; , \tag{1.18}$$

for $n = 0, 1, 2, \ldots$.

Formulas (1.17)–(1.18) can be proved easily by induction on $n \in \mathbb{N}_0$. A basic property of Hahn's operator relies upon the fact it maps a polynomial of degree n into one of degree $n-1$. Indeed, since $D_{q,\omega} x^n = \sum_{k=0}^{n-1} (qx + \omega)^k x^{n-1-k}$, applying the binomial formula to $(qx + \omega)^k$, we obtain

$$D_{q,\omega} x^n = \sum_{k=0}^{n-1} [n, k]_{q,\omega} x^{n-1-k} = [n]_q x^{n-1} + (\text{lower degree terms}) \; , \tag{1.19}$$

where the number $[n,k]_{q,\omega}$ is defined by

$$[n,k]_{q,\omega} := \omega^k \sum_{j=0}^{n-1-k} \binom{k+j}{j} q^j \quad (n,k=0,1,\ldots)\,.$$

We adopt the convention that an empty sum equals zero, hence

$$[n,k]_{q,\omega} = 0 \quad \text{if} \quad n \le k\,.$$

We also point out the following useful representations:

$$[n,k]_{q,\omega} = \frac{\omega^k}{k!}\frac{\mathrm{d}^k}{\mathrm{d}q^k}\left(\sum_{j=k}^{n-1} q^j\right) = \frac{\omega^k}{k!}\frac{\mathrm{d}^k}{\mathrm{d}q^k}\left(\frac{q^n-q^k}{q-1}\right) = \frac{\omega^k}{k!}\frac{\mathrm{d}^k}{\mathrm{d}q^k}\Big([n]_q-[k]_q\Big)\,.$$

In particular, for $k \in \{0,1,2\}$, we compute

$$\begin{aligned}
[n,0]_{q,\omega} &= [n]_q\,,\\
[n,1]_{q,\omega} &= \big(n[n-1]_q-(n-1)[n]_q\big)\,\omega_0\,,\\
[n,2]_{q,\omega} &= \big(n(n-1)[n-2]_q-2n(n-2)[n-1]_q+(n-2)(n-1)[n]_q\big)\,\omega_0^2/2\,,
\end{aligned}$$

where ω_0 is given by (1.8). Taking $\omega = 0$ in (1.19) we see that D_q fulfills

$$D_q x^n = [n]_q x^{n-1} \quad (n=0,1,\ldots)\,. \tag{1.20}$$

The usefulness of this property relies upon the following fact: if $\mathbf{u} \in \mathscr{P}^*$, $\phi \in \mathscr{P}_2$, and $\psi \in \mathscr{P}_1$, then $\mathbf{u}$ satisfies the functional equation $D_q(\phi\mathbf{u}) = \psi\mathbf{u}$ if and only if the sequence of moments $(u_n := \langle \mathbf{u}, x^n\rangle)_{n\ge 0}$ satisfies the homogeneous second order linear difference equation

$$\Big(\psi'(0)q^n + \frac{\phi''(0)}{2}[n]_q\Big)u_{n+1} + \Big(\psi(0)q^n + \phi'(0)[n]_q\Big)u_n + \phi(0)[n]_q u_{n-1} = 0 \quad (n=0,1,2,\ldots)\,. \tag{1.21}$$

Of course, taking into account (1.19), the analogous to property (1.20) is no longer true if D_q is replaced by $D_{q,\omega}$ ($\omega \neq 0$). Hence, one can not expect that the moments corresponding to a functional $\mathbf{u}$ fulfilling $D_{q,\omega}(\phi\mathbf{u}) = \psi\mathbf{u}$ —being $\mathbf{u}$, ϕ, and ψ as above— satisfy a second order difference equation like (1.21). Häcker replaced the power basis $(x^n)_{n\ge 0}$ by a different polynomial basis, $(X_n)_{n\ge0} \equiv (X_n(\cdot;q,\omega))_{n\ge 0}$ defined as follows.

Proposition 1.2.4 *[22]*
Let define a simple set of polynomials $(X_n)_{n\ge 0}$ by

$$X_0(x) := 0, \quad X_{n+1}(x) = q^{-n}\big(x-\omega[n]_q\big)X_n(x) \quad (n=0,1,2,\ldots)\,.$$

Then we have

$$D_{q,\omega}X_n = q^{1-n}[n]_q X_{n-1} \quad (n=0,1,2,\ldots)\,. \tag{1.22}$$

For our purposes it is more convenient to use a basis (of $\mathscr{P}$) of monic polynomials, namely $(Y_n)_{n\geq 0} \equiv (Y_n(\cdot;q,\omega))_{n\geq 0}$, where $Y_n := q^{\binom{n}{2}} X_n$, so that

$$Y_0(x)=1\,,\quad Y_{n+1}(x)=\big(x-\omega[n]_q\big)Y_n(x)=\prod_{j=0}^{n}\big(x-\omega[j]_q\big)\quad (n=0,1,2,\dots)\,. \tag{1.23}$$

Clearly, $(Y_n)_{n\geq 0}$ fulfills the desired property:

$$D_{q,\omega}Y_n(x)=[n]_qY_{n-1}(x)\quad (n=0,1,2,\dots)\,. \tag{1.24}$$

Finally, using (1.24) it is straightforward (e.g. by using Mathematica, or by induction on $n\in\mathbb{N}_0$) to show the following result.

Proposition 1.2.5 *Let* $\mathbf{u}\in\mathscr{P}^*$ *be a linear functional. Then* $\mathbf{u}$ *satisfies the functional equation* $D_{q,\omega}(\phi\mathbf{u})=\psi\mathbf{u}$, *where* $\phi\in\mathscr{P}_2$ *and* $\psi\in\mathscr{P}_1$, *if and only if the sequence of moments with respect to the basis* $(Y_n)_{n\geq 0}$, $(y_n:=\langle\mathbf{u},Y_n\rangle)_{n\geq 0}$ *defined in (1.23), fulfills*

$$d_ny_{n+1}+\big(e_n+\omega[n]_qd_{n-1}\big)y_n+[n]_q(\phi(0)+\omega e_{n-1})y_{n-1}=0\quad (n=0,1,2,\dots)\,, \tag{1.25}$$

where $(d_n)_{n\geq 0}$ *and* $(e_n)_{n\geq 0}$ *are sequences of complex numbers given by*

$$d_n=\psi'(0)q^n+\frac{\phi''(0)}{2}[n]_q\,,\quad e_n=\psi(0)q^n+\Big(\omega d_n+\phi'(0)\Big)[n]_q\,.$$

1.3 Some operators on a nonuniform lattice (NUL)

A nonuniform lattice (NUL) is a mapping $x(s)$, $s\in\mathbb{C}$, given by

$$x(s):=\begin{cases}\mathfrak{c}_1q^{-s}+\mathfrak{c}_2q^s+\mathfrak{c}_3 & \text{if}\quad q\neq 1\,,\\ \mathfrak{c}_4s^2+\mathfrak{c}_5s+\mathfrak{c}_6 & \text{if}\quad q=1\,,\end{cases} \tag{1.26}$$

where $q>0$ (fixed) and $\mathfrak{c}_j$ $(1\leq j\leq 6)$ are constants in $\mathbb{C}$, that may depend on q, such that $(\mathfrak{c}_1,\mathfrak{c}_2)\neq(0,0)$ if $q\neq 1$, and $(\mathfrak{c}_4,\mathfrak{c}_5,\mathfrak{c}_6)\neq(0,0,0)$ if $q=1$. In the case $q=1$, the lattice is called quadratic if $\mathfrak{c}_4\neq 0$, and it is called linear if $\mathfrak{c}_4=0$; and in the case $q\neq 1$, it is called $q-$quadratic if $\mathfrak{c}_1\mathfrak{c}_2\neq 0$, and $q-$linear if $\mathfrak{c}_1\mathfrak{c}_2=0$ (cf. [7]). Notice that

$$\frac{x\big(s+\frac12\big)+x\big(s-\frac12\big)}{2}=\alpha x(s)+\beta\,,$$

where α and β are given by

$$\alpha:=\frac{q^{1/2}+q^{-1/2}}{2}\,,\quad \beta:=\begin{cases}(1-\alpha)\mathfrak{c}_3 & \text{if}\quad q\neq 1\,,\\ \mathfrak{c}_4/4 & \text{if}\quad q=1\,.\end{cases} \tag{1.27}$$

The lattice $x(s)$ fulfills (cf. [7]):

$$\frac{x(s+n)+x(s)}{2}=\alpha_n x_n(s)+\beta_n\,,\quad x(s+n)-x(s)=\gamma_n \nabla x_{n+1}(s),\quad n=0,1,2,\ldots$$

where $x_\mu(s):=x\big(s+\frac{\mu}{2}\big)$, $\nabla f(s):=f(s)-f(s-1)$, and $(\alpha_n)_{n\geq 0}$, $(\beta_n)_{n\geq 0}$, and $(\gamma_n)_{n\geq 0}$ are sequences of numbers given by the following system of difference equations

$$\alpha_0=1\,,\quad \alpha_1=\alpha\,,\quad \alpha_{n+1}-2\alpha\alpha_n+\alpha_{n-1}=0 \tag{1.28}$$

$$\beta_0=0\,,\quad \beta_1=\beta\,,\quad \beta_{n+1}-2\beta_n+\beta_{n-1}=2\beta\alpha_n \tag{1.29}$$

$$\gamma_0=0\,,\quad \gamma_1=1\,,\quad \gamma_{n+1}-\gamma_{n-1}=2\alpha_n \quad n=1,2,3,\ldots\,. \tag{1.30}$$

The explicit solutions of these difference equations are

$$\alpha_n=\frac{q^{n/2}+q^{-n/2}}{2}\,, \tag{1.31}$$

$$\beta_n=\begin{cases}\beta\left(\dfrac{q^{n/4}-q^{-n/4}}{q^{1/4}-q^{-1/4}}\right)^2 & \text{if}\quad q\neq 1\\ \beta\, n^2 & \text{if}\quad q=1\,,\end{cases} \tag{1.32}$$

$$\gamma_n=\begin{cases}\dfrac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}} & \text{if}\quad q\neq 1\\ n & \text{if}\quad q=1\,.\end{cases} \tag{1.33}$$

These formulas may be easily checked (alternatively, see [7]). We also point out the following useful relations:

$$\gamma_{n+1}-2\alpha\gamma_n+\gamma_{n-1}=0\,, \tag{1.34}$$

$$\alpha_n+\gamma_{n-1}=\alpha\gamma_n\,, \tag{1.35}$$

$$(2\alpha^2-1)\alpha_n+(\alpha^2-1)\gamma_{n-1}=\alpha\alpha_{n+1}\,, \tag{1.36}$$

$$\gamma_{2n}=2\alpha_n\gamma_n\,, \tag{1.37}$$

$$\alpha_n^2+(\alpha^2-1)\gamma_n^2=\alpha_{2n}=2\alpha_n^2-1\,, \tag{1.38}$$

$$\alpha_{n-1}-\alpha\alpha_n=(1-\alpha^2)\gamma_n\,, \tag{1.39}$$

$$\alpha+\alpha_n\gamma_n=\alpha_{n-1}\gamma_{n+1}\,, \tag{1.40}$$

$$1+\alpha_{n+1}\gamma_n=\alpha_n\gamma_{n+1}\,, \tag{1.41}$$

valid for each $n=0,1,2,\ldots$ (provided we define $\alpha_{-1}:=\alpha$ and $\gamma_{-1}:=-1$, in consistence with (1.31) and (1.33)).

Definition 1.8 *Consider the lattice $x(s)$ given by (1.26). We define two operators $\mathrm{D}_x:\mathscr{P}\to\mathscr{P}$ (called the $x-$derivative operator on $\mathscr{P}$) and $\mathrm{S}_x:\mathscr{P}\to\mathscr{P}$ (called the $x-$averaging operator on $\mathscr{P}$),* *so that* $\deg(\mathrm{D}_x f) = \deg f - 1$, $\deg(\mathrm{S}_x f) = \deg f$, *and*

$$\mathrm{D}_x f(x(s)) = \frac{f\big(x(s+\frac{1}{2})\big) - f\big(x(s-\frac{1}{2})\big)}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})} \,, \tag{1.42}$$

and

$$\mathrm{S}_x f(x(s)) = \frac{f\left(x\big(s+\frac{1}{2}\big)\right) + f\left(x\big(s-\frac{1}{2}\big)\right)}{2} \tag{1.43}$$

for each $f \in \mathscr{P}$ *(see [6, 7, 17]). Further, we set* $D_x f = f'$ *and* $S_x f = f$*, for all* $f \in \mathscr{P}$*, whenever* $x(s) = \mathfrak{c}_6$.

The operators D_x and S_x on $\mathscr{P}$ induce two operators on the dual space $\mathscr{P}^*$:

Definition 1.9 *We define* $\mathbf{D}_x : \mathscr{P}^* \to \mathscr{P}^*$ *and* $\mathbf{S}_x : \mathscr{P}^* \to \mathscr{P}^*$ *by*

$$\langle \mathbf{D}_x \mathbf{u}, f\rangle := -\langle \mathbf{u}, \mathrm{D}_x f\rangle \,, \quad \langle \mathbf{S}_x \mathbf{u}, f\rangle := \langle \mathbf{u}, \mathrm{S}_x f\rangle \quad (f \in \mathscr{P},\ \mathbf{u} \in \mathscr{P}^*) \,. \tag{1.44}$$

For each $\mathbf{u} \in \mathscr{P}^*$*, the functional* $\mathbf{D}_x\mathbf{u} \in \mathscr{P}^*$ *is called the* $x-$*derivative of* $\mathbf{u}$*, while* $\mathbf{S}_x\mathbf{u} \in \mathscr{P}^*$ *is called the* $x-$*average of* $\mathbf{u}$ *(see [17]).*

Hereafter, $z := x(s)$ being the lattice (1.26), we define polynomials $\mathtt{U}_1$ and $\mathtt{U}_2$ by

$$\mathtt{U}_1(z) := (\alpha^2 - 1)z + \beta(\alpha + 1) \,, \tag{1.45}$$

$$\mathtt{U}_2(z) := (\alpha^2 - 1)z^2 + 2\beta(\alpha + 1)z + \delta \,, \tag{1.46}$$

$\delta \equiv \delta_x$ being a constant with respect to the lattice, given by

$$\delta := \left(\frac{x(0) + x(1) - 2\beta(\alpha+1)}{2\alpha}\right)^2 - x(0)x(1) \,. \tag{1.47}$$

It is also straightforward to verify that

$$\delta = \begin{cases} (\alpha^2 - 1)\big(\mathfrak{c}_3^2 - 4\mathfrak{c}_1\mathfrak{c}_2\big) & \text{if} \quad q \neq 1 \,, \\ \frac{1}{4}\mathfrak{c}_5^2 - \mathfrak{c}_4\mathfrak{c}_6 & \text{if} \quad q = 1 \,, \end{cases} \tag{1.48}$$

and

$$\mathtt{U}_1(z) = \begin{cases} (\alpha^2 - 1)\big(z - \mathfrak{c}_3\big) & \text{if} \quad q \neq 1 \,, \\ \frac{1}{2}\mathfrak{c}_4 & \text{if} \quad q = 1 \,, \end{cases} \tag{1.49}$$

hence, we deduce

$$\mathtt{U}_2(z) = \begin{cases} (\alpha^2 - 1)\big((z - \mathfrak{c}_3)^2 - 4\mathfrak{c}_1\mathfrak{c}_2\big) & \text{if} \quad q \neq 1 \,, \\ \mathfrak{c}_4(z - \mathfrak{c}_6) + \frac{1}{4}\mathfrak{c}_5^2 & \text{if} \quad q = 1 \,. \end{cases} \tag{1.50}$$

By direct computations, we may check that the polynomials introduced in (1.45)–(1.46) satisfy the following relations:

$$\mathrm{D}_x \mathtt{U}_1 = \alpha^2 - 1\,, \qquad \mathrm{S}_x \mathtt{U}_1 = \alpha \mathtt{U}_1\,, \tag{1.51}$$

$$\mathrm{D}_x \mathtt{U}_2 = 2\alpha \mathtt{U}_1\,, \qquad \mathrm{S}_x \mathtt{U}_2 = \alpha^2 \mathtt{U}_2 + \mathtt{U}_1^2\,. \tag{1.52}$$

1.3.1 Some properties

We start by pointing out some properties.

Lemma 1.3.1 *Let $f,g \in \mathscr{P}$ and $\mathbf{u} \in \mathscr{P}^*$. Then the following properties hold:*

$$\mathrm{D}_x\big(fg\big) = \big(\mathrm{D}_x f\big)\big(\mathrm{S}_x g\big) + \big(\mathrm{S}_x f\big)\big(\mathrm{D}_x g\big)\,, \tag{1.53}$$

$$\mathrm{S}_x\big(fg\big) = \big(\mathrm{D}_x f\big)\big(\mathrm{D}_x g\big)\mathtt{U}_2 + \big(\mathrm{S}_x f\big)\big(\mathrm{S}_x g\big)\,, \tag{1.54}$$

$$\mathrm{S}_x \mathrm{D}_x f = \alpha \mathrm{D}_x \mathrm{S}_x f - \mathrm{D}_x\big(\mathtt{U}_1 \mathrm{D}_x f\big)\,, \tag{1.55}$$

$$\mathrm{S}_x^2 f = \alpha^{-1}\mathrm{S}_x\big(\mathtt{U}_1 \mathrm{D}_x f\big) + \alpha^{-1}\mathtt{U}_2 \mathrm{D}_x^2 f + f\,, \tag{1.56}$$

$$f\mathrm{S}_x g = \mathrm{S}_x\Big(\big(\mathrm{S}_x f - \alpha^{-1}\mathtt{U}_1\,\mathrm{D}_x f\big)g\Big) - \alpha^{-1}\mathtt{U}_2 \mathrm{D}_x\big(g\mathrm{D}_x f\big)\,, \tag{1.57}$$

$$f\mathrm{D}_x g = \mathrm{D}_x\Big(\big(\mathrm{S}_x f - \alpha^{-1}\mathtt{U}_1\,\mathrm{D}_x f\big)g\Big) - \alpha^{-1}\mathrm{S}_x\big(g\mathrm{D}_x f\big)\,, \tag{1.58}$$

$$\mathbf{D}_x(f\mathbf{u}) = \big(\mathrm{S}_x f - \alpha^{-1}\mathtt{U}_1 \mathrm{D}_x f\big)\,\mathbf{D}_x \mathbf{u} + \alpha^{-1}\big(\mathrm{D}_x f\big)\mathbf{S}_x \mathbf{u}\,, \tag{1.59}$$

$$\mathbf{S}_x(f\mathbf{u}) = \big(\mathrm{S}_x f - \alpha^{-1}\mathtt{U}_1 \mathrm{D}_x f\big)\,\mathbf{S}_x \mathbf{u} + \alpha^{-1}\big(\mathrm{D}_x f\big)\mathbf{D}_x(\mathtt{U}_2 \mathbf{u})\,, \tag{1.60}$$

$$\mathbf{S}_x(f\mathbf{u}) = \big(\alpha \mathtt{U}_2 - \alpha^{-1}\mathtt{U}_1^2\big)(\mathrm{D}_x f)\mathbf{D}_x \mathbf{u} + \big(\mathrm{S}_x f + \alpha^{-1}\mathtt{U}_1 \mathrm{D}_x f\big)\mathbf{S}_x \mathbf{u}\,, \tag{1.61}$$

$$\mathbf{D}_x^2\big(\mathtt{U}_2 \mathbf{u}\big) = \alpha \mathbf{S}_x^2 \mathbf{u} + \mathbf{D}_x\big(\mathtt{U}_1 \mathbf{S}_x \mathbf{u}\big) - \alpha \mathbf{u}\,, \tag{1.62}$$

$$\mathbf{D}_x^2\big(\mathtt{U}_2 \mathbf{u}\big) = (2\alpha - \alpha^{-1})\mathbf{S}_x^2 \mathbf{u} + \alpha^{-1}\mathtt{U}_1 \mathbf{D}_x \mathbf{S}_x \mathbf{u} - \alpha \mathbf{u}\,, \tag{1.63}$$

$$\mathbf{D}_x \mathbf{S}_x \mathbf{u} = \alpha \mathbf{S}_x \mathbf{D}_x \mathbf{u} + \mathbf{D}_x\big(\mathtt{U}_1 \mathbf{D}_x \mathbf{u}\big)\,. \tag{1.64}$$

Proof The reader may encounter properties (1.53)–(1.60) in [50, Propositions 5–7]. To prove (1.61), set $f = \mathtt{U}_2$ in (1.59) and then use (1.52) to obtain

$$\mathbf{D}_x(\mathtt{U}_2 \mathbf{u}) = \big(\alpha^2 \mathtt{U}_2 - \mathtt{U}_1^2\big)\mathbf{D}_x \mathbf{u} + 2\mathtt{U}_1 \mathbf{S}_x \mathbf{u}\,.$$

Replacing this expression in the right-hand side of (1.60) we obtain (1.61). Next, taking arbitrarily $f \in \mathscr{P}$, we have

$$\begin{aligned}\langle \mathbf{D}_x^2\big(\mathtt{U}_2 \mathbf{u}\big), f\rangle &= \langle \mathbf{u}, \mathtt{U}_2 \mathrm{D}_x^2 f\rangle = \langle \mathbf{u}, \alpha \mathrm{S}_x^2 f - \mathrm{S}_x(\mathtt{U}_1 \mathrm{D}_x f) - \alpha f\rangle \\ &= \langle \alpha \mathbf{S}_x^2 \mathbf{u} + \mathbf{D}_x\big(\mathtt{U}_1 \mathbf{S}_x \mathbf{u}\big) - \alpha \mathbf{u}, f\rangle\,,\end{aligned}$$

where the second equality holds by (1.56). This proves (1.62). Setting $f = \mathtt{U}_1$ in (1.59) and replacing therein $\mathbf{u}$ by $\mathbf{S}_x \mathbf{u}$, and taking into account (1.51), we deduce

$$\mathbf{D}_x\big(\mathtt{U}_1 \mathbf{S}_x \mathbf{u}\big) = \alpha^{-1}\mathtt{U}_1 \mathbf{D}_x \mathbf{S}_x \mathbf{u} + (\alpha - \alpha^{-1})\mathbf{S}_x^2 \mathbf{u}\,.$$

Substituting this into the right-hand side of (1.62) we obtain (1.63). Finally, (1.64) follows easily from (1.44) and (1.55).

Proposition 1.3.2 *Let us consider the q-quadratic lattice* $x(s) = \mathfrak{c}_1 q^{-s} + \mathfrak{c}_2 q^s + \mathfrak{c}_3$. *Then the following relations hold.*

$$
\begin{aligned}
\mathrm{D}_x z^n &= \gamma_n z^{n-1} + u_n z^{n-2} + v_n z^{n-3} + \cdots\,, &(1.65)\\
\mathrm{S}_x z^n &= \alpha_n z^n + \widehat{u}_n z^{n-1} + \widehat{v}_n z^{n-2} + \cdots\,, &(1.66)
\end{aligned}
$$

where α_n *and* γ_n *are given by* (1.31) *and* (1.33)*, respectively, and*

$$
\begin{aligned}
u_n &:= \big(n\gamma_{n-1} - (n-1)\gamma_n\big)\mathfrak{c}_3\,, &(1.67)\\
v_n &:= \big(n\gamma_{n-2} - (n-2)\gamma_n\big)\mathfrak{c}_1\mathfrak{c}_2 &(1.68)\\
&\quad + \tfrac{1}{2}\big(n(n-1)\gamma_{n-2} - 2n(n-2)\gamma_{n-1} + (n-1)(n-2)\gamma_n\big)\mathfrak{c}_3^2\,,\\
\widehat{u}_n &:= n(\alpha_{n-1} - \alpha_n)\mathfrak{c}_3\,, &(1.69)\\
\widehat{v}_n &:= n(\alpha_{n-2} - \alpha_n)\mathfrak{c}_1\mathfrak{c}_2 + n(n-1)(\alpha-1)\alpha_{n-1}\mathfrak{c}_3^2\,. &(1.70)
\end{aligned}
$$

for $n = 0, 1, 2, \ldots$.

Proof The proof is done by mathematical induction on $n \in \mathbb{N}_0$. For $n = 0$, we have $D_x z^0 = 0$ and $S_x z^0 = 1$. So we have $\gamma_0 = 0$, $\alpha_0 = 1$ and $u_0 = v_0 = \widehat{u}_0 = \widehat{v}_0 = 0$. Then (1.65)–(1.66) is true for $n = 0$. Now suppose that (1.65)–(1.66) are true for all integers less than or equal to a fixed n. Then by using this hypothesis together with (1.53)–(1.54), we have

$$
\begin{aligned}
D_x z^{n+1} &= D_x(zz^n) = D_x z^n\, S_x z + S_x z^n\, D_x z\\
&= (\alpha z + \beta) D_x z^n + S_x z^n\\
&= (\alpha_n + \alpha\gamma_n) z^n + (\alpha u_n + \widehat{u}_n + \beta\gamma_n) z^{n-1} + (\alpha v_n + \widehat{v}_n + \beta u_n) z^{n-2} + \cdots\,.
\end{aligned}
$$

In a similar way we also have

$$
\begin{aligned}
S_x z^{n+1} &= S_x(zz^n) = \mathtt{U}_2(z) D_x z\, D_x z^n + S_x z^n\, S_x z\\
&= \mathtt{U}_2(z) D_x z^n + (\alpha z + \beta) S_x z^n\\
&= \big(\alpha\alpha_n + (\alpha^2-1)\gamma_n\big) z^{n+1} + \big[(\alpha^2-1)(u_n - 2\gamma_n\mathfrak{c}_3) + \alpha\widehat{u}_n + \beta\alpha_n\big] z^n\\
&\quad + \big[(\alpha^2-1)\big(v_n - 2u_n\mathfrak{c}_3 + (\mathfrak{c}_3^2 - 4\mathfrak{c}_1\mathfrak{c}_2)\gamma_n\big) + \alpha\widehat{v}_n + \beta\widehat{u}_n\big] z^{n-1} + \cdots\,.
\end{aligned}
$$

Using (1.34)–(1.41), we finally obtain

$$
\begin{aligned}
D_x z^{n+1} &= \gamma_{n+1} z^n + u_{n+1} z^{n-1} + v_{n+1} z^{n-2} + \cdots\,,\\
S_x z^{n=1} &= \alpha_{n+1} z^{n+1} + \widehat{u}_{n+1} z^n + \widehat{v}_{n+1} z^{n-1} + \cdots\,.
\end{aligned}
$$

Thus (1.65)–(1.66) are proved for all $n \in \mathbb{N}_0$.

1.3.2 A Leibniz-type formula

Here we state a Leibniz-type formula, involving the $x-$derivative operator, for the left multiplication of a functional by a polynomial. For this, we need the following results.

Lemma 1.3.3 *Let $f \in \mathscr{P}$. Then*

$$D_x^n S_x f = \alpha_n S_x D_x^n f + \gamma_n \mathtt{U}_1 D_x^{n+1} f \quad (n = 0, 1, 2, \ldots) . \tag{1.71}$$

Proof By mathematical induction on n. (1.71) is satisfied for $n = 0$. Suppose that for all positive integers less than or equal to a fixed integer n, (1.71) is true. Then by using successively (1.71) firstly for n and secondly for $n = 1$ with f replaced by $D_x^n f$, and (1.53) to $D_x(\mathtt{U}_1 D_x^{n+1} f)$ we have

$$\begin{aligned} D_x^{n+1} S_x f &= D_x \left(D_x^n S_x f\right) = D_x \left(\alpha_n S_x D_x^n f + \gamma_n \mathtt{U}_1 D_x^{n+1} f\right) \\ &= \alpha_n D_x S_x (D_x^n f) + \gamma_n D_x \left(\mathtt{U}_1 D_x^{n+1} f\right) \\ &= \alpha_n \left(\alpha S_x D_x^{n+1} f + \mathtt{U}_1 D_x^{n+2} f\right) + \gamma_n \left((\alpha^2 - 1) S_x D_x^{n+1} f + \alpha \mathtt{U}_1 D_x^{n+2} f\right) \\ &= \left(\alpha \alpha_n + (\alpha^2 - 1)\gamma_n\right) S_x D_x^{n+1} f + (\alpha_n + \alpha \gamma_n) \mathtt{U}_1 D_x^{n+2} f . \end{aligned}$$

Using (1.34)–(1.39), we see that (1.71) is true whenever n is replaced by $n+1$. Hence (1.71) is true for all n.

There is a functional version of (1.71).

Lemma 1.3.4 *Let $\mathbf{u} \in \mathscr{P}^*$. Then*

$$\alpha \mathbf{D}_x^n \mathbf{S}_x \mathbf{u} = \alpha_{n+1} \mathbf{S}_x \mathbf{D}_x^n \mathbf{u} + \gamma_n \mathtt{U}_1 \mathbf{D}_x^{n+1} \mathbf{u} \quad (n = 0, 1, 2, \ldots) . \tag{1.72}$$

Proof We prove (1.72) by mathematical induction on n. Since $\alpha_1 = \alpha$ and $\gamma_0 = 0$, then (1.72) is trivial for $n = 0$. For $n = 1$, (1.72) is obtained multiplying both sides of (1.64) by α and taking into account that, by (1.59) and (1.51), the equality $\alpha \mathbf{D}_x\big(\mathtt{U}_1 \mathbf{D}_x \mathbf{u}\big) = \mathtt{U}_1 \mathbf{D}_x^2 \mathbf{u} + (\alpha^2 - 1) \mathbf{S}_x \mathbf{D}_x \mathbf{u}$ holds, and recalling also that $\alpha_2 = 2\alpha^2 - 1$ and $\gamma_1 = 1$. Suppose now (induction hypothesis) that property (1.72) holds for a fixed integer $n \in \mathbb{N}$. Then, we have

$$\alpha \mathbf{D}_x^{n+1} \mathbf{S}_x \mathbf{u} = \mathbf{D}_x\big(\alpha \mathbf{D}_x^n \mathbf{S}_x \mathbf{u}\big) = \alpha_{n+1} \mathbf{D}_x \mathbf{S}_x \mathbf{D}_x^n \mathbf{u} + \gamma_n \mathbf{D}_x\big(\mathtt{U}_1 \mathbf{D}_x^{n+1} \mathbf{u}\big) . \tag{1.73}$$

Considering (1.72) for $n = 1$ and replacing therein $\mathbf{u}$ by $\mathbf{D}_x^n \mathbf{u}$, we obtain

$$\mathbf{D}_x \mathbf{S}_x \mathbf{D}_x^n \mathbf{u} = \alpha^{-1} \alpha_2 \mathbf{S}_x \mathbf{D}_x^{n+1} \mathbf{u} + \alpha^{-1} \gamma_n \mathtt{U}_1 \mathbf{D}_x^{n+2} \mathbf{u} . \tag{1.74}$$

Moreover, using again (1.59) and (1.51), we deduce

$$\mathbf{D}_x\big(\mathtt{U}_1 \mathbf{D}_x^{n+1} \mathbf{u}\big) = \alpha^{-1} \mathtt{U}_1 \mathbf{D}_x^{n+2} \mathbf{u} + \alpha^{-1} (\alpha^2 - 1) \mathbf{S}_x \mathbf{D}_x^{n+1} \mathbf{u} . \tag{1.75}$$

Putting (1.74) and (1.75) into the right-hand side of (1.73) and taking into account (1.35) and (1.36), we obtain (1.72) with n replaced by $n+1$. This proves (1.72).

Next, we introduce the operator $\mathrm{T}_{n,k} : \mathscr{P} \to \mathscr{P}$ ($n = 0,1,2\ldots; k = 0,1,2\ldots,n$), defined for each $f \in \mathscr{P}$ as follows: if $n = k = 0$, set

$$\mathrm{T}_{0,0} f := f \; ; \tag{1.76}$$

and if $n \geq 1$ and $0 \leq k \leq n$, define recurrently

$$\mathrm{T}_{n,k} f := \mathrm{S}_x \mathrm{T}_{n-1,k} f - \frac{\gamma_{n-k}}{\alpha_{n-k}} \mathrm{U}_1 \mathrm{D}_x \mathrm{T}_{n-1,k} f + \frac{1}{\alpha_{n+1-k}} \mathrm{D}_x \mathrm{T}_{n-1,k-1} f \,, \tag{1.77}$$

with the conventions $\mathrm{T}_{n,k} f := 0$ whenever $k > n$ or $k < 0$. Note that

$$\deg \mathrm{T}_{n,k} f \leq \deg f - k \; .$$

We are ready to state the following.

Proposition 1.3.5 (Leibniz-type formula) *Let* $\mathbf{u} \in \mathscr{P}^*$ *and* $f \in \mathscr{P}$. *Then*

$$\mathbf{D}_x^n \big(f \mathbf{u} \big) = \sum_{k=0}^{n} \mathrm{T}_{n,k} f \, \mathbf{D}_x^{n-k} \mathbf{S}_x^k \mathbf{u} \quad (n = 0,1,2,\ldots), \tag{1.78}$$

where $\mathrm{T}_{n,k} f$ *is a polynomial defined by* (1.76)–(1.77).

Proof The proof is done by mathematical induction on n. Clearly, (1.78) is true if $n = 0$. Suppose now that (1.78) holds for a fixed nonnegative integer n. Then

$$\mathbf{D}_x^{n+1} \big(f \mathbf{u} \big) = \mathbf{D}_x \big(\mathbf{D}_x^n (f \mathbf{u}) \big) = \sum_{k=0}^{n} \mathbf{D}_x \big(\mathrm{T}_{n,k} f \mathbf{D}_x^{n-k} \mathbf{S}_x^k \mathbf{u} \big) \; . \tag{1.79}$$

Notice that, by (1.72),

$$\mathbf{S}_x \mathbf{D}_x^{n-k} \mathbf{S}_x^k \mathbf{u} = \frac{1}{\alpha_{n+1-k}} \Big(\alpha \mathbf{D}_x^{n-k} \mathbf{S}_x^{k+1} \mathbf{u} - \gamma_{n-k} \mathrm{U}_1 \mathbf{D}_x^{n+1-k} \mathbf{S}_x^k \mathbf{u} \Big) \; . \tag{1.80}$$

Therefore, using successively (1.59), (1.80), (1.35), and (1.77), we may write

$$\begin{aligned}
&\mathbf{D}_x \big(\mathrm{T}_{n,k} f \mathbf{D}_x^{n-k} \mathbf{S}_x^k \mathbf{u} \big) \\
&\quad = \big(\mathrm{S}_x \mathrm{T}_{n,k} f - \alpha^{-1} \mathrm{U}_1 \mathrm{D}_x \mathrm{T}_{n,k} f \big) \mathbf{D}_x^{n+1-k} \mathbf{S}_x^k \mathbf{u} + \alpha^{-1} \mathrm{D}_x \mathrm{T}_{n,k} f \mathbf{S}_x \mathbf{D}_x^{n-k} \mathbf{S}_x^k \mathbf{u} \\
&\quad = \Big(\mathrm{S}_x \mathrm{T}_{n,k} f - \frac{\gamma_{n+1-k}}{\alpha_{n+1-k}} \mathrm{U}_1 \mathrm{D}_x \mathrm{T}_{n,k} f \Big) \mathbf{D}_x^{n+1-k} \mathbf{S}_x^k \mathbf{u} + \frac{\mathrm{D}_x \mathrm{T}_{n,k} f}{\alpha_{n+1-k}} \mathbf{D}_x^{n-k} \mathbf{S}_x^{k+1} \mathbf{u} \\
&\quad = \Big(\mathrm{T}_{n+1,k} f - \frac{\mathrm{D}_x \mathrm{T}_{n,k-1} f}{\alpha_{n+2-k}} \Big) \mathbf{D}_x^{n+1-k} \mathbf{S}_x^k \mathbf{u} + \frac{\mathrm{D}_x \mathrm{T}_{n,k} f}{\alpha_{n+1-k}} \mathbf{D}_x^{n-k} \mathbf{S}_x^{k+1} \mathbf{u} \, .
\end{aligned}$$

Substituting this expression in the right-hand side of (1.79) and then applying the method of telescoping sums, we get

$$\mathbf{D}_x^{n+1} \big(f \mathbf{u} \big) = \sum_{k=0}^{n} \mathrm{T}_{n+1,k} f \mathbf{D}_x^{n+1-k} \mathbf{S}_x^k \mathbf{u} + \frac{\mathrm{D}_x \mathrm{T}_{n,n} f}{\alpha_1} \mathbf{S}_x^{n+1} \mathbf{u} - \frac{\mathrm{D}_x \mathrm{T}_{n,-1} f}{\alpha_{n+2}} \mathbf{D}_x^{n+1} \mathbf{u} \; .$$

Finally, since $\mathrm{T}_{n,-1}f=0$ and $\frac{1}{\alpha_1}\mathbf{D}_x\mathrm{T}_{n,n}f=\mathrm{T}_{n+1,n+1}f$ (this last equality follows from (1.77) taking therein $k=n$ and in the resulting expression shifting n into $n+1$), we obtain (1.78) with n replaced by $n+1$. Thus (1.78) is proved.

Corollary 1.3.6 *Let x be the $q-$quadratic NUL $x(s):=\mathfrak{c}_1q^{-s}+\mathfrak{c}_2q^s+\mathfrak{c}_3$, $(s\in\mathbb{C};\ q>0)$. Let $\mathbf{u}\in\mathscr{P}^*$ and $f\in\mathscr{P}_2$. Set $f(z)=az^2+bz+c$, with $a,b,c\in\mathbb{C}$. Then*

$$\begin{aligned}\mathbf{D}_x^n(f\mathbf{u})=&\left(\frac{a\alpha}{\alpha_n\alpha_{n-1}}(z-\mathfrak{c}_3)^2+\frac{f'(\mathfrak{c}_3)}{\alpha_n}(z-\mathfrak{c}_3)+f(\mathfrak{c}_3)+\frac{4a(1-\alpha^2)\gamma_n\mathfrak{c}_1\mathfrak{c}_2}{\alpha_{n-1}}\right)\mathbf{D}_x^n\mathbf{u}\\&+\frac{\gamma_n}{\alpha_n}\left(\frac{a(\alpha_n+\alpha\alpha_{n-1})}{\alpha_{n-1}^2}(z-\mathfrak{c}_3)+f'(c_3)\right)\mathbf{D}_x^{n-1}\mathbf{S}_x\mathbf{u}\\&+\frac{a\gamma_n\gamma_{n-1}}{\alpha_{n-1}^2}\mathbf{D}_x^{n-2}\mathbf{S}_x^2\mathbf{u}\end{aligned}\tag{1.81}$$

for each $n=0,1,2,\ldots$. In particular,

$$\mathbf{D}_x^n\big((bz+c)\mathbf{u}\big)=\left(\frac{b}{\alpha_n}(z-\beta_n)+c\right)\mathbf{D}_x^n\mathbf{u}+\frac{b\gamma_n}{\alpha_n}\mathbf{D}_x^{n-1}\mathbf{S}_x\mathbf{u}\quad(n=0,1,2,\ldots)\,.\tag{1.82}$$

The proof of identities (1.81)–(1.82) relies upon the Leibniz formula (1.78), (1.59) and (1.72), by a straightforward computation. Alternatively, we may apply mathematical induction on n as follows.

Proof We only prove (1.81) since (1.82) is the particular case of (1.81) where $a=0$. Let's define $g(z)=f(z-\mathfrak{c}_3)=a(z-\mathfrak{c}_3)^2+b(z-\mathfrak{c}_3)+c$. We want to show that

$$(T_{n,0}g)(z)=g\left(\frac{z-\mathfrak{c}_3}{\alpha_n}+\mathfrak{c}_3\right)+\frac{a\gamma_n}{\alpha_{n-1}}\mathtt{U}_2\left(\frac{z-\mathfrak{c}_3}{\alpha_n}+\mathfrak{c}_3\right),\tag{1.83}$$

$$(T_{n,1}g)(z)=\frac{\gamma_n}{\alpha_n}\left(\frac{a(\alpha_n+\alpha\alpha_{n-1})}{\alpha_{n-1}^2}(z-\mathfrak{c}_3)+b\right),\tag{1.84}$$

$$(T_{n,2}g)(z)=\frac{a\gamma_n\gamma_{n-1}}{\alpha_{n-1}^2},\tag{1.85}$$

for $n=0,1,2,\ldots$, where $T_{n,k}f$ is defined by (1.76)–(1.77). Note that we have

$$(T_{n,0}g)(z)=\frac{\alpha a}{\alpha_n\alpha_{n-1}}(z-\mathfrak{c}_3)^2+\frac{b}{\alpha_n}(z-\mathfrak{c}_3)+c+\frac{4a(1-\alpha^2)\gamma_n}{\alpha_{n-1}}\mathfrak{c}_1\mathfrak{c}_2\,.\tag{1.86}$$

We proceed by induction on $n\in\mathbb{N}_0$. For $n=0$ in (1.83)–(1.85), we find $T_{0,0}g=g$ and $T_{0,1}g=0=T_{0,2}g$ which agree with (1.76)–(1.77). Now suppose that (1.83)–(1.85) is true for all positive integers up to a fixed n. Then by (1.77), we have

$$T_{n+1,0}g=S_x(T_{n,0}g)-\frac{\gamma_{n+1}}{\alpha_{n+1}}\mathtt{U}_1D_x(T_{n,0}g)\,,\tag{1.87}$$

$$T_{n+1,1}g=S_x(T_{n,1}g)-\frac{\gamma_n}{\alpha_n}\mathtt{U}_1D_x(T_{n,1}g)+\frac{1}{\alpha_{n+1}}D_x(T_{n,0}g)\,,\tag{1.88}$$

$$T_{n+1,2}g=S_x(T_{n,2}g)+\frac{1}{\alpha_n}D_x(T_{n,1}g)\,.\tag{1.89}$$

Using the following identities $S_x\left((z-\mathfrak{c}_3)^2\right)=(2\alpha^2-1)(z-\mathfrak{c}_3)^2+4(1-\alpha^2)\mathfrak{c}_1\mathfrak{c}_2,\ D_x\left((z-\mathfrak{c}_3)^2\right)=2\alpha(z-\mathfrak{c}_3)$, and $S_x\left((z-\mathfrak{c}_3)\right)=\alpha(z-\mathfrak{c}_3)$, we find

$$
\begin{aligned}
S_x(T_{n,0}g)(z) &= \frac{\alpha(2\alpha^2-1)a}{\alpha_n\alpha_{n-1}}(z-\mathfrak{c}_3)^2+\frac{\alpha b}{\alpha_n}(z-\mathfrak{c}_3)+c+\frac{4a(1-\alpha^2)(\alpha+\alpha_n\gamma_n)}{\alpha_n\alpha_{n-1}}\mathfrak{c}_1\mathfrak{c}_2\,,\\
S_x(T_{n,1}g)(z) &= \frac{\gamma_n}{\alpha_n}\left(\frac{\alpha a(\alpha_n+\alpha\alpha_{n-1})}{\alpha_{n_1}^2}(z-\mathfrak{c}_3)+b\right)\,,\\
D_x(T_{n,0}g)(z) &= \frac{1}{\alpha_n}\left(\frac{2\alpha^2 a}{\alpha_{n-1}}(z-\mathfrak{c}_3)+b\right)\,,\\
D_x(T_{n,1}g)(z) &= \frac{a\gamma_n(\alpha_n+\alpha\alpha_{n-1})}{\alpha_n\alpha_{n-1}^2}\,.
\end{aligned}
$$

Therefore from (1.87) we use (1.34)–(1.41) to obtain

$$
\begin{aligned}
(T_{n+1,0}g)(z) &= \frac{\alpha a}{\alpha_n\alpha_{n-1}}\left(2\alpha^2-1+2\alpha\frac{(1-\alpha^2)\gamma_{n+1}}{\alpha_{n+1}}\right)(z-\mathfrak{c}_3)^2+\frac{b(\alpha\alpha_{n+1}+(1-\alpha^2)\gamma_{n+1})}{\alpha_n\alpha_{n+1}}(z-\mathfrak{c}_3)\\
&\quad +c+\frac{4a(1-\alpha^2)(\alpha+\alpha_n\gamma_n)}{\alpha_n\alpha_{n-1}}\mathfrak{c}_1\mathfrak{c}_2\,,\\
&= \frac{\alpha a}{\alpha_n\alpha_{n+1}}(z-\mathfrak{c}_3)^2+\frac{b}{\alpha_{n+1}}(z-\mathfrak{c}_3)+c+\frac{4a(1-\alpha^2)\gamma_{n+1}}{\alpha_n}\mathfrak{c}_1\mathfrak{c}_2\,.
\end{aligned}
$$

Hence (1.83) holds for all n. Similarly, from (1.88), we use again (1.34)–(1.41) and the identity $\alpha_{n+1}\gamma_n(\alpha_n+\alpha\alpha_{n-1})+2\alpha^2\alpha_n=\alpha_{n-1}\gamma_{n+1}(\alpha_{n+1}+\alpha\alpha_n)$ to have

$$
\begin{aligned}
(T_{n+1,1}g)(z) &= \frac{a}{\alpha_n\alpha_{n-1}}\left(\frac{\gamma_n(\alpha_n+\alpha\alpha_{n-1})}{\alpha_n}+\frac{2\alpha^2}{\alpha_{n+1}}\right)(z-\mathfrak{c}_3)+\frac{b}{\alpha_n}\left(\gamma_n+\frac{1}{\alpha_{n+1}}\right)\\
&= \frac{\gamma_{n+1}}{\alpha_{n+1}}\left(\frac{a(\alpha_{n+1}+\alpha\alpha_n)}{\alpha_n^2}(z-\mathfrak{c}_3)+b\right)\,.
\end{aligned}
$$

Hence (1.84) holds for $n=0,1,2,\ldots$. Finally from (1.89) it is obvious that (1.85) holds for n replaced by $n+1$ and consequently for all n. Thus (1.83)–(1.85) hold and therefore (1.81) follows.

1.4 Outline of the thesis

This thesis fits into the theory of Orthogonal Polynomials and Special Functions, in the framework of Approximation Theory and Classical Analysis. In what follows, we describe summarily the organization of this thesis highlighting the main contributions in each chapter.

I. Let $\mathbf{u}$ be a nonzero linear functional acting on the space of polynomials $\mathscr{P}$. Let $\mathbf{D}_{q,\omega}$ be a Hahn operator acting on the dual space of polynomials $\mathscr{P}'$. Suppose that there exist polynomials ϕ and ψ, with $\deg\phi\leq 2$ and $\deg\psi\leq 1$, so that the functional equation

$$\mathbf{D}_{q,\omega}(\phi\mathbf{u})=\psi\mathbf{u}$$

holds, where the involved operations are defined in the distributional sense. We state necessary and sufficient conditions, involving only the coefficients of ϕ and ψ, such that $\mathbf{u}$ is regular, that is, there exists a sequence of orthogonal polynomials with respect to **u**. In addition, the coefficients of the three-term recurrence relation (TTRR) satisfied by the corresponding monic OPS are given. A key step in the proofs of these results relies upon the fact that a distributional Rodrigues-type formula holds without assuming that **u** is regular. This is achieved in Chapter 2. The results of this chapter are published in [4].

II. Let M and N be fixed non-negative integer numbers and let π_N be a polynomial of degree N. Suppose that $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are two OPS such that

$$\pi_N(x)\,P_{n+m}^{(m)}(x)=\sum_{j=n-M}^{n+N} r_{n,j}Q_{j+k}^{(k)}(x)\quad (n=0,1,\ldots)\,, \tag{1.90}$$

where $r_{n,j}$ are complex numbers independent of x and $f^{(k)}(x)=\frac{\mathrm{d}^k}{\mathrm{d}x^k}f(x)$ for each $f\in\mathscr{P}$. It is shown that under some natural constraints, $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are semiclassical OPS. That is, there exist nonzero polynomials ϕ_1, ϕ_2, ψ_1 and ψ_2 such that the corresponding functionals **u** and **v** fulfill the functional equations

$$\mathbf{D}(\phi_1\mathbf{u})=\psi_1\mathbf{u},\qquad \mathbf{D}(\phi_2\mathbf{v})=\psi_2\mathbf{v}.$$

Moreover we show that **u** and **v** are related by a rational modification in the distributional sense, meaning that $P\mathbf{u}=Q\mathbf{v}$ for some nonzero polynomials $P,Q\in\mathscr{P}$. This leads us to introduce the concept of π_N-coherent pairs with index M and order (m,k). This is one of the achievements of Chapter 3, published in [10].

III. Chapter 3 also extends the previous concept to the one of π_N-(q,ω)-coherent pairs with index M and order (m,k), which appears in the framework of discrete OPS by replacing in (1.90) the ordinary derivative by the discrete Hahn's operator $D_{q,\omega}$. This leads to the (structure) relation

$$\pi_N(x)\,D_{q,\omega}^m P_{n+m}(x)=\sum_{j=n-M}^{n+N} r_{n,j}D_{q,\omega}^k Q_{j+k}(x)\quad (n=0,1,\ldots)\,.$$

Again, in this situation, it is shown that under some natural constraints, $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are semiclassical OPS (with respect to $D_{q,\omega}$) and the corresponding functionals are related by a (distributional) rational modification. Some examples of application are given, recovering in a more simple way some known results in the literature about the subject. Our results enable us to describe in a unified way all the classical OPS with respect to Jackson's operator, which appear as special or limiting cases of a four parametric family of q-polynomials. These results are available in [5].

IV. Let's consider now that **u** is a functional on $\mathscr{P}$ satisfying the more general functional equation

$$\mathbf{D}_x(\phi\mathbf{u})=\mathbf{S}_x(\psi\mathbf{u})\,,$$

where $\mathbf{D}_x$ and $\mathbf{S}_x$ are the operators defined on $\mathscr{P}'$ as before. We state necessary and sufficient conditions for the regularity of **u**, giving, in addition, closed formulas for the coefficients of the TTRR of the corresponding monic OPS, as well as a Rodrigues-type formula. Some examples are given to point out the power of our formulas in the framework of classical OPS on nonuniform lattices. In particular, our results enable us to derive in a simple way the coefficients of the TTRR of the Racah polynomials as well as the ones for the Askey-Wilson polynomials. The results of this chapter are contained in a manuscript to be submitted for publication soon [11].

V. Let $(P_n)_{n\geq 0}$ be a monic OPS and π a polynomial of a degree at most two such that

$$\pi(x)P_n'(x) = (a_nx+b_n)P_n(x)+c_nP_{n-1}(x) \quad (n=0,1,2\ldots)\ ,$$

for some complex sequences of coefficients a_n, b_n and c_n. It is well known that the only OPS that fulfill the above structure relation are the classical OPS (Hermite, Laguerre, Bessel and Jacobi). This result is referred in the literature as Al-Salam and Chihara characterization of classical OPS (see [13]). In [26] M. E. H. Ismail consider the same type of structure relation, replacing the standard derivative by the Askey-Wilson operator, so that (even more generally), we may consider

$$\pi(x)D_xP_n(x) = (a_nx+b_n)P_n(x)+c_nP_{n-1}(x) \quad (n=0,1,2\ldots)\ .$$

Ismail posed the problem of characterizing all OPS fulfilling this structure relation, and he conjectured that the solutions are the continuous $q-$Jacobi polynomials, the Al-Salam Chihara polynomials, or special or limiting cases of them. The case where the polynomial π is a constant was proved by Al-Salam [2]. In Chapter 5, using the main results of Chapter 4, we give a positive answer to Ismail's conjecture, for any polynomial π. The results of this chapter are contained in a manuscript to be submitted for publication soon [12].

Finally in Chapter 6, we present some further directions of research in the framework of the problems considered in this thesis.

Chapter 2

Classical orthogonal polynomials related to Hahn's operator

2.1 Preliminaries

2.1.1 Introduction

The (very) classical OPS of Hermite, Laguerre, Jacobi, and Bessel, constitute the most studied class of OPS. In the framework of regular orthogonality, these OPS are defined as orthogonal with respect to a moment linear functional $\mathbf{u} : \mathscr{P} \to \mathbb{C}$ such that there exist two nonzero polynomials $\phi \in \mathscr{P}_2$ and $\psi \in \mathscr{P}_1$ so that $\mathbf{u}$ satisfies the functional equation

$$\mathbf{D}(\phi\mathbf{u}) = \psi\mathbf{u} \,, \tag{2.1}$$

where the functional $\mathbf{Du}$ is defined as in (1.3). Hermite and Laguerre functionals (corresponding to the Hermite and Laguerre OPS) appear in (2.1) taking $\phi \equiv \text{const.} \neq 0$ and $\deg\phi = 1$, respectively. If $\deg\phi = 2$ we obtain a Jacobi functional whenever the zeros of ϕ are distinct, and a Bessel functional if ϕ has a double zero. A natural question arises: if $\mathbf{u}$ is a nonzero linear functional defined on $\mathscr{P}$ satisfying (2.1), with $\phi \in \mathscr{P}_2$ and $\psi \in \mathscr{P}_1$, and if at least one among ϕ and ψ is not the zero polynomial, to determine necessary and sufficient conditions, involving only the coefficients of ϕ and ψ, such that $\mathbf{u}$ is regular (i.e., there exists an OPS with respect to $\mathbf{u}$). This question has been answered by Marcellán and Petronilho in the following

Theorem 2.1.1 [38, Lemma 2 and Theorem 2] *Let $\mathbf{u} \in \mathscr{P}' \setminus \{\mathbf{0}\}$. Suppose that (2.1) holds where $\phi \in \mathscr{P}_2$, $\psi \in \mathscr{P}_1$, and at least one of ϕ and ψ is not the zero polynomial. Write*

$$\phi(x) := ax^2 + bx + c \,, \quad \psi(x) := dx + e \,, \quad d_n := d + an \,, \quad e_n := e + bn \,.$$

$(a,b,c,d,e \in \mathbb{C};\ |a|+|b|+|c|+|d|+|e| \neq 0$.) Then, $\mathbf{u}$ is regular if and only if

$$d_n \neq 0 \,, \quad \phi\Big(-\frac{e_n}{d_{2n}}\Big) \neq 0 \,, \quad \forall n \in \mathbb{N}_0 \,. \tag{2.2}$$

Under these conditions, the monic OPS $(P_n)_{n\geq 0}$ with respect to $\mathbf{u}$ satisfies the three-term recurrence relation (1.1) with coefficients (1.2) given by

$$\beta_n = \frac{ne_{n-1}}{d_{2n-2}} - \frac{(n+1)e_n}{d_{2n}}\,, \quad \gamma_{n+1} = -\frac{(n+1)d_{n-1}}{d_{2n-1}d_{2n+1}}\phi\Big(-\frac{e_n}{d_{2n}}\Big) \quad (n=0,1,\ldots)\,. \tag{2.3}$$

In addition, the following (distributional) Rodrigues formula holds

$$P_n\mathbf{u} = k_n\,\mathbf{D}^n\big(\phi^n\mathbf{u}\big)\,, \quad k_n := \prod_{j=0}^{n-1} d_{n+j-1}^{-1} \quad (n=0,1,\ldots)\,. \tag{2.4}$$

The aim of this chapter is to state a $(q,\omega)-$analogue of Theorem 2.1.1, replacing in the functional equation (2.1) the derivative operator $\mathbf{D}$ by the (distributional) Hahn's operator, denoted by $\mathbf{D}_{q,\omega}$ (defined as in (1.6)).

2.1.2 Preliminaries results

Given a nonnegative integer number k and a monic polynomial P_n of degree n, we denote by $P_n^{[k]} \equiv P_n^{[k]}(\cdot;q,\omega)$ the monic polynomial of degree n defined by

$$P_n^{[k]}(x) := \frac{D_{q,\omega}^k P_{n+k}(x)}{\prod_{j=1}^k [n+j]_q} = \frac{[n]_q!}{[n+k]_q!} D_{q,\omega}^k P_{n+k}(x) \quad (k,n=0,1,\ldots)\,. \tag{2.5}$$

If $k=0$, it is understood that $D_{q,\omega}^0 f = f$ and that empty product equals one. Set

$$\phi(x) := ax^2+bx+c\,, \quad \psi(x) := dx+e\,, \tag{2.6}$$

$$d_n \equiv d_n(q) := \psi' q^n + \tfrac{1}{2}\phi''\,[n]_q = dq^n + a[n]_q\,, \quad e_n \equiv e_n(q,\omega) := eq^n + (\omega d_n + b)[n]_q\,. \tag{2.7}$$

Definition 2.1 *(ϕ,ψ) is called a $(q,\omega)-$admissible pair if*

$$\phi\in\mathscr{P}_2\,, \quad \psi\in\mathscr{P}_1\,, \quad \textit{and} \quad d_n\neq 0, \quad \forall n\in\mathbb{N}_0\,, \tag{2.8}$$

where d_n is given by (2.7).

Definition 2.2 *A linear regular functional $\boldsymbol{u}\in\mathscr{P}^*$ is a $D_{q,\omega}$-semiclassical (or (q,ω)-semiclassical) functional if it is regular and there exist $\phi,\psi\in\mathscr{P}$, with deg $\psi\geq 1$, such that the following functional equation holds.*

$$\mathbf{D}_{q,\omega}(\phi\mathbf{u}) = \psi\boldsymbol{u}\,. \tag{2.9}$$

The class of a $D_{q,\omega}$-semiclassical functional $\boldsymbol{u}$, denoted by $\mathfrak{s}(\mathbf{u})$, is the unique non-negative integer number defined by

$$\mathfrak{s}(\mathbf{u}) := \min_{(\phi,\psi)\in\mathscr{A}_{\mathbf{u}}} \max\big\{\deg\phi-2, \deg\psi-1\big\}\,,$$

where $\mathscr{A}_{\mathbf{u}}$ is the set of all pairs of nonzero polynomials (ϕ,ψ) fulfilling the functional equation (2.9). When $\mathfrak{s}(\mathbf{u})=0$, $\mathbf{u}$ is said to be a $D_{q,\omega}$-classical (or (q,ω)-classical) functional. We also say that the corresponding OPS is $D_{q,\omega}$-semiclassical of class $\mathfrak{s}(\mathbf{u})$ or $D_{q,\omega}$-classical, respectively.

Remark 2.1.1 *Note that*

$$\mathbf{D}_{q,\omega}(\phi \boldsymbol{u})=\psi \boldsymbol{u} \;\Leftrightarrow\; \mathbf{D}_{1/q,-\omega/q}(\widehat{\phi}\boldsymbol{u})=\psi \boldsymbol{u}\;,$$

where $\widehat{\phi}(x):=q^{-1}\Big[\phi(x)+\Big((q-1)x+\omega\Big)\psi(x)\Big]$. Consequently, a regular linear functional $\boldsymbol{u}$ is $D_{q,\omega}$-semiclassical if and only if it is $D_{1/q,-\omega/q}$-semiclassical.

Remark 2.1.2 *It is worth mentioning that $D_{q,0}-$classical OPS were extensively studied in [32] and an introduction to the study of D_q-semiclassical OPS has been addressed in [31]. Moreover, an extensive study of $D_{q,0}-$classical OPS was made in [1].*

Let $\mathbf{u}\in\mathscr{P}^*$ be a linear functional satisfying the functional equation (2.9) where $\phi\in\mathscr{P}_2$ and $\psi\in\mathscr{P}_1$. We also set

$$\mathbf{u}^{[0]}:=\mathbf{u}\;,\quad \mathbf{u}^{[k]}:=\mathbf{L}_{q,\omega}\big(\phi\mathbf{u}^{[k-1]}\big)=L_{q,\omega}\phi\,\mathbf{L}_{q,\omega}\mathbf{u}^{[k-1]}\quad (k=1,2,\ldots)\;, \tag{2.10}$$

where the last equality holds by (1.14). Iterating (2.10) and taking into account (1.11) yields

$$\mathbf{u}^{[k]}=\Big(\prod_{j=1}^{k}L_{q,\omega}^{j}\phi\Big)\mathbf{L}_{q,\omega}^{k}\mathbf{u}=\Phi(\cdot;k)\mathbf{L}_{q,\omega}^{k}\mathbf{u}\quad (k=0,1,\ldots)\;, \tag{2.11}$$

where

$$\Phi(x;k):=\prod_{j=1}^{k}\phi(q^jx+\omega[j]_q)\;. \tag{2.12}$$

Proposition 2.1.2 *[15, Theorem 3.1]*
The functional $\mathbf{u}^{[k]}$ defined in (2.11) fulfils the functional equation

$$\mathbf{D}_{q,\omega}\big(\phi\mathbf{u}^{[k]}\big)=\psi^{[k]}\mathbf{u}^{[k]}\quad (k=0,1,\ldots)\;, \tag{2.13}$$

where $\psi^{[k]}\in\mathscr{P}_1$ is defined by

$$\psi^{[0]}:=\psi\;,\quad \psi^{[k]}:=D_{q,\omega}\phi+qL_{q,\omega}\psi^{[k-1]}\quad (k=1,2,3,\ldots)\;. \tag{2.14}$$

We point out that equality (2.13) was stated in [15] under the assumption that $\mathbf{u}$ is a regular functional, but inspection of the proof given therein shows that the equality remains true without such assumption.

Corollary 2.1.3 *The polynomial $\psi^{[k]}$ defined in (2.14) is explicitly given by*

$$\psi^{[k]}(x)=d_{2k}x+e_k\quad (k=0,1,2,\ldots)\;, \tag{2.15}$$

where d_k and e_k are defined as in (2.7).

Proof Apply mathematical induction on $k \in \mathbb{N}_0$.

Formula (2.15) has not been observed in [15]. It will play a central role along this chapter.

Lemma 2.1.4 *Let $\mathbf{u} \in \mathscr{P}^* \setminus \{0\}$. Suppose that $\mathbf{u}$ satisfies (2.9), where $\phi \in \mathscr{P}_2$ and $\psi \in \mathscr{P}_1$. Let $(Q_n)_{n\geq 0}$ be any simple set of polynomials and define*

$$\begin{aligned} R_{n+1}(x) \quad &:= \phi(x) D^*_{q,\omega} Q_n(x) + q\psi(x) Q_n(x) \\ &= a_n q^{1-n} d_n x^{n+1} + \text{(lower degree terms)} , \end{aligned} \tag{2.16}$$

where $a_n \in \mathbb{C} \setminus \{0\}$ is the leading coefficient of Q_n and d_n is defined as in (2.7). Then the following functional equation holds:

$$\mathbf{D}^*_{q,\omega}\big(Q_n \mathbf{u}^{[1]}\big) = R_{n+1}\mathbf{u} \quad (n = 0, 1, \ldots) . \tag{2.17}$$

Moreover, $(R_n)_{n\geq 0}$ is a simple set of polynomials if and only if (ϕ, ψ) is a $(q,\omega)-$admissible pair, provided that we define $R_0(x) := 1$.

Proof Let $n \in \mathbb{N}_0$ and take arbitrarily $f \in \mathscr{P}$. Then

$$\begin{aligned} \big\langle \mathbf{D}^*_{q,\omega}\big(Q_n \mathbf{u}^{[1]}\big), f \big\rangle &= -q\langle \mathbf{L}_{q,\omega}(\phi\mathbf{u}), Q_n D_{q,\omega} f\rangle = -\big\langle \phi\mathbf{u}, \big(L^*_{q,\omega} Q_n\big)\big(L^*_{q,\omega} D_{q,\omega} f\big)\big\rangle \\ &= -\big\langle \phi\mathbf{u}, \big(L^*_{q,\omega} Q_n\big)\big(D^*_{q,\omega} f\big)\big\rangle . \end{aligned}$$

Now, using relation (1.15) with $D^*_{q,\omega}$ instead of $D_{q,\omega}$, we obtain

$$\begin{aligned} \big\langle \mathbf{D}^*_{q,\omega}\big(Q_n \mathbf{u}^{[1]}\big), f \big\rangle &= -\langle \phi\mathbf{u}, D^*_{q,\omega}(f Q_n) - f D^*_{q,\omega} Q_n\rangle \\ &= q\big\langle \mathbf{D}_{q,\omega}(\phi\mathbf{u}), Q_n f\big\rangle + \langle \phi \mathbf{D}^*_{q,\omega} Q_n, f\rangle \\ &= \big\langle \mathbf{u}, \big(q\psi Q_n + \phi D^*_{q,\omega} Q_n\big) f\big\rangle = \langle R_{n+1}\mathbf{u}, f\rangle . \end{aligned}$$

This proves (2.17). Moreover, taking into account (1.19), we have

$$\begin{aligned} D^*_{q,\omega} Q_n(x) &= a_n D_{1/q,-\omega/q} x^n + \text{(lower degree terms)} \\ &= a_n q^{1-n} [n]_q x^{n-1} + \text{(lower degree terms)} , \end{aligned}$$

where we also took into account that $[n]_{q^{-1}} = q^{1-n}[n]_q$. Hence

$$R_{n+1}(x) = \big(a a_n q^{1-n}[n]_q + q a_n d\big) x^{n+1} + \text{(lower degree terms)} ,$$

and so we obtain the expression for R_{n+1} given in (2.16). Thus, $\deg R_{n+1} = n+1$ for each $n = 0, 1, \ldots$ if and only if $d_n \neq 0$ for each $n = 0, 1, \ldots$, i.e., if and only if (ϕ, ψ) is a $(q,\omega)-$admissible pair. This concludes the proof.

In the statement of the next lemma, which is interesting for its own sake, we emphasize that neither the given functional **u** needs to be regular nor the sequence $(P_n)_{n\geq 0}$ needs to be an OPS. Under the assumption that **u** is regular and satisfies (2.9), formula (2.18) in bellow may be derived in a very simple way (see Section 2.2.2 below).

Lemma 2.1.5 *[4]*
Let $\mathbf{u} \in \mathscr{P}^* \setminus \{0\}$. *Suppose that* $\mathbf{u}$ *satisfies* (2.9), *where* (ϕ, ψ) *is a* $(q, \omega)-$*admissible pair. Then the Rodrigues-type formula*

$$P_n\mathbf{u} = k_n \mathbf{D}^n_{1/q,-\omega/q}\, \mathbf{u}^{[n]} \quad (n = 0, 1, \ldots) \tag{2.18}$$

holds in $\mathscr{P}^*$, *where* $\mathbf{u}^{[n]}$ *is defined as in (2.11),*

$$k_n := q^{n(n-3)/2} \prod_{j=0}^{n-1} d^{-1}_{n+j-1} , \tag{2.19}$$

and $(P_n)_{n\geq 0}$ *is a simple set of monic polynomials given by the three-term recurrence relation (1.1) with coefficients (1.2) given by*

$$\beta_n := \omega[n]_q + \frac{[n]_q e_{n-1}}{d_{2n-2}} - \frac{[n+1]_q e_n}{d_{2n}} , \tag{2.20}$$

$$\gamma_{n+1} := -\frac{q^n[n+1]_q d_{n-1}}{d_{2n-1} d_{2n+1}} \phi\Big(-\frac{e_n}{d_{2n}}\Big) \quad (n = 0, 1, \ldots) , \tag{2.21}$$

where e_n *and* d_n *defined as in (2.7).*

Proof Since (ϕ, ψ) is a $(q, \omega)-$admissible pair, then $d_n \neq 0$ for each $n = 0, 1, \ldots$. Hence the sequence $(P_n)_{n\geq 0}$ given by (1.1) and with coefficients given by (2.21) is well defined. For simplicity, we set $\mathbf{H}_{q,\omega} := \mathbf{D}^*_{q,\omega} := \mathbf{D}_{1/q,-\omega/q}$, and so (2.18) reads as

$$P_n\mathbf{u} = k_n \mathbf{H}^n_{q,\omega}\, \mathbf{u}^{[n]} \quad (n = 0, 1, \ldots) . \tag{2.22}$$

Notice that the second relation in (1.13) can be rewritten as

$$\mathbf{H}_{q,\omega}\, \mathbf{L}_{q,\omega} = q\mathbf{D}_{q,\omega} , \tag{2.23}$$

while, setting $H_{q,\omega} := D^*_{q,\omega}$, the Leibniz rule (1.18) applied to $\mathbf{D}^*_{q,\omega}$ gives

$$\mathbf{H}^n_{q,\omega}(f\mathbf{u}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} L^{*k}_{q,\omega}\big(H^{n-k}_{q,\omega} f\big)\, \mathbf{H}^k_{q,\omega}\mathbf{u} \quad (f \in \mathscr{P}) . \tag{2.24}$$

We will prove (2.22) by mathematical induction on n. For $n = 0$, (2.22) becomes a trivial equality. For $n = 1$, we use (2.10) and (2.23) to deduce

$$\mathbf{H}_{q,\omega}\, \mathbf{u}^{[1]} = \mathbf{H}_{q,\omega}\mathbf{L}_{q,\omega}\big(\phi\mathbf{u}\big) = q\mathbf{D}_{q,\omega}\big(\phi\mathbf{u}\big) = q\psi\mathbf{u} .$$

Therefore, since $P_1(x) = x - \beta_0 = x - (-e_0/d_0) = x + e/d = d^{-1}\psi(x)$, and so $q\psi = qdP_1 = k_1^{-1}P_1$, we obtain (2.22) for $n = 1$. Assume now that (2.22) holds for given consecutive numbers $n-1$ and n

($n \in \mathbb{N}$), i.e., suppose that (induction hypothesis)

$$P_{n-1}\mathbf{u} = k_{n-1}\mathbf{H}_{q,\omega}^{n-1}\,\mathbf{u}^{[n-1]}\,, \quad P_n\mathbf{u} = k_n\mathbf{H}_{q,\omega}^{n}\,\mathbf{u}^{[n]}\,. \tag{2.25}$$

We need to show that

$$P_{n+1}\mathbf{u} = k_{n+1}\mathbf{H}_{q,\omega}^{n+1}\,\mathbf{u}^{[n+1]}\,. \tag{2.26}$$

To prove (2.26), we start by noting that

$$\mathbf{H}_{q,\omega}^{n+1}\,\mathbf{u}^{[n+1]} = q\mathbf{H}_{q,\omega}^{n}\big(\psi^{[n]}\mathbf{u}^{[n]}\big)\,. \tag{2.27}$$

Indeed, using successively (2.10) and (2.23), we have

$$\mathbf{H}_{q,\omega}^{n+1}\,\mathbf{u}^{[n+1]} = \mathbf{H}_{q,\omega}^{n}\Big(\mathbf{H}_{q,\omega}\mathbf{L}_{q,\omega}(\phi\mathbf{u}^{[n]})\Big) = q\mathbf{H}_{q,\omega}^{n}\Big(\mathbf{D}_{q,\omega}\big(\phi\mathbf{u}^{[n]}\big)\Big)\,,$$

and so (2.27) follows taking into account (2.13). Next, by (2.24) with $f = \psi^{[n]} = d_{2n}x + e_n$,

$$\mathbf{H}_{q,\omega}^{n}\big(\psi^{[n]}\mathbf{u}^{[n]}\big) = \big(L_{q,\omega}^{*n}\,\psi^{[n]}\big)\mathbf{H}_{q,\omega}^{n}\mathbf{u}^{[n]} + [n]_{q^{-1}}d_{2n}\mathbf{H}_{q,\omega}^{n-1}\mathbf{u}^{[n]}\,.$$

Replacing this into (2.27) and using the second identity in (2.25), we deduce

$$[n]_{q^{-1}}d_{2n}\mathbf{H}_{q,\omega}^{n-1}\mathbf{u}^{[n]} = q^{-1}\mathbf{H}_{q,\omega}^{n+1}\,\mathbf{u}^{[n+1]} - k_n^{-1}\big(L_{q,\omega}^{*n}\,\psi^{[n]}\big)P_n\mathbf{u}\,. \tag{2.28}$$

Taking into account both identities appearing in (2.25), we may change n into $n-1$ in the preceding reasoning, to obtain

$$[n-1]_{q^{-1}}d_{2n-2}\mathbf{H}_{q,\omega}^{n-2}\mathbf{u}^{[n-1]} = \Big(q^{-1}k_n^{-1}P_n - k_{n-1}^{-1}\big(L_{q,\omega}^{*n-1}\,\psi^{[n-1]}\big)P_{n-1}\Big)\mathbf{u}\,. \tag{2.29}$$

Next, by the analogue of (1.16) for $\mathbf{D}_{q,\omega}^{*}$, we have

$$\begin{aligned}\mathbf{H}_{q,\omega}\big(\psi^{[n]}\mathbf{u}^{[n]}\big) &= \big(D_{q,\omega}^{*}\psi^{[n]}\big)\mathbf{L}_{q,\omega}^{*}\mathbf{u}^{[n]} + \psi^{[n]}\mathbf{H}_{q,\omega}\mathbf{u}^{[n]}\\ &= d_{2n}\mathbf{L}_{q,\omega}^{*}\mathbf{L}_{q,\omega}\big(\phi\mathbf{u}^{[n-1]}\big) + \psi^{[n]}\mathbf{H}_{q,\omega}\mathbf{L}_{q,\omega}\big(\phi\mathbf{u}^{[n-1]}\big)\\ &= \big(d_{2n}\phi + q\psi^{[n]}\psi^{[n-1]}\big)\mathbf{u}^{[n-1]}\,,\end{aligned} \tag{2.30}$$

where in the last equality we used (1.9), (2.23), and (2.13). From (2.27) and (2.30), we obtain

$$\mathbf{H}_{q,\omega}^{n+1}\,\mathbf{u}^{[n+1]} = q\mathbf{H}_{q,\omega}^{n-1}\big(\theta_2(\cdot;n)\mathbf{u}^{[n-1]}\big)\,, \tag{2.31}$$

where $\theta_2(x;n) := d_{2n}\phi + q\psi^{[n]}\psi^{[n-1]}$. Since $\deg\theta_2(\cdot;n) \le 2$, applying the Leibniz formula (2.24) to the right-hand side of (2.31), we obtain

$$\begin{aligned}\mathbf{H}_{q,\omega}^{n+1}\,\mathbf{u}^{[n+1]} &= qL_{q,\omega}^{*n-1}\big(\theta_2(\cdot;n)\big)\mathbf{H}_{q,\omega}^{n-1}\mathbf{u}^{[n-1]}\\ &\quad + q[n-1]_{q^{-1}}L_{q,\omega}^{*n-2}\big(D_{q,\omega}^{*}\theta_2(\cdot;n)\big)\mathbf{H}_{q,\omega}^{n-2}\mathbf{u}^{[n-1]}\\ &\quad + \frac{q[n-1]_{q^{-1}}[n-2]_{q^{-1}}}{[2]_{q^{-1}}}L_{q,\omega}^{*n-3}\big(D_{q,\omega}^{*2}\theta_2(\cdot;n)\big)\mathbf{H}_{q,\omega}^{n-3}\mathbf{u}^{[n-1]}\,.\end{aligned} \tag{2.32}$$

Now, since $\phi(x) = ax^2 + bx + c$, $\psi^{[k]} = d_{2k}x + e_k$, and the relations

$$d_{k+1} = a + qd_k \ , \quad e_{k+1} = b + qe_k + \omega d_{2k+1} \ , \quad d_{2k+2} + qd_{2k} = (1+q)d_{2k+1} \tag{2.33}$$

hold for each $k = 0, 1, \ldots$, we show that $\theta_2(\cdot;n)$ is given explicitly by

$$\theta_2(x;n) = d_{2n}d_{2n-1}x^2 + d_{2n-1}\big((1+q)e_n - \omega d_{2n}\big)x + cd_{2n} + qe_ne_{n-1} \ .$$

(Hence, $\deg \theta_2(\cdot;n) = 2$.) From this and taking into account (1.19), we compute

$$\begin{aligned} D^*_{q,\omega}\big(\theta_2(\cdot;n)\big) &= [2]_{q^{-1}}d_{2n-1}\big(d_{2n}x + qe_n - \omega d_{2n}\big) \ , \\ D^{*2}_{q,\omega}\big(\theta_2(\cdot;n)\big) &= [2]_{q^{-1}}d_{2n-1}d_{2n} \ . \end{aligned}$$

Moreover, by (1.11),

$$L^{*k}_{q,\omega}1 = 1 \ , \quad L^{*k}_{q,\omega}x = q^{-k}\big(x - \omega[k]_q\big) \ , \quad L^{*k}_{q,\omega}x^2 = q^{-2k}\big(x^2 - 2\omega[k]_qx + \omega^2[k]_q^2\big)$$

for each $k = 0, 1, \ldots$, hence we deduce

$$\begin{aligned} L^{*n-1}_{q,\omega}\big(\theta_2(\cdot;n)\big) = q^{2-2n}&d_{2n}d_{2n-1}x^2 \\ &+ q^{1-n}d_{2n-1}\Big((1+q)e_n - \omega d_{2n}\big([n]_{q^{-1}} + q^{-1}[n-1]_{q^{-1}}\big)\Big)x \\ &+ \omega^2q^{1-n}d_{2n}d_{2n-1}[n-1]_q[n]_{q^{-1}} \\ &\quad + qe_n\big(e_{n-1} - \omega(1+q)d_{2n-1}q^{-n}[n-1]_q\big) + cd_{2n} \ , \end{aligned} \tag{2.34}$$

$$L^{*n-2}_{q,\omega}\big(D^*_{q,\omega}\theta_2(\cdot;n)\big) = [2]_{q^{-1}}d_{2n-1}\big(d_{2n}q^{2-n}x + qe_n - \omega d_{2n}[n-1]_{q^{-1}}\big) \ , \tag{2.35}$$

$$L^{*n-3}_{q,\omega}\big(D^{*2}_{q,\omega}\theta_2(\cdot;n)\big) = [2]_{q^{-1}}d_{2n-1}d_{2n} \ . \tag{2.36}$$

Relation (2.36) allow us to rewrite (2.32) as

$$\begin{aligned} q[n-1]_{q^{-1}}&[n-2]_{q^{-1}}d_{2n-1}d_{2n}\mathbf{H}^{n-3}_{q,\omega}\mathbf{u}^{[n-1]} \\ &= \mathbf{H}^{n+1}_{q,\omega}\mathbf{u}^{[n+1]} - qL^{*n-1}_{q,\omega}\big(\theta_2(\cdot;n)\big)\mathbf{H}^{n-1}_{q,\omega}\mathbf{u}^{[n-1]} \\ &\quad - q[n-1]_{q^{-1}}L^{*n-2}_{q,\omega}\big(D^*_{q,\omega}\theta_2(\cdot;n)\big)\mathbf{H}^{n-2}_{q,\omega}\mathbf{u}^{[n-1]} \ . \end{aligned} \tag{2.37}$$

On the other hand,

$$\begin{aligned} \mathbf{H}^{n-1}_{q,\omega}\mathbf{u}^{[n]} &= \mathbf{H}^{n-2}_{q,\omega}\big(\mathbf{H}_{q,\omega}\mathbf{L}_{q,\omega}\big(\phi\mathbf{u}^{[n-1]}\big) = q\mathbf{H}^{n-2}_{q,\omega}\big(\mathbf{D}_{q,\omega}\big(\phi\mathbf{u}^{[n-1]}\big) = q\mathbf{H}^{n-2}_{q,\omega}\big(\psi^{[n-1]}\mathbf{u}^{[n-1]}\big) \\ &= qL^{*n-2}_{q,\omega}\big(\psi^{[n-1]}\big)\mathbf{H}^{n-2}_{q,\omega}\mathbf{u}^{[n-1]} + q[n-2]_{q^{-1}}L^{*n-3}_{q,\omega}\big(D^*_{q,\omega}\psi^{[n-1]}\big)\big)\mathbf{H}^{n-3}_{q,\omega}\mathbf{u}^{[n-1]} \ , \end{aligned}$$

where in the last equality we used once again the Leibniz formula. As a consequence, since $L^{*n-3}_{q,\omega}\big(D^*_{q,\omega}\psi^{[n-1]}\big)\big) = d_{2n-2}$, we obtain

$$q[n-2]_{q^{-1}}d_{2n-2}\mathbf{H}^{n-3}_{q,\omega}\mathbf{u}^{[n-1]} = \mathbf{H}^{n-1}_{q,\omega}\mathbf{u}^{[n]} - qL^{*n-2}_{q,\omega}\big(\psi^{[n-1]}\big)\mathbf{H}^{n-2}_{q,\omega}\mathbf{u}^{[n-1]} \ . \tag{2.38}$$

Substituting in (2.37) the expression for $\mathbf{H}_{q,\omega}^{n-3}\mathbf{u}^{[n-1]}$ given by (2.38), and then taking into account (2.28) and (2.29), as well as the first equation in (2.25), we deduce

$$\left(1-\frac{q^{-1}[n-1]_{q^{-1}}}{[n]_{q^{-1}}}\frac{d_{2n-1}}{d_{2n-2}}\right)\mathbf{H}_{q,\omega}^{n+1}\,\mathbf{u}^{[n+1]}=\Big(A(\cdot;n)P_n+B(\cdot;n)P_{n-1}\Big)\mathbf{u}\,, \tag{2.39}$$

where $A(\cdot;n)$ and $B(\cdot;n)$ are polynomials given by

$$\begin{aligned}A(x;n):=\frac{k_n^{-1}}{d_{2n-2}}\Bigg\{&-\frac{[n-1]_{q^{-1}}d_{2n-1}\big(L_{q,\omega}^{*n}\psi^{[n]}\big)}{[n]_{q^{-1}}}\\&+L_{q,\omega}^{*n-2}\big(D_{q,\omega}^{*}\theta_2(\cdot;n)\big)-\frac{d_{2n-1}d_{2n}}{d_{2n-2}}\big(L_{q,\omega}^{*n-2}\psi^{[n-1]}\big)\Bigg\}\end{aligned} \tag{2.40}$$

and

$$\begin{aligned}B(x;n):=\frac{qk_{n-1}^{-1}}{d_{2n-2}}\Bigg\{&d_{2n-2}L_{q,\omega}^{*n-1}\big(\theta_2(\cdot;n)\big)-\big(L_{q,\omega}^{*n-1}\psi^{[n-1]}\big)\times\\&\times\left(L_{q,\omega}^{*n-2}\big(D_{q,\omega}^{*}\theta_2(\cdot;n)\big)-\frac{d_{2n-1}d_{2n}}{d_{2n-2}}\big(L_{q,\omega}^{*n-2}\psi^{[n-1]}\big)\right)\Bigg\}\,.\end{aligned} \tag{2.41}$$

Now, taking into account (2.34) and (2.35), as well as the relations

$$L_{q,\omega}^{*n}\psi^{[n]}(x)=q^{-n}d_{2n}x+e_n-\omega[n]_q q^{-n}d_{2n}\,, \tag{2.42}$$

$$L_{q,\omega}^{*n-2}\psi^{[n-1]}(x)=q^{2-n}d_{2n-2}x+e_{n-1}-\omega[n-2]_q q^{2-n}d_{2n-2}\,, \tag{2.43}$$

and also making use of the identities

$$k_n^{-1}=\frac{q^{n-1}d_{n-1}}{d_{2n}d_{2n-1}}k_{n+1}^{-1}\,,\quad k_{n-1}^{-1}=\frac{q^{2n-3}d_{n-1}d_{n-2}}{d_{2n}d_{2n-1}d_{2n-2}d_{2n-3}}k_{n+1}^{-1}\,, \tag{2.44}$$

it is straightforward to verify that

$$A(x;n)=k_{n+1}^{-1}\frac{q^{n-1}d_{n-1}}{[n]_q d_{2n-2}}\big(x-\beta_n\big)\,,\quad B(x;n)=-k_{n+1}^{-1}\frac{q^{n-1}d_{n-1}}{[n]_q d_{2n-2}}\gamma_n\,, \tag{2.45}$$

where β_n and γ_n are given by (2.20)–(2.21). Since computations on how to obtain (2.45) from (2.40)–(2.41) are straightforward without any technical aspect, we refer this to the Appendix A.1. Alternatively one can use Wolfram Mathematica.

Finally, replacing these expressions for $A(\cdot;n)$ and $B(\cdot;n)$ in the right-hand side of (2.39), and taking into account (1.1) and the identity

$$1-\frac{q^{-1}[n-1]_{q^{-1}}}{[n]_{q^{-1}}}\frac{d_{2n-1}}{d_{2n-2}}=\frac{q^{n-1}d_{n-1}}{[n]_q d_{2n-2}}\,,$$

(2.26) follows.

Lemma 2.1.6 in bellow can be easily proved (see [15, Lemma 3.1]).

Lemma 2.1.6 *Let* $\mathbf{u} \in \mathscr{P}^*$*. Suppose that* $\mathbf{u}$ *is regular and fulfills* (2.9)*, with* $\phi \in \mathscr{P}_2$ *and* $\psi \in \mathscr{P}_1$*. If at least one of the polynomials* ϕ *and* ψ *is not the zero polynomial, then none of these polynomials can be the zero polynomial and, moreover,* $\deg \psi = 1$.

The statement of the next lemma is given in [15, Lemma 3.5]. However the proof of the $(q,\omega)-$admissibility condition given therein is incorrect. Indeed, in [15, Lemma 3.5–(i)], it is stated that $\lambda_{n+1,0} \neq 0$ for all $n = 0,1,2,\ldots$, where $\lambda_{n+1,0} := q^{-1}a[n]_{q^{-1}} + d$, and so, since $\lambda_{n+1,0} = q^{-n}d_n$, [15, Lemma 3.5–(i)] is equivalent to say that (ϕ,ψ) is a (q,ω)-admissibility pair. The argument used in the proof of [15, Lemma 3.5–(i)] is based on relations (3.79) given therein, which can be written (in out notation) as

$$\mathbf{D}_{q,\omega}(\phi \mathbf{u}) = \psi \mathbf{u} \Leftrightarrow -\frac{\lambda_{n+1,0}}{[n+1]_q} M_{n+1} = \sum_{j=0}^{n} f_j M_j, \ \ \forall n \in \mathbb{N}_0 \ ,$$

where $(f_j)_{j\geq 0}$ is a sequence of numbers and $M_j := \langle \mathbf{u}, x^j \rangle$ for each $j = 0,1,2,\ldots$. After stating this in [15, p.56], the author says that "Since $\mathbf{u}$ is regular, to have all its moments given in the unique way by the previous ones, it is necessary to have $\lambda_{n+1,0} \neq 0$ for all $n = 0,1,2,\ldots$". Clearly, this sentence would be correct if "necessary" is replaced by "sufficient". But in that case the argument is not valid to deduce the (q,ω)-admissibility condition. For sake of completeness, we present a proof following the ideas presented in [38].

Lemma 2.1.7 *Let* $\mathbf{u} \in \mathscr{P}^*$*. Suppose that* $\mathbf{u}$ *is regular and satisfies* (2.9)*, where* $\phi \in \mathscr{P}_2$*,* $\psi \in \mathscr{P}_1$*, and at least one of the polynomials* ϕ *and* ψ *is not the zero polynomial. Then* (ϕ,ψ) *is a* $(q,\omega)-$*admissible pair and* $\mathbf{u}^{[k]}$ *is regular for each* $k \in \mathbb{N}$*. Moreover, if* $(P_n)_{n\geq 0}$ *is the monic OPS with respect to* $\mathbf{u}$*, then* $\big(P_n^{[k]}\big)_{n\geq 0}$ *is the monic OPS with respect to* $\mathbf{u}^{[k]}$.

Proof We start by considering the case $k = 1$. Set $Q_n := P_n^{[1]} = D_{q,\omega}P_{n+1}/[n+1]_q$ and let R_{n+1} be the corresponding polynomial defined by (2.16). Let m and n be arbitrary integers, with $m \leq n$. Then, by Lemma 2.1.4,

$$\begin{aligned} [m+1]_q \langle \mathbf{u}^{[1]}, Q_n Q_m \rangle &= -\big\langle \mathbf{D}^*_{q,\omega}\big(Q_n \mathbf{u}^{[1]}\big), P_{m+1} \big\rangle = -q^{-1} \langle R_{n+1}\mathbf{u}, P_{m+1} \rangle \\ &= -q^{-n} d_n \langle \mathbf{u}, P^2_{n+1} \rangle \delta_{m,n} \ , \end{aligned}$$

hence we obtain

$$\big\langle \mathbf{u}^{[1]}, P_n^{[1]} P_m^{[1]} \big\rangle = -\frac{q^{-n} d_n}{[n+1]_q} \langle \mathbf{u}, P^2_{n+1} \rangle \delta_{m,n} \quad (m,n = 0,1,\ldots) \ . \tag{2.46}$$

Next, let $s := \deg \phi \in \{0,1,2\}$. Then

$$0 \neq \big\langle \mathbf{u}, \phi\big(L^*_{q,\omega} P_n^{[1]}\big) P_{n+s} \big\rangle = \big\langle \phi \mathbf{u}, L^*_{q,\omega}\big(P_n^{[1]} L_{q,\omega} P_{n+s}\big) \big\rangle = q \big\langle \mathbf{u}^{[1]}, P_n^{[1]} L_{q,\omega} P_{n+s} \big\rangle \ . \tag{2.47}$$

Since $L_{q,\omega}P_{n+s}(x)=\sum_{m=0}^{n+s}c_{n,m}P_m^{[1]}(x)$ for some coefficients $c_{n,m}\equiv c_{n,m}(s;q,\omega)\in\mathbb{C}$, from (2.46) and (2.47) we deduce

$$0\neq\sum_{m=0}^{n+s}c_{n,m}\big\langle\mathbf{u}^{[1]},P_n^{[1]}P_m^{[1]}\big\rangle=-\frac{q^{-n}d_nc_{n,n}}{[n+1]_q}\langle\mathbf{u},P_{n+1}^2\rangle\quad(n=0,1,\ldots)\,. \tag{2.48}$$

This implies $d_n\neq0$ (and also $c_{n,n}\neq0$) for each $n=0,1,\ldots$, which means that (ϕ,ψ) is a $(q,\omega)-$admissible pair. Thus, it follows from (2.46) that $\big(P_n^{[1]}\big)_{n\geq0}$ is a monic OPS with respect to $\mathbf{u}^{[1]}$. This proves the last statement in the theorem for $k=1$. Now, by (2.13), $\mathbf{u}^{[1]}$ fulfills $\mathbf{D}_{q,\omega}\big(\phi\mathbf{u}^{[1]}\big)=\psi^{[1]}\mathbf{u}^{[1]}$, hence, since $P_n^{[2]}=D_{q,\omega}P_{n+1}^{[1]}/[n+1]_q$ and, by (2.15), $\psi^{[1]}(x)=d_2x+e_1$, from (2.46) with $\mathbf{u}$, ψ, and $(P_n)_{n\geq0}$ replaced (respectively) by $\mathbf{u}^{[1]}$, $\psi^{[1]}$, and $(P_n^{[1]})_{n\geq0}$, we deduce, for every $n,m\in\mathbb{N}_0$,

$$\langle\mathbf{u}^{[2]},P_n^{[2]}P_m^{[2]}\rangle=-\frac{q^{-n}d_n^{[1]}}{[n+1]_q}\langle\mathbf{u}^{[1]},\big(P_{n+1}^{[1]}\big)^2\rangle\delta_{nm}$$

where $d_n^{[1]}$ is defined as in (2.7) corresponding to the pair $(\phi,\psi^{[1]})$, so that

$$d_n^{[1]}:=\big(\psi^{[1]}\big)'q^n+\tfrac{1}{2}\phi''\,[n]_q=d_2q^n+a[n]_q=d_{n+2}\,.$$

Therefore, and taking into account once again (2.46), we obtain

$$\langle\mathbf{u}^{[2]},P_n^{[2]}P_m^{[2]}\rangle=q^{-(2n+1)}\frac{d_{n+1}d_{n+2}}{[n+1]_q[n+2]_q}\langle\mathbf{u},P_{n+2}^2\rangle\delta_{nm}\quad(n,m\in\mathbb{N}_0)\,,$$

and so $(P_n^{[2]})_{n\geq0}$ is a monic OPS with respect to $\mathbf{u}^{[2]}$. Arguing by induction, we prove

$$\langle\mathbf{u}^{[k]},P_n^{[k]}P_m^{[k]}\rangle=(-1)^kq^{-k(2n+k-1)/2}\Big(\prod_{j=1}^{k}\frac{d_{n+k+j-2}}{[n+j]_q}\Big)\langle\mathbf{u},P_{n+k}^2\rangle\delta_{nm}\quad(k,n,m\in\mathbb{N}_0)\,, \tag{2.49}$$

hence $(P_n^{[k]})_{n\geq0}$ is a monic OPS with respect to $\mathbf{u}^{[k]}$, for each $k\in\mathbb{N}_0$. This completes the proof.

We have shown that if a nonzero linear functional $\mathbf{u}\in\mathscr{P}'$ satisfies (2.9), where (ϕ,ψ) given by (2.6)–(2.7) is a (q,ω)-admissible pair, then the Rodrigues-type formula (2.18) holds, where $(P_n)_{n\geq0}$ is a simple set of monic polynomials defined by the three-term recurrence relation (1.1) with coefficients β_n and γ_n given by (2.20)–(2.21). In the next section, we give necessary and sufficient conditions for the regularity of such functional $\mathbf{u}$. We also show that under those regularity conditions, the above mentioned polynomial sequence $(P_n)_{n\geq0}$ is the monic OPS with respect to $\mathbf{u}$.

2.2 Regularity conditions

2.2.1 Main theorem

Theorem 2.2.1 *Let $q,\omega \in \mathbb{C}$ fulfilling* (1.5). *Let $\mathbf{u} \in \mathscr{P}' \setminus \{\mathbf{0}\}$. Suppose that the functional equation (2.9) holds; with $\phi \in \mathscr{P}_2$, $\psi \in \mathscr{P}_1$, and at least one of ϕ and ψ is not the zero polynomial.*

Then, $\mathbf{u}$ is regular if and only if

$$d_n \neq 0\,, \quad \phi\Big(-\frac{e_n}{d_{2n}}\Big) \neq 0\,, \quad \forall n \in \mathbb{N}_0\,, \tag{2.50}$$

where ϕ, d_n and e_n are given by relations (2.6)–(2.7). Under these conditions, the monic OPS $(P_n)_{n\geq 0} \equiv (P_n(\cdot;q,\omega))_{n\geq 0}$ with respect to $\mathbf{u}$ satisfies the three-term recurrence relation (1.1) *with coefficients β_n and γ_n given by* (2.20)–(2.21).

In addition, the Rodrigues-type formula

$$P_n\mathbf{u} = k_n\,\mathbf{D}^n_{1/q,-\omega/q}\Big(\Phi(\cdot;n)\mathbf{L}^n_{q,\omega}\mathbf{u}\Big) \quad (n=0,1,\ldots) \tag{2.51}$$

holds in $\mathscr{P}^*$, where k_n and $\Phi(\cdot;n)$ are defined in (2.19) and (2.12), respectively.

Proof Suppose that $\mathbf{u}$ is regular. Let $n \in \mathbb{N}_0$. Since $\mathbf{u}$ satisfies (2.9), Lemma 2.1.7 ensures that (ϕ,ψ) is a $(q,\omega)-$admissible pair, and so $d_n \neq 0$. Moreover, $\mathbf{u}^{[n]}$ is regular and $\big(P_j^{[n]}\big)_{j\geq 0}$ is the corresponding monic OPS, which fulfills a three-term recurrence relation:

$$P^{[n]}_{j+1}(x) = (x-\beta^{[n]}_j)P^{[n]}_j(x) - \gamma^{[n]}_j P^{[n]}_{j-1}(x) \quad (j=0,1,\ldots)\,, \tag{2.52}$$

where $P^{[n]}_{-1}(x)=0$, being $\beta^{[n]}_j \in \mathbb{C}$ and $\gamma^{[n]}_j \in \mathbb{C}\setminus\{0\}$ for each j. Let us compute $\gamma^{[n]}_1$. We first show that (for $n=0$) the coefficient $\gamma_1 \equiv \gamma^{[0]}_1$, appearing in the three-term recurrence relation for $(P_j)_{j\geq 0}$, is given by

$$\gamma_1 = -\frac{1}{dq+a}\,\phi\left(-\frac{e}{d}\right)\,. \tag{2.53}$$

This may be proved taking $n=0$ and $n=1$ in the relation $\langle \mathbf{D}_{q,\omega}(\phi\mathbf{u}),x^n\rangle = \langle \psi\mathbf{u},x^n\rangle$. Indeed, setting $u_n := \langle \mathbf{u},x^n\rangle$, for $n=0$ we obtain $0 = du_1+eu_0$, and for $n=1$ we find $-q^{-1}(au_2+bu_1+cu_0) = du_2+eu_1$. Therefore,

$$u_1 = -\frac{e}{d}u_0\,, \quad u_2 = -\frac{1}{dq+a}\left[-(qe+b)\frac{e}{d}+c\right]u_0\,. \tag{2.54}$$

On the other hand, since $P_1(x) = x-\beta_0 = x-u_1/u_0$, we also have

$$\gamma_1 = \frac{\langle \mathbf{u},P_1^2\rangle}{u_0} = \frac{u_2u_0-u_1^2}{u_0^2} = \frac{u_2}{u_0} - \left(\frac{u_1}{u_0}\right)^2\,. \tag{2.55}$$

Substituting u_1 and u_2 given by (2.54) into (2.55) yields (2.53). Now, since equation (2.13) is of the same type as (2.9), with the same polynomial ϕ and being ψ replaced by $\psi^{[n]}$, we see that $\gamma^{[n]}_1$ may be obtained replacing in (2.53) the coefficients d and e of $\psi(x) = dx+e$ by the corresponding coefficients of $\psi^{[n]}(x) = d_{2n}x + e_n$. Hence,

$$\gamma_1^{[n]} = -\frac{1}{d_{2n}q+a}\phi\left(-\frac{e_n}{d_{2n}}\right) = -\frac{1}{d_{2n+1}}\phi\left(-\frac{e_n}{d_{2n}}\right) . \tag{2.56}$$

Since $\mathbf{u}^{[n]}$ is regular, then $\gamma_1^{[n]} \neq 0$, hence $\phi\left(-\frac{e_n}{d_{2n}}\right) \neq 0$. Thus, (2.50) holds.

Conversely, suppose that (2.50) holds. Then, by Favard's theorem, the sequence $(P_n)_{n\geq 0}$ defined by the three-term recurrence relation (1.1) with coefficients given in (2.20) – (2.21) is a monic OPS. We claim that $(P_n)_{n\geq 0}$ is an OPS with respect to $\mathbf{u}$. To prove this sentence we only need to show that (see e.g. [13, Chapter I, Exercise 4.14] or [56, Corollary 6.2])

$$\langle \mathbf{u}, 1\rangle \neq 0 , \quad \langle \mathbf{u}, P_n\rangle = 0 \quad (n = 1, 2, \ldots) . \tag{2.57}$$

Suppose that $\langle \mathbf{u}, 1\rangle = 0$. Since the functional equation (2.9) is equivalent to the second order difference equation (1.25) fulfilled by the moments $y_n := \langle \mathbf{u}, Y_n\rangle$ (with d_n and e_n defined as in relations (2.6)–(2.7)), and noting that for $n = 0$, (1.25) yields $dy_1 + ey_0 = 0$, we get $y_1 = 0$ (because $y_0 = \langle \mathbf{u}, 1\rangle = 0$ and $d = d_0 \neq 0$); hence $y_0 = y_1 = 0$ and so it follows recurrently from (1.25) that $y_n = 0$ for each $n \in \mathbb{N}_0$. Therefore $\mathbf{u} = \mathbf{0}$, in contradiction with the hypothesis. Thus, $\langle \mathbf{u}, 1\rangle \neq 0$. On the other hand, by Lemma 2.1.5, for each $n \geq 1$ we may write

$$\langle \mathbf{u}, P_n\rangle = \langle P_n\mathbf{u}, 1\rangle = -qk_n\left\langle \mathbf{D}^{n-1}_{1/q,-\omega/q}\mathbf{u}^{[n]}, D_{q,\omega}1\right\rangle = 0 .$$

Thus (2.57) is proved, hence $\mathbf{u}$ is regular and $(P_n)_{n\geq 0}$ is the corresponding monic OPS. Finally, the Rodrigues-type formula (2.51) follows from Lemma 2.1.5 and (2.11), concluding the proof of Theorem 2.2.1.

2.2.2 Final remarks

1. Under the assumption that $\mathbf{u}$ is regular, the Rodrigues-type formula (2.51) appears in Médem et al. [52] for $\omega = 0$ and $q \neq 1$, and in Salto [59] for $q = 1$ and $\omega \neq 0$. However, we proved a more general result (cf. Lemma 2.1.5), showing that (2.51) holds without assuming the regularity of $\mathbf{u}$, provided that $(P_n)_{n\geq 0}$ is a simple set of polynomials defined by (1.1) with coefficients given by (2.20)–(2.21), which we see is well defined requiring only (the admissibility condition) $d_n \neq 0$ for each $n = 0, 1, \ldots$. It is worth mentioning that this (non trivial) fact is known for the continuous case [38, Lemma 2], but for the $(q, \omega)-$case we did not found a reference in the available literature.

2. As expected, taking $\omega = 0$ and letting $q \to 1$ in Theorem 2.2.1 yields Theorem 2.1.1.

3. We highlight that Häcker [22, Theorem 1.4 (p. 26)] gave regularity conditions different from (2.50), considering a definition of $\mathbf{D}_{q,\omega}$ in the sense discussed in the previous chapter. Häcker's approach is different from ours, since his results are derived from the analysis of a discrete Sturm-Liouville problem, while our proof of Theorem 2.2.1 uses appropriate modifications of some ideas appearing in [38], based in the McS thesis [57] and obtained independently of Häcker's results. Indeed, as we said at the beginning of this thesis, our approach is supported on the algebraic theory of orthogonal polynomials developed by Maroni [46].

4. As we mentioned before, there are some advantages in defining $\mathbf{D}_{q,\omega}$ as in (1.6). For instance, in the regularity condition (2.50) as well as in the expression for γ_n given by (2.21), the polynomial appearing therein is precisely ϕ. The same does not hold in the formulas given in Häcker thesis (cf. [22, Section 2.4]).

5. Since $-e_n/d_{2n}$ is the unique zero of $\psi^{[n]}(x) = d_{2n}x + e_n$, the regularity conditions (2.50) for $\mathbf{u}$ given in Theorem 2.2.1 may be restated as follows: $\mathbf{u}$ *is regular if and only if* (ϕ, ψ) *is a* $(q,\omega)-$*admissibe pair and* $\psi^{[n]} \nmid \phi$ *for each* $n = 0,1,\ldots$. Thus, comparing with [22, Theorem 1.4], we see once again that it is advantageous to define $\mathbf{D}_{q,\omega}$ as in (1.6).

6. It may seems somehow intricate the way how formulas (2.20) and (2.21) appear on the course of the proof of Theorem 2.2.1. In fact, they were given in the proof of the sufficiency of the condition, hence without assuming *a priori* the regularity of $\mathbf{u}$ (as a matter of fact, they were used to prove the regularity of $\mathbf{u}$). Assuming the regularity of $\mathbf{u}$, there is a more transparent way to obtain those formulas. Indeed, going back to the end of the proof of the necessity of the condition on Theorem 2.2.1, we may deduce (2.20) and (2.21) as follows. First, from (2.49), we may write

$$\gamma_j^{[n]} = \frac{\left\langle \mathbf{u}^{[n]}, \left(P_j^{[n]}\right)^2 \right\rangle}{\left\langle \mathbf{u}^{[n]}, \left(P_{j-1}^{[n]}\right)^2 \right\rangle} = \frac{q^{-n}[j]_q d_{j+2n-2}}{[j+n]_q d_{j+n-2}} \frac{\langle \mathbf{u}, P_{j+n}^2 \rangle}{\langle \mathbf{u}, P_{j+n-1}^2 \rangle} = \frac{q^{-n}[j]_q d_{j+2n-2}}{[j+n]_q d_{j+n-2}} \gamma_{j+n}$$

for every $j = 1,2,\ldots$ and $n = 0,1,\ldots$. Taking $j = 1$ and using (2.56), we obtain

$$\gamma_{n+1} = \frac{q^n[n+1]_q d_{n-1}}{d_{2n-1}} \gamma_1^{[n]} = -\frac{q^n[n+1]_q d_{n-1}}{d_{2n-1}d_{2n+1}} \phi\left(-\frac{e_n}{d_{2n}}\right) .$$

This proves (2.21). To prove (2.20), set $P_n^{[k]}(x) = x^n + t_n^{[k]} x^{n-1} + (\text{lower degree terms})$, for each $k = 0,1,\ldots$. It is well known (see e.g. [13, Theorem 4.2-(d)]) that

$$t_n^{[k]} = -\sum_{j=0}^{n-1} \beta_j^{[k]} \quad (k = 0,1,\ldots;\ n = 1,2,\ldots) .$$

Using (1.19), and recalling that $P_n^{[0]} = P_n$, we deduce

$$\begin{aligned} D_{q,\omega}P_{n+1}(x) &= D_{q,\omega}(x^{n+1}) + t_{n+1}^{[0]} D_{q,\omega}(x^n) + (\text{lower degree terms}) \\ &= [n+1]_q x^n + \left\{ \big((n+1)[n]_q - n[n+1]_q\big)\omega_0 + t_{n+1}^{[0]}[n]_q \right\} x^{n-1} + (\text{lower degree terms}) , \end{aligned}$$

hence, since $P_n^{[1]}(x) := D_{q,\omega}P_{n+1}(x)/[n+1]_q$, we obtain

$$t_n^{[1]} = \left(\frac{(n+1)[n]_q}{[n+1]_q} - n \right) \omega_0 + t_{n+1}^{[0]} \frac{[n]_q}{[n+1]_q} \quad (n = 1,2,\ldots) .$$

Rewrite this equality as

$$\frac{t_{n+1}^{[0]} + (n+1)\omega_0}{[n+1]_q} = \frac{t_n^{[1]} + n\omega_0}{[n]_q} \quad (n = 1,2,\ldots) .$$

Applying successively this relation, yields

$$\frac{t_{n+1}^{[0]}+(n+1)\omega_0}{[n+1]_q}=\frac{t_1^{[n]}+1\cdot\omega_0}{[1]_q}=-\beta_0^{[n]}+\omega_0 \quad (n=1,2,\ldots)\,,$$

hence

$$t_{n+1}^{[0]}=\big([n+1]_q-(n+1)\big)\omega_0-[n+1]_q\beta_0^{[n]} \quad (n=0,1,\ldots)\,.$$

(Note that this equality is trivial if $n=0$.) Therefore,

$$\beta_n=\beta_n^{[0]}=t_n^{[0]}-t_{n+1}^{[0]}=\big([n]_q-[n+1]_q+1\big)\omega_0+[n]_q\beta_0^{[n-1]}-[n+1]_q\beta_0^{[n]}\,.$$

This proves (2.20), since $\beta_0=u_1/u_0=-e/d$, hence $\beta_0^{[n]}=-e_n/d_{2n}$, and taking into account that $([n]_q-[n+1]_q+1)\omega_0=[n]_q\omega$.
Now, suppose that $\mathbf{u}\in\mathscr{P}^*$ is regular and satisfies the functional equation (2.9). Then the Rodrigues-type formula (2.51) is a simple consequence of the relation between the dual basis $(\mathbf{a}_n)_{n\geq 0}$ and $\big(\mathbf{a}_n^{[k]}\big)_{n\geq 0}$ associated to the monic OPS $(P_n)_{n\geq 0}$ and $(P_n^{[k]})_{n\geq 0}$ ($k=0,1,\ldots$), respectively. To see why this holds, we may write (in the sense of the weak dual topology in $\mathscr{P}'$):

$$\mathbf{D}^k_{1/q,-\omega/q}\big(\mathbf{a}_n^{[k]}\big)=\sum_{j=0}^{\infty}\langle\mathbf{D}^k_{1/q,-\omega/q}\big(\mathbf{a}_n^{[k]}\big),P_j\rangle\mathbf{a}_j \quad (n=0,1,\ldots)\,.$$

Since $\langle\mathbf{D}^k_{1/q,-\omega/q}\big(\mathbf{a}_n^{[k]}\big),P_j\rangle=0$ if $j<k$ and, if $j\geq k$,

$$\langle\mathbf{D}^k_{1/q,-\omega/q}\big(\mathbf{a}_n^{[k]}\big),P_j\rangle=(-q)^k\langle\mathbf{a}_n^{[k]},D^k_{q,\omega}P_j\rangle=(-q)^k\frac{[j]_q!}{[j-k]_q!}\langle\mathbf{a}_n^{[k]},P_{j-k}^{[k]}\rangle\,,$$

we deduce

$$\mathbf{D}^k_{1/q,-\omega/q}\big(\mathbf{a}_n^{[k]}\big)=(-q)^k\,\frac{[n+k]_q!}{[n]_q!}\,\mathbf{a}_{n+k} \quad (n,k=0,1,\ldots)\,. \tag{2.58}$$

Taking $n=0$ and then replacing k by n, we obtain

$$\mathbf{D}^n_{1/q,-\omega/q}\big(\mathbf{a}_0^{[n]}\big)=(-q)^n[n]_q!\mathbf{a}_n \quad (n=0,1,\ldots)\,.$$

Therefore, since $\mathbf{a}_0^{[n]}=\langle\mathbf{u}^{[n]},1\rangle^{-1}\mathbf{u}^{[n]}$ and $\mathbf{a}_n=\langle\mathbf{u},P_n^2\rangle^{-1}P_n\mathbf{u}$ (see [46, 48]), we deduce

$$\mathbf{D}^n_{1/q,-\omega/q}\big(\mathbf{u}^{[n]}\big)=(-q)^n[n]_q!\frac{\langle\mathbf{u}^{[n]},1\rangle}{\langle\mathbf{u},P_n^2\rangle}P_n\mathbf{u} \quad (n=0,1,\ldots)\,.$$

Finally, taking into account (2.11) and (2.49), (2.51) follows.

Chapter 3

Another extension of coherent pairs of measures

3.1 Introduction

In the framework of the theory of orthogonal polynomials, the concept of coherent pair of measures as well as its multiple generalizations have been a subject of increasing research interest along the last decades. This concept was introduced by Iserles et al. [25] motivated by the theory of polynomial approximation with respect to certain Sobolev inner products. In [27, 29], the notion of $(M,N)-$coherent pair of order (m,k) were introduced as extensions of most of the concepts of coherence up to that time. More precisely, given two monic OPS, $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$, we say that $\big((P_n)_{n\geq 0},(Q_n)_{n\geq 0}\big)$ is an $(M,N)-$coherent pair of order (m,k) if there exist two non-negative integer numbers M and N, and sequences of complex numbers $(a_{n,j})_{n\geq 0}$ $(j=0,1,\ldots,M)$ and $(b_{n,j})_{n\geq 0}$ $(j=0,1,2,\ldots,N)$ such that, under natural assumptions on the coefficients $a_{n,j}$ and $b_{n,j}$, the structure relation

$$\sum_{j=0}^{M} a_{n,j}P_{n-j}^{[m]}(x) = \sum_{j=0}^{N} b_{n,j}Q_{n-j}^{[k]}(x) \quad (n=0,1,\ldots)$$

holds. We use the notation

$$P_n^{[m]}(x) := \frac{1}{(n+1)_m}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_{n+m}(x)$$

($Q_n^{[k]}$ is defined in the same way), where for any positive real number α, $(\alpha)_n$ denotes the Pochhammer symbol defined by

$$(\alpha)_0 := 1\,, \quad (\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1) \quad \text{if} \quad n\in\mathbb{N}\,.$$

Note that $P_n^{[m]}$ is a normalization of the derivative of order m of P_{n+m} defined so that it becomes a monic polynomial of degree n. Let $\mathbf{u}$ and $\mathbf{v}$ be the moment regular functionals with respect to which $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are orthogonal. It follows from the results in [27–29, 58] that if $m=k$ then $\mathbf{u}$ and $\mathbf{v}$ are connected by a rational transformation (in the distributional sense), i.e., there exist nonzero polynomials Φ and Ψ such that $\Phi\mathbf{u}=\Psi\mathbf{v}$. Otherwise if $m\neq k$ then $\mathbf{u}$ and $\mathbf{v}$ are still connected by a rational transformation and, in addition, they are semiclassical functionals, i.e., there exist nonzero polynomials Φ_1, Ψ_1, Φ_2, and Ψ_2 such that

$$D(\Phi_1 \mathbf{u}) = \Psi_1 \mathbf{u} \,, \quad D(\Phi_2 \mathbf{v}) = \Psi_2 \mathbf{v} \,.$$

In this chapter we modify the left-hand side of the above structure relation, and consider the following one:

$$\pi_N(x) P_n^{[m]}(x) = \sum_{j=n-M}^{n+N} c_{n,j} Q_j^{[k]}(x) \quad (n = 0, 1, 2, \ldots) \,, \tag{3.1}$$

where M and N are fixed non-negative integer numbers, π_N is a monic polynomial of degree N (hence $c_{n,n+N} = 1$ for each n), and we consider the convention $Q_j \equiv 0$ if $j < 0$. Further, we will assume that the following condition holds:

$$c_{n,n-M} \neq 0 \quad \text{if} \quad n \geq M \,. \tag{3.2}$$

Maroni and Sfaxi [45] considered the case $(m,k) = (0,1)$ and called the pair $\big((P_n)_{n\geq 0}, (Q_n)_{n\geq 0}\big)$ fulfilling the structure relation (3.1) whenever $(m,k) = (0,1)$ a π_N-coherent pair with index M. This motivates the following.

Definition 3.1 *Let M and N be non-negative integer numbers and let π_N be a monic polynomial of degree N. If $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are two monic OPS such that their normalized derivatives of orders m and k (respectively) satisfy* (3.1)–(3.2)*, we call $\big((P_n)_{n\geq 0}, (Q_n)_{n\geq 0}\big)$, as well as the corresponding pair $(\mathbf{u}, \mathbf{v})$ of regular functionals, a π_N-coherent pair with index M and order (m,k).*

Besides [45], many other instances of the structure relation (3.1) were considered previously by several authors. For instance, the case $N = 0$ (i.e., $\pi_N \equiv 1$ and M, m, and k being arbitrary) fits into the theory of $(M,0)-$coherent pairs of order (m,k), described at the begin of this introduction. Also, whenever $(m,k) = (1,0)$ and $(P_n)_{n\geq 0} \equiv (Q_n)_{n\geq 0}$, (3.1) becomes a characterization of semiclassical OPS due to Maroni [44, 46]. Note that for $N \leq 2$ and $M = 0$, this reduces to the well known Al-Salam-Chihara characterization of the classical OPS [3]. The case $k = 0$ (M, N and m being arbitrary) was considered by Bonan et al. [8] in the framework of orthogonality in the positive-definite sense, i.e., whenever the orthogonality of each of the involved OPS is considered with respect to positive Borel measures. In the special case $m = 1$, a complementary approach to the case considered in [8] was presented in [39], in the framework of the so-called regular (or formal) orthogonality. A relevant reference concerning finite-type relations between OPS is Maroni's article [40].

It is a remarkable fact that in all the previous works the involved OPS and their corresponding regular moment linear functionals are semiclassical. Thus, a major question is to analyze whether the OPS involved in a π_N-coherent pair with index M and order (m,k) are semiclassical, and in such case to determine the relations between the corresponding regular moment linear functionals. This will be treated in the next section. In Section 3.3 we analyze the case of a discrete variable obtained from (3.1) by replacing the derivative operator by the discrete Hahn operator defined by (1.4). The last section is devoted to applications where we present alternative approaches to some results due to Griffin (see [21]), Datta and Griffin (see [14]) which fit into this notion of coherence pair of measures.

3.2 π_N-coherent pairs with index M and order (m,k): the continuous case

In this section we establish the semiclassical character of the OPS and their associated regular functionals involved in a π_N-coherent pair with index M and order (m,k). As in the previous chapter, our approach is based upon the algebraic theory of orthogonal polynomials developed by Maroni [43, 46].

Lemma 3.2.1 *Let $\big((P_n)_{n\geq 0},(Q_n)_{n\geq 0}\big)$ be a π_N-coherent pair with index M and order (m,k), so that* (3.1)–(3.2) *hold. Set*

$$\psi(x;n):=\sum_{j=n-N}^{n+M}\frac{(-1)^m(j+1)_m\,c_{j,n}}{\langle\mathbf{u},P_{m+j}^2\rangle}P_{m+j}(x)\,, \tag{3.3}$$

$$\phi(x;n,j):=\frac{(-1)^k(n+1)_k}{\langle\mathbf{v},Q_{n+k}^2\rangle}\sum_{\ell=0}^{N-j}\binom{k+N}{\ell}\binom{N-\ell}{N-j-\ell}\pi_N^{(\ell)}(x)Q_{n+k}^{(N-j-\ell)}(x)\,, \tag{3.4}$$

for all $n=0,1,\ldots$, *and* $j=0,1,2,\ldots,N$, *so that*

$$\deg\psi(\cdot;n)=m+n+M\,,\quad \deg\phi(\cdot;n,j)=k+n+j\,.$$

Let $\mathbf{u}$ *and* $\mathbf{v}$ *be the regular functionals with respect to which* $(P_n)_{n\geq 0}$ *and* $(Q_n)_{n\geq 0}$ *are orthogonal. Then the following functional equations hold:*

$$\psi(\cdot;n)\mathbf{u}=D^{m-k-N}\left(\sum_{j=0}^{N}\phi(\cdot;n,j)D^j\mathbf{v}\right)\qquad \text{if}\quad m\geq k+N\,, \tag{3.5}$$

$$D^{k+N-m}\big(\psi(\cdot;n)\mathbf{u}\big)=\sum_{j=0}^{N}\phi(\cdot;n,j)D^j\mathbf{v}\qquad \text{if}\quad m<k+N\,, \tag{3.6}$$

for all $n=0,1,\ldots$.

Proof Let $(\mathbf{a}_n)_{n\geq 0}$, $(\mathbf{b}_n)_{n\geq 0}$, $(\mathbf{a}_n^{[m]})_{n\geq 0}$, and $(\mathbf{b}_n^{[k]})_{n\geq 0}$ be the dual basis corresponding to the simple sets of polynomials $(P_n)_{n\geq 0}$, $(Q_n)_{n\geq 0}$, $(P_n^{[m]})_{n\geq 0}$ and $(Q_n^{[k]})_{n\geq 0}$, respectively. Then

$$\pi_N\mathbf{b}_n^{[k]}=\sum_{j=0}^{+\infty}\big\langle\pi_N\mathbf{b}_n^{[k]},P_j^{[m]}\big\rangle\,\mathbf{a}_j^{[m]}\quad(n=0,1,2,\ldots)$$

(in the sense of the weak dual topology in $\mathscr{P}'$). From (3.1), we have

$$\begin{aligned}\big\langle\pi_N\mathbf{b}_n^{[k]},P_j^{[m]}\big\rangle&=\big\langle b_n^{[k]},\pi_N P_j^{[m]}\big\rangle=\sum_{\ell=j-M}^{j+N}c_{j,\ell}\big\langle b_n^{[k]},Q_\ell^{[k]}\big\rangle\\&=\begin{cases}c_{j,n} & \text{if}\quad n-N\leq j\leq n+M\\ 0 & \text{otherwise}\,.\end{cases}\end{aligned}$$

Hence

$$\pi_N \mathbf{b}_n^{[k]} = \sum_{j=n-N}^{n+M} c_{j,n} \mathbf{a}_j^{[m]} \quad (n = 0,1,2,\ldots) \,. \tag{3.7}$$

Considering the m-th derivative on both sides of this equation and taking into account that $D^m\big(\mathbf{a}_j^{[m]}\big) = (-1)^m (j+1)_m \mathbf{a}_{j+m}$, we obtain

$$D^m\big(\pi_N \mathbf{b}_n^{[k]}\big) = \psi(\cdot;n)\mathbf{u} \quad (n = 0,1,2,\ldots) \,, \tag{3.8}$$

where $\psi(\cdot;n)$ is defined by (3.3). Notice that the condition (3.2) ensures that $\deg \psi(\cdot,n) = M+m+n$ for each $n = 0,1,2,\ldots$. Using the Leibniz rule for the derivative of the left product of a functional by a polynomial, and taking into account that $\pi_N^{(j)} = 0$ if $j > N$, as well as

$$D^k\big(\mathbf{b}_n^{[k]}\big) = (-1)^k (n+1)_k \mathbf{b}_{n+k} = (-1)^k (n+1)_k \langle \mathbf{v}, Q_{n+k}^2 \rangle^{-1} Q_{n+k} \mathbf{v} \,,$$

we deduce

$$\begin{aligned}
D^{k+N}\big(\pi_N \mathbf{b}_n^{[k]}\big) &= \frac{(-1)^k (n+1)_k}{\langle \mathbf{v}, Q_{n+k}^2 \rangle} \sum_{j=0}^{N} \binom{k+N}{j} \pi_N^{(j)} D^{N-j}(Q_{n+k}\mathbf{v}) \\
&= \frac{(-1)^k (n+1)_k}{\langle \mathbf{v}, Q_{n+k}^2 \rangle} \sum_{j=0}^{N} \sum_{\ell=0}^{N-j} \binom{k+N}{j} \binom{N-j}{\ell} \pi_N^{(j)} Q_{n+k}^{(\ell)} D^{N-j-\ell}\mathbf{v} \\
&= \frac{(-1)^k (n+1)_k}{\langle \mathbf{v}, Q_{n+k}^2 \rangle} \sum_{j=0}^{N} \sum_{\ell=j}^{N} \binom{k+N}{j} \binom{N-j}{\ell-j} \pi_N^{(j)} Q_{n+k}^{(\ell-j)} D^{N-\ell}\mathbf{v} \\
&= \frac{(-1)^k (n+1)_k}{\langle \mathbf{v}, Q_{n+k}^2 \rangle} \sum_{\ell=0}^{N} \sum_{j=0}^{\ell} \binom{k+N}{j} \binom{N-j}{\ell-j} \pi_N^{(j)} Q_{n+k}^{(\ell-j)} D^{N-\ell}\mathbf{v} \\
&= \sum_{\nu=0}^{N} \left(\frac{(-1)^k (n+1)_k}{\langle \mathbf{v}, Q_{n+k}^2 \rangle} \sum_{j=0}^{N-\nu} \binom{k+N}{j} \binom{N-j}{N-\nu-j} \pi_N^{(j)} Q_{n+k}^{(N-\nu-j)} \right) D^{\nu}\mathbf{v} \,.
\end{aligned}$$

Hence, by (3.4), we obtain

$$D^{k+N}\big(\pi_N \mathbf{b}_n^{[k]}\big) = \sum_{j=0}^{N} \phi(\cdot;n,j) D^j \mathbf{v} \,. \tag{3.9}$$

If $m \geq k+N$, we rewrite (3.8) as

$$\psi(\cdot;n)\mathbf{u} = D^{m-k-N} D^{k+N}\big(\pi_N \mathbf{b}_n^{[k]}\big) \quad (n = 0,1,2\ldots) \,, \tag{3.10}$$

and (3.5) follows from (3.9) and (3.10). If $m < k+N$, writing

$$D^{k+N}\big(\pi_N \mathbf{b}_n^{[k]}\big) = D^{k-m+N} D^m\big(\pi_N \mathbf{b}_n^{[k]}\big) \quad (n = 0,1,2\ldots) \,,$$

we see that (3.6) follows from (3.8) and (3.9). This ends the proof.

Our problem will be separated into three cases depending on m, k and N.

3.2.1 Case $m \geq k+N$

Theorem 3.2.2 *Let $\big((P_n)_{n\geq 0},(Q_n)_{n\geq 0}\big)$ be a π_N−coherent pair with index M and order (m,k), so that* (3.1)–(3.2) *holds. Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be the regular functionals with respect to which $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are orthogonal. Suppose $m \geq k+N$. Assume further that $m > k$ whenever $N = 0$. For each $i = 0,\ldots,m-k$ and $n = 0,1,2,\ldots$, let*

$$\varphi(x;n,i) := \sum_{\substack{j+\ell=i\\ 0\leq j\leq N\\ 0\leq \ell\leq M}} \binom{m-k-N}{\ell} \big(\phi(x;n,j)\big)^{(m-k-N-\ell)}, \tag{3.11}$$

$\phi(\cdot;n,j)$ being the polynomial introduced in (3.4). *Let $\mathscr{A}(x)$ be the polynomial matrix of order $m-k+1$ defined by*

$$\mathscr{A}(x) := \big[\varphi(x;n,j)\big]_{n,j=0}^{m-k}.$$

Let $\mathscr{A}_1(x)$ (resp., $\mathscr{A}_2(x)$) be the matrix obtained by replacing the first (resp., the second) column of $\mathscr{A}(x)$ by $\big[\psi(x;0),\psi(x;1),\cdots,\psi(x;m-k)\big]^t$, and set

$$A(x) := \det \mathscr{A}(x), \quad A_1(x) := \det \mathscr{A}_1(x), \quad A_2(x) := \det \mathscr{A}_2(x).$$

Assume that the polynomial $A(x)$ does not vanishes identically. Then

$$A\mathbf{v} = A_1\mathbf{u}, \quad AD\mathbf{v} = A_2\mathbf{u}, \tag{3.12}$$

hence $\mathbf{u}$ *and* $\mathbf{v}$ *are semiclassical functionals related by a rational transformation. Moreover,* $\mathbf{u}$ *and* $\mathbf{v}$ *fulfill the following equations:*

$$D(AA_1\mathbf{u}) = \big((AA_1)'+AA_2\big)\mathbf{u}, \quad D(AA_1\mathbf{v}) = \big(2A'A_1+AA_2\big)\mathbf{v}. \tag{3.13}$$

Proof By (3.5) and the Leibniz rule, we have

$$\psi(\cdot;n)\mathbf{u} = \sum_{j=0}^{N}\sum_{\ell=0}^{m-k-N}\binom{m-k-N}{\ell}\big(\phi(\cdot;n,j)\big)^{(m-k-N-\ell)}D^{j+\ell}\mathbf{v}.$$

This may be rewritten as

$$\psi(\cdot;n)\mathbf{u} = \sum_{i=0}^{m-k}\varphi(\cdot;n,i)D^i\mathbf{v} \quad (n = 0,1,2\cdots), \tag{3.14}$$

where $\varphi(\cdot;n,i)$ is the polynomial introduced in (3.11). Taking $n = 0,1,2\ldots,m-k$ in (3.14) we obtain a system with $m-k+1$ equations that can be written as

$$\begin{pmatrix} \psi(x;0)\mathbf{u}\\ \psi(x;1)\mathbf{u}\\ \vdots\\ \psi(x;m-k)\mathbf{u}\end{pmatrix} = \mathscr{A}(x)\begin{pmatrix}\mathbf{v}\\ D\mathbf{v}\\ \vdots\\ D^{m-k}\mathbf{v}\end{pmatrix}.$$

Solving for **v** and $D\mathbf{v}$ we obtain (3.12). Finally, (3.13) follows from (3.12).

Remark 3.2.1 *If $m = k$ and $N = 0$, then* **u** *and* **v** *are still related by a rational transformation, but we cannot ensure that they are semiclassical (see [27, 28]).*

Now, we consider the second case.

3.2.2 Case $m < k+N$

Theorem 3.2.3 *Let $\big((P_n)_{n\geq 0}, (Q_n)_{n\geq 0}\big)$ be a π_N-coherent pair with index M and order (m,k), so that (3.1)–(3.2) holds. Let* **u** *and* **v** *be the regular functionals with respect to which $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are orthogonal. Assume further that $m < k+N$. For each $j = 0,\ldots,k-m+N$ and $n = 0,1,\ldots$, set*

$$\xi(x;n,j) := \binom{k-m+N}{j}\big(\psi(x;n)\big)^{(k-m+N-j)}, \tag{3.15}$$

$\psi(\cdot;n)$ being the polynomial introduced in (3.3). Let $\mathscr{B}(x) := \big[b_{i,j}(x)\big]_{i,j=0}^{k-m+2N}$ be the polynomial matrix of order $k-m+2N+1$ defined by

$$b_{i,j}(x) := \begin{cases} \phi(x;i,j) & \text{if} \quad 0 \leq j \leq N, \\ -\xi(x;i,j-N) & \text{if} \quad N+1 \leq j \leq k-m+2N, \end{cases}$$

$\phi(\cdot;i,j)$ being the polynomial given by (3.4). Let $\mathscr{B}_1(x)$ (resp., $\mathscr{B}_2(x)$ and $\mathscr{B}_{N+2}(x)$) be the matrix obtained by replacing the first (resp., the second and the $(N+2)$-th) column of $\mathscr{B}(x)$ by $\big[\xi(x;0,0),\xi(x;1,0),\cdots,\xi(x;m-k+2N,0)\big]^t$, and set

$$B(x) := \det\mathscr{B}(x), \quad B_j(x) := \det\mathscr{B}_j(x), \quad j \in \{1,2,N+2\}.$$

Assume that the polynomial $B(x)$ does not vanishes identically. Then

$$B\mathbf{v} = B_1\mathbf{u}, \quad BD\mathbf{v} = B_2\mathbf{u}, \quad BD\mathbf{u} = B_{N+2}\mathbf{u}, \tag{3.16}$$

hence **u** *and* **v** *are semiclassical functionals related by a rational transformation. Moreover,* **u** *and* **v** *fulfill the following equations:*

$$D(B\mathbf{u}) = \big(B' + B_{N+2}\big)\mathbf{u}, \quad D(BB_1\mathbf{v}) = \big((BB_1)' + BB_2\big)\mathbf{v}. \tag{3.17}$$

Proof By the Leibniz rule, we can rewrite (3.5) as

$$\sum_{j=0}^{k-m+N} \xi(\cdot;n,j)D^j\mathbf{u} = \sum_{j=0}^{N} \phi(\cdot;n,j)D^j\mathbf{v} \quad (n = 0,1,\ldots).$$

Taking $n=0,1,2\ldots,k-m+2N$, we obtain the following system of $k-m+2N+1$ equations:

$$\begin{pmatrix} \xi(x;0,0)\mathbf{u} \\ \xi(x;1,0)\mathbf{u} \\ \vdots \\ \xi(x;k-m+N,0)\mathbf{u} \\ \xi(x;k-m+N+1,0)\mathbf{u} \\ \vdots \\ \xi(x;k-m+2N,0)\mathbf{u} \end{pmatrix} = \mathscr{B}(x) \begin{pmatrix} \mathbf{v} \\ D\mathbf{v} \\ \vdots \\ D^N\mathbf{v} \\ D\mathbf{u} \\ \vdots \\ D^{k-m+N}\mathbf{u} \end{pmatrix} .$$

The theorem follows by solving this system for $\mathbf{v}$, $D\mathbf{v}$, and $D\mathbf{u}$.

3.2.3 Case $k=0$

In this case, we may state a finer result.

Theorem 3.2.4 *Let $\big((P_n)_{n\geq 0},(Q_n)_{n\geq 0}\big)$ be a π_N-coherent pair with index M and order $(m,0)$, so that the structure relation*

$$\pi_N(x)P_n^{[m]}(x) = \sum_{j=n-M}^{n+N} c_{n,j}Q_j(x) \quad (n=0,1,2,\ldots)$$

holds, where M and N are fixed non-negative integer numbers, π_N is a monic polynomial of degree N, and $c_{n,n-M}\neq 0$ if $n\geq M$. Assume further that $m\geq 1$ if $N=0$. Let $\mathbf{u}$ and $\mathbf{v}$ be the regular functionals with respect to which $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are (respectively) orthogonal. Then $\mathbf{u}$ and $\mathbf{v}$ are semiclassical functionals related by a rational transformation. More precisely, setting

$$\Phi(x;j) := \frac{\langle \mathbf{v},Q_j^2\rangle \psi(x;j) - \sum_{\ell=0}^{j-1}\binom{m}{\ell}Q_j^{(\ell)}(x)\Phi(x;\ell)}{j!\binom{m}{j}} \quad (j=0,1,2,\ldots,m)\ , \tag{3.18}$$

$\psi(\cdot;j)$ being the polynomial introduced in (3.3), then $\deg\Phi(\cdot;0)=M+m$, $\deg\Phi(\cdot;j)\leq M+m+j$ for each $j=1,\ldots,m$, and the following holds:

$$D\big(\Phi(\cdot;1)\mathbf{u}\big) = \Phi(\cdot;0)\mathbf{u} \tag{3.19}$$

$$\pi_N\mathbf{v} = \Phi(\cdot;m)\mathbf{u} \tag{3.20}$$

$$D\big(\Phi(\cdot;m)\pi_N\mathbf{v}\big) = \big(\Phi(\cdot;m)'+\Phi(\cdot;m-1)\big)\pi_N\mathbf{v}\ . \tag{3.21}$$

Moreover, $\mathfrak{s}(\mathbf{u})\leq M+m-1$ and $\mathfrak{s}(\mathbf{v})\leq N+M+2(m-1)$.

Proof Since $k=0$ then $\mathbf{b}_n^{[k]}\equiv\mathbf{b}_n^{[0]}=\mathbf{b}_n=\langle\mathbf{v},Q_n^2\rangle^{-1}Q_n\mathbf{v}$ for each $n=0,1,2,\ldots$, hence relation (3.8) may be rewritten as

$$D^m\big(Q_n\pi_N\mathbf{v}\big) = \langle\mathbf{v},Q_n^2\rangle\,\psi(\cdot;n)\mathbf{u} \quad (n=0,1,2,\ldots)\ , \tag{3.22}$$

where $\psi(\cdot;n)$ is defined by (3.3). Taking $n=0$, we obtain

$$D^m\big(\pi_N\mathbf{v}\big)=\Phi(\cdot;0)\mathbf{u}\,. \tag{3.23}$$

Taking $n=1$ in (3.22) and then applying the Leibniz rule, we deduce

$$\langle\mathbf{v},Q_1^2\rangle\,\psi(\cdot;1)\mathbf{u}=D^m\big(Q_1\pi_N\mathbf{v}\big)=mD^{m-1}\big(\pi_N\mathbf{v}\big)+Q_1D^m\big(\pi_N\mathbf{v}\big)\,.$$

Hence, by (3.23), we have

$$D^{m-1}\big(\pi_N\mathbf{v}\big)=\Phi(\cdot;1)\mathbf{u}\,. \tag{3.24}$$

Thus (3.19) follows from (3.23) and (3.24). This proves that $\mathbf{u}$ is semiclassical of class $\mathfrak{s}(\mathbf{u})\leq M+m-1$. We conclude pursuing with the described procedure, so that by taking successively $n=0,1,\ldots,m$ in (3.22), we conclude that the following relations hold:

$$D^{m-j}\big(\pi_N\mathbf{v}\big)=\Phi(\cdot;j)\mathbf{u}\quad(j=0,1,2,\ldots,m)\,. \tag{3.25}$$

In particular, for $j=m$ we obtain (3.20), hence $\mathbf{u}$ and $\mathbf{v}$ are related by a rational transformation. Next, setting $j=m-1$ in (3.25) we obtain

$$D\big(\pi_N\mathbf{v}\big)=\Phi(\cdot;m-1)\mathbf{u}\,. \tag{3.26}$$

Since $D\big(\Phi(\cdot;m)\pi_N\mathbf{v}\big)=\Phi(\cdot;m)'\pi_N\mathbf{v}+\Phi(\cdot;m)D\big(\pi_N\mathbf{v}\big)$, we obtain (3.21) using (3.26) and (3.20). Thus $\mathbf{v}$ is semiclassical of class $\mathfrak{s}(\mathbf{v})\leq N+M+2m-2$, and the theorem is proved.

In the case $m=1$, Theorem 3.2.4 was partially proved in [39]. Note that the functional equation (3.21) (for $m=1$) was not given therein. The results stated in [28] for the continuous $(M,N)-$coherent pairs of order (m,k) were extended in [27] to the setting of discrete OPS. In a similar way, the results proved in this section may be extended to the discrete OPS, replacing the derivative operator D by the Hahn operator $D_{q,\omega}$ defined in (1.4). This is the objective of the next section.

3.3 π_N-coherent pairs with index M and order (m,k): the discrete case

In this section, we consider (3.1) redefining the derivatives as "discrete" derivatives,

$$S_n^{[m]}:=\frac{[n]_q!}{[n+m]_q!}\,D_{q,\omega}^mS_{n+m}\,, \tag{3.27}$$

where $D_{q,\omega}$ is the Hahn operator defined in (1.4)–(1.5). This leads to the concept of discrete π_N-(q,ω)-coherent pair with index M and order (m,k), defined as in Definition 3.1 with the obvious modification; that is, replacing in (3.1) the standard derivatives by the discrete ones (3.27). Taking into account the Leibniz formula (1.18), proves of our results in this section are similar to the ones on the previous section for the continuous case and because of this, we present only the results with few details in their proofs.

Lemma 3.3.1 *Let* $\big((P_n)_{n\geq 0},(Q_n)_{n\geq 0}\big)$ *be a* π_N*-*(q,ω)*-coherent pair with index* M *and order* (m,k)*, so that* (3.1) *and* (3.2) *hold. Set*

$$\psi(x;n) := \sum_{j=n-N}^{n+M} \frac{(-q)^m[j+m]_q!}{[j]_q!\langle \mathbf{u}, P_{m+j}^2\rangle} c_{j,n} P_{m+j}(x)\ , \tag{3.28}$$

$$\begin{aligned}\phi(x;n,j) := &\frac{(-q)^k[n+k]_q!}{[n]_q!\langle \mathbf{v}, Q_{n+k}^2\rangle} \sum_{\ell=0}^{N-j} \begin{bmatrix} k+n \\ \ell \end{bmatrix}_{q^{-1}} \begin{bmatrix} N-\ell \\ N-j-\ell \end{bmatrix}_{q^{-1}} \\ &\times L_{1/q,-\omega/q}^{k+N-\ell}\left(D_{1/q,-\omega/q}^{\ell}\pi_N\right)(x)\, L_{1/q,-\omega/q}^{j}\left(D_{1/q,-\omega/q}^{n-j-\ell}Q_{n+k}\right)(x)\ ,\end{aligned} \tag{3.29}$$

for all $n=0,1,2,\ldots$ *and* $j=0,1,\ldots,N$*, so that*

$$\deg\psi(\cdot;n) = m+n+M\ , \quad \deg\phi(\cdot;n,j) = k+n+j\ .$$

Let $\mathbf{u}$ *and* $\mathbf{v}$ *be the regular functionals with respect to which* $(P_n)_{n\geq 0}$ *and* $(Q_n)_{n\geq 0}$ *are orthogonal. Then the following functional equations hold:*

$$\psi(\cdot;n)\mathbf{u} = \mathbf{D}_{1/q,-\omega/q}^{m-k-N}\Big(\sum_{j=0}^{N}\phi(\cdot;n,j)\mathbf{D}_{1/q,-\omega/q}^{j}\mathbf{v}\Big) \qquad \text{if} \quad m\geq k+N, \tag{3.30}$$

$$\mathbf{D}_{1/q,-\omega/q}^{k+N-m}\big(\psi(\cdot;n)\mathbf{u}\big) = \sum_{j=0}^{N}\phi(\cdot;n,j)\mathbf{D}_{1/q,-\omega/q}^{j}\mathbf{v} \qquad \text{if} \quad m<k+N, \tag{3.31}$$

for all $n=0,1,2,\ldots$.

Proof Let $(\mathbf{a}_n)_{n\geq 0}$, $(\mathbf{b}_n)_{n\geq 0}$, $(\mathbf{a}_n^{[m]})_{n\geq 0}$, and $(\mathbf{b}_n^{[k]})_{n\geq 0}$ be the dual basis corresponding to the simple sets of polynomials $(P_n)_{n\geq 0}$, $(Q_n)_{n\geq 0}$, $(P_n^{[m]})_{n\geq 0}$ and $(Q_n^{[k]})_{n\geq 0}$, respectively. Then

$$\pi_N\mathbf{b}_n^{[k]} = \sum_{j=n-N}^{n+M} c_{j,n}\mathbf{a}_j^{[m]}, \quad n=0,1,2,\ldots\ . \tag{3.32}$$

From (2.58) we get

$$\mathbf{D}_{1/q,-\omega/q}^{m}\big(\pi_N\mathbf{b}_n^{[k]}\big) = \psi(\cdot;n)\mathbf{u}, \quad n=0,1,2,\ldots\ . \tag{3.33}$$

By Leibniz's formula (1.18), and since $D_{1/q,-\omega/q}^{j}\pi_N=0$ for $j>N$, we deduce

$$\begin{aligned}&\mathbf{D}_{1/q,-\omega/q}^{k+N}\big(\pi_N\mathbf{b}_n^{[k]}\big) \\ &= \frac{(-q)^k[n+k]_q!}{[n]_q!\langle \mathbf{v}, Q_{n+k}^2\rangle}\sum_{j=0}^{N}\begin{bmatrix} k+N \\ j \end{bmatrix}_{q^{-1}} L_{1/q,-\omega/q}^{k+N-j}\left(D_{1/q,-\omega/q}^{j}\pi_N\right)\mathbf{D}_{1/q,-\omega/q}^{N-j}(Q_{n+k}\mathbf{v})\ .\end{aligned}$$

Applying once again Leibniz's formula (1.18) to $\mathbf{D}_{1/q,-\omega/q}^{N-j}(Q_{n+k}\mathbf{v})$, after some straightforward calculations, we find

$$\mathbf{D}_{1/q,-\omega/q}^{k+N}\big(\pi_N\mathbf{b}_n^{[k]}\big) = \sum_{j=0}^{N}\phi(\cdot;n,j)\mathbf{D}_{1/q,-\omega/q}^{j}\mathbf{v}\ . \tag{3.34}$$

For $m \geq k+N$, rewriting (3.33) as

$$\psi(\cdot;n)\mathbf{u} = \mathbf{D}^{m-k-N}_{1/q,-\omega/q}\mathbf{D}^{k+N}_{1/q,-\omega/q}\big(\pi_N \mathbf{b}^{[k]}_n\big), \quad n = 0,1,\dots ,$$

and using (3.34), (3.30) follows. For $m < k+N$, writing

$$\mathbf{D}^{k+N}_{1/q,-\omega/q}\big(\pi_N \mathbf{b}^{[k]}_n\big) = \mathbf{D}^{k-m+N}_{1/q,-\omega/q}\mathbf{D}^{m}_{q,\omega}\big(\pi_N \mathbf{b}^{[k]}_n\big) , \quad n = 0,1,\dots ,$$

and using (3.33) and (3.34), we obtain (3.31) and the proof is complete.

3.3.1 Case $m \geq k+N$

Theorem 3.3.2 (Case $m \geq k+N$) *Let $\big((P_n)_{n\geq 0},(Q_n)_{n\geq 0}\big)$ be a π_N-(q,ω)-coherent pair with index M and order (m,k), so that* (3.1) *and* (3.2) *hold. Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be the regular functionals with respect to which $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are orthogonal. Suppose $m \geq k+N$. Assume further that $m > k$ whenever $N = 0$. For each $i = 0,\dots,m-k$ and $n = 0,1,\dots$, let*

$$\varphi(x;n,i) := \sum_{\substack{j+\ell=i\\ 0\leq j\leq N\\ 0\leq \ell\leq M}} \begin{bmatrix} m-k-N \\ j \end{bmatrix}_{q^{-1}} L^{j}_{1/q,-\omega/q}\big(D^{m-k-N-j}_{1/q,-\omega/q}\phi(.;n,\ell)\big)(x) , \tag{3.35}$$

$\phi(\cdot;n,j)$ being the polynomial introduced in (3.29). *Let $\mathscr{A}(x)$ be the polynomial matrix of order $m-k+1$ defined by*

$$\mathscr{A}(x) := \big[\varphi(x;n,j)\big]^{m-k}_{n,j=0} .$$

Let $\mathscr{A}_1(x)$ (resp., $\mathscr{A}_2(x)$) be the matrix obtained by replacing the first (resp., the second) column of $\mathscr{A}(x)$ by $\big[\psi(x;0),\psi(x;1),\dots,\psi(x;m-k)\big]^t$, and set

$$A(x) := \det\mathscr{A}(x) , \quad A_1(x) := \det\mathscr{A}_1(x) , \quad A_2(x) := \det\mathscr{A}_2(x) .$$

Assume that the polynomial $A(x)$ does not vanishes identically. Then

$$A\mathbf{v} = A_1\mathbf{u} , \quad A\mathbf{D}_{1/q,-\omega/q}\mathbf{v} = A_2\mathbf{u} , \tag{3.36}$$

hence $\mathbf{u}$ *and* $\mathbf{v}$ *are $D_{q,\omega}$-semiclassical functionals related by a rational transformation. Moreover,* $\mathbf{u}$ *and* $\mathbf{v}$ *fulfill the following equations:*

$$\mathbf{D}_{1/q,-\omega/q}(A_1 L_{q,\omega}(A)\mathbf{u}) = \big(qA_1 D_{q,\omega}(A) + A_1 D_{1/q,-\omega/q}(A) + A_2 L_{1/q,-\omega/q}(A)\big)\mathbf{u} , \tag{3.37}$$

$$\mathbf{D}_{1/q,-\omega/q}(L_{q,\omega}(AA_1)\mathbf{v}) = \big(qD_{q,\omega}(AA_1) + AA_2\big)\mathbf{v} . \tag{3.38}$$

Proof Combining (3.30) and the Leibniz formula (1.18), we get

$$\psi(\cdot;n)\mathbf{u} = \sum_{\ell=0}^{N}\sum_{j=0}^{m-k-N} \begin{bmatrix} m-k-N \\ j \end{bmatrix}_{q^{-1}} L^{j}_{1/q,-\omega/q}\big(D^{m-k-N-j}_{1/q,-\omega/q}\phi(\cdot;n,\ell)\big)\mathbf{D}^{j+\ell}_{1/q,-\omega/q}\mathbf{v} .$$

This may be rewritten as

$$\psi(\cdot;n)\mathbf{u} = \sum_{\ell=0}^{m-k} \varphi(\cdot;n,\ell)\mathbf{D}^{\ell}_{1/q,-\omega/q}\mathbf{v}, \quad n = 0,1,\ldots\,, \tag{3.39}$$

where $\varphi(\cdot;n,i)$ is the polynomial introduced in (3.35). Taking $n = 0,1,\ldots,m-k$ in (3.39) we obtain a system with $m-k+1$ equations that can be written as

$$\begin{pmatrix} \psi(x;0)\mathbf{u} \\ \psi(x;1)\mathbf{u} \\ \vdots \\ \psi(x;m-k)\mathbf{u} \end{pmatrix} = \mathscr{A}(x) \begin{pmatrix} \mathbf{v} \\ \mathbf{D}_{1/q,-\omega/q}\mathbf{v} \\ \vdots \\ \mathbf{D}^{m-k}_{1/q,-\omega/q}\mathbf{v} \end{pmatrix}.$$

Solving for $\mathbf{v}$ and $\mathbf{D}_{1/q,-\omega/q}\mathbf{v}$ we obtain (3.36). Finally, one can remark that $D_{1/q,-\omega/q}L_{q,\omega} = qD_{q,\omega}$. Hence (3.37) and (3.38) follow from (3.36).

Remark 3.3.1 *Remark* (3.2.1) *is also valid in the present context. That is, if $m = k$ and $N = 0$, then* $\mathbf{u}$ *and* $\mathbf{v}$ *are connected by a rational transformation but they are not necessary $D_{q,\omega}$-semiclassical.*

3.3.2 Case $m < k+N$

Theorem 3.3.3 (Case $m < k+N$) *Let $\big((P_n)_{n\geq 0},(Q_n)_{n\geq 0}\big)$ be a π_N-(q,ω)-coherent pair with index M and order (m,k), so that* (3.1) *and* (3.2) *hold. Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be the regular functionals with respect to which $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are orthogonal. Assume further that $m < k+N$. For each $j = 0,\ldots,k-m+N$ and $n = 0,1,\ldots$, set*

$$\xi(x;n,j) := \begin{bmatrix} k+N-m \\ j \end{bmatrix}_{q^{-1}} L^{j}_{1/q,-\omega/q}\big(D^{k+N-m-j}_{1/q,-\omega/q}\psi\big)(x;n)\,, \tag{3.40}$$

$\psi(\cdot;n)$ being the polynomial introduced in (3.28). *Let $\mathscr{B}(x) := \big[b_{i,j}(x)\big]_{i,j=0}^{k-m+2N}$ be the polynomial matrix of order $k-m+2N+1$ defined by*

$$b_{i,j}(x) := \begin{cases} \phi(x;i,j) & \text{if} \quad 0 \leq j \leq N\,, \\ -\xi(x;i,j-N) & \text{if} \quad N+1 \leq j \leq k-m+2N\,, \end{cases}$$

$\phi(\cdot;i,j)$ being the polynomial given by (3.29). *Let $\mathscr{B}_1(x)$ (resp., $\mathscr{B}_2(x)$ and $\mathscr{B}_{N+2}(x)$) be the matrix obtained by replacing the first (resp., the second and the $(N+2)$-th) column of $\mathscr{B}(x)$ by $\big[\xi(x;0,0),\xi(x;1,0),\ldots,\xi(x;m-k+2N,0)\big]^t$, and set*

$$B(x) := \det\mathscr{B}(x)\,, \quad B_j(x) := \det\mathscr{B}_j(x)\,, \quad j \in \{1,2,N+2\}\,.$$

Assume that the polynomial $B(x)$ does not vanishes identically. Then

$$B\mathbf{v} = B_1\mathbf{u}\,, \quad B\mathbf{D}_{1/q,-\omega/q}\mathbf{v} = B_2\mathbf{u}\,, \quad B\mathbf{D}_{1/q,-\omega/q}\mathbf{u} = B_{N+2}\mathbf{u}\,, \tag{3.41}$$

hence **u** *and* **v** *are* $D_{q,\omega}$*-semiclassical functionals related by a rational transformation. Moreover,* **u** *and* **v** *fulfill the following equations:*

$$\mathbf{D}_{1/q,-\omega/q}(L_{q,\omega}(BB_1)\mathbf{v}) = \big(qD_{q,\omega}(BB_1)+BB_2\big)\mathbf{v} \tag{3.42}$$

$$\mathbf{D}_{1/q,-\omega/q}(L_{q,\omega}B\mathbf{u}) = \big(qD_{q,\omega}B+B_{N+2}\big)\mathbf{u}\,. \tag{3.43}$$

Proof Using the Leibniz formula (1.18) we can rewrite (3.30) as

$$\sum_{j=0}^{k-m+N}\xi(\cdot;n,j)\mathbf{D}^{j}_{1/q,-\omega/q}\mathbf{u} = \sum_{j=0}^{N}\phi(\cdot;n,j)\mathbf{D}^{j}_{1/q,-\omega/q}\mathbf{v}, \quad n=0,1,\dots\,.$$

Taking $n=0,1,\dots,k-m+2N$, we obtain the following system of $k-m+2N+1$ equations:

$$\begin{pmatrix} \xi(x;0,0)\mathbf{u} \\ \xi(x;1,0)\mathbf{u} \\ \vdots \\ \xi(x;k-m+N,0)\mathbf{u} \\ \xi(x;k-m+N+1,0)\mathbf{u} \\ \vdots \\ \xi(x;k-m+2N,0)\mathbf{u} \end{pmatrix} = \mathscr{B}(x)\begin{pmatrix} \mathbf{v} \\ \mathbf{D}_{1/q,-\omega/q}\mathbf{v} \\ \vdots \\ \mathbf{D}^{N}_{1/q,-\omega/q}\mathbf{v} \\ \mathbf{D}_{1/q,-\omega/q}\mathbf{u} \\ \vdots \\ \mathbf{D}^{k-m+N}_{1/q,-\omega/q}\mathbf{u} \end{pmatrix}.$$

The theorem follows by solving this system for $\mathbf{v}$, $\mathbf{D}_{1/q,-\omega/q}\mathbf{v}$, and $\mathbf{D}_{1/q,-\omega/q}\mathbf{u}$.

3.3.3 Case $k=0$

Theorem 3.3.4 (Case $k=0$) *Let* $\big((P_n)_{n\geq 0},(Q_n)_{n\geq 0}\big)$ *be a* π_N*-* (q,ω)*-coherent pair with index* M *and order* $(m,0)$*, so that the structure relation*

$$\pi_N(x)P_n^{[m]}(x) = \sum_{j=n-M}^{n+N} c_{n,j}Q_j(x), \quad (n=0,1,\dots)\,,$$

holds, where M *and* N *are fixed non-negative integer numbers,* π_N *is a monic polynomial of degree* N*, and* $c_{n,n-M}\neq 0$ *if* $n\geq M$*. Assume further that* $m\geq 1$ *if* $N=0$*. Let* **u** *and* **v** *be the regular functionals with respect to which* $(P_n)_{n\geq 0}$ *and* $(Q_n)_{n\geq 0}$ *are orthogonal. Then* **u** *and* **v** *are* $D_{q,\omega}$*-semiclassical functionals related by a rational transformation. More precisely, setting*

$$\Phi(x;j) := \frac{\langle \mathbf{v},Q_j^2\rangle\psi(x;j) - \displaystyle\sum_{l=0}^{j-1}\begin{bmatrix} m \\ l\end{bmatrix}_{q^{-1}} L^{m-l}_{1/q,-\omega/q}\big(D^{l}_{1/q,-\omega/q}Q_j\big)(x)\Phi(x;l)}{[j]_{q^{-1}}!\begin{bmatrix} m \\ j\end{bmatrix}_{q^{-1}}}, \tag{3.44}$$

$j=0,1,\ldots$, $\psi(\cdot;j)$ being the polynomial introduced in (3.28)*, then* $\deg\Phi(\cdot;0)=M+m$*,* $\deg\Phi(\cdot;j)\leq M+m+j$ *for each* $j=1,\ldots,m$*, and the following holds:*

$$\mathbf{D}_{1/q,-\omega/q}\big(\Phi(\cdot;1)\mathbf{u}\big)=\Phi(\cdot;0)\mathbf{u}\,, \tag{3.45}$$

$$\pi_N\mathbf{v}=\Phi(\cdot;m)\mathbf{u}\,, \tag{3.46}$$

$$\mathbf{D}_{1/q,-\omega/q}\big(L_{q,\omega}\Phi(\cdot;m))\pi_N\mathbf{v}\big)=\big(qD_{q,\omega}\Phi(\cdot;m)+\Phi(\cdot;m-1)\big)\,\pi_N\mathbf{v}\,. \tag{3.47}$$

Moreover, $\mathfrak{s}(\mathbf{u})\leq M+m-1$ *and* $\mathfrak{s}(\mathbf{v})\leq N+M+2(m-1)$.

Proof Relation (3.33) in this case may be rewritten as

$$\mathbf{D}^m_{1/q,-\omega/q}\big(Q_n\pi_N\mathbf{v}\big)=\langle\mathbf{v},Q_n^2\rangle\,\psi(\cdot;n)\mathbf{u},\quad n=0,1,\ldots\,, \tag{3.48}$$

where $\psi(\cdot;n)$ is defined by (3.28). Taking $n=0$, we obtain

$$\mathbf{D}^m_{1/q,-\omega/q}\big(\pi_N\mathbf{v}\big)=\Phi(\cdot;0)\mathbf{u}\,. \tag{3.49}$$

Taking $n=1$ in (3.48) and then applying Leibniz's formula (1.18), we deduce

$$\begin{aligned}\langle\mathbf{v},Q_1^2\rangle\,\psi(\cdot;1)\mathbf{u}&=\mathbf{D}^m_{1/q,-\omega/q}\big(Q_1\pi_N\mathbf{v}\big)\\&=[m]_{q^{-1}}\mathbf{D}^{m-1}_{1/q,-\omega/q}\big(\pi_N\mathbf{v}\big)+L^m_{1/q,-\omega/q}Q_1\mathbf{D}^m_{1/q,-\omega/q}\big(\pi_N\mathbf{v}\big)\,.\end{aligned}$$

Hence, by (3.49), we have

$$\mathbf{D}^{m-1}_{1/q,-\omega/q}\big(\pi_N\mathbf{v}\big)=\Phi(\cdot;1)\mathbf{u}\,. \tag{3.50}$$

Thus (3.45) follows from (3.49) and (3.50). This proves that $\mathbf{u}$ is $D_{q,\omega}$-semiclassical of class $\mathfrak{s}(\mathbf{u})\leq M+m-1$. We conclude pursuing with the described procedure, so that by taking successively $n=0,1,\ldots,m$ in (3.48), the following relation holds:

$$\mathbf{D}^{m-j}_{1/q,-\omega/q}\big(\pi_N\mathbf{v}\big)=\Phi(\cdot;j)\mathbf{u},\quad j=0,1,\ldots,m\,. \tag{3.51}$$

In particular, for $j=m$ we obtain (3.46), hence $\mathbf{u}$ and $\mathbf{v}$ are related by a rational transformation. Setting $j=m-1$ in (3.51), we obtain

$$\mathbf{D}_{1/q,-\omega/q}\big(\pi_N\mathbf{v}\big)=\Phi(\cdot;m-1)\mathbf{u}\,. \tag{3.52}$$

Since $\mathbf{D}_{1/q,-\omega/q}\big(L_{q,\omega}(\Phi(\cdot;m))\pi_N\mathbf{v}\big)=qD_{q,\omega}(\Phi(\cdot;m))\pi_N\mathbf{v}+\Phi(\cdot;m)\mathbf{D}_{1/q,-\omega/q}\big(\pi_N\mathbf{v}\big)$, we obtain (3.47) using (3.52) and (3.46). Thus $\mathbf{v}$ is $D_{q,\omega}$-semiclassical of class $\mathfrak{s}(\mathbf{v})\leq N+M+2m-2$, and the theorem is proved.

Remark 3.3.2 *Actually, as we just did for discrete OPS, similar results can be obtained for discrete OPS on a nonuniform lattice involving the operators D_x and S_x defined in* (1.42)–(1.43).

3.4 Applications

3.4.1 Continuous variable

Let $(P_n)_{n\geq 0}$ be a monic OPS with respect to a positive Borel measure. Suppose that $(P_n)_{n\geq 0}$ satisfies the differential-difference equation

$$\pi(x)P_n'(x) = b_n P_n(x) + (c_n x + d_n)P_{n-1}(x) \quad (n = 0, 1, 2, \dots)\,, \tag{3.53}$$

where $\pi(x)$ is a monic polynomial of degree 1 and $(b_n)_{n\geq 0}$, $(c_n)_{n\geq 0}$, and $(d_n)_{n\geq 0}$ are sequences of real numbers, with $c_n \neq 0$ for each $n = 1, 2, 3, \dots$. We assume

$$\pi(x) = x\,.$$

OPS characterized by equation (3.53) have been studied recently in [21]. Here we give an alternative approach based on the general results presented in the previous sections. $(P_n)_{n\geq 0}$ is characterized by a three-term recurrence relation (1.1) where $(\beta_n)_{n\geq 0}$ and $(\gamma_n)_{n\geq 1}$ are sequences of real numbers such that $\gamma_n > 0$ for each $n \geq 1$. We set $\gamma_0 := 0$. Using (1.1), we rewrite (3.53) as

$$xP_n^{[1]}(x) = P_{n+1}(x) + r_n P_n(x) + s_n P_{n-1}(x) \quad (n = 0, 1, 2, \dots)\,, \tag{3.54}$$

where

$$r_n := \frac{c_{n+1}\beta_n + d_{n+1}}{n+1}\,, \quad s_n := \frac{c_{n+1}\gamma_n}{n+1} \quad (n = 0, 1, 2, \dots)\,.$$

Notice that $s_n \neq 0$ for each $n = 1, 2, 3, \dots$. Comparing (3.54) with (3.1), we have

$$N = M = m = 1\,,\ k = 0\,,\ c_{n,n+1} = 1\,,\ c_{n,n} = r_n\,,\ c_{n,n-1} = s_n\,. \tag{3.55}$$

Thus $\big((P_n)_{n\geq 0}, (P_n)_{n\geq 0}\big)$ is a π_1-coherent pair with index 1 and order $(1,0)$, where $\pi_1(x) = x$. By Theorem 3.2.4, the functional $\mathbf{u}$ with respect to which $(P_n)_{n\geq 0}$ is orthogonal satisfies the relations

$$D\big(\Phi(\cdot;1)\mathbf{u}\big) = \Phi(\cdot;0)\mathbf{u} \tag{3.56}$$

$$x\mathbf{u} = \Phi(\cdot;1)\mathbf{u}\,. \tag{3.57}$$

Since $\mathbf{u}$ is regular, then (3.57) implies

$$\Phi(x;1) = x\,. \tag{3.58}$$

On the other hand, by (3.18) and using the relations (1.2), we have

$$\Phi(x;0) := -\frac{r_0}{\gamma_1}P_1(x) - \frac{2s_1}{\gamma_1\gamma_2}P_2(x)\,. \tag{3.59}$$

From (3.54) for $n = 0,1,2$, and taking into account (1.2), we deduce

$$\begin{aligned} &r_0 = \beta_0 , \quad r_1 = \tfrac{1}{2}(\beta_0+\beta_1) , \quad r_2 = \tfrac{1}{3}(\beta_0+\beta_1+\beta_2) , \\ &s_1 = \gamma_1 + \tfrac{1}{2}\beta_0(\beta_0-\beta_1) , \quad \beta_0(s_2-\gamma_2) = (\beta_0\beta_1-\gamma_1)(r_2-\beta_2) , \\ &s_2 = \tfrac{1}{3}\left(\beta_0^2+\beta_1^2-(\beta_0+\beta_1)\beta_2+2(\gamma_1+\gamma_2)\right) . \end{aligned} \tag{3.60}$$

Therefore, taking into account (3.58)–(3.60) and (1.2), (3.56) reduces to

$$D\big(x\mathbf{u}\big) = (-2ax^2+bx+c+1)\mathbf{u} , \tag{3.61}$$

where

$$\begin{aligned} a &:= \frac{s_1}{\gamma_1\gamma_2} = \frac{2\gamma_1+(\beta_0-\beta_1)\beta_0}{2\gamma_1\gamma_2} , \\ b &:= \frac{\big(2\gamma_1+(\beta_0-\beta_1)\beta_0\big)(\beta_0+\beta_1)-\beta_0\gamma_2}{\gamma_1\gamma_2} , \\ c &:= \frac{\beta_0^2\gamma_2-\big(2\gamma_1+(\beta_0-\beta_1)\beta_0\big)(\beta_0\beta_1-\gamma_1)}{\gamma_1\gamma_2} - 1 . \end{aligned}$$

Using (3.60), and assuming $s_1 > 0$, we deduce

$$\begin{aligned} &\beta_0 = r_0 , \quad \beta_1 = 2r_1-r_0 , \quad \gamma_1 = s_1 - r_0(r_0-r_1) , \\ &\gamma_2 = \frac{s_1(3s_2-2s_1)+2r_1\big(s_1(2r_0-r_1)-r_0r_1(r_0-r_1)\big)}{2s_1+r_0r_1} . \end{aligned} \tag{3.62}$$

(Notice that $2s_1+r_0r_1 \neq 0$; indeed, using $\gamma_1 = s_1 - r_0(r_0-r_1)$, we have $2s_1+r_0r_1 = \gamma_1+s_1+r_0^2 > 0$.) Thus a, b, and c may be written only in terms of r_0, r_1, s_1, and s_2. Hereafter we impose the (integrability) conditions

$$a > 0 , \quad c > -1 . \tag{3.63}$$

(Note that the condition $a > 0$ is equivalent to $s_1 > 0$ in equation (3.54), or to $c_2 > 0$ in equation equation (3.53).) Let w be a solution of

$$xw'(x) = (-2ax^2+bx+c)w(x) , \quad x \in \mathbb{R}\setminus\{0\} . \tag{3.64}$$

Solving this equation imposing (without loss of generality) w to be right-continuous at $x = 0$, we find

$$w(x) = \begin{cases} K_1|x|^c e^{-ax^2+bx} & \text{if} \quad x < 0 , \\ K_2|x|^c e^{-ax^2+bx} & \text{if} \quad x \geq 0 , \end{cases} \tag{3.65}$$

K_1 and K_2 being real constants. Requiring, in addition, K_1 and K_2 to be non-negative and no simultaneously equal to zero, w becomes a weight function, i.e., a non-negative and integrable function which does not vanishes identically and having finite moments of all orders. Now, define a functional $\mathbf{w}$ by

$$\langle \mathbf{w}, f\rangle := \kappa \int_{\mathbb{R}} f(x)w(x)\,\mathrm{d}x \quad (f \in \mathscr{P}) ,$$

where κ is a normalization constant chosen so that $\langle \mathbf{w},1\rangle=\langle \mathbf{u},1\rangle$. Using (3.64) and integration by parts, together with the rules of the distributional calculus, we show that $D(x\mathbf{w})=(-2ax^2+bx+c+1)\mathbf{w}$ on $\mathscr{P}'$, hence $\mathbf{w}$ fulfills the same functional equation (3.61) as $\mathbf{u}$. This is equivalent to saying that the sequences of moments $(u_n)_{n\geq 0}$ and $(w_n)_{n\geq 0}$ of $\mathbf{u}$ and $\mathbf{w}$ (defined by $u_n:=\langle \mathbf{u},x^n\rangle$ and $w_n:=\langle \mathbf{w},x^n\rangle$) are solutions of the second order linear difference equation

$$-2av_{n+2}+(n+b)v_{n+1}+(c+1)v_n=0 \quad (n=0,1,2,\cdots)\,.$$

Now we show that we may choose K_1 and K_2 so that $\mathbf{u}=\mathbf{w}$. Indeed, since by definition of $\mathbf{w}$ the condition $u_0=w_0$ holds, we only need to show that we may choose K_1 and K_2 so that $u_1=w_1$. Indeed,

$$\kappa^{-1}w_1=\int_{\mathbb{R}} xw(x)\,\mathrm{d}x=K_1\int_{-\infty}^{0} x|x|^c e^{-ax^2+bx}\,\mathrm{d}x+K_2\int_0^{+\infty} x^{c+1}e^{-ax^2+bx}\,\mathrm{d}x\,,$$

and making the change of variables $x\mapsto -x$ on the first integral, we obtain

$$w_1=\kappa\Big(K_2\int_0^{+\infty} x^{c+1}e^{-ax^2+bx}\,\mathrm{d}x-K_1\int_0^{+\infty} x^{c+1}e^{-ax^2-bx}\,\mathrm{d}x\Big)\,.$$

On the other hand, from $P_1(x)=x-\beta_0$, we have $u_1=\beta_0u_0=r_0w_0$, i.e.,

$$u_1=\kappa r_0\Big(K_2\int_0^{+\infty} x^{c}e^{-ax^2+bx}\,\mathrm{d}x+K_1\int_0^{+\infty} x^{c}e^{-ax^2-bx}\,\mathrm{d}x\Big)\,.$$

Therefore, in order to have $u_1=w_1$, we need to impose

$$r_0=\frac{K_2\int_0^{+\infty} x^{c+1}e^{-ax^2+bx}\,\mathrm{d}x-K_1\int_0^{+\infty} x^{c+1}e^{-ax^2-bx}\,\mathrm{d}x}{K_1\int_0^{+\infty} x^{c}e^{-ax^2-bx}\,\mathrm{d}x+K_2\int_0^{+\infty} x^{c}e^{-ax^2+bx}\,\mathrm{d}x}\,.$$

Assuming without loss of generality that $K_2>0$, and setting $M=K_1/K_2$, this is achieved provided that

$$M=\frac{\int_0^{+\infty} x^{c+1}e^{-ax^2+bx}\,\mathrm{d}x-r_0\int_0^{+\infty} x^{c}e^{-ax^2+bx}\,\mathrm{d}x}{\int_0^{+\infty} x^{c+1}e^{-ax^2-bx}\,\mathrm{d}x+r_0\int_0^{+\infty} x^{c}e^{-ax^2-bx}\,\mathrm{d}x}\,. \tag{3.66}$$

Thus, up to a positive constant factor, $\mathbf{u}$ admits the integral representation

$$\langle \mathbf{u},f\rangle:=\int_{\mathbb{R}} f(x)w(x)\,\mathrm{d}x \quad (f\in\mathscr{P})\,.$$

We remark that w is a.e. on $\mathbb{R}$ the unique weight function with respect to which $(P_n)_{n\geq 0}$ is a monic OPS. This is an immediate consequence of the fact that the moment problem associated to the distribution function with weight w is determined, as we may see easily taking into account Riesz uniqueness criterium (see e.g. [20, Theorem II-5.2]). Finally, set

$$\mathbf{u}^{(M,t,c)}:=h_{\sqrt{a}}\mathbf{u}\,,\quad t:=b/\sqrt{a}\,, \tag{3.67}$$

meaning that $\langle \mathbf{u}^{(M,t,c)}, x^n\rangle := \langle \mathbf{u}, \left(\sqrt{a}\,x\right)^n\rangle$ for each $n = 0,1,2,\ldots$. Note that making the change of variables $x \to x/\sqrt{a}$ in the integrals appearing in (3.66) we obtain

$$M = \frac{\int_0^{+\infty} \left(x - \sqrt{a}\,r_0\right) x^c e^{-x^2+tx}\,\mathrm{d}x}{\int_0^{+\infty} \left(x + \sqrt{a}\,r_0\right) x^c e^{-x^2-tx}\,\mathrm{d}x}\,. \tag{3.68}$$

Since $\mathbf{u}$ fulfils (3.61) then $\mathbf{u}^{(M,t,c)}$ satisfies

$$D\big(x\mathbf{u}^{(M,t,c)}\big) = (-2x^2 + tx + c + 1)\mathbf{u}^{(M,t,c)}\,.$$

Let $(P_n^{(M,t,c)})_{n\geq 0}$ be the monic OPS with respect to $\mathbf{u}^{(M,t,c)}$. Then (3.67) implies

$$P_n(x) := \frac{1}{(\sqrt{a})^n}\,P_n^{(M,t,c)}\big(\sqrt{a}\,x\big) \quad (n = 0,1,2,\ldots)\,. \tag{3.69}$$

Moreover, up to a constant factor, $\mathbf{u}^{(M,t,c)}$ admits the integral representation

$$\langle \mathbf{u}^{(M,t,c)}, f\rangle := \int_{\mathbb{R}} f(x) w^{(M,t,c)}(x)\,\mathrm{d}x \quad (f \in \mathscr{P})\,,$$

where

$$w^{(M,t,c)}(x) := \frac{a^{c/2}}{K_2}\,w\Big(\frac{x}{\sqrt{a}}\Big) = \begin{cases} M|x|^c e^{-x^2+tx} & \text{if} \quad x < 0\,, \\ |x|^c e^{-x^2+tx} & \text{if} \quad x \geq 0\,. \end{cases} \tag{3.70}$$

In conclusion, we have the following results.

Theorem 3.4.1 *Let $(P_n)_{n\geq 0}$ be a monic OPS with respect to a positive-definite linear functional and fulfills (3.54), where $(r_n)_{n\geq 0}$ and $(s_n)_{n\geq 1}$ are sequences of real numbers such that $s_n \neq 0$ for each $n = 1,2,3,\ldots$. Then $(P_n)_{n\geq 0}$ is given by (3.69) —$(P_n^{(M,t,c)})_{n\geq 0}$ being the unique monic OPS with respect to the weight function $w^{(M,t,c)}$ defined by the right-hand side of (3.70), provided that conditions (3.63) hold for each choice of the (real) parameters r_0, r_1, s_1, and s_2.*

Corollary 3.4.2 *Under the assumption of the previous theorem, if $r_0 = r_1 = 0$, $s_1 = 1/2$, and $s_2 = 1$, then up to an affine change of the variable, $(P_n)_{n\geq 0}$ is the Hermite monic OPS.*

Proof
Since $r_0 = r_1 = 0$, $s_1 = 1/2$ and $s_2 = 1$, we obtain $a = 1$, $t = c = 0$, and $M = 1$, hence $w^{(1,0,0)}(x) = e^{-x^2}$ and the corollary follows.

Finally, we note that (3.70), (3.69), and (3.68) agree, respectively, with (2.27), (2.29), and (2.30) in [21].

3.4.2 Discrete variable

The interest of the results presented in Section 3.3 will be illustrated by an exhaustive analysis of the π_N-(q,w)-coherent pairs of index $M = 0$ and order $(m,k) = (1,0)$, considering $N \leq 2$ and $P_n = Q_n$

for each $n = 0, 1, \ldots$. This means that we focus on the structure relation

$$\pi_N(x) D_{q,\omega} P_{n+1}(x) = [n+1]_q \sum_{j=n}^{n+N} c_{n,j} P_j(x), \quad n = 0, 1, \ldots \,, \tag{3.71}$$

where π_N is a monic polynomial of degree $N \in \{0, 1, 2\}$. We assume that the $c_{n,j}$ are complex parameters subject to the conditions $c_{n,n} \neq 0$ for each $n = 0, 1, 2, \ldots$. Our aim is to describe all the monic OPS $(P_n)_{n\geq 0}$ fulfilling (3.71). We prove in a rather simple way that, up to affine changes of variable depending of the pair (q, ω), the only monic OPS satisfying (3.71) are the monic $q-$classical OPS given in Table 3.1. This is a $(q, \omega)-$analogue of the well known characterization of classical OPS (Hermite, Laguerre, Jacobi, and Bessel) due to Al-Salam and Chihara [3]. See also [32, 40].

Table 3.1 Monic $q-$classical OPS

Name	Notation (P_n)	Restrictions	Reference
Al-Salam-Carlitz	$U_n^{(a)}(\cdot\|q)$	$a \neq 0$	[34, (14.24.4)]
Big-q-Laguerre	$L_n(\cdot; a, b\|q)$	$ab \neq 0\,;\; a, b \notin \Lambda$	[34, (14.11.4)]
Little-q-Laguerre	$L_n(\cdot; a\|q)$	$a \neq 0\,;\; a \notin \Lambda$	[34, (14.20.4)]
——	$l_n(\cdot; a\|q)$	$a \neq 0$	[51, Table 2]
Big-q-Jacobi	$P_n(\cdot; a, b, c\|q)$	$ac \neq 0\,;\; a, b, c, ab, abc^{-1} \notin \Lambda$	[34, (14.5.4)]
Little-q-Jacobi	$P_n(\cdot; a, b\|q)$	$a \neq 0\,;\; a, b, ab \notin \Lambda$	[34, (14.12.4)]
q-Bessel	$B_n(\cdot; a\|q)$	$a \neq 0\,;\; -a \notin \Lambda$	[34, (14.22.4)]
——	$j_n(\cdot; a, b\|q)$	$ab \neq 0\,;\; a \notin \Lambda$	[51, Table 2]

In Table 1 we have set $\Lambda := \{q^{-n} \,:\, n = 1, 2, \ldots\}$. We will show that the possible families $(P_n)_{n\geq 0}$ fulfilling (3.71) may be related (up to affine changes of the variable) to one of the following two OPS:

(I) The monic OPS $(L_n(x; a, b, c|q))_{n\geq 0}$ given by (1.1), where

$$\beta_n = \big(a + b - c(q^{n+1} + q^n - 1)\big)\, q^n \,,$$
$$\gamma_{n+1} = -\big(a - cq^{n+1}\big)\,(b - cq^{n+1})(1 - q^{n+1})q^n$$

for each $n = 0, 1, 2, \ldots$, and $a, b, c \in \mathbb{C}$ are parameters subject to the regularity conditions

$$a \neq cq^n \,, \quad b \neq cq^n$$

for each $n = 1, 2, \ldots$. Although there are three parameters in the definition of $L_n(x; a, b, c|q)$, we note that, without loss of generality, if $c \neq 0$ then, up to an affine change of variables, we may reduce to the case $c = 1$. Indeed, the relation

$$L_n(x; a, b, c|q) = c^n L_n(x/c; a/c, b/c, 1|q)$$

holds for each $n = 0, 1, 2, \ldots$. Moreover, if $c = 0$ (and so $ab \neq 0$, by the regularity conditions), then up to the affine change of variable $x \mapsto bx$, we may reduce to the case $b = 1$, taking into account that the relation

$$L_n(x;a,b,0|q) = b^n L_n(x/b;a/b,1,0|q)$$

holds for each $n = 0,1,2,\ldots$.

(II) The monic OPS $(J_n(x;a,b,c,d|q))_{n\geq 0}$ given by (1.1), where

$$\beta_n = q^n \frac{[a(b+d)+c(b+1)](1+dq^{2n+1}) - [c(b+d)+ad(b+1)](1+q)q^n}{(1-dq^{2n})(1-dq^{2n+2})},$$

$$\gamma_{n+1} = -\frac{q^n(1-q^{n+1})(1-bq^{n+1})(1-dq^{n+1})(a-cq^{n+1})(b-dq^{n+1})(c-adq^{n+1})}{(1-dq^{2n+1})(1-dq^{2n+2})^2(1-dq^{2n+3})}$$

for each $n = 0,1,2,\ldots$, where $a,b,c,d \in \mathbb{C}$ fulfil the regularity conditions

$$b \neq q^{-n}, \quad d \neq q^{-n}, \quad a \neq cq^n, \quad b \neq dq^n, \quad c \neq adq^n$$

for each $n = 1,2,\ldots$.

Remark 3.4.1 *Note that $L_n(\cdot;a,b,c|q)$ is a special or limiting case of $J_n(\cdot;a,b,c,d|q)$ for each $n = 0,1,2,\ldots$. Indeed,*

$$L_n\big(x;a,b,c|q\big) = \begin{cases} J_n(x;ab/c,c/b,b,0|q) & \text{if} \quad bc \neq 0, \\ J_n(x;ab/c,c/a,a,0|q) & \text{if} \quad ac \neq 0, \end{cases}$$

$$L_n(x;0,0,c|q) = \lim_{b\to 0} J_n(x;0,c/b,b,0|q) \quad \text{if} \quad c \neq 0,$$

$$L_n(x;a,1,0|q) = \lim_{b\to 0} J_n(x;a/b,b,1,0|q).$$

Remark 3.4.2 *The $q-$classical OPS (see Table 3.1), up to affine transformations of the variable, can be obtained from the monic OPS given in* **(I)** *and* **(II)**. *Indeed:*

$$U_n^{(a)}(x) = L_n(x;a,1,0|q)$$

$$L_n(x;a,b|q) = (abq)^n L_n\big(x/(abq);1/a,1/b,1|q\big)$$

$$L_n(x;a|q) = L_n\big(x;0,1,a|q\big)$$

$$l_n(x;a|q) = L_n\big(x;0,0,-a|q\big)$$

$$P_n(x;a,b,c|q) = \begin{cases} q^n J_n(x/q;1,a,c,ab|q), & \text{if} \quad b \neq 0 \\ (acq)^n L_n(x/(acq);1/a,1/c,1|q), & \text{if} \quad b = 0 \end{cases}$$

$$P_n(x;a,b|q) = \begin{cases} J_n(x;0,a,1,ab|q), & \text{if} \quad b \neq 0 \\ a^n L_n(x/a;1/a,0,1|q), & \text{if} \quad b = 0 \end{cases}$$

$$B_n(x;a|q) = J_n(x;0,0,1,-a/q|q)$$

$$j_n(x;a,b|q) = q^n J_n\big(x/q;b,0,0,a/q|q\big),$$

where in each case the parameters are subject to the restrictions given in Table 3.1.

Remark 3.4.3 *The converse of the statement in Remark 3.4.2 is also true, that is, the monic OPS in* **(I)** *and* **(II)** *can be obtained from the $q-$classical OPS. Indeed:*

(i) *If* $c=0$ *in the definition of* $L_n(\cdot;a,b,c|q)$ *(and so* $ab\neq 0$*, by the regularity conditions), we obtain (monic) Al-Salam-Carlitz polynomials:*

$$L_n(x;a,b,0|q)=b^n U_n^{(a/b)}(x/b|q)\ .$$

Consider now $c\neq 0$*. If* $ab\neq 0$*, we obtain Big* $q-$*Laguerre polynomials:*

$$L_n(x;a,b,c|q)=(ab/(cq))^n L_n\left(cqx/(ab);c/a,c/b\middle|q\right)\ ;$$

if $ab=0$ *and* $|a|+|b|\neq 0$*, we obtain Little* $q-$*Laguerre polynomials:*

$$L_n\big(x;a,b,c|q\big)=\begin{cases} b^n L_n(x/b;c/b|q) & \text{if} \quad a=0 \text{ and } b\neq 0\,,\\ a^n L_n(x/a;c/a|q) & \text{if} \quad a\neq 0 \text{ and } b=0\,;\end{cases}$$

and if $a=b=0$*, we obtain one of the monic OPS given by Medem and Álvarez-Nodarse in [51, Table 2]:*

$$L_n(x;0,0,c|q)=l_n(x;-c|q)\,.$$

(ii) *If* $d=0$ *in the definition of* $J_n(\cdot;a,b,c,d|q)$*, the regularity conditions imply* $bc\neq 0$*, and we obtain Little* $q-$*Laguerre polynomials if* $a=0$ *and Big* $q-$*Laguerre polynomials if* $a\neq 0$*, according to* (i) *and the relation*

$$J_n(x;a,b,c,0|q)=L_n\big(x;ab,c,bc|q\big)\ .$$

Consider now $d\neq 0$*. If* $abc\neq 0$*, we obtain Big q-Jacobi polynomials:*

$$J_n(x;a,b,c,d|q)=(a/q)^n P_n(qx/a;b,d/b,c/a|q)\ ;$$

if only one among a, b, and c is zero, then we obtain Little q-Jacobi polynomials:

$$J_n\big(x;a,b,c,d|q\big)=\begin{cases} c^n P_n(x/c;b,d/b|q) & \text{if} \quad a=0 \text{ and } bc\neq 0\,,\\ c^n P_n(x/c;ad/c,c/a|q) & \text{if} \quad b=0 \text{ and } ac\neq 0\,,\\ (ab)^n P_n(x/(ab);d/b,b|q) & \text{if} \quad c=0 \text{ and } ab\neq 0\,;\end{cases}$$

if $a=b=0$ *(and so* $c\neq 0$*, by regularity), we obtain q-Bessel polynomials:*

$$J_n\big(x;0,0,c,d|q\big)=c^n B_n(x/c;-dq|q)\ ;$$

and if $b=c=0$ *(and so* $a\neq 0$*, by regularity) we obtain the other monic OPS given by Medem and Álvarez-Nodarse in [51, Table 2]:*

$$J_n(x;a,0,0,d|q)=q^{-n}j_n(qx;qd,a|q)\ .$$

(There are no additional cases, since the condition $d\neq 0$ *together with the regularity conditions for* $(J_n(\cdot;a,b,c,d|q))_{n\geq 0}$ *imply* $(a,c)\neq(0,0)$*.)*

Remark 3.4.4 *Note that the $q-$classical OPS are (up to affine changes of the variable) special or limiting cases of the polynomials J_n.*

Theorem 3.4.3 *A monic OPS $(P_n)_{n\geq 0}$ satisfies* (3.53) *if and only if, up to an affine transformation of the variable, it is a $q-$classical monic OPS.*

Proof In the analysis of the structure relation (3.71) we consider the three possible cases, according to the degree of the (monic) polynomial π_N, $N \in \{0,1,2\}$.

CASE I: $N = 0$. Then $\pi_0(x) = 1$ and so (3.71) becomes

$$D_{q,\omega}P_{n+1}(x) = [n+1]_q P_n(x)\ , \quad n = 0,1,\dots\ .$$

From (3.45), (3.46), and (1.1), we see that $\mathbf{u}$ satisfies the functional equation

$$\mathbf{D}_{1/q,-\omega/q}\mathbf{u} = -\frac{q}{\gamma_1}(x-\beta_0)\mathbf{u}\ .$$

Let a and b be the two roots of the quadratic equation

$$z^2 + (\omega_0 - \beta_0)z + \gamma_1/(q-1) = 0\ .$$

Note that $ab \neq 0$, $\gamma_1 = ab(q-1)$, and $\beta_0 = a+b+\omega_0$, where

$$\omega_0 := \frac{\omega}{1-q}\ .$$

Using (2.20) and (2.21), the recurrence coefficients for the monic OPS $\big(P_n\big)_{n\geq 0}$ are

$$\beta_n = \omega_0 + (a+b)q^n\ , \quad \gamma_{n+1} = -ab(1-q^{n+1})q^n\ , \quad n = 0,1,\dots\ .$$

This means that

$$P_n(x) = L_n\left(x-\omega_0; a,b,0|q\right)\ , \quad n = 0,1,\dots\ .$$

(Thus, according to Remark 3.4.3, in this case, up to affine transformations of the variable, we obtain Al-Salam-Carlitz polynomials.)

CASE II: $N = 1$. Writing $\pi_1(x) = x - \omega_0 + c$, $c \in \mathbb{C}$, (3.71) becomes

$$(x-\omega_0+c)D_{q,\omega}P_{n+1}(x) = [n+1]_q P_{n+1}(x) + [n+1]_q c_{n,n} P_n(x)\ , \quad n = 0,1,\dots\ .$$

Setting $n = 0$ gives $c_{0,0} = c + \beta_0 - \omega_0$, and so condition (3.2) implies $c + \beta_0 \neq \omega_0$. By (3.45), (3.46), and (1.1), we obtain the functional equation

$$\mathbf{D}_{1/q,-\omega/q}\Big((x-\omega_0+c)\mathbf{u}\Big) = -\frac{q(\beta_0-\omega_0+c)}{\gamma_1}(x-\beta_0)\mathbf{u}\ .$$

Setting $\alpha = -q(\beta_0 - \omega_0 + c)/\gamma_1$ (hence $\alpha \neq 0$) and $\beta = \alpha(\omega_0 - \beta_0)$, the above functional equation becomes

$$\mathbf{D}_{1/q,-\omega/q}\Big((x-\omega_0+c)\mathbf{u}\Big) = \big(\alpha(x-\omega_0)+\beta\big)\mathbf{u}\,,$$

hence, using (2.20) and (2.21), we find

$$\beta_n = \omega_0 - \frac{\big(\beta(1-q)+(1+q)(1-q^n)\big)q^n}{\alpha(1-q)}\,, \tag{3.72}$$

$$\gamma_{n+1} = \frac{q^{n+1}(1-q^{n+1})\Big(\alpha c(1-q)+(q+\beta(1-q))q^n - q^{2n+1}\Big)}{\alpha^2(1-q)^2} \tag{3.73}$$

for each $n = 0,1,\ldots$. Let a and b be the zeros of the polynomial

$$\theta_2(z) := -q^{-1}z^2 + \big(1-\beta(1-q^{-1})\big)z + \alpha c(1-q)\,,$$

so that $\theta_2(z) = -q^{-1}(z-a)(z-b)$. Then $a+b = q+\beta(1-q)$ and $ab = c\alpha q(1-q)$. Therefore, setting $r := 1/(\alpha(q-1))$, we have $r \neq 0$ and, from (3.72) and (3.73),

$$\beta_n = \omega_0 + rq^n\big(a+b+1-q^n-q^{n+1}\big)\,,$$
$$\gamma_{n+1} = -r^2q^n\big(1-q^{n+1}\big)\big(a-q^{n+1}\big)\big(b-q^{n+1}\big)$$

for each $n = 0,1,\ldots$. This means that

$$P_n(x) = r^n L_n\left(\frac{x-\omega_0}{r};a,b,1\middle|q\right) = L_n\left(x-\omega_0;ar,br,r\middle|q\right)\,.$$

(Therefore, in this case, up to affine transformations of the variable, we obtain Big-q-Laguerre polynomials if $ab \neq 0$, Little-q-Laguerre polynomials if $ab = 0$ and a and b do not vanish simultaneously, and the OPS $(l_n)_{n\geq 0}$ if $a = b = 0$.)

CASE III: $N = 2$. Then we may write $\pi_2(x) = (x-\omega_0-r)(x-\omega_0-s)$, with $r,s \in \mathbb{C}$, and (3.71) becomes

$$(x-\omega_0-r)(x-\omega_0-s)D_{q,\omega}P_{n+1}(x) = [n+1]_q\Big(P_{n+2}(x)+c_{n,n+1}P_{n+1}(x)+c_{n,n}P_n(x)\Big)\,,$$

for each $n = 0,1,\ldots$. From (3.45) and (3.46), we deduce

$$\mathbf{D}_{1/q,-\omega/q}\Big((x-\omega_0-r)(x-\omega_0-s)\mathbf{u}\Big) = \Big(\alpha(x-\omega_0)+\beta\Big)\mathbf{u}\,,$$

where $\alpha := -q\big(\gamma_1+\pi_2(\beta_0)\big)/\gamma_1$ and $\beta = -\alpha(\beta_0-\omega_0)$. The regularity of $\mathbf{u}$ implies $\alpha \neq 0$. Since $(1-q^{-1})d_n = 1+(-1+\alpha(1-q^{-1}))q^{-n}$, then we will distinguish two sub-cases, depending whether $(d_n)_{n\geq 0}$ is a constant sequence or not.

CASE III.a) If $\alpha = 1/(1-q^{-1})$, then $d_n = \alpha$ for all n. Let $c := (q-1)\beta + q(r+s)$. By using (2.20)–(2.21) we find

$$P_n(x) = L_n\left(x-\omega_0;r,s,c|q^{-1}\right)\,,\quad n = 0,1,\ldots.$$

(Therefore, in this case, we obtain Al-Salam-Carlitz polynomials if $c = 0$, i.e., $\beta_0 = \omega_0 + r + s$, Big-$q$-Laguerre polynomials if $rs \neq 0$, Little-q-Laguerre polynomials if $rs = 0$ and r and s do not vanish simultaneously, and the OPS $(l_n)_{n\geq 0}$ if $r = s = 0$.)

CASE III.b) If $\alpha \neq 1/(1-q^{-1})$, then $(d_n)_{n\geq 0}$ is not a constant sequence. Let

$$u := q(q+\alpha(1-q)) \,, \quad \lambda := (r+s)q - \beta(1-q) \,.$$

Note that $u \neq 0$ (since $\alpha \neq 1/(1-q^{-1})$). Note also that $d_n = (1-uq^{-n-2})/(1-q^{-1})$ for each $n = 0,1,\ldots$ and so, since $d_n \neq 0$, we obtain $u \neq q^n$ for each $n = 0,1,\ldots$. Therefore, using (2.20)–(2.21), we obtain

$$\beta_n = \omega_0 + q^{-n}\frac{(\lambda+r+s)(1+uq^{-2n-1}) - (1+q^{-1})(\lambda+ru+su)q^{-n}}{(1-uq^{-2n})(1-uq^{-2n-2})} \tag{3.74}$$

and

$$\gamma_{n+1} = -\frac{q^{-n}(1-q^{-n-1})(1-uq^{-n-1})\varphi(q^{-n-1};r,s)\varphi(q^{-n-1};s,r)}{(1-uq^{-2n-1})(1-uq^{-2n-2})^2(1-uq^{-2n-3})} \tag{3.75}$$

for each $n = 0,1,\ldots$, where

$$\varphi(z;x,y) := xuz^2 - \lambda z + y \,.$$

If $r = \lambda = 0$ then $\varphi(z;r,s) = s$ and $\varphi(z;s,r) = suz^2$. Then, from (3.74)–(3.75) we obtain $s \neq 0$ and

$$P_n(x) = J_n\left(x-\omega_0;0,0,s,u|q^{-1}\right) \,, \quad n = 0,1,\ldots \,.$$

(This means that, in this case, the P_n's are q-Bessel polynomials.) If $r = 0$ and $\lambda \neq 0$, define $a = \lambda/u$ and $b = us/\lambda$; and if $r \neq 0$ (λ being zero or not), define $a = (\lambda+\sqrt{\Delta})/(2u)$ and $b = (\lambda-\sqrt{\Delta})/(2r)$, where $\Delta := \lambda^2 - 4rsu$ alternatively, we may choose $a = (\lambda-\sqrt{\Delta})/(2u)$ and $b = (\lambda+\sqrt{\Delta})/(2r)$). These choices of a and b (in either cases $r = 0$ and $\lambda \neq 0$, or $r \neq 0$) give $s = ab$ and $\lambda = au + br$, and so

$$\varphi(z;r,s) = (rz-a)(uz-b) \,, \quad \varphi(z;s,r) = (auz-r)(bz-1) \,.$$

Therefore, using (3.74)–(3.75), we obtain

$$P_n(x) = J_n\left(x-\omega_0;a,b,r,u|q^{-1}\right) \,, \quad n = 0,1,\ldots \,.$$

(In this case, if $r = 0$ (and so $a \neq 0$) we obtain Little-q-Jacobi polynomials if $b \neq 0$ and the OPS $(j_n)_{n\geq 0}$ if $b = 0$; and if $r \neq 0$, we obtain Big-q-Jacobi polynomials if $a, b \neq 0$, Little-q-Jacobi polynomials if $ab = 0$ and a and b do not vanish simultaneously, and q-Bessel polynomials if $a = b = 0$.)

Remark 3.4.5 *In [?] the authors claim: "We show that the only orthogonal polynomials satisfying a $q-$difference equation of the form $\pi(x)D_qP_n(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x)$, where $\pi(x)$ is a polynomial of degree 2, are the Al-Salam Carlitz 1, little and big $q-$Laguerre, the little and big $q-$Jacobi, and the $q-$Bessel polynomials. This is a $q-$analog of the work carried out in [1]." However, according to Theorem 3.4.3 for $\omega = 0$, there are two additional families of monic OPS (given in Table 3.1) that also satisfy the above $q-$difference equation. Therefore, Theorem 3.4.3 for $\omega = 0$ is the true $q-$analogue of the work by Al-Salam and Chihara [3].*

This section highlights that the concept of coherent pair of measures, besides its own theoretical interest, is a useful tool to deal with specific algebraic problems in the theory of orthogonal polynomials. It is worth mentioning that several problems and conjectures related with this type of structure relations remain unsolved (see [26, Section 24.7.1]). Indeed it is in this direction that we pursue our investigation (in the next chapters) in the framework of classical OPS with respect to the operator D_x defined in (1.42). These remarks will be helpful to solve Conjecture [26, 24.7.8], which fits into the theory of self coherent pairs of measures (on nonuniform lattices).

Chapter 4

Classical orthogonal polynomials on nonuniform lattices

4.1 Introduction

In this chapter we consider a nonuniform lattice given by (1.26), i.e.,

$$x(s) := \begin{cases} \mathfrak{c}_1 q^{-s} + \mathfrak{c}_2 q^s + \mathfrak{c}_3 & \text{if} \quad q \neq 1 \,, \\ \mathfrak{c}_4 s^2 + \mathfrak{c}_5 s + \mathfrak{c}_6 & \text{if} \quad q = 1 \,, \end{cases} \tag{1.26}$$

where $q > 0$ and $\mathfrak{c}_j$ $(1 \leq j \leq 6)$ are constants in $\mathbb{C}$, that may depend on q, such that $(\mathfrak{c}_1, \mathfrak{c}_2) \neq (0,0)$ if $q \neq 1$, and $(\mathfrak{c}_4, \mathfrak{c}_5, \mathfrak{c}_6) \neq (0,0,0)$ if $q = 1$. We consider the notations given in Section 1.3 of Chapter 1. In particular, the numbers α, β, α_n, β_n, γ_n, and δ introduced in (1.27)—(1.33) and (1.47)—(1.48), as well as the operators D_x and S_x (on $\mathscr{P}$) and $\mathbf{D}_x$ and $\mathbf{S}_x$ (on $\mathscr{P}^*$) introduced in (1.42)—(1.44), together with the polynomials $\mathtt{U}_1$ and $\mathtt{U}_2$ given by (1.45)—(1.46) (or (1.49)—(1.50)), will play a fundamental role along this chapter. Our aim is primarily to obtain regularity results similar to the ones stated in Chapter 2 for Hanh's operator.

Definition 4.1 *Let $x(s)$ be the NUL given by* (1.26) *and let $\mathbf{u} \in \mathscr{P}^*$. The functional $\mathbf{u}$ is called $x-$classical if it is regular and there exist $\phi \in \mathscr{P}_2$ and $\psi \in \mathscr{P}_1$, ϕ and ψ not vanishing everywhere simultaneously, such that*

$$\mathbf{D}_x(\phi \mathbf{u}) = \mathbf{S}_x(\psi \mathbf{u}) \,. \tag{4.1}$$

An OPS with respect to a $x-$classical functional will be called a $x-$classical OPS (or a classical OPS on the NUL x).

As far as we know, Definition 4.1 was introduced in [17]. It is an extension of the definition of (very) classical functional for continuous OPS (i.e., Jacobi, Laguerre, Hermite, and Bessel functionals) deeply studied by many authors, specially Maroni (see [42, 47, 48]). We will refer to the functional equation (4.1) as $x-$Geronimus–Pearson functional equation on the NUL x, or, simply, $x-$GP functional equation. The principal goal of this chapter is to state necessary and sufficient conditions, involving only the polynomials ϕ and ψ (or, equivalently, their coefficients), such that a given functional $\mathbf{u} \in \mathscr{P}^*$

satisfying the $x-$GP functional equation (4.1) becomes regular. This will extend to OPS on NUL previous known results for continuous OPS (associated with the very classical OPS) and for OPS with respect to Hahn's operator (cf. Chapter 2 and [4, 38]).

The structure of the chapter is the following. In Section 4.2 we prove several preliminary results, including a functional version on NUL of the Rodrigues-type formula. This formula holds for functionals that are solutions of the $x-$GP equation, even without assuming regularity. Indeed, the existence of such formula only requires the admissibility of the pair (ϕ,ψ) appearing in the $x-$GP equation, in a sense to be defined later. In Section 4.3 we state our main results, presenting the necessary and sufficient regularity conditions mentioned above, and giving explicit formulas (in a closed form) for the recurrence coefficients appearing in the TTRR satisfied by the monic OPS with respect to $\mathbf{u}$. Finally, in order to illustrate the power of such formulas, in Section 4.4 we revisit the Racah and the Askey-Wilson polynomials, computing in a simple way the corresponding coefficients of the TTRR from the functional equation fulfilled by the associated regular functional.

4.2 Preliminaries

Along this chapter, we will denote by $P_n^{[k]}$ the monic polynomial of degree n defined by

$$P_n^{[k]}(z) := \frac{D_x^k P_{n+k}(z)}{\prod_{j=1}^k \gamma_{n+j}} = \frac{\gamma_n!}{\gamma_{n+k}!} D_x^k P_{n+k}(z) \quad (k,n=0,1,2,\ldots)\,. \tag{4.2}$$

As usual we understood that $D_x^0 f = f$, the empty product is one and $\gamma_0! := 1$, $\gamma_{n+1}! := \gamma_1 \ldots \gamma_n \gamma_{n+1}$.

Definition 4.2 *Let $\phi \in \mathscr{P}_2$ and $\psi \in \mathscr{P}_1$. (ϕ,ψ) is called an $x-$admissible pair if*

$$d_n := \tfrac{1}{2}\,\gamma_n\,\phi'' + \alpha_n \psi' \neq 0 \quad (n=0,1,2,\ldots)\,.$$

This is an extension of the corresponding definition for the continuous case, as well as for the case involving the Hahn operator (see Chapter 2, see also [4]).

4.2.1 Properties of higher order x-derivative

Following [17], given $\mathbf{u} \in \mathscr{P}^*$, $\phi \in \mathscr{P}_2$, and $\psi \in \mathscr{P}_1$, we define recursively polynomials $\phi^{[k]} \in \mathscr{P}_2$ and $\psi^{[k]} \in \mathscr{P}_1$ (for each $k=0,1,2,\ldots$) by

$$\phi^{[0]} := \phi\,, \quad \psi^{[0]} := \psi\,, \tag{4.3}$$

$$\phi^{[k+1]} := \mathrm{S}_x \phi^{[k]} + \mathtt{U}_1 \mathrm{S}_x \psi^{[k]} + \alpha \mathtt{U}_2 \mathrm{D}_x \psi^{[k]}\,, \tag{4.4}$$

$$\psi^{[k+1]} := \mathrm{D}_x \phi^{[k]} + \alpha \mathrm{S}_x \psi^{[k]} + \mathtt{U}_1 \mathrm{D}_x \psi^{[k]}\,, \tag{4.5}$$

and functionals $\mathbf{u}^{[k]} \in \mathscr{P}^*$ by

$$\mathbf{u}^{[0]} := \mathbf{u}\,, \quad \mathbf{u}^{[k+1]} := \mathbf{D}_x\big(\mathtt{U}_2 \psi^{[k]} \mathbf{u}^{[k]}\big) - \mathbf{S}_x\big(\phi^{[k]} \mathbf{u}^{[k]}\big)\,. \tag{4.6}$$

Note that $\mathbf{u}^{[k]}$ may be regarded as the higher order $x-$derivative of $\mathbf{u}$. In the next result we give explicit expressions for the polynomials $\phi^{[k]}$ and $\psi^{[k]}$ defined by (4.3)–(4.5).

Proposition 4.2.1 *Consider a $q-$quadratic NUL, i.e., $x(s) := \mathfrak{c}_1 q^{-s} + \mathfrak{c}_2 q^s + \mathfrak{c}_3$ ($s \in \mathbb{C}$; $q > 0$; $q \neq 1$). Let $\phi \in \mathscr{P}_2$ and $\psi \in \mathscr{P}_1$, and write*

$$\phi(z) = az^2 + bz + c\,, \quad \psi(z) = dz + e\,,$$

where $a,b,c,d,e \in \mathbb{C}$. Then the polynomials $\phi^{[k]}$ and $\psi^{[k]}$ defined by (4.3)–(4.5) *are given by*

$$\psi^{[k]}(z) = \big(a\gamma_{2k} + d\alpha_{2k}\big)(z - \mathfrak{c}_3) + \phi'(\mathfrak{c}_3)\gamma_k + \psi(\mathfrak{c}_3)\alpha_k\,, \tag{4.7}$$

$$\begin{aligned}\phi^{[k]}(z) = {}& \big(d(\alpha^2-1)\gamma_{2k} + a\alpha_{2k}\big)\big((z-\mathfrak{c}_3)^2 - 2\mathfrak{c}_1\mathfrak{c}_2\big) \\ & + \big(\phi'(\mathfrak{c}_3)\alpha_k + \psi(\mathfrak{c}_3)(\alpha^2-1)\gamma_k\big)(z-\mathfrak{c}_3) + \phi(\mathfrak{c}_3) + 2a\mathfrak{c}_1\mathfrak{c}_2\,,\end{aligned} \tag{4.8}$$

for each $k = 0, 1, 2 \ldots$.

<u>Proof</u> Set

$$\phi^{[k]}(z) = a^{[k]}z^2 + b^{[k]}z + c^{[k]}\,, \quad \psi^{[k]}(z) = d^{[k]}z + e^{[k]} \tag{4.9}$$

where $a^{[k]}, b^{[k]}, c^{[k]}, d^{[k]}, e^{[k]} \in \mathbb{C}$. Clearly, by (4.3),

$$a^{[0]} = a\,, \quad b^{[0]} = b\,, \quad c^{[0]} = c\,, \quad d^{[0]} = d\,, \quad e^{[0]} = e\,.$$

In order to determine $a^{[k]}$, $b^{[k]}$, $c^{[k]}$, $d^{[k]}$, and $e^{[k]}$ for each $k = 1, 2, 3 \ldots$, we proceed as follows. Firstly we replace in (4.4) and in (4.5) the expressions of $\phi^{[k]}$, $\phi^{[k+1]}$, $\psi^{[k]}$, and $\psi^{[k+1]}$ given by (4.9); and then, in the two resulting identities, using (1.65)–(1.66) together with (1.45) and (1.46), after identification of the coefficients of the polynomials appearing in both sides of each of those identities, we obtain a system with five difference equations, given by

$$a^{[k+1]} = (2\alpha^2 - 1)a^{[k]} + 2\alpha(\alpha^2-1)d^{[k]} \tag{4.10}$$

$$b^{[k+1]} = \alpha b^{[k]} + (\alpha^2-1)e^{[k]} + 2\beta(2\alpha+1)a^{[k]} + \beta(\alpha+1)(4\alpha-1)d^{[k]} \tag{4.11}$$

$$c^{[k+1]} = c^{[k]} + \widehat{v}_2 a^{[k]} + \beta b^{[k]} + \beta(\alpha+1)e^{[k]} + \big(\beta^2(\alpha+1) + \alpha\delta\big)d^{[k]} \tag{4.12}$$

$$d^{[k+1]} = 2\alpha a^{[k]} + (2\alpha^2-1)d^{[k]} \tag{4.13}$$

$$e^{[k+1]} = b^{[k]} + \alpha e^{[k]} + 2\beta a^{[k]} + \beta(2\alpha+1)d^{[k]} \tag{4.14}$$

for each $k = 0, 1, 2, \ldots$. The explicit solution of this system is

$$a^{[k]} = d(\alpha^2-1)\gamma_{2k} + a\alpha_{2k}\,, \tag{4.15}$$

$$b^{[k]} = \psi(\mathfrak{c}_3)(\alpha^2-1)\gamma_k + \phi'(\mathfrak{c}_3)\alpha_k - 2\mathfrak{c}_3\big(d(\alpha^2-1)\gamma_{2k} + a\alpha_{2k}\big)\,, \tag{4.16}$$

$$\begin{aligned}c^{[k]} = {}& \phi(\mathfrak{c}_3) + 2a\mathfrak{c}_1\mathfrak{c}_2 - \mathfrak{c}_3\big(\psi(\mathfrak{c}_3)(\alpha^2-1)\gamma_k + \phi'(\mathfrak{c}_3)\alpha_k\big) \\ & + (\mathfrak{c}_3^2 - 2\mathfrak{c}_1\mathfrak{c}_2)\big(d(\alpha^2-1)\gamma_{2k} + a\alpha_{2k}\big)\,,\end{aligned} \tag{4.17}$$

$$d^{[k]} = a\gamma_{2k} + d\alpha_{2k}\,, \tag{4.18}$$

$$e^{[k]} = \phi'(\mathfrak{c}_3)\gamma_k + \psi(\mathfrak{c}_3)\alpha_k - \mathfrak{c}_3\big(a\gamma_{2k} + d\alpha_{2k}\big) \tag{4.19}$$

for each $k=0,1,2,\ldots$. This can be easily proved by induction on k. Finally (4.7)–(4.8) is obtained by replacing (4.15)–(4.19) in (4.9).

Lemma 4.2.2 in bellow is proved in [17]. We point out that the proof given in [17] assumes that $\mathbf{u}$ is a regular functional. However, inspection of the proof given therein shows that the result remains unchanged without such assumption.

Lemma 4.2.2 *Let* $\mathbf{u}\in\mathscr{P}^*$. *Suppose that there exist* $\phi\in\mathscr{P}_2$ *and* $\psi\in\mathscr{P}_1$ *such that (4.1) holds. Then* $\mathbf{u}^{[k]}$ *fulfills the functional equation*

$$\mathbf{D}_x\big(\phi^{[k]}\mathbf{u}^{[k]}\big)=\mathbf{S}_x\big(\psi^{[k]}\mathbf{u}^{[k]}\big)\quad (k=0,1,2,\ldots)\,. \tag{4.20}$$

The next result gives some additional functional equations fulfilled by $\mathbf{u}^{[k]}$.

Lemma 4.2.3 *Let* $\mathbf{u}\in\mathscr{P}^*$ *be a functional satisfying (4.1) for some* $\phi\in\mathscr{P}_2$ *and* $\psi\in\mathscr{P}_1$. *Then the relations*

$$\mathbf{D}_x\big(\mathbf{u}^{[k+1]}\big)=-\alpha\psi^{[k]}\mathbf{u}^{[k]}\,, \tag{4.21}$$

$$\mathbf{S}_x\big(\mathbf{u}^{[k+1]}\big)=-\alpha\big(\alpha\phi^{[k]}+\mathtt{U}_1\psi^{[k]}\big)\mathbf{u}^{[k]}\,, \tag{4.22}$$

$$2\mathtt{U}_1\mathbf{u}^{[k+1]}=\mathbf{S}_x\big(\mathtt{U}_2\psi^{[k]}\mathbf{u}^{[k]}\big)-\mathbf{D}_x\big(\mathtt{U}_2\phi^{[k]}\mathbf{u}^{[k]}\big) \tag{4.23}$$

hold for each $k=0,1,2,\ldots$.

Proof Using (1.63) and (4.20), we deduce

$$\begin{aligned}\mathbf{D}_x^2\big(\mathtt{U}_2\psi^{[k]}\mathbf{u}^{[k]}\big)&=(2\alpha-\alpha^{-1})\mathbf{S}_x^2\big(\psi^{[k]}\mathbf{u}^{[k]}\big)+\alpha^{-1}\mathtt{U}_1\mathbf{D}_x\mathbf{S}_x\big(\psi^{[k]}\mathbf{u}^{[k]}\big)-\alpha\psi^{[k]}\mathbf{u}^{[k]}\\&=(2\alpha-\alpha^{-1})\mathbf{S}_x\mathbf{D}_x\big(\phi^{[k]}\mathbf{u}^{[k]}\big)+\alpha^{-1}\mathtt{U}_1\mathbf{D}_x^2\big(\phi^{[k]}\mathbf{u}^{[k]}\big)-\alpha\psi^{[k]}\mathbf{u}^{[k]}\\&=\mathbf{D}_x\mathbf{S}_x\big(\phi^{[k]}\mathbf{u}^{[k]}\big)-\alpha\psi^{[k]}\mathbf{u}^{[k]}\,,\end{aligned}$$

where the last equality follows from (1.72) for $n=1$ and taking into account that $\alpha_2=2\alpha^2-1$ and $\gamma_1=1$. Therefore, by the definition of $\mathbf{u}^{[k+1]}$, we obtain

$$\mathbf{D}_x\mathbf{u}^{[k+1]}=\mathbf{D}_x^2\big(\mathtt{U}_2\psi^{[k]}\mathbf{u}^{[k]}\big)-\mathbf{D}_x\mathbf{S}_x\big(\phi^{[k]}\mathbf{u}^{[k]}\big)=-\alpha\psi^{[k]}\mathbf{u}^{[k]}\,.$$

This proves (4.21). Next by (1.59) and (1.60), we may write

$$\begin{aligned}\mathbf{D}_x\big(\mathtt{U}_2\phi^{[k]}\mathbf{u}^{[k]}\big)&=\big(\mathrm{S}_x\mathtt{U}_2-\alpha^{-1}\mathtt{U}_1\mathrm{D}_x\mathtt{U}_2\big)\,\mathbf{D}_x\big(\phi^{[k]}\mathbf{u}^{[k]}\big)+\alpha^{-1}\big(\mathrm{D}_x\mathtt{U}_2\big)\mathbf{S}_x\big(\phi^{[k]}\mathbf{u}^{[k]}\big)\,,\\ \mathbf{S}_x\big(\mathtt{U}_2\psi^{[k]}\mathbf{u}^{[k]}\big)&=\big(\mathrm{S}_x\mathtt{U}_2-\alpha^{-1}\mathtt{U}_1\mathrm{D}_x\mathtt{U}_2\big)\,\mathbf{S}_x\big(\psi^{[k]}\mathbf{u}^{[k]}\big)+\alpha^{-1}\big(\mathrm{D}_x\mathtt{U}_2\big)\mathbf{D}_x\big(\mathtt{U}_2\psi^{[k]}\mathbf{u}^{[k]}\big)\,.\end{aligned}$$

After subtracting these two equalities and taking into account (4.20), as well as the relation $\alpha^{-1}\mathrm{D}_x\mathtt{U}_2=2\mathtt{U}_1$ (cf. (1.52)), we get (4.23). To prove (4.22), note first that, by the definition of $\mathbf{u}^{[k+1]}$,

$$\alpha_2\mathbf{S}_x\mathbf{u}^{[k+1]}=\alpha_2\mathbf{S}_x\mathbf{D}_x\big(\mathtt{U}_2\psi^{[k]}\mathbf{u}^{[k]}\big)-\alpha_2\mathbf{S}_x^2\big(\phi^{[k]}\mathbf{u}^{[k]}\big)\,. \tag{4.24}$$

Using again (1.72) for $n=1$, we have

$$\alpha_2\mathbf{S}_x\mathbf{D}_x\big(\mathtt{U}_2\psi^{[k]}\mathbf{u}^{[k]}\big)=\alpha\mathbf{D}_x\mathbf{S}_x\big(\mathtt{U}_2\psi^{[k]}\mathbf{u}^{[k]}\big)-\mathtt{U}_1\mathbf{D}_x^2\big(\mathtt{U}_2\psi^{[k]}\mathbf{u}^{[k]}\big)$$

and by (1.63), we also have

$$\alpha_2 \mathbf{S}_x^2\big(\phi^{[k]}\mathbf{u}^{[k]}\big) = -\mathtt{U}_1\mathbf{D}_x\mathbf{S}_x\big(\phi^{[k]}\mathbf{u}^{[k]}\big) + \alpha^2\phi^{[k]}\mathbf{u}^{[k]} + \alpha\mathbf{D}_x^2\big(\mathtt{U}_2\phi^{[k]}\mathbf{u}^{[k]}\big) \,.$$

Substituting these two expressions into the right-hand side of (4.24), we get

$$(2\alpha^2-1)\mathbf{S}_x\mathbf{u}^{[k+1]} = \alpha\mathbf{D}_x\mathbf{S}_x\big(\mathtt{U}_2\psi^{[k]}\mathbf{u}^{[k]}\big) - \alpha\mathbf{D}_x^2\big(\mathtt{U}_2\phi^{[k]}\mathbf{u}^{[k]}\big) - \mathtt{U}_1\mathbf{D}_x\mathbf{u}^{[k+1]} - \alpha^2\phi^{[k]}\mathbf{u}^{[k]} \,. \tag{4.25}$$

Next, by taking $f=\mathtt{U}_1$ and replacing $\mathbf{u}$ by $\mathbf{u}^{[k+1]}$ in (1.59), and then using (1.51) and (4.21), we derive

$$\mathbf{D}_x\big(\mathtt{U}_1\mathbf{u}^{[k+1]}\big) = -\mathtt{U}_1\psi^{[k]}\mathbf{u}^{[k]} + (\alpha-\alpha^{-1})\mathbf{S}_x\mathbf{u}^{[k+1]} \,.$$

Multiplying both sides of this equality by 2α and combining the resulting equality with the one obtained by applying the operator $\mathbf{D}_x$ to both sides of (4.23), we get

$$(2\alpha^2-2)\mathbf{S}_x\mathbf{u}^{[k+1]} = \alpha\mathbf{D}_x\mathbf{S}_x\big(\mathtt{U}_2\psi^{[k]}\mathbf{u}^{[k]}\big) - \alpha\mathbf{D}_x^2\big(\mathtt{U}_2\phi^{[k]}\mathbf{u}^{[k]}\big) + 2\alpha\mathtt{U}_1\psi^{[k]}\mathbf{u}^{[k]} \,. \tag{4.26}$$

Finally, subtracting (4.26) to (4.25), and taking into account (4.21), (4.22) follows.

4.2.2 Rodrigues-type formula on NUL

In the proposition in bellow we establish a functional version of the Rodrigues-type formula on NUL.

Proposition 4.2.4 (Rodrigues-type formula) *Let $x(s)$ be a $q-$quadratic NUL, i.e.*

$$x(s) := \mathfrak{c}_1 q^{-s} + \mathfrak{c}_2 q^{s} + \mathfrak{c}_3 \quad (s\in\mathbb{C};\ q>0;\ q\neq 1) \,.$$

Let $\mathbf{u}\in\mathscr{P}^*$ and suppose that there exists an $x-$admissible pair (ϕ,ψ) such that $\mathbf{u}$ fulfills the $x-GP$ functional equation (4.1). Set

$$d_n := \tfrac{1}{2}\phi''(\mathfrak{c}_3)\gamma_n + \psi'(\mathfrak{c}_3)\alpha_n \,, \quad e_n := \phi'(\mathfrak{c}_3)\gamma_n + \psi(\mathfrak{c}_3)\alpha_n \,, \tag{4.27}$$

for each $n=0,1,2,\ldots$. Then

$$R_n\mathbf{u} = \mathbf{D}_x^n\mathbf{u}^{[n]} \tag{4.28}$$

for each $n=0,1,\cdots$, where $\mathbf{u}^{[n]}$ is the functional defined by (4.6) and $(R_n)_{n\geq 0}$ is a simple set of polynomials given by the TTRR

$$R_{n+1}(z) = (a_n z - s_n)R_n(z) - t_nR_{n-1}(z) \tag{4.29}$$

for each $n=0,1,\ldots$, with initial conditions $R_{-1}=0$ and $R_0=1$, and $(a_n)_{n\geq 0}$, $(s_n)_{n\geq 0}$, and $(t_n)_{n\geq 1}$ are sequences of complex numbers defined by

$$a_n := -\frac{\alpha\, d_{2n}d_{2n-1}}{d_{n-1}}\,, \tag{4.30}$$

$$s_n := a_n\left(\mathfrak{c}_3+\frac{\gamma_n e_{n-1}}{d_{2n-2}}-\frac{\gamma_{n+1}e_n}{d_{2n}}\right)\,, \tag{4.31}$$

$$t_n := a_n\,\frac{\alpha\,\gamma_n d_{2n-2}}{d_{2n-1}}\phi^{[n-1]}\left(\mathfrak{c}_3-\frac{e_{n-1}}{d_{2n-2}}\right)\,, \tag{4.32}$$

$\phi^{[n-1]}$ *being given by* (4.8). *(It is understood that $a_0:=-\alpha d$ and $s_0:=\alpha e$.)*

Proof We apply mathematical induction on n. If $n=0$, (4.28) is trivial. If $n=1$, (4.28) follows from (4.21), since $R_1=-\alpha\psi$. Assume now (induction hypothesis) that (4.28) holds for two consecutive nonnegative integer numbers, i.e., the relations

$$R_{n-1}\mathbf{u}=\mathbf{D}_x^{n-1}\mathbf{u}^{[n-1]}\,,\quad R_n\mathbf{u}=\mathbf{D}_x^{n}\mathbf{u}^{[n]} \tag{4.33}$$

hold for some fixed $n\in\mathbb{N}$. We need to prove that $R_{n+1}\mathbf{u}=\mathbf{D}_x^{n+1}\mathbf{u}^{[n+1]}$. Notice first that, by (4.7) and (4.27), we have

$$\psi^{[k]}(z)=d_{2k}(z-\mathfrak{c}_3)+e_k\,, \tag{4.34}$$

for each $k=0,1,\ldots$. By (4.21) and the Leibniz formula in Proposition 1.3.5, we may write

$$\begin{aligned}\mathbf{D}_x^{n+1}\mathbf{u}^{[n+1]}&=\mathbf{D}_x^{n}\mathbf{D}_x\mathbf{u}^{[n+1]}=-\alpha\mathbf{D}_x^{n}(\psi^{[n]}\mathbf{u}^{[n]})\\&=-\alpha\mathrm{T}_{n,0}\,\psi^{[n]}\mathbf{D}_x^{n}\mathbf{u}^{[n]}-\alpha\mathrm{T}_{n,1}\,\psi^{[n]}\mathbf{D}_x^{n-1}\mathbf{S}_x\mathbf{u}^{[n]}\,.\end{aligned}$$

From (1.82) we have $\mathrm{T}_{n,1}\,\psi^{[n]}=d_{2n}\gamma_n/\alpha_n$, and so, using also (4.33),

$$\mathbf{D}_x^{n-1}\mathbf{S}_x\mathbf{u}^{[n]}=-\frac{\alpha_n}{\alpha d_{2n}\gamma_n}\Big(\mathbf{D}_x^{n+1}\mathbf{u}^{[n+1]}+\alpha\big(\mathrm{T}_{n,0}\,\psi^{[n]}\big)R_n\mathbf{u}\Big)\,. \tag{4.35}$$

Shifting n into $n-1$, and using again the induction hypothesis (4.33), we obtain

$$\mathbf{D}_x^{n-2}\mathbf{S}_x\mathbf{u}^{[n-1]}=-\frac{\alpha_{n-1}}{\alpha d_{2n-2}\gamma_{n-1}}\Big(R_n+\alpha\big(\mathrm{T}_{n-1,0}\,\psi^{[n-1]}\big)R_{n-1}\Big)\,\mathbf{u}\,. \tag{4.36}$$

Next, using (4.21), (1.59), and (4.22), we deduce

$$\begin{aligned}\mathbf{D}_x^{n+1}\mathbf{u}^{[n+1]}&=-\alpha\mathbf{D}_x^{n}\big(\psi^{[n]}\mathbf{u}^{[n]}\big)=-\alpha\mathbf{D}_x^{n-1}\big(\mathbf{D}_x(\psi^{[n]}\mathbf{u}^{[n]})\big)\\&=-\mathbf{D}_x^{n-1}\Big(\big(\alpha\mathrm{S}_x\psi^{[n]}-\mathtt{U}_1\mathrm{D}_x\psi^{[n]}\big)\mathbf{D}_x\mathbf{u}^{[n]}+\mathrm{D}_x\psi^{[n]}\mathbf{S}_x\mathbf{u}^{[n]}\Big)\\&=\mathbf{D}_x^{n-1}\big(\xi_2(\cdot;n)\mathbf{u}^{[n-1]}\big)\,,\end{aligned} \tag{4.37}$$

where $\xi_2(\cdot;n)$ is a polynomial of degree 2, given by

$$\xi_2(z;n)=\alpha^2\big(\psi^{[n-1]}\mathrm{S}_x\psi^{[n]}+\phi^{[n-1]}\mathrm{D}_x\psi^{[n]}\big)(z)\,. \tag{4.38}$$

The following identities may be proved by a straightforward computation:

$$d_{2n-1} - \alpha d_{2n-2} = a^{[n-1]} \, ,$$
$$d_{2n-2}\big(e_n - 2\alpha \mathfrak{c}_3 d_{2n}\big) + d_{2n}\big(b^{[n-1]} + \alpha e_{n-1}\big) = 2d_{2n-1}(\alpha e_n - \mathfrak{c}_3 d_{2n})$$

for each $n = 1, 2, \ldots$. (The second one is achieved by using equation (4.14).) Using these relations, together with (4.9), (4.34), (1.65), and (1.66), we deduce

$$\begin{aligned} \xi_2(z;n) &= \alpha^2 d_{2n} d_{2n-1} z^2 + 2\alpha^2 d_{2n-1}(\alpha e_n - \mathfrak{c}_3 d_{2n}) z \\ &\quad + \alpha^2 \big(d_{2n} c^{[n-1]} + (e_{n-1} - \mathfrak{c}_3 d_{2n-2})(e_n - \alpha \mathfrak{c}_3 d_{2n})\big) \, . \end{aligned} \tag{4.39}$$

Since $\deg \xi_2(\cdot;n) = 2$, using again Proposition 1.3.5, we may write

$$\begin{aligned} \mathbf{D}_x^{n-1}\big(\xi_2(\cdot;n)\mathbf{u}^{[n-1]}\big) &= \mathrm{T}_{n-1,0}\xi_2(\cdot;n)\mathbf{D}_x^{n-1}\mathbf{u}^{[n-1]} + \mathrm{T}_{n-1,1}\xi_2(\cdot;n)\mathbf{D}_x^{n-2}\mathrm{S}_x\mathbf{u}^{[n-1]} \\ &\quad + \mathrm{T}_{n-1,2}\xi_2(\cdot;n)\mathbf{D}_x^{n-3}\mathbf{S}_x^2\mathbf{u}^{[n-1]} \, . \end{aligned} \tag{4.40}$$

Therefore, since, by (1.82), $\mathrm{T}_{n-1,2}\xi_2(\cdot;n) = \alpha^2\gamma_{n-1}\gamma_{n-2}d_{2n}d_{2n-1}/\alpha_{n-2}^2$, combining equations (4.40), (4.37), (4.36), and (4.33), we obtain

$$\begin{aligned} \mathbf{D}_x^{n-3}\mathbf{S}_x^2\mathbf{u}^{[n-1]} &= \frac{\alpha_{n-2}^2}{\alpha^2\gamma_{n-1}\gamma_{n-2}d_{2n}d_{2n-1}} \Big\{ \mathbf{D}_x^{n+1}\mathbf{u}^{[n+1]} - \big(\mathrm{T}_{n-1,0}\xi_2(\cdot;n)\big)R_{n-1}\mathbf{u} \\ &\quad + \frac{\alpha_{n-1}\mathrm{T}_{n-1,1}\xi_2(\cdot;n)}{\alpha\gamma_{n-1}d_{2n-2}} \Big(R_n + \alpha\big(\mathrm{T}_{n-1,0}\psi^{[n-1]}\big)R_{n-1}\Big)\mathbf{u} \Big\} \, . \end{aligned} \tag{4.41}$$

On the other hand, by (4.22),

$$\mathbf{S}_x\mathbf{u}^{[n]} = \eta_2(\cdot;n)\mathbf{u}^{[n-1]} \, , \quad \eta_2(z;n) := -\alpha\big(\alpha\phi^{[n-1]} + \mathtt{U}_1\psi^{[n-1]}\big)(z) \, . \tag{4.42}$$

Therefore, once again by Leibniz's formula and (4.33), and taking into account that $\eta_2(\cdot;n)$ is a polynomial of degree at most two, we may write

$$\begin{aligned} \mathbf{D}_x^{n-1}\mathbf{S}_x\mathbf{u}^{[n]} &= \mathbf{D}_x^{n-1}\big(\eta_2(\cdot;n)\mathbf{u}^{[n-1]}\big) \\ &= \big(\mathrm{T}_{n-1,0}\eta_2(\cdot;n)\big)R_{n-1}\mathbf{u} + \mathrm{T}_{n-1,1}\eta_2(\cdot;n)\mathbf{D}_x^{n-2}\mathbf{S}_x\mathbf{u}^{[n-1]} \\ &\quad + \mathrm{T}_{n-1,2}\eta_2(\cdot;n)\mathbf{D}_x^{n-3}\mathbf{S}_x^2\mathbf{u}^{[n-1]} \, . \end{aligned} \tag{4.43}$$

Note that $\eta_2(\cdot;n)$ is given explicitly by

$$\begin{aligned} \eta_2(z;n) &= \alpha(\alpha d_{2n-1} - d_{2n})z^2 - \alpha\Big(\alpha b^{[n-1]} + (\alpha^2 - 1)(e_{n-1} - 2\mathfrak{c}_3 d_{2n-2})\Big)z \\ &\quad - \alpha\big(\alpha c^{[n-1]} + \beta(\alpha+1)(e_{n-1} - \mathfrak{c}_3 d_{2n-2})\big) \, . \end{aligned} \tag{4.44}$$

Hence, using (1.82), $\mathrm{T}_{n-1,2}\eta_2(\cdot;n) = \alpha\gamma_{n-1}\gamma_{n-2}(\alpha d_{2n-1} - d_{2n})/\alpha_{n-2}^2$. Therefore, substituting (4.35), (4.36), and (4.41) in (4.43), we obtain

$$\mathbf{D}_x^{n+1}\mathbf{u}^{[n+1]} = \big(A(\cdot;n)R_n + B(\cdot;n)R_{n-1}\big)\mathbf{u} \, , \tag{4.45}$$

where $A(\cdot;n)$ and $B(\cdot;n)$ are polynomials depending on n, given by

$$\varepsilon_n A(z;n) = \frac{\alpha_n (T_{n,0}\psi^{[n]})(z)}{\gamma_n d_{2n}} - \frac{\alpha_{n-1}(T_{n-1,1}\eta_2)(z;n)}{\alpha\gamma_{n-1}d_{2n-2}} + \frac{\alpha_{n-1}(\alpha d_{2n-1}-d_{2n})(T_{n-1,1}\xi_2)(z;n)}{\alpha^2\gamma_{n-1}d_{2n}d_{2n-1}d_{2n-2}}\,, \tag{4.46}$$

and

$$\begin{aligned}\varepsilon_n B(z;n) = (T_{n-1,0}\eta_2)(z;n) &- \frac{\alpha_{n-1}(T_{n-1,0}\psi^{[n-1]})(z)(T_{n-1,1}\eta_2)(z;n)}{\gamma_{n-1}d_{2n-2}}\\ &+ \frac{(d_{2n}-\alpha d_{2n-1})(T_{n-1,0}\xi_2)(z;n)}{\alpha d_{2n}d_{2n-1}}\\ &+ \frac{\alpha_{n-1}(\alpha d_{2n-1}-d_{2n})(T_{n-1,1}\xi_2)(z;n)(T_{n-1,0}\psi^{[n-1]})(z)}{\alpha\gamma_{n-1}d_{2n}d_{2n-1}d_{2n-2}}\,,\end{aligned} \tag{4.47}$$

where

$$\varepsilon_n = \frac{d_{2n}-\alpha d_{2n-1}}{\alpha d_{2n}d_{2n-1}} - \frac{\alpha_n}{\alpha\gamma_n d_{2n}} = -\frac{d_{n-1}}{\alpha\gamma_n d_{2n}d_{2n-1}}\,.$$

Note that $\gamma_n d_{2n} - (\alpha+\alpha_n)d_{2n-1} = -d_{n-1}$ for each $n = 0,1,2,\ldots$. By straightforward computation using (1.81)-(1.82) we arrive at

$$A(z;n) = -\alpha\frac{d_{2n}d_{2n-1}}{d_{n-1}}(z-\mathfrak{c}_3) + \frac{\alpha\gamma_n d_{2n}d_{2n-1}e_{n-1}}{d_{2n-2}d_{n-1}} - \frac{\alpha\gamma_{n+1}d_{2n-1}e_n}{d_{n-1}} = a_n z - s_n\,. \tag{4.48}$$

Similarly, $B(z;n)$ reduces to the following constant:

$$B(z;n) = \alpha^2\frac{\gamma_n d_{2n}d_{2n-1}}{d_{n-1}}\phi^{[n-1]}\left(\mathfrak{c}_3 - \frac{e_{n-1}}{d_{2n-2}}\right) = -t_n \tag{4.49}$$

for each $n = 0,1,2,\ldots$, where a_n, s_n, and t_n are given by (4.30)–(4.32). Since their computations are rather technical, we provide more details on the derivation of (4.48)–(4.49) in the Appendix A.2. Hence (4.45) reduces to $\mathbf{D}_x^{n+1}\mathbf{u}^{[n+1]} = R_{n+1}\mathbf{u}$. This completes the proof.

The next result is virtually proved in [38, Theorem 2].

Lemma 4.2.5 *Let* $\mathbf{u}\in\mathscr{P}^*$ *be regular. Suppose that there is* $(\phi,\psi)\in\mathscr{P}_2\times\mathscr{P}_1\setminus\{(0,0)\}$ *so that* (4.1) *holds. Then neither* ϕ *nor* ψ *is the zero polynomial, and* $\deg\psi = 1$.

The statement of the next lemma is given in [17, Proposition 4]. However the proof of the $x-$admissibility condition given therein is incorrect. For sake of completeness, we present a proof following the ideas presented in [4, 38].

Lemma 4.2.6 *Let* $\mathbf{u}\in\mathscr{P}^*$. *Suppose that* $\mathbf{u}$ *is regular and satisfies* (4.1), *where* $\phi\in\mathscr{P}_2$, $\psi\in\mathscr{P}_1\setminus\mathscr{P}_0$. *Then* (ϕ,ψ) *is a* $x-$*admissible pair and* $\mathbf{u}^{[k]}$ *is regular for each* $k\in\mathbb{N}$. *Moreover, if* $(P_n)_{n\geq 0}$ *is the monic OPS with respect to* $\mathbf{u}$, *then* $\left(P_n^{[k]}\right)_{n\geq 0}$ *is the monic OPS with respect to* $\mathbf{u}^{[k]}$.

Proof Suppose that $\mathbf{u}$ is regular. Set $\phi(z) = az^2+bz+c$ and $\psi(z) = dz+e$, with $a,b,c,d,e\in\mathbb{C}$. If $\deg\phi\in\{0,1\}$ then $d_n = d\alpha_n$, hence $d_n\neq 0$ for each $n = 0,1,\ldots$, since $d\neq 0$ (see Lemma (4.2.5)). Assume now that $\deg\phi = 2$. Then $d_n = a\gamma_n + d\alpha_n$ and $a\neq 0$. To prove that $d_n\neq 0$, we start by showing that

$$\left\langle \mathbf{u}, \left(\mathtt{U}_2\psi \mathrm{D}_x P_n^{[1]} + \phi \mathrm{S}_x P_n^{[1]}\right) P_{n+2} \right\rangle = -\left\langle \mathbf{u}^{[1]}, \left(\mathrm{S}_x P_{n+2} + \alpha^{-1}\mathtt{U}_1 \mathrm{D}_x P_{n+2}\right) P_n^{[1]} \right\rangle \tag{4.50}$$

for each $n = 0, 1, 2, \ldots$.

Indeed, we have

$$\begin{aligned}
&\left\langle \mathbf{u}, \left(\mathtt{U}_2\psi \mathrm{D}_x P_n^{[1]} + \phi \mathrm{S}_x P_n^{[1]}\right) P_{n+2} \right\rangle \\
&\quad = \left\langle \mathtt{U}_2\psi\mathbf{u}, P_{n+2}\mathrm{D}_x P_n^{[1]} \right\rangle + \left\langle \phi\mathbf{u}, P_{n+2}\mathrm{S}_x P_n^{[1]} \right\rangle \\
&\quad = \left\langle \mathtt{U}_2\psi\mathbf{u}, \mathrm{D}_x\left((\mathrm{S}_x P_{n+2} - \alpha^{-1}\mathtt{U}_1\mathrm{D}_x P_{n+2})P_n^{[1]}\right) - \alpha^{-1}\mathrm{S}_x\left(P_n^{[1]}\mathrm{D}_x P_{n+2}\right) \right\rangle \\
&\qquad + \left\langle \phi\mathbf{u}, \mathrm{S}_x\left((\mathrm{S}_x P_{n+2} - \alpha^{-1}\mathtt{U}_1\mathrm{D}_x P_{n+2})P_n^{[1]}\right) - \alpha^{-1}\mathtt{U}_2\mathrm{D}_x\left(P_n^{[1]}\mathrm{D}_x P_{n+2}\right) \right\rangle \\
&\quad = -\left\langle \mathbf{u}^{[1]}, \left(\mathrm{S}_x P_{n+2} - \alpha^{-1}\mathtt{U}_1\mathrm{D}_x P_{n+2}\right)P_n^{[1]} \right\rangle \\
&\qquad - \alpha^{-1}\left\langle \mathbf{S}_x(\mathtt{U}_2\psi\mathbf{u}) - \mathbf{D}_x(\mathtt{U}_2\phi\mathbf{u}), P_n^{[1]}\mathrm{D}_x P_{n+2} \right\rangle ,
\end{aligned}$$

where the second equality holds by (1.58) and (1.57). Therefore, using (4.23) for $n = 0$, we obtain (4.50). Now, on the one hand, $\mathtt{U}_2\psi \mathrm{D}_x P_n^{[1]} + \phi \mathrm{S}_x P_n^{[1]}$ is a polynomial of degree at most $n+2$, being the coefficient of z^{n+2} equal to $(\alpha^2 - 1)d\gamma_n + a\alpha_n$. Hence, since the relations

$$(\alpha^2 - 1)d\gamma_n + a\alpha_n = d_{n+1} - \alpha d_n = \alpha d_n - d_{n-1} \quad (n = 1, 2, \ldots)$$

hold, we get

$$\mathtt{U}_2\psi \mathrm{D}_x P_n^{[1]} + \phi \mathrm{S}_x P_n^{[1]} = \left(\alpha d_n - d_{n-1}\right) z^{n+2} + (\text{lower degree terms})$$

for each $n = 1, 2, \ldots$. Consequently,

$$\left\langle \mathbf{u}, \left(\mathtt{U}_2\psi \mathrm{D}_x P_n^{[1]} + \phi \mathrm{S}_x P_n^{[1]}\right) P_{n+2} \right\rangle = (\alpha d_n - d_{n-1})\langle \mathbf{u}, P_{n+2}^2\rangle \quad (n = 1, 2, \ldots) . \tag{4.51}$$

On the other hand, since $\mathrm{S}_x P_{n+2} + \alpha^{-1}\mathtt{U}_1\mathrm{D}_x P_{n+2} = \sum_{j=0}^{n+2} c_{n,j}P_j^{[1]}$ for some coefficients $c_{n,0}, \ldots, c_{n,n+2} \in \mathbb{C}$, and taking the particular case where $k = 1$ in the following equation

$$\left\langle \mathbf{u}^{[k]}, P_n^{[k]} P_m^{[k]} \right\rangle = \alpha \frac{d_n^{[k-1]}}{\gamma_{n+1}} \langle \mathbf{u}, (P_{n+1}^{[k-1]})^2 \rangle \delta_{n,m} \quad (0 \le m \le n\, ; \, , n = 0, 1, \ldots) , \tag{4.52}$$

(see [17, Proof of Theorem 5–step 1.1]) we obtain

$$\left\langle \mathbf{u}^{[1]}, \left(\mathrm{S}_x P_{n+2} + \alpha^{-1}\mathtt{U}_1\mathrm{D}_x P_{n+2}\right) P_n^{[1]} \right\rangle = \frac{\alpha c_{n,n} d_n}{\gamma_{n+1}} \langle \mathbf{u}, P_{n+1}^2 \rangle \quad (n = 1, 2, \ldots) . \tag{4.53}$$

Substituting (4.51) and (4.53) into (4.50), and since $C_{n+2} = \langle \mathbf{u}, P_{n+2}^2\rangle / \langle \mathbf{u}, P_{n+1}^2\rangle$, we deduce

$$\alpha\left(1 + \frac{c_{n,n}}{\gamma_{n+1}C_{n+2}}\right) d_n = d_{n-1} \quad (n = 1, 2, 3, \ldots) .$$

This implies that

$$\prod_{j=1}^{n}\left[\alpha^{j}\left(1+\frac{c_{j,j}}{\gamma_{j+1}C_{j+2}}\right)\right]d_n=d_0=d\neq 0\,.$$

Therefore $d_n\neq 0$ for each $n=0,1,2\ldots$, and consequently, by (4.52) for $k=1$, $(P_n^{[1]})_{n\geq 0}$ is the monic OPS with respect to $\mathbf{u}^{[1]}$. This proves the last statement in the lemma for $k=1$. Since $d_n^{[k]}=a^{[k]}\gamma_n+\alpha_n d^{[k]}$ for $n,k=0,1,2,\ldots$, it is easy to see, using (4.15)–(4.18), that $d_n^{[k]}=d_{n+2k}$, for $n,k=0,1,2,\ldots$. Thus the desired result is obtained from (4.52).

4.3 Regularity conditions

In this section we state our main results: given the nonuniform lattice (1.26), we state necessary and sufficient conditions for which a functional $\mathbf{u}\in\mathscr{P}^*$ satisfying (4.1) is regular and, in such a case, we describe the associated monic OPS.

Theorem 4.3.1 *Consider the NUL*

$$x(s)=\mathfrak{c}_1q^{-s}+\mathfrak{c}_2q^{s}+\mathfrak{c}_3\quad (s\in\mathbb{C}\,;\ q>0\,;\ q\neq 1)\,.$$

Let $\mathbf{u}\in\mathscr{P}^*$ *and suppose that there exist* $(\phi,\psi)\in\mathscr{P}_2\times\mathscr{P}_1\setminus\{(0,0)\}$ *such that the functional equation* (4.1) *holds, that is:*

$$\mathbf{D}_x(\phi\mathbf{u})=\mathbf{S}_x(\psi\mathbf{u})\,. \tag{4.1}$$

Set $\phi(z):=az^2+bz+c$ *and* $\psi(z):=dz+e$ $(a,b,c,d,e\in\mathbb{C})$. *If* $\mathbf{u}$ *is regular then* (ϕ,ψ) *is an* $x-$*admissible pair and* $\psi^{[n]}\nmid\phi^{[n]}$ *for each* $n=0,1,2,\ldots$, *i.e., the conditions*

$$d_n\neq 0\,,\quad \phi^{[n]}\left(\mathfrak{c}_3-\frac{e_n}{d_{2n}}\right)\neq 0\,,\quad \forall n\in\mathbb{N}_0 \tag{4.54}$$

hold, where d_n *and* e_n *are given by* (4.27), *and* $\phi^{[n]}$ *and* $\psi^{[n]}$ *are given by* (4.7)–(4.8).

Proof Suppose that $\mathbf{u}$ is regular. Then $d_n\neq 0$ for $n=0,1,2,\ldots$. Indeed $\mathbf{u}$ satisfies (4.1) and Lemma 4.2.6 ensures that (ϕ,ψ) is a $x-$admissible pair. In addition $\big(P_j^{[n]}\big)_{j\geq 0}$ is the monic OPS with respect to $\mathbf{u}^{[n]}$ and so the following TTRR holds.

$$P_{j+1}^{[n]}(z)=(z-B_j^{[n]})P_j^{[n]}(z)-C_j^{[n]}P_{j-1}^{[n]}(z)\quad (j=0,1,2,\ldots)\,, \tag{4.55}$$

where $P_{-1}^{[n]}(z)=0$, being $B_j^{[n]}\in\mathbb{C}$ and $C_{j+1}^{[n]}\in\mathbb{C}\setminus\{0\}$ for each $j=0,1,2,\ldots$. Let us compute $C_1^{[n]}$. We first show that (for $n=0$) the coefficient $C_1\equiv C_1^{[0]}$, appearing in the TTRR for $(P_j)_{j\geq 0}$, is given by

$$C_1=-\frac{1}{d\alpha+a}\,\phi\left(-\frac{e}{d}\right)=-\frac{1}{d_1}\,\phi\left(\mathfrak{c}_3-\frac{e_0}{d_0}\right)\,. \tag{4.56}$$

This may be proved taking $n=0$ and $n=1$ in the relation $\langle\mathbf{D}_x(\phi\mathbf{u}),z^n\rangle=\langle\mathbf{S}_x(\psi\mathbf{u}),z^n\rangle$. Indeed, setting $u_n:=\langle\mathbf{u},z^n\rangle$, for $n=0$ we obtain $0=du_1+eu_0$, and for $n=1$ we find $au_2+bu_1+cu_0=$

$-d\alpha u_2-(e\alpha+d\beta)u_1-e\beta u_0$. Therefore,

$$u_1=-\frac{e}{d}u_0\,,\quad u_2=-\frac{1}{d\alpha+a}\left[-(b+e\alpha)\frac{e}{d}+c\right]u_0\,. \tag{4.57}$$

On the other hand, since $P_1(z)=z-B_0^{[0]}=z-u_1/u_0$, we also have

$$C_1=\frac{\langle \mathbf{u},P_1^2\rangle}{u_0}=\frac{u_2u_0-u_1^2}{u_0^2}=\frac{u_2}{u_0}-\left(\frac{u_1}{u_0}\right)^2\,. \tag{4.58}$$

Substituting u_1 and u_2 given by (4.57) into (4.58) yields (4.56). Since equation (4.20) is of the same type as (4.1), with polynomials ϕ and ψ replaced by $\phi^{[n]}$ and $\psi^{[n]}$, respectively, we see that $C_1^{[n]}$ may be obtained replacing in (4.56) ϕ and $\psi(z)=dz+e$ by $\phi^{[n]}$ and $\psi^{[n]}(z)=d_{2n}(z-\mathfrak{c}_3)+e_n$, respectively. Hence,

$$C_1^{[n]}=-\frac{1}{d_{2n}\alpha+a^{[n]}}\phi^{[n]}\left(\mathfrak{c}_3-\frac{e_n}{d_{2n}}\right)=-\frac{1}{d_{2n+1}}\phi^{[n]}\left(\mathfrak{c}_3-\frac{e_n}{d_{2n}}\right)\,. \tag{4.59}$$

Since $\mathbf{u}^{[n]}$ is regular, then $C_1^{[n]}\neq 0$, hence $\phi^{[n]}\left(\mathfrak{c}_3-\frac{e_n}{d_{2n}}\right)\neq 0$. Thus, (4.54) holds.

The converse of Theorem 4.3.1 is given by the following

Theorem 4.3.2 *Consider the NUL*

$$x(s)=\mathfrak{c}_1q^{-s}+\mathfrak{c}_2q^s+\mathfrak{c}_3\quad (s\in\mathbb{C};\ q>0;\ q\neq 1)\,.$$

Let $\mathbf{u}\in\mathscr{P}^*\setminus\{\mathbf{0}\}$ *and suppose that there exist* $(\phi,\psi)\in\mathscr{P}_2\times\mathscr{P}_1\setminus\{(0,0)\}$ *such that the functional equation* (4.1) *holds. Assume that conditions* (4.54) *hold. Then* $\mathbf{u}$ *is regular and the corresponding monic OPS* $(P_n)_{n\geq 0}$ *satisfies the TTRR*

$$P_{n+1}(z)=(z-B_n)P_n(z)-C_nP_{n-1}(z)\quad (n=0,1,2,\dots)\,, \tag{4.60}$$

with $P_{-1}(z)=0$, *being* B_n *and* C_{n+1} *given by*

$$B_n:=\mathfrak{c}_3+\frac{\gamma_ne_{n-1}}{d_{2n-2}}-\frac{\gamma_{n+1}e_n}{d_{2n}}\,, \tag{4.61}$$

$$C_{n+1}:=-\frac{\gamma_{n+1}d_{n-1}}{d_{2n-1}d_{2n+1}}\phi^{[n]}\left(\mathfrak{c}_3-\frac{e_n}{d_{2n}}\right) \tag{4.62}$$

for each $n=0,1,2\dots$. *Moreover, the Rodrigues-type formula*

$$P_n\mathbf{u}=k_n\mathbf{D}_x^n\mathbf{u}^{[n]} \tag{4.63}$$

holds for each $n=0,1,2\dots$, *where*

$$k_n:=(-\alpha)^{-n}\prod_{j=1}^n d_{n+j-2}^{-1}\,,\quad n=0,1,2,\dots\,. \tag{4.64}$$

Proof Define a sequence of monic polynomials $(P_n)_{n\geq 0}$ by setting $P_{-1}(z) := 0$ and $P_0(z) := 1$, and satisfying the TTRR (4.60)–(4.62). By assumption, $C_{n+1} \neq 0$ for each $n = 0,1,2,\ldots$. Therefore $(P_n)_{n\geq 0}$ is a monic OPS (by Favard's theorem). Let's show that it is the monic OPS with respect to $\mathbf{u}$. For this we only need to prove that $u_0 \neq 0$ and $\langle \mathbf{u}, P_n\rangle = 0$ for every $n = 1,2,3,\ldots$ ($u_n := \langle \mathbf{u}, z^n\rangle$). Firstly we show that $u_0 \neq 0$. Suppose that $u_0 = 0$. Since (4.1) holds, then $\langle D_x(\phi\mathbf{u}) - S_x(\psi\mathbf{u}), z^n\rangle = 0$ for $n = 0,1,2,\ldots$. This implies that

$$d_n u_{n+1} + s_n u_n + f_n u_{n-1} + \sum_{l=0}^{n-2} a_{n,l} u_l = 0\,, \qquad n = 0,1,2,\ldots \tag{4.65}$$

for some complex numbers s_n, f_n and $a_{n,l}$, $l = 0,1,2,\ldots,n-2$. For $n = 0$ in (4.65), we have $d_0 u_1 = 0$ and since $d_n \neq 0$ for all n, we find $u_1 = 0$. For $n = 2$, with the same arguments we also find $u_2 = 0$ and proceeding in this way we have $u_n = 0$ for $n = 0,1,2,\ldots$, which is impossible since $\mathbf{u} \neq \mathbf{0}$. This shows that $u_0 \neq 0$. Secondly, note that, from Proposition 4.2.4, we may write $P_n(z) = k_n R_n(z)$ for each $n = 0,1,2,\ldots$, where $k_n^{-1} = (-\alpha)^n \prod_{j=1}^{n} d_{n+j-2}$. Therefore, using (4.28), we obtain

$$\langle \mathbf{u}, P_n\rangle = k_n\,\langle \mathbf{u}, R_n\rangle = k_n\,\langle R_n\mathbf{u}, 1\rangle = k_n \left\langle D_x^n \mathbf{u}^{[n]}, 1\right\rangle = (-1)^n k_n \left\langle \mathbf{u}^{[n]}, D_x^n\, 1\right\rangle = 0$$

for each $n = 1,2,\ldots$. Hence $(P_n)_{n\geq 0}$ is the monic OPS with respect to $\mathbf{u}$. By Proposition 4.2.4, the proof is concluded.

The following corollary gives the asymptotic behavior of the sequence $(B_n)_{n\geq 0}$ appearing in (4.61). This result will be very helpful in the next chapter.

Corollary 4.3.3 *Under the same assumptions of Theorem 4.3.2,*

$$S_n := \sum_{j=0}^{n-1} (B_j - \mathfrak{c}_3) = -\frac{\gamma_n e_{n-1}}{d_{2n-2}}, \quad n = 0,1,2,\ldots. \tag{4.66}$$

In addition, setting $u := (q^{1/2} - q^{-1/2})^{-1}$*, the following holds:*

a) For $0 < q < 1$ *and* $d - 2au \neq 0$*, we have*

$$\lim_{n\to\infty} (B_n - \mathfrak{c}_3) = \lim_{n\to\infty} q^{\pm n/2}(B_n - \mathfrak{c}_3) = 0\,, \tag{4.67}$$

$$\lim_{n\to\infty} q^{-n}(B_n - \mathfrak{c}_3) = -\frac{q^{-1/2}\Big(\psi(\mathfrak{c}_3) - 4\alpha u^2 \phi'(\mathfrak{c}_3)\Big)}{u(d-2au)}\,, \tag{4.68}$$

$$S := \sum_{j=0}^{\infty} (B_j - \mathfrak{c}_3) = \frac{\psi(\mathfrak{c}_3) - 2u\phi'(\mathfrak{c}_3)}{(q-1)(d-2au)}\,. \tag{4.69}$$

b) For $1<q<\infty$ and $d+2au\neq 0$, we have

$$\lim_{n\to\infty}(B_n-\mathfrak{c}_3)=\lim_{n\to\infty}q^{\pm n/2}(B_n-\mathfrak{c}_3)=0\,, \tag{4.70}$$

$$\lim_{n\to\infty}q^n(B_n-\mathfrak{c}_3)=\frac{q^{1/2}\Big(\psi(\mathfrak{c}_3)-4\alpha u^2\phi'(\mathfrak{c}_3)\Big)}{u(d+2au)} \tag{4.71}$$

$$S=\sum_{j=0}^{\infty}(B_j-\mathfrak{c}_3)=\frac{\psi(\mathfrak{c}_3)+2u\phi'(\mathfrak{c}_3)}{(q^{-1}-1)(d+2au)}\,. \tag{4.72}$$

Proof From Theorem 4.3.2, we have

$$B_n-\mathfrak{c}_3=\frac{\gamma_n e_{n-1}}{d_{2n-2}}-\frac{\gamma_{n+1}e_n}{d_{2n}}\quad (n=0,1,2,\ldots)\,,$$

and so (4.66) holds. If $0<q<1$ and $d-2au\neq 0$ then we see that

$$\frac{\gamma_{n+1}e_n}{d_{2n}}=\frac{\theta_1 q^{2n}+\theta_2 q^n+\theta_3}{(d+2au)q^n+(d-2au)},\quad n=0,1,2,\ldots\,,$$

where

$$\theta_1:=uq^{1/2}\Big(\psi(\mathfrak{c}_3)+2u\phi'(\mathfrak{c}_3)\Big),\;\;\theta_2:=\psi(\mathfrak{c}_3)-4\alpha u^2\phi'(\mathfrak{c}_3),\;\;\theta_3:=-uq^{-1/2}\Big(\psi(\mathfrak{c}_3)-2u\phi'(\mathfrak{c}_3)\Big)\,.$$

Taking the limit as $n\to\infty$, equations (4.67)–(4.69) hold. Similarly we deduce (4.70)–(4.72).

We finish this section by considering the quadratic lattice $x(s)=\mathfrak{c}_4 s^2+\mathfrak{c}_5 s+\mathfrak{c}_6$. Recall that, here $\mathfrak{c}_4=4\beta$. For this lattice, the system of equations (4.10)–(4.14) becomes

$$\begin{aligned}
&a^{[n+1]}=a^{[n]},\;\; d^{[n+1]}=2a^{[n]}+d^{[n]},\;\; b^{[n+1]}=b^{[n]}+6\beta(a^{[n]}+d^{[n]})\,,\\
&e^{[n+1]}=e^{[n]}+b^{[n]}+\beta(2a^{[n]}+3d^{[n]})\,,\\
&c^{[n+1]}=c^{[n]}+\beta(b^{[n]}+2e^{[n]})+\beta^2 d^{[n]}+\left(\beta^2-4\beta\mathfrak{c}_6+\frac{\mathfrak{c}_5^2}{4}\right)\left(a^{[n]}+d^{[n]}\right)\,.
\end{aligned}$$

By solving this system using the initial conditions $a^{[0]}=a$, $b^{[0]}=b$, $c^{[0]}=c$, $d^{[0]}=d$ and $e^{[0]}=e$, we obtain

$$\begin{aligned}
&a^{[n]}=a,\;\; b^{[n]}=b+6\beta n(an+d),\;\; d^{[n]}=2an+d\,,\\
&e^{[n]}=bn+e+2d\beta n^2+\beta n^2(2an+d)\,,\\
&c^{[n]}=\phi(\beta n^2)+2\beta n\psi(\beta n^2)-n\left(4\beta\mathfrak{c}_6-\frac{\mathfrak{c}_5^2}{4}\right)(an+d)\,,
\end{aligned}$$

for $n=0,1,2,\ldots$. Thus, by a limit process on the previous results, way may infer and then to prove the following result for the quadratic lattice.

Theorem 4.3.4 *Let $x(s)$ be the (quadratic) NUL $x(s)=4\beta s^2+\mathfrak{c}_5 s+\mathfrak{c}_6$. Let $\mathbf{u}\in\mathscr{P}^*\setminus\{\mathbf{0}\}$ and suppose that there exist $(\phi,\psi)\in\mathscr{P}_2\times\mathscr{P}_1\setminus\{(0,0)\}$ such that the functional equation* (4.1) *holds.*

Set $\phi(z) := az^2+bz+c$, $\psi(z) := dz+e$ $(a,b,c,d,e \in \mathbb{C})$. Then $\mathbf{u}$ *is regular if and only if*

$$d_n \neq 0\,, \quad \phi^{[n]}\left(-\beta n^2 - \frac{e_n}{d_{2n}}\right) \neq 0\,, \quad \forall n \in \mathbb{N}_0\,, \tag{4.73}$$

where $d_n := an+d$, $e_n := bn+e+2d\beta n^2$ and

$$\phi^{[n]}(z) = az^2 + (b+6\beta n d_n)z + \phi(\beta n^2) + 2\beta n \psi(\beta n^2) - \frac{n}{4}\left(16\beta \mathfrak{c}_6 - \mathfrak{c}_5^2\right) d_n\,,$$

for $n = 0,1,2,\ldots$.

Moreover the monic OPS $(P_n)_{n\geq 0}$ with respect to $\mathbf{u}$ *satisfies the TTRR* (4.60) *with*

$$B_n = \frac{ne_{n-1}}{d_{2n-2}} - \frac{(n+1)e_n}{d_{2n}} - 2\beta n(n-1)\,, \tag{4.74}$$

$$C_{n+1} = -\frac{(n+1)d_{n-1}}{d_{2n-1}d_{2n+1}}\phi^{[n]}\left(-\beta n^2 - \frac{e_n}{d_{2n}}\right), \;\; n = 0,1,2,\ldots\,. \tag{4.75}$$

In addition, the following Rodrigues-type formula holds

$$P_n\mathbf{u} = k_n \mathbf{D}_x^n \mathbf{u}^{[n]}, \quad k_n := (-1)^n \prod_{j=1}^{n} d_{n+j-2}^{-1} \qquad (n = 0,1,2,\ldots)\,. \tag{4.76}$$

Remark 4.3.1 *More generally, under the regularity conditions* (4.54) *and* (4.73)*, the recurrence coefficients for the TTRR satisfied by the sequence of x-derivatives $(P_n^{[k]})_{n\geq 0}$ are given by the following relations*

$$P_{n+1}^{[k]}(z) = (z - B_n^{[k]})P_n^{[k]}(z) - C_n^{[k]}P_{n-1}^{[k]}(z) \qquad (n,k = 0,1,2,\ldots)\,,$$

with $P_{-1}^{[k]}(z) = 0$, being $B_n^{[k]}$ and $C_{n+1}^{[k]}$ given by

- *for NUL lattices $x(s) = \mathfrak{c}_1 q^{-s} + \mathfrak{c}_2 q^s + \mathfrak{c}_3$ (with the notations of Theorem 4.3.2)*

$$B_n^{[k]} = \mathfrak{c}_3 + \frac{\gamma_n e_{n+k-1}}{d_{2n+2k-2}} - \frac{\gamma_{n+1} e_{n+k}}{d_{2n+2k}}\,,$$

$$C_{n+1}^{[k]} = -\frac{\gamma_{n+1} d_{n+2k-1}}{d_{2n+2k-1}d_{2n+2k+1}}\phi^{[n+k]}\left(\mathfrak{c}_3 - \frac{e_{n+k}}{d_{2n+2k}}\right) \quad (n,k = 0,1,2,\ldots)\,;$$

- *for NUL lattices $x(s) = 4\beta s^2 + \mathfrak{c}_5 s + \mathfrak{c}_6$ (with the notations of Theorem 4.3.4)*

$$B_n^{[k]} = \frac{ne_{n+k-1}}{d_{2n+2k-2}} - \frac{(n+1)e_{n+k}}{d_{2n+2k}} - 2\beta\left((n+k)^2 - n - \frac{1}{2}k^2\right)\,,$$

$$C_{n+1}^{[k]} = -\frac{(n+1)d_{n+2k-1}}{d_{2n+2k-1}d_{2n+2k+1}}\phi^{[n+k]}\left(-\beta(n+k)^2 - \frac{e_{n+k}}{d_{2n+2k}}\right) \quad (n,k = 0,1,2,\ldots)\,.$$

These expressions are obtained using ideas presented in Chapter 2, Section 2.2.2.

4.4 Applications

4.4.1 The very classical OPS

Consider the particular quadratic lattice $x(s)=\mathfrak{c}_6$. Thus $\beta=\mathfrak{c}_5=0$. From Theorem 4.3.4 we recover [38, Lemma 2 and Theorem 2] stated in Theorem 2.1.1 for the very classical OPS.

4.4.2 The Racah polynomials

Consider the quadratic lattice $x(s)=s(s+a+b+1)$. Let $\mathbf{u}\in\mathscr{P}^*$ satisfying (4.1) with

$$\begin{aligned}\phi(z)&=2z^2+[(a+b+2c+3)d+c(a-b+3)+2(a+b+ab+2)]z\\&\quad+(1+a)(1+d)(a+b+1)(b+c+1),\\ \psi(z)&=2(d+c+2)z+2(1+a)(1+d)(b+c+1)\,.\end{aligned}$$

Here $a,b,c,d\in\mathbb{C}$. The regularity conditions for $\mathbf{u}$ given by (4.73) read as

$$(n+a+1)(n+c+1)(n+d+1)(n+d+c)(n+b+c+1)(n+c+d-a+1)(n+d-b+1)\neq 0$$

for each $n=0,1,2,\ldots$. Let $(P_n)_{n\geq 0}$ be the monic OPS with respect to $\mathbf{u}$. Using Theorem 4.3.4, we see that $(P_n)_{n\geq 0}$ satisfies the TTRR (4.60). Hence applying (4.74)–(4.75), we obtain

$$\begin{aligned}B_n&=-\frac{(n+a+1)(n+d+1)(n+b+c+1)(n+d+c+1)}{(2n+d+c+1)(2n+d+c+2)}-\frac{n(n+c)(n+d+c-a)(n+d-b)}{(2n+d+c)(2n+d+c+1)},\\ C_{n+1}&=\frac{(n+1)(n+a+1)(n+c+1)(n+d+1)(n+d+c+1)(n+b+c+1)(n+c+d-a+1)(n+d-b+1)}{(2n+d+c+1)(2n+d+c+2)^2(2n+d+c+3)}\end{aligned}$$

for each $n=0,1,2,\ldots$. Therefore,

$$P_n(z)=R_n(z;d,c,a,b),\quad n=0,1,2,\ldots\,,$$

where $(R_n(.;d,c,a,b))_{n\geq 0}$ is the monic OPS of the Racah polynomial (see [34, p.190]).

4.4.3 The Askey-Wilson polynomials

Consider the q-quadratic lattice $x(s)=\mathfrak{c}_1q^{-s}+\mathfrak{c}_2q^s+\mathfrak{c}_3$ ($q>0$; $q\neq 1$). Let $\mathbf{u}$ be a linear functional on $\mathscr{P}$ satisfying (4.1), where ϕ and ψ are given by

$$\begin{aligned}\phi(z)&=2(1+abcd)(z-\mathfrak{c}_3)^2-2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(a+b+c+d+abc+abd+acd+bcd)(z-\mathfrak{c}_3)\\&\quad+4(ab+ac+ad+bc+bd+cd-abcd-1)\mathfrak{c}_1\mathfrak{c}_2\,,\end{aligned}$$

$$\psi(z)=\frac{4q^{1/2}}{q-1}\Big((abcd-1)(z-\mathfrak{c}_3)+\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(a+b+c+d-abc-abd-acd-bcd)\Big),$$

where $a,b,c,d\in\mathbb{C}$ (with $a\neq 0$). Acccording to Theorem 4.3.1, $\mathbf{u}$ is regular if and only if

$$(1-abcdq^n)(1-abq^n)(1-acq^n)(1-adq^n)(1-bcq^n)(1-bdq^n)(1-cdq^n)\mathfrak{c}_1\mathfrak{c}_2\neq 0$$

for each $n = 0, 1, 2, \ldots$. Assuming that this conditions hold, by Theorem 4.3.2 the corresponding monic OPS $(P_n)_{n\geq 0}$ satisfies the TTRR (4.60) , where (by (4.61)–(4.62)),

$$B_n = \mathfrak{c}_3 + 2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}\left[a + \frac{1}{a} - \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})} - \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-1})(1-abcdq^{2n-2})}\right]$$

(if $a = 0$, we define B_n by continuity, taking the limit as $a \to 0$ in the preceding expression), and

$$C_{n+1} = \frac{\mathfrak{c}_1\mathfrak{c}_2(1-abq^n)(1-acq^n)(1-adq^n)(1-bcq^n)(1-bdq^n)(1-cdq^n)(1-q^{n+1})(1-abcdq^{n-1})}{(1-abcdq^{2n-1})(1-abcdq^{2n})^2(1-abcdq^{2n+1})},$$

for $n = 0, 1, 2, \ldots$. Hence

$$P_n(z) = 2^n(\mathfrak{c}_1\mathfrak{c}_2)^{n/2} Q_n\left(\frac{z-\mathfrak{c}_3}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}; a, b, c, d|q\right), \quad n = 0, 1, 2, \ldots,$$

where $(Q_n(.; a, b, c, d|q))_{n\geq 0}$ is the monic OPS of the Askey-Wilson polynomials (see [34, (14.1.5)]).

Chapter 5

On a characterization of continuous $q-$Jacobi and Al-Salam Chihara polynomials

The purpose of this chapter is to give a positive answer to a conjecture posed by M. E. H. Ismail concerning a characterization of the continuous $q-$Jacobi and Al-Salam Chihara polynomials (see [26, Conjecture 24.7.8]). The proof makes use of some results stated in the previous chapters.

5.1 The conjecture

Let π be a nonzero polynomial of degree at most 2 and consider three sequences of numbers $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, and $(c_n)_{n\geq 0}$. Al-Salam and Chihara [3] proved that the only OPS, say $(P_n)_{n\geq 0}$, that satisfy

$$\pi(x)\,DP_n(x) = (a_n x + b_n)P_n(x) + c_n P_{n-1}(x)\ , \tag{5.1}$$

are those of Hermite, Laguerre, Jacobi, and Bessel (here D denotes the standard derivative with respect to x). Consider now (5.1) with D replaced by the Askey–Wilson operator,

$$(\mathscr{D}_q f)(x) := \frac{\breve{f}\big(q^{1/2}z\big) - \breve{f}\big(q^{-1/2}z\big)}{\breve{e}\big(q^{1/2}z\big) - \breve{e}\big(q^{-1/2}z\big)} \quad (z = e^{i\theta})\ , \tag{5.2}$$

where $\breve{f}(z) := f\big((z+1/z)/2\big) = f(\cos\theta)$ for each polynomial f and $e(x) := x$. Here $0<q<1$ and θ is not necessarily a real number (see [26, p. 300]).

Conjecture 5.1.1 [26, Conjecture 24.7.8] *Let $(P_n)_{n\geq 0}$ be a monic OPS and π be a polynomial of degree at most 2 which does not depend on n. If $(P_n)_{n\geq 0}$ satisfies*

$$\pi(z)\mathscr{D}_q P_n(z) = (a_n z + b_n)P_n(z) + c_n P_{n-1}(z)\ , \tag{5.3}$$

then $(P_n)_{n\geq 0}$ are continuous $q-$Jacobi polynomials, Al-Salam-Chihara polynomials, or special or limiting cases of them.

M. E. H. Ismail himself proved that the continuous $q-$Jacobi polynomials indeed satisfy (5.3) for suitable polynomial π and parameters a_n, b_n, and c_n (cf. [26, Theorem 15.5.2]). On another hand, Al-Salam [2] proved that the conjecture is true whenever $\pi(x)\equiv 1$, by characterizing the Rogers $q-$Hermite polynomials, $P_n(x):=H_n(x|q)$, as the only OPS that fulfill $\mathscr{D}_q P_n = c_n P_{n-1}$ for every $n=1,2,\ldots$.

We recall that the monic continuous $q-$Jacobi polynomials, $\widehat{P}_n^{(a,b)}(x|q)$, depend on two complex parameters a and b, and they are characterized by the TTRR

$$\begin{aligned} x\widehat{P}_n^{(a,b)}(x|q) &= \widehat{P}_{n+1}^{(a,b)}(x|q)+\tfrac{1}{2}\left(q^{(2a+1)/4}+q^{-(2a+1)/4}-y_n(a,b)-z_n(a,b)\right)\widehat{P}_n^{(a,b)}(x|q) \\ &\quad +\tfrac{1}{4}y_{n-1}(a,b)z_n(a,b)\,\widehat{P}_{n-1}^{(a,b)}(x|q) \end{aligned} \tag{5.4}$$

$(n=0,1,\ldots)$, being

$$y_n(a,b):=\frac{(1-q^{n+a+1})(1-q^{n+a+b+1})(1+q^{n+(a+b+1)/2})(1+q^{n+(a+b+2)/2})}{q^{(2a+1)/4}(1-q^{2n+a+b+1})(1-q^{2n+a+b+2})}\,, \tag{5.5}$$

$$z_n(a,b):=\frac{q^{(2a+1)/4}(1-q^n)(1-q^{n+b})(1+q^{n+(a+b)/2})(1+q^{n+(a+b+1)/2})}{(1-q^{2n+a+b})(1-q^{2n+a+b+1})}\,, \tag{5.6}$$

and subject to the restrictions $(1-q^{n+a})(1-q^{n+b})(1-q^{n+a+b})\neq 0$ for each $n=0,1,2,\ldots$, while the monic Al-Salam-Chihara polynomials, $Q_n(x;c,d|q)$, which also depend on two complex parameters c and d, are characterized by

$$\begin{aligned} xQ_n(x;c,d|q) &= Q_{n+1}(x;c,d|q)+\tfrac{1}{2}\,(c+d)q^n\,Q_n(x;c,d|q) \\ &\quad +\tfrac{1}{4}\,(1-cdq^{n-1})(1-q^n)\,Q_{n-1}(x;c,d|q) \end{aligned} \tag{5.7}$$

$(n=0,1,\ldots)$, provided we define $\widehat{P}_{-1}^{(a,b)}(x|q)=Q_{-1}(x;c,d|q)=0$ (see e.g. [26]). Further, up to normalization, the Rogers $q-$Hermite polynomials are the special case $c=d=0$ of the Al-Salam-Chihara polynomials.

Taking $e^{i\theta}=q^s$, $\mathscr{D}_q$ reads

$$\mathscr{D}_q f(x(s))=\frac{f\big(x(s+\frac{1}{2})\big)-f\big(x(s-\frac{1}{2})\big)}{x(s+\frac{1}{2})-x(s-\frac{1}{2})}\,,\quad x(s)=\tfrac{1}{2}(q^{-s}+q^s)\,. \tag{5.8}$$

Recently, Kenfack-Nangho and Jordaan [33] used (5.8) to answer another conjecture posed by M. E. H. Ismail [26, Conjecture 24.7.9] concerning a characterization of the Askey–Wilson polynomials. We also mention the related work [30] by the same authors, where the Bochner-type equation is used.

Throughout this chapter we denote by $x(s)$ the q-quadratic lattice defined by

$$x(s)=\mathfrak{c}_1 q^{-s}+\mathfrak{c}_2 q^s+\mathfrak{c}_3\,, \tag{5.9}$$

where $q\in(0,+\infty)\setminus\{1\}$ and $\mathfrak{c}_1$, $\mathfrak{c}_2$, and $\mathfrak{c}_3$ are real constants so that $(\mathfrak{c}_1,\mathfrak{c}_2)\neq(0,0)$. We will prove Conjecture 5.1.1 for the general operator D_x (note that $\mathscr{D}_q$ is obtained from D_x by taking $\mathfrak{c}_3=0$ and

$\mathfrak{c}_1 = \mathfrak{c}_2 = 1/2$). Henceforth, we assume that there exists a monic OPS $(P_n)_{n\geq 0}$ satisfying

$$\pi(x)D_xP_n(z) = (a_nz+b_n)P_n(z)+c_nP_{n-1}(z) \quad (n=0,1,2,\ldots)\ , \tag{5.10}$$

where π is a nonzero polynomial of degree at most 2 and $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, and $(c_n)_{n\geq 0}$ are sequences of complex numbers such that $c_n \neq 0$ for all $n = 1,2,3,\ldots$. Our aim is to determine all such OPS $(P_n)_{n\geq 0}$. In this chapter B_n and C_n denote the coefficients of the TTRR fulfilled by $(P_n)_{n\geq 0}$, so that

$$zP_n(z) = P_{n+1}(z)+B_nP_n(z)+C_nP_{n-1}(z) \quad (n=0,1,2,\ldots)\ , \tag{5.11}$$

with $P_{-1}(z) = 0$, being $B_n \in \mathbb{C}$ and $C_{n+1} \neq 0$ for each $n = 0,1,2,\ldots$.

5.2 Preliminary results

We start by showing that all monic OPS satisfying (5.10) are x-classical and then we prove that the coefficients of the associated TTRR satisfy a system of non linear equations. This system will be solved (in the next section) considering three cases, according with the degree of the polynomial π.

Theorem 5.2.1 *Let $\mathbf{u} \in \mathscr{P}^*$ be a regular functional such that its corresponding monic OPS $(P_n)_{n\geq 0}$ satisfies* (5.10) *subject to the condition $c_n \neq 0$, for each $n = 1,2,\ldots$. Then $\mathbf{u}$ is x-classical. Moreover $D_x(\phi\mathbf{u}) = S_x(\psi\mathbf{u})$, with ψ and ϕ the polynomials given by*

$$\psi(z) = z-B_0, \quad \phi(z) = (\mathfrak{a}z-\mathfrak{b})(z-B_0)-(\mathfrak{a}+\alpha)C_1\ , \tag{5.12}$$

where

$$\mathfrak{a} := \frac{(a_2C_2+c_2)C_1}{(a_1C_1+c_1)C_2}-\alpha, \quad \mathfrak{b} := \beta-B_0+(\mathfrak{a}+\alpha)B_1-\frac{b_1+a_1B_1}{c_1+a_1C_1}C_1\ . \tag{5.13}$$

(Here, B_0, B_1, C_1, and C_2 are coefficients of the TTRR (5.11) *satisfied by $(P_n)_{n\geq 0}$, and c_1, c_2, b_1, a_1, and a_2 are coefficients appearing in the structure relation* (5.10).*)*

Proof Let $(\mathbf{a}_n)_{n\geq 0}$ be the dual basis associated to the monic OPS $(P_n)_{n\geq 0}$. We claim that

$$\mathbf{D}_x(\pi\mathbf{u}) = R_1\mathbf{u}, \quad R_1(z) := -\frac{a_1C_1+c_1}{C_1}(z-B_0),\ \text{ with }\ a_1C_1+c_1 \neq 0\ . \tag{5.14}$$

Indeed let $j \in \mathbb{N}_0$. Using (5.10) and (5.11), we deduce

$$\left\langle \mathbf{D}_x(\pi\mathbf{a}_0),P_j\right\rangle = -\left\langle \mathbf{a}_0,\pi D_xP_j\right\rangle = -a_j\delta_{0,j+1}-(a_jB_j+b_j)\delta_{0,j}-(c_j+a_jC_j)\delta_{1,j}\ .$$

Taking $n=0$ in (5.10), we find $a_0 = b_0 = 0$, and since $\left\langle \mathbf{u},P_n^2\right\rangle \mathbf{a}_n = P_n\mathbf{u}$ and $C_{n+1} = \left\langle \mathbf{u},P_{n+1}^2\right\rangle / \left\langle \mathbf{u},P_n^2\right\rangle$, we obtain

$$\mathbf{D}_x(\pi\mathbf{a}_0) = \sum_{j=0}^{\infty} \left\langle \mathbf{D}_x(\pi\mathbf{a}_0),P_j\right\rangle \mathbf{a}_j = -(c_1+a_1C_1)\mathbf{a}_1\ .$$

If $c_1+a_1C_1 = 0$, then $\mathbf{D}_x(\pi\mathbf{u}) = 0$, hence $0 = \langle\mathbf{D}_x(\pi\mathbf{u}),f\rangle = -\langle\pi\mathbf{u},D_xf\rangle$, $\forall f \in \mathscr{P}$. This implies $\pi\mathbf{u} = \mathbf{0}$. But this is impossible, since $\pi \neq 0$ and $\mathbf{u}$ is regular. So $c_1+a_1C_1 \neq 0$. Hence (5.14) holds.

Applying D_x to both sides of (5.11), and using (1.53), yields

$$S_xP_n(z) = -(\alpha z+\beta)D_xP_n(z)+D_xP_{n+1}(z)+B_nD_xP_n(z)+C_nD_xP_{n-1}(z)\ .$$

Multiplying both sides of this equality by $\pi(z)$ and using (5.10) and (5.11), we obtain

$$\pi(z)S_xP_n(z) = r_n^{[1]}P_{n+2}(z)+r_n^{[2]}P_{n+1}(z)+r_n^{[3]}P_n(z)+r_n^{[4]}P_{n-1}(z)+r_n^{[5]}P_{n-2}(z) \tag{5.15}$$

for each $n = 0,1,2,\ldots$, where

$$\begin{aligned}
&r_n^{[1]} := a_{n+1}-\alpha a_n\ , \quad r_n^{[2]} := g_{n+1}-\alpha g_n+a_n(B_n-\alpha B_{n+1}-\beta)\ ,\\
&r_n^{[3]} := s_{n+1}-\alpha s_n+g_n((1-\alpha)B_n-\beta)+a_{n-1}C_n-\alpha a_nC_{n+1}\ ,\\
&r_n^{[4]} := (g_{n-1}-\alpha g_n)C_n+s_n(B_n-\beta-\alpha B_{n-1})\ , \quad r_n^{[5]} := C_ns_{n-1}-\alpha C_{n-1}s_n\ ,
\end{aligned}$$

and $g_n = b_n+a_nB_n$, $s_n = c_n+a_nC_n$. For a fixed $j\in\mathbb{N}_0$, using (5.15) we obtain

$$\left\langle \mathbf{S}_x(\pi\mathbf{a}_0),P_j\right\rangle = \left\langle \mathbf{a}_0,\pi S_xP_j\right\rangle = r_j^{[1]}\delta_{0,j+2}+r_j^{[2]}\delta_{0,j+1}+r_j^{[3]}\delta_{0,j}+r_j^{[4]}\delta_{0,j-1}+r_j^{[5]}\delta_{0,j-2}\ .$$

Therefore,

$$\mathbf{S}_x(\pi\mathbf{a}_0) = \sum_{j=0}^{\infty}\left\langle \mathbf{S}_x(\pi\mathbf{a}_0),P_j\right\rangle \mathbf{a}_j = r_0^{[3]}\mathbf{a}_0+r_1^{[4]}\mathbf{a}_1+r_2^{[5]}\mathbf{a}_2\ ,$$

and so

$$\mathbf{S}_x(\pi\mathbf{u}) = R_2\mathbf{u}, \quad R_2(z) := r_0^{[3]}+\frac{r_1^{[4]}}{C_1}P_1(z)+\frac{r_2^{[5]}}{C_2C_1}P_2(z)\ . \tag{5.16}$$

Next, on the first hand, applying successively (5.16), (1.72) and (5.14), we obtain

$$\begin{aligned}
\mathbf{D}_x(R_2\mathbf{u}) = \mathbf{D}_x\mathbf{S}_x(\pi\mathbf{u}) &= \frac{2\alpha^2-1}{\alpha}\mathbf{S}_x\mathbf{D}_x(\pi\mathbf{u})+\frac{\mathtt{U}_1}{\alpha}\mathbf{D}_x^2(\pi\mathbf{u})\\
&= \frac{2\alpha^2-1}{\alpha}\mathbf{S}_x(R_1\mathbf{u})+\frac{\mathtt{U}_1}{\alpha}\mathbf{D}_x(R_1\mathbf{u})\ .
\end{aligned} \tag{5.17}$$

On the other hand, using (1.59) with $f = \mathtt{U}_1$, we obtain

$$\mathbf{D}_x(\mathtt{U}_1R_1\mathbf{u}) = \left(S_x\mathtt{U}_1-\frac{\mathtt{U}_1}{\alpha}D_x\mathtt{U}_1\right)\mathbf{D}_x(R_1\mathbf{u})+\alpha^{-1}D_x\mathtt{U}_1\mathbf{S}_x(R_1\mathbf{u}) = \frac{\mathtt{U}_1}{\alpha}\mathbf{D}_x(R_1\mathbf{u})+\frac{\alpha^2-1}{\alpha}\mathbf{S}_x(R_1\mathbf{u})\ .$$

Hence the following relation holds:

$$\frac{\mathtt{U}_1}{\alpha}\mathbf{D}_x(R_1\mathbf{u}) = \mathbf{D}_x(\mathtt{U}_1R_1\mathbf{u})-\frac{\alpha^2-1}{\alpha}\mathbf{S}_x(R_1\mathbf{u})\ . \tag{5.18}$$

Thus, combining (5.17) and (5.18), we obtain

$$\mathbf{D}_x\Big((R_2-\mathtt{U}_1R_1)\mathbf{u}\Big) = \mathbf{S}_x(\alpha R_1\mathbf{u})\ .$$

This leads us to define

$$\psi(x) := z - B_0, \quad \phi(z) := -\frac{C_1}{\alpha(c_1 + a_1 C_1)}\Big(R_2(z) - \mathtt{U}_1(z) R_1(z)\Big) .$$

Clearly, $\deg\psi = 1$, $\deg\phi \leq 2$ and $\mathbf{D}_x(\phi\mathbf{u}) = \mathbf{S}_x(\psi\mathbf{u})$. Finally, since $a_0 = 0 = b_0$ (this is achieved by taking $n = 0$ in (5.10)), and setting (without lost of generality) $c_0 := 0$ and $C_0 := 0$, we have

$$\mathfrak{a} := \frac{1}{2}\phi''(0) = -\frac{C_1}{\alpha(c_1 + a_1 C_1)}\left(\frac{r_2^{[5]}}{C_1 C_2} + (\alpha^2 - 1)\frac{c_1 + a_1 C_1}{C_1}\right) = \frac{(c_2 + a_2 C_2)C_1}{(c_1 + a_1 C_1)C_2} - \alpha .$$

Similarly, we also have

$$\phi'(0) = -\mathfrak{a}B_0 - \beta + B_0 - (\mathfrak{a} + \alpha)B_1 + \frac{b_1 + a_1 B_1}{c_1 + a_1 C_1}C_1 ,$$
$$\phi(0) = -(\mathfrak{a} + \alpha)C_1 - B_0\left(-\beta + B_0 - (\mathfrak{a} + \alpha)B_1 + \frac{b_1 + a_1 B_1}{c_1 + a_1 C_1}C_1\right) .$$

Hence the desired result is proved.

Theorem 5.2.2 *Let $(P_n)_{n\geq 0}$ be a monic OPS satisfying* (5.10). *Then the coefficients B_n and C_n of the TTRR* (5.11) *satisfied by $(P_n)_{n\geq 0}$ fulfill the following system of difference equations:*

$$a_{n+2} - 2\alpha a_{n+1} + a_n = 0 , \tag{5.19}$$

$$t_{n+2} - 2\alpha t_{n+1} + t_n = 0 , \quad t_n := c_n/C_n = k_1 q^{n/2} + k_2 q^{-n/2} , \tag{5.20}$$

$$r_{n+3}(B_{n+2} - \mathfrak{c}_3) - (r_{n+2} + r_{n+1})(B_{n+1} - \mathfrak{c}_3) + r_n(B_n - \mathfrak{c}_3) = 0 , \quad r_n := t_n + a_n - a_{n-1} , \tag{5.21}$$

$$\begin{aligned}&(r_{n+1} + r_{n+2})(C_{n+1} - \mathfrak{c}_1\mathfrak{c}_2) - 2(1+\alpha)r_n(C_n - \mathfrak{c}_1\mathfrak{c}_2) + (r_{n-1} + r_{n-2})(C_{n-1} - \mathfrak{c}_1\mathfrak{c}_2)\\ &\quad = r_n\left[(B_n - \mathfrak{c}_3)^2 - 2\alpha(B_n - \mathfrak{c}_3)(B_{n-1} - \mathfrak{c}_3) + (B_{n-1} - \mathfrak{c}_3)^2\right] ,\end{aligned} \tag{5.22}$$

$$\begin{aligned}&\Big(2(1-\alpha)(a_n B_n + b_n) - 4\beta a_n\Big)B_n^2 + (t_{n+1} + a_{n+1} - a_{n+2})B_{n+1}C_{n+1} + (t_n + a_{n-1} - a_{n-2})B_{n-1}C_n\\ &\quad + \Big[(2a_n - a_{n+2} - a_{n-1})C_{n+1} + (2a_n - a_{n+1} - a_{n-2})C_n + (1-2\alpha)(c_n + c_{n+1}) - 4\beta b_n\\ &\quad + (\beta^2 - \delta)a_n\Big]B_n + 2\Big(b_n - \alpha b_{n+1} - \beta(a_n + a_{n+1} + t_{n+1})\Big)C_{n+1}\\ &\quad + 2\Big(b_n - \alpha b_{n-1} - \beta(a_{n-1} + a_n + t_n)\Big)C_n = b_n(\delta - \beta^2) ,\end{aligned} \tag{5.23}$$

where $\beta = (1-\alpha)\mathfrak{c}_3$ and $\delta = (\alpha^2 - 1)(\mathfrak{c}_3^2 - 4\mathfrak{c}_1\mathfrak{c}_2)$. Equation (5.19) *holds for $n = 0, 1, 2, \ldots$;* (5.22) *holds for $n = 2, 3, 4, \ldots$; and* (5.20), (5.21)*, and* (5.23) *hold for $n = 1, 2, 3, \ldots$.*

In addition, the following relations hold for $n = 0, 1, 2, \ldots$*:*

$$c_n = \gamma_n, \quad B_n = \mathfrak{c}_3 + (B_0 - \mathfrak{c}_3)(\gamma_{n+1} - \gamma_n), \qquad \text{if } \deg\pi = 0\,; \tag{5.24}$$

$$b_n = \gamma_n, \quad c_n = (b_n - b_{n-1}) \sum_{j=0}^{n-1} (B_j - \mathfrak{c}_3) + \pi(\mathfrak{c}_3) b_n, \qquad \text{if } \deg\pi = 1\,; \tag{5.25}$$

$$a_n = \gamma_n, \quad b_n = (a_n - a_{n-1}) \sum_{j=0}^{n-1} (B_j - \mathfrak{c}_3) + \pi'(\mathfrak{c}_3/2) a_n, \qquad \text{if } \deg\pi = 2\,. \tag{5.26}$$

Proof Applying the operator S_x to both sides of (5.11) and using (1.54), we deduce

$$\mathtt{U}_2(z) D_x P_n(z) + (\alpha z + \beta) S_x P_n(z) = S_x P_{n+1}(z) + B_n S_x P_n(z) + C_n S_x P_{n-1}(z)\,.$$

Multiplying both sides of this equality by $\pi(z)$ and then using successively (1.50), (5.10), (5.15), and (5.11), we obtain a vanishing linear combination of the polynomials P_{n+3}, P_{n+2},..., P_{n-3}. Thus, setting

$$t_n := c_n / C_n, \quad n = 1, 2, 3, \ldots\,, \tag{5.27}$$

after straightforward computations we obtain (5.19)–(5.20) together with the following equations:

$$\begin{aligned}&(a_{n+1}-a_{n+2})B_{n+1}+(a_n-a_{n-1})B_n+b_{n+2}-2\alpha b_{n+1}+b_n\\&\quad=2\beta(a_n+a_{n+1})\,,\end{aligned}\tag{5.28}$$

$$\begin{aligned}&(a_{n+1}-a_{n+2}-t_{n+2})B_{n+1}+(a_n-a_{n-1}+t_{n+1}+t_n)B_n\\&\quad-t_{n-1}B_{n-1}+b_{n+1}-2\alpha b_n+b_{n-1}=2\beta(t_{n+1}+t_n+a_{n+1}+a_n)\,,\end{aligned}\tag{5.29}$$

$$\begin{aligned}&(a_{n+1}-a_{n+2})B_{n+1}^2+2(1-\alpha)a_nB_n^2+(a_n-a_{n-1})B_nB_{n+1}\\&\quad+(b_{n+1}+b_n-2\alpha b_{n+1}-2\beta a_n-2\beta a_{n+1})B_{n+1}\\&\quad+(b_{n+1}+b_n-2\alpha b_n-4\beta a_n)B_n+(a_n-a_{n+2})C_{n+1}+(a_n-a_{n-2})C_n\\&\quad+c_{n+2}-2\alpha c_{n+1}+c_n=a_n(\delta-\beta^2)+2\beta(b_n+b_{n+1})\,,\end{aligned}\tag{5.30}$$

$$\begin{aligned}&[2(1-\alpha)a_n+t_n]B_n^2+(t_n+a_{n-1}-a_{n-2})B_{n-1}^2\\&\quad+(a_n-t_{n-1}-t_{n+1}-a_{n+1})B_nB_{n-1}\\&\quad+(b_n+b_{n-1}-2\alpha b_n-2\beta t_n-4\beta a_n)B_n\\&\quad+(b_{n-1}+b_n-2\alpha b_{n-1}-2\beta a_{n-1}-2\beta t_n-2\beta a_n)B_{n-1}\\&\quad+(a_n-a_{n+2}-t_{n+2}-t_{n+1})C_{n+1}+[2(1+\alpha)t_n+a_n-a_{n-2}]\,C_n\\&\quad-(t_{n-2}+t_{n-1})C_{n-1}+c_{n+1}-2\alpha c_n+c_{n-1}\\&\qquad=(t_n+a_n)(\delta-\beta^2)+2\beta(b_n+b_{n-1})\,,\end{aligned}\tag{5.31}$$

$$\begin{aligned}&2(1-\alpha)a_nB_n^3+[2(1-\alpha)b_n-4\beta a_n]B_n^2\\&\quad+\big[(2a_n-a_{n+2}-a_{n-1})C_{n+1}+(2a_n-a_{n+1}-a_{n-2})C_n\\&\qquad+c_{n+1}-2\alpha c_n+c_n-2\alpha c_{n+1}-4\beta b_n+\beta^2a_n-\delta a_n\big]\,B_n\\&\quad+(c_{n+1}+a_{n+1}C_{n+1}-a_{n+2}C_{n+1})B_{n+1}\\&\quad+(c_n+a_{n-1}C_n-a_{n-2}C_n)B_{n-1}\\&\quad+2(b_n-\alpha b_{n+1}-\beta a_{n+1}-\beta a_n)C_{n+1}\\&\quad+2(b_n-\alpha b_{n-1}-\beta a_{n-1}-\beta a_n)C_n=2\beta(c_n+c_{n+1})+b_n(\delta-\beta^2)\,.\end{aligned}\tag{5.32}$$

(5.21) (respectively (5.22)) is obtained by shifting n to $n+1$ in (5.29) (respectively (5.31))) and combine it with (5.28) (respectively (5.30)) and by using (5.19)–(5.20). (5.23) is obtained by using (5.19), (5.20) and (5.28). Now suppose that $deg\pi=2$. Using (5.11), we may write

$$P_n(z)=z^n-z^{n-1}\sum_{j=0}^{n-1}B_j+w_nz^{n-2}+\cdots\,,$$

for some complex sequence $(w_n)_{n\geq 0}$. Using (1.65), we compare the two first coefficients of higher power of n in both side of (5.10) to deduce (5.26). Equations (5.24)–(5.25) are obtained in a similar way. This completes the proof.

Remark 5.2.1 *According to Theorem 5.2.2, the coefficients B_n and C_n of the TTRR* (5.11) *of any monic OPS $(P_n)_{n\geq 0}$ fulfilling* (5.10) *must fulfill* (5.19)–(5.23)*. However, for each concrete polynomial π appearing in* (5.10)*, we need to take into account some initial conditions which will be specified in the proof of the conjecture in all situations according to the degree of π. Indeed, for instance, it is* *clear that*

$$B_n = \mathfrak{c}_3, \qquad C_{n+1} = \mathfrak{c}_1\mathfrak{c}_2 \quad (n = 0,1,2,\ldots) ,$$

provide a solution of the system (5.19)–(5.23) *if* $deg\pi = 0$. *The corresponding monic OPS is*

$$P_n(x) = 2^n(\mathfrak{c}_1\mathfrak{c}_2)^{n/2}\widehat{U}_n\left(\frac{z-\mathfrak{c}_3}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}\right) \quad (n = 0,1,2,\ldots) ,$$

where $(\widehat{U}_n)_{n\geq 0}$ *is the monic Chebyschev polynomials of the second kind. However this sequence* $(P_n)_{n\geq 0}$ *does not provide a solution of* (5.10) *(see* (5.38) *below).*

The system of equations (5.19)–(5.23) is non-linear and so, in general it is not easy to solve it. Nevertheless, in view of Theorem 5.2.1, an OPS satisfying (5.10) is x-classical and so the results presented in the previous chapter will be useful to find the explicit expressions for the coefficients of the TTRR satisfied by the OPS under analysis (see Theorem 4.3.2 and Corollary 4.3.3). We will see that some patterns appear associated with the system of equations (5.19)–(5.23) which will allow us to solve the system for each possible case of the degree of the polynomial π.

Recall that from (5.20), we have

$$t_n = \frac{c_n}{C_n} = k_1 q^{n/2} + k_2 q^{-n/2} \quad (n = 1,2,3,\ldots) , \tag{5.33}$$

where k_1 and k_2 are two complex numbers. Since $c_n \neq 0$, for $n = 1,2,3,\ldots$, then k_1 and k_2 cannot vanish simultaneously. Recall also that we defined $c_0 = C_0 = 0$, and so we define

$$t_0 := k_1 + k_2 ,$$

by compatibility with (5.33).

5.3 Proof of the conjecture

In this section, we prove that the only monic OPS $(P_n)_{n\geq 0}$ satisfying (5.10), where $deg\pi \leq 2$ and subject to the condition $c_n \neq 0$, for $n = 1,2,3,\ldots$, are, up to an affine transformation of the variable, the continuous $q-$Jacobi polynomials and some special cases of the Al-Salam-Chihara polynomials. The proof will be done by considering separately the cases $deg\pi = 0$, $deg\pi = 1$ and $deg\pi = 2$.

5.3.1 Case $deg\pi = 0$

For this case (5.10) becomes

$$D_x P_n(z) = c_n P_{n-1}(z) \quad (n = 0,1,2,\ldots) . \tag{5.34}$$

As we mentioned at the introduction of this chapter, this case was solved by Al-Salam in [2] for the case where the lattice is given by $x(s) = (q^{-s} + q^s)/2$ (i.e where $D_x \equiv \mathscr{D}_q$). Here we present a different proof without any specialisation on the lattice. Indeed, we are going to use the results presented in the previous chapter. The following proposition holds.

Proposition 5.3.1 *Up to an affine transformation of the variable, the only monic OPS* $(P_n)_{n\geq 0}$ *satisfying* (5.34) *are the monic q-Hermite polynomials of Rogers.*

Proof Let $(P_n)_{n\geq 0}$ be a monic OPS satisfying (5.34). Since (5.34) is obtained from (5.10) by taking $\pi(z)=1$, then $a_n=0=b_n$, for $n=0,1,2,\ldots$, and so, equation (5.21) reads as

$$t_{n+3}(B_{n+2}-\mathfrak{c}_3)-(t_{n+2}+t_{n+1})(B_{n+1}-\mathfrak{c}_3)+t_n(B_n-\mathfrak{c}_3)=0 \quad (n=0,1,2,\ldots)\,, \tag{5.35}$$

where, taking into account (5.20) and (5.24),

$$t_n=\frac{\gamma_n}{C_n}=k_1q^{n/2}+k_2q^{-n/2},\quad k_1=\frac{(1+q)C_1-C_2}{q^{1/2}(q-1)C_1C_2},\quad k_2=\frac{(q^{-1}+1)C_1-C_2}{q^{-1/2}(q^{-1}-1)C_1C_2}\,. \tag{5.36}$$

Again from (5.24) we have $B_2-\mathfrak{c}_3=(4\alpha^2-2\alpha-1)(B_0-\mathfrak{c}_3)$ and $B_1-\mathfrak{c}_3=(2\alpha-1)(B_0-\mathfrak{c}_3)$. This satisfies (5.35) for $n=0$ if and only if $B_0=\mathfrak{c}_3$. This equivalence is straightforward taking into account that $t_n\neq 0$ for all n, and (5.20) holds. Hence (5.24) reduces to

$$B_n=\mathfrak{c}_3 \quad (n=0,1,2,\ldots)\,. \tag{5.37}$$

In addition, from (4.61) in Theorem 4.3.2 we obtain

$$\frac{\gamma_{n+1}e_n}{d_{2n}}=\frac{\gamma_n e_{n-1}}{d_{2n-2}} \quad (n=0,1,2,\ldots)\,.$$

Since $e_n=\phi'(\mathfrak{c}_3)\gamma_n+\psi(\mathfrak{c}_3)\alpha_n$, we find $e_0=0$ (because $\gamma_0=0$, and from (5.12), $\psi(\mathfrak{c}_3)=\mathfrak{c}_3-B_0=0$) and so $e_n=0$, for $n=0,1,2,\ldots$, and consequently $\phi'(\mathfrak{c}_3)=0$. Then, from (5.12) we obtain $\mathfrak{b}=\mathfrak{a}\mathfrak{c}_3$. Taking $n=3$ in (5.34) and using (1.65)–(1.68), we obtain

$$C_2=2(2\alpha^2-1)(C_1-\mathfrak{c}_1\mathfrak{c}_2)+2\mathfrak{c}_1\mathfrak{c}_2\,. \tag{5.38}$$

Therefore, using (5.12), we have $\phi(z)=\mathfrak{a}(z-\mathfrak{c}_3)^2-(\mathfrak{a}+\alpha)C_1$ and $\psi(z)=z-\mathfrak{c}_3$, where

$$\mathfrak{a}=\frac{2\alpha C_1}{C_2}-\alpha=\frac{2\alpha(1-\alpha^2)(C_1-\mathfrak{c}_1\mathfrak{c}_2)}{(2\alpha^2-1)(C_1-\mathfrak{c}_1\mathfrak{c}_2)+\mathfrak{c}_1\mathfrak{c}_2}\,. \tag{5.39}$$

Taking into account (5.37), and since in this case $a_n=b_n=0$ $(n=0,1,2,\ldots)$, (5.22) reduces to

$$(t_{n+1}+t_{n+2})(C_{n+1}-\mathfrak{c}_1\mathfrak{c}_2)-2(1+\alpha)t_n(C_n-\mathfrak{c}_1\mathfrak{c}_2)+(t_{n-1}+t_{n-2})(C_{n-1}-\mathfrak{c}_1\mathfrak{c}_2)=0\,. \tag{5.40}$$

Next, define $\theta_n:=t_n+t_{n+1}=\overline{a}q^{n/2}+\overline{b}q^{-n/2}$, where $\overline{a}:=k_1(1+q^{1/2})$ and $\overline{b}:=k_2(1+q^{-1/2})$. By setting $K_c:=\Big(\theta_2(C_2-\mathfrak{c}_1\mathfrak{c}_2)-\theta_0(C_1-\mathfrak{c}_1\mathfrak{c}_2)\Big)/(1-q^{-1/2})$, we see that (5.40) reads as

$$\theta_{n+1}(C_{n+1}-\mathfrak{c}_1\mathfrak{c}_2)-\theta_{n-1}(C_n-\mathfrak{c}_1\mathfrak{c}_2)=\theta_n(C_n-\mathfrak{c}_1\mathfrak{c}_2)-\theta_{n-2}(C_{n-1}-\mathfrak{c}_1\mathfrak{c}_2)\,.$$

By applying this relation successively, we obtain

$$\theta_{n+1}(C_{n+1}-\mathfrak{c}_1\mathfrak{c}_2)-\theta_{n-1}(C_n-\mathfrak{c}_1\mathfrak{c}_2)=K_c(1-q^{-1/2})\,.$$

Multiplying this equation by θ_n and applying a telescoping process to the resulting equation, we obtain

$$C_{n+1}=\mathfrak{c}_1\mathfrak{c}_2+\frac{1}{\theta_{n+1}\theta_n}\left(\theta_0\theta_1(C_1-\mathfrak{c}_1\mathfrak{c}_2)+K_c(1-q^{-1/2})\sum_{l=1}^{n}\theta_l\right)$$

for each $n=0,1,2,\ldots$. Therefore,

$$C_{n+1}=\mathfrak{c}_1\mathfrak{c}_2+\frac{\theta_0\theta_1(C_1-\mathfrak{c}_1\mathfrak{c}_2)q^{n/2}+K_c\left(\overline{a}q^n+(\overline{b}-\overline{a}q^{-1/2})q^{n/2}-\overline{b}q^{-1/2}\right)}{(\overline{a}q^n+\overline{b})(\overline{a}q^{n+1}+\overline{b})}q^{(n+1)/2} \tag{5.41}$$

for each $n=0,1,2,\ldots$. We claim that

$$\left(C_2-(1+q)C_1\right)\left(C_2-(1+q^{-1})C_1\right)=0\,. \tag{5.42}$$

Indeed, suppose that (5.42) does not hold. Then, by (5.36), we would have $k_1k_2\neq 0$, and we may write

$$C_{n+1}=\frac{\gamma_{n+1}}{t_{n+1}}=\frac{u(q^{n+1}-1)}{k_1q^{n+1}+k_2},\quad u^{-1}=q^{1/2}-q^{-1/2}\ \ (n=0,1,2,\ldots)\,. \tag{5.43}$$

Assume without lost of generality that $0<q<1$. Then taking successively limits as $n\to\infty$ in the expressions for C_{n+1} and $q^{-(n+1)/2}(C_{n+1}-\mathfrak{c}_1\mathfrak{c}_2)$ given by (5.41) and (5.43), we obtain $u+k_2\mathfrak{c}_1\mathfrak{c}_2=0$ and $K_c=0$. Now, the equality between (5.41) and (5.43) implies

$$\begin{aligned}2(1+\alpha)(u-k_1\mathfrak{c}_1\mathfrak{c}_2)k_1^2q^{2n+2}+k_1\Big(4(1+\alpha)^2(u-k_1\mathfrak{c}_1\mathfrak{c}_2)k_2-\theta_0\theta_1(C_1-\mathfrak{c}_1\mathfrak{c}_2)\Big)q^{n+1}\\ +k_2\Big(2(1+\alpha)(u-k_1\mathfrak{c}_1\mathfrak{c}_2)k_2-\theta_0\theta_1(C_1-\mathfrak{c}_1\mathfrak{c}_2)\Big)=0\,,\end{aligned}$$

for each $n=0,1,2,\ldots$. This implies $u-k_1\mathfrak{c}_1\mathfrak{c}_2=0$ and $\theta_0\theta_1(C_1-\mathfrak{c}_1\mathfrak{c}_2)=0$. (For the case where $1<q<\infty$ we proceed in a similar way, taking limits as $n\to\infty$ on both expressions for C_{n+1} and $q^{(n+1)/2}(C_{n+1}-\mathfrak{c}_1\mathfrak{c}_2)$, and we obtain the same result.) Consequently, $C_{n+1}=\mathfrak{c}_1\mathfrak{c}_2$ for each $n=0,1,2,\ldots$, which contradicts (5.38). Hence (5.42) holds and so $k_1k_2=0$.

Suppose that $k_1=0$, i.e., $C_2=(1+q)C_1$. Then, from (5.38), we find $C_1=(1-q)\mathfrak{c}_1\mathfrak{c}_2$ and so, from (5.39), we obtain $\mathfrak{a}=-1/(2u)$. Since $\mathfrak{b}=\mathfrak{a}\mathfrak{c}_3$ and $B_0=\mathfrak{c}_3$, using (4.62) in Theorem 4.3.2 we obtain

$$C_{n+1}=(1-q^{n+1})\mathfrak{c}_1\mathfrak{c}_2\quad(n=0,1,2,\ldots)\,. \tag{5.44}$$

Since $a_n=b_n=0$ for all n, one easily see that the expressions for B_n and C_{n+1} given by (5.37) and (5.44) satisfy (5.23) and so the system of equations (5.19)–(5.23) is satisfied.

Similarly, if $k_2=0$, i.e. $C_2=(1+q^{-1})\mathfrak{c}_1\mathfrak{c}_2$, we obtain $\mathfrak{a}=1/(2u)$ and

$$C_{n+1}=(1-q^{-n-1})\mathfrak{c}_1\mathfrak{c}_2\quad(n=0,1,2,\ldots)\,, \tag{5.45}$$

which together with (5.37) also fulfill the system of equations (5.19)–(5.23). Thus

$$\phi(z) = \pm\frac{1}{2u}\left((z-\mathfrak{c}_3)^2 - \mathfrak{c}_1\mathfrak{c}_2\right) \quad \textit{and} \quad \psi(z) = z - \mathfrak{c}_3\,.$$

Hence, taking into account the TTRR (5.7) for the Al-Salam Chihara polynomials, we conclude that

$$P_n(z) = 2^n(\mathfrak{c}_1\mathfrak{c}_2)^{n/2}Q_n\left(\frac{z-\mathfrak{c}_3}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}};0,0\Big|q\right) \quad \textit{or} \quad P_n(z) = 2^n(\mathfrak{c}_1\mathfrak{c}_2)^{n/2}Q_n\left(\frac{z-\mathfrak{c}_3}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}};0,0\Big|q^{-1}\right)\,,$$

so that $(P_n)_{n\geq 0}$ is a special case of the monic Al-Salam Chihara polynomials. As a matter of fact, in this case, $(P_n)_{n\geq 0}$ is the sequence of monic q-Hermite polynomials of Rogers. Thus the proof is complete in the case $\deg\pi = 0$.

5.3.2 Case $\deg\pi = 1$

In this case (5.10) can be rewritten as

$$(z-\mathfrak{c}_3-r)D_xP_n(z) = b_nP_n(z) + c_nP_{n-1}(z) \quad (n=0,1,2,\ldots)\,, \tag{5.46}$$

where $r\in\mathbb{C}$. We start by stating a preliminary result.

Lemma 5.3.2 *Let $(P_n)_{n\geq 0}$ be a monic OPS satisfying* (5.46) *and* (5.11). *Then*

$$\left(c_2C_1 - q^{-1/2}c_1C_2\right)\left(c_2C_1 - q^{1/2}c_1C_2\right) = 0\,. \tag{5.47}$$

Proof Since $(P_n)_{n\geq 0}$ satisfies (5.46), then $a_n = 0$ for each $n = 0,1,2,\ldots$, hence (5.35) holds, and by (5.20) and (5.25), we have

$$t_n = \frac{c_n}{C_n} = k_1q^{n/2} + k_2q^{-n/2}, \quad k_1 = \frac{c_2C_1 - q^{-1/2}c_1C_2}{(q-1)C_1C_2}, \quad k_2 = \frac{c_2C_1 - q^{1/2}c_1C_2}{(q^{-1}-1)C_1C_2}\,. \tag{5.48}$$

Suppose that (5.47) does not hold. This means that $k_1k_2 \neq 0$. Taking successively $n=1$ and $n=2$ in (5.46), and using (5.11), (1.65)–(1.68), we have $b_1 = 1$, $b_2 = 2\alpha$ and

$$\begin{aligned}
&r + \mathfrak{c}_3 = B_0 - c_1\,,\\
&c_2 = (2\alpha-1)(B_1+B_0-2\mathfrak{c}_3) - 2\alpha r\,,\\
&(r+\mathfrak{c}_3)(B_1+B_0-2\beta) = -c_2B_0 + 2\alpha(B_0B_1 - C_1)\,.
\end{aligned}$$

Hence, the first equation, and the one obtained from the last one (using the two previous ones) give

$$c_1 = B_0 - \mathfrak{c}_3 - r, \quad B_1 = \mathfrak{c}_3 + (2\alpha-1)(B_0-\mathfrak{c}_3) + 2\alpha\frac{C_1}{c_1}\,. \tag{5.49}$$

We claim that

$$B_n = \mathfrak{c}_3 - \frac{t_0t_1\psi(\mathfrak{c}_3)}{t_nt_{n+1}}, \quad \textit{with} \quad \psi(\mathfrak{c}_3)\neq 0 \quad (n=0,1,2,\ldots)\,. \tag{5.50}$$

Indeed, writing (5.35) as $t_{n+3}(B_{n+2}-\mathfrak{c}_3)-t_{n+1}(B_{n+1}-\mathfrak{c}_3)=t_{n+2}(B_{n+1}-\mathfrak{c}_3)-t_n(B_n-\mathfrak{c}_3)$ and proceeding in a recurrent way, we find by setting $K_b := \Big(t_2(B_1-\mathfrak{c}_3)-t_0(B_0-\mathfrak{c}_3)\Big)/(1-q^{-1/2})$, $t_{n+3}(B_{n+2}-\mathfrak{c}_3)=t_{n+1}(B_{n+1}-\mathfrak{c}_3)+K_b(1-q^{-1/2})$. Multiplying both sides of this equation by t_{n+2} and proceed again in a recurrent way, we obtain

$$B_n=\mathfrak{c}_3+\frac{1}{t_{n+1}t_n}\left(t_0t_1(B_0-\mathfrak{c}_3)+K_b(1-q^{-1/2})\sum_{j=1}^{n}t_j\right),\ \ n=0,1,2,\dots\,.$$

Then

$$B_n=\mathfrak{c}_3+\frac{t_0t_1(B_0-\mathfrak{c}_3)q^{n/2}+K_b\Big(k_1q^n+(k_2-k_1q^{-1/2})q^{n/2}-k_2q^{-1/2}\Big)}{(k_1q^n+k_2)(k_1q^{n+1}+k_2)}q^{(n+1)/2}\,, \tag{5.51}$$

for $n=0,1,2,\dots$. Without loss of generality, we assume $0<q<1$. Since $k_2\neq 0$, then we obtain $\lim_{n\to\infty}q^{-n/2}(B_n-\mathfrak{c}_3)=-K_b/k_2$ and consequently we have $K_b=0$ by applying (4.67). This holds because the condition $d-2au\neq 0$ in (4.67) is fulfilled in the present situation. Indeed we have for the present case $d=1$ and $a=\mathfrak{a}$. So $d-2au=1-2\mathfrak{a}u=2q^{-1/2}k_2/t_1\neq 0$. (For $1<q<\infty$, we proceed in a similar way using the fact that $d+2au=1+2\mathfrak{a}u=2q^{1/2}k_1/t_1\neq 0$ and (4.70) to show that $K_b=0$). This implies that $B_1-\mathfrak{c}_3=t_0(B_0-\mathfrak{c}_3)/t_2$. If $B_0=\mathfrak{c}_3$, then we find $B_1=\mathfrak{c}_3$ which is in contradiction with the second equation in (5.49). Then (5.50) is proved.

Note that, from (5.13) and using (5.48), we obtain

$$\mathfrak{a}=\frac{c_2C_1}{c_1C_2}-\alpha=\frac{t_2}{t_1}-\alpha=\frac{k_1q^{1/2}-k_2q^{-1/2}}{2ut_1}\,, \tag{5.52}$$

since $a_n=0$, for $n=0,1,2,\dots$ and $u^{-1}=q^{1/2}-q^{-1/2}$. Using (5.50), we obtain

$$S_n=\sum_{j=0}^{n-1}(B_j-\mathfrak{c}_3)=\frac{t_1(B_0-\mathfrak{c}_3)\gamma_n}{t_n}\quad(n=0,1,2,\dots)\,.$$

Thus using (4.66) we have

$$t_1\psi(\mathfrak{c}_3)d_{2n-2}=t_n\big(\phi'(\mathfrak{c}_3)\gamma_{n-1}+\psi(\mathfrak{c}_3)\alpha_{n-1}\big)\quad(n=0,1,2,\dots)\,.$$

This gives the following equations:

$$(2\mathfrak{a}ut_1+k_2q^{-1/2})\psi(\mathfrak{c}_3)=2uk_1q^{1/2}\phi'(\mathfrak{c}_3)\,,$$
$$(2\mathfrak{a}ut_1-k_1q^{1/2})\psi(\mathfrak{c}_3)=2uk_2q^{-1/2}\phi'(\mathfrak{c}_3)\,.$$

Taking into account that $k_1k_2\neq 0$ and using (5.52), this implies that

$$\Big|\psi(\mathfrak{c}_3)+2u\phi'(\mathfrak{c}_3)\Big|+\Big|\psi(\mathfrak{c}_3)-2u\phi'(\mathfrak{c}_3)\Big|=0\,,$$

which is impossible because we proved in (5.50) that $\psi(\mathfrak{c}_3)\neq 0$. This concludes the proof.

Proposition 5.3.3 *Up to an affine transformation of the variable, the only OPS* $(P_n)_{n\geq 0}$ *satisfying* (5.46) *are the (special) monic Al-Salam Chihara polynomials* (5.7) *with parameters c and d both nonzero and satisfying* $c/d = q^{\pm 1/2}$.

Proof Note that (5.47) is equivalent to $k_1k_2 = 0$. Suppose that $k_1 = 0$. Then by (5.48), we have $t_n = k_2q^{-n/2}$, for $n = 0,1,2,\ldots$, where $k_2 = q^{1/2}c_1/C_1$. We claim that

$$B_n = \mathfrak{c}_3 + (B_0 - \mathfrak{c}_3)q^n = \mathfrak{c}_3 + (r + c_1)q^n, \quad n = 0,1,2,\ldots \,. \tag{5.53}$$

Indeed (5.21) reduces to

$$q^{-1/2}(B_{n+2} - \mathfrak{c}_3) + (1+q^{1/2})(B_{n+1} - \mathfrak{c}_3) + q(B_n - \mathfrak{c}_3) = 0, \;\; n = 0,1,2,\ldots \,.$$

Note that q and $q^{1/2}$ are the solutions of the associated characteristic equation (for the discrete variable $B_n - \mathfrak{c}_3$), hence we find

$$B_n = \mathfrak{c}_3 + vq^n + sq^{n/2}, \quad n = 0,1,2,\ldots\,, \tag{5.54}$$

for some $v, s \in \mathbb{C}$. Moreover, since $k_1 = 0$, from (5.52) we have $\mathfrak{a} = -1/(2u)$. Hence, by (5.12),

$$\phi(z) = -\frac{1}{2u}\Big((z + 2\mathfrak{b}u)\,(z - B_0) + 2uq^{-1/2}C_1 \Big) \quad \text{and} \quad \psi(z) = z - B_0\,. \tag{5.55}$$

Therefore, using (4.61) in Theorem 4.3.2, we obtain

$$B_n = \mathfrak{c}_3 + q^{(2n-1)/2}\Big(2\alpha u(\mathfrak{b} - \mathfrak{a}\mathfrak{c}_3)(q^n - 1) + q^{1/2}(B_0 - \mathfrak{c}_3) \Big), \;\; n = 0,1,2,\ldots \,. \tag{5.56}$$

Comparing both expressions for B_n given by (5.54) and (5.56), we find $s = 0$, $\mathfrak{b} = \mathfrak{a}\mathfrak{c}_3$ and $v = B_0 - \mathfrak{c}_3$. Hence using the first equation in (5.49), (5.53) follows. As consequence, taking $n = 1$ in (5.53) and comparing the result with the expression for B_1 given by (5.49), we obtain

$$C_1 = (q^{1/2} - 1)(r + c_1)c_1\,. \tag{5.57}$$

Since $C_n = c_n/t_n$, from (5.25) and (5.53), we find

$$C_{n+1} = \frac{C_1}{(q-1)c_1}\left(1 - q^{n+1}\right)\left(r - \frac{r + c_1}{1 + q^{1/2}}(1 + q^{(2n+1)/2}) \right), \quad n = 0,1,2,\ldots \,. \tag{5.58}$$

Taking into account that is $\mathfrak{a} = -1/(2u)$ and $\mathfrak{b} = \mathfrak{a}\mathfrak{c}_3$, using (4.62) in Theorem 4.3.2, we also have

$$C_{n+1} = \left(1 - q^{n+1}\right)\left(\mathfrak{c}_1\mathfrak{c}_2(1 - q^n) + \frac{C_1}{1-q}q^n \right), \;\; n = 0,1,2,\ldots \,. \tag{5.59}$$

If $c_1 = rq^{1/2}$ then (5.58) becomes $C_{n+1} = C_1(1 - q^{n+1})q^n/(1-q)$ which is incompatible with (5.59), since $\mathfrak{c}_1\mathfrak{c}_2 \neq 0$. Then $c_1 \neq rq^{1/2}$. Comparing the expressions for C_{n+1} given in (5.58) and (5.59) yields

$$C_1 = (1-q)\frac{(1+q^{1/2})c_1}{c_1 - q^{1/2}r}\mathfrak{c}_1\mathfrak{c}_2\,. \tag{5.60}$$

Therefore combining this expression of C_1 with the one given by (5.57), we see that $r+c_1$ is a solution of one of the following quadratic equations

$$Z^2-(1+q^{-1/2})c_1Z-2(1+\alpha)\mathfrak{c}_1\mathfrak{c}_2=0\,; \tag{5.61}$$

$$Z^2-(1+q^{1/2})rZ+2(1+\alpha)q^{1/2}\mathfrak{c}_1\mathfrak{c}_2=0\,. \tag{5.62}$$

Suppose that $r+c_1$ is a solution of (5.61). Let c and d be two complex numbers defined by

$$(c,d)\ or\ (d,c)\in\left\{\left(\frac{c_1-\sqrt{\Delta}}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}},\frac{c_1-\sqrt{\Delta}}{2q^{1/2}\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}\right),\ \left(\frac{c_1+\sqrt{\Delta}}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}},\frac{c_1+\sqrt{\Delta}}{2q^{1/2}\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}\right)\right\}\,,$$

where $\Delta=c_1^2+4q^{1/2}\mathfrak{c}_1\mathfrak{c}_2$. Note that $cd\neq 0$. Set $Z_1:=\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(c+d)$ and $Z_2:=-\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(c^{-1}+d^{-1})$. Then Z_1 and Z_2 are solutions of (5.61). (If $r+c_1$ is a solution of (5.62), in this case we define c and d by

$$(c,d)\ or\ (d,c)\in\left\{\left(\frac{r-\sqrt{\Delta}}{2q^{-1/2}\sqrt{\mathfrak{c}_1\mathfrak{c}_2}},\frac{r-\sqrt{\Delta}}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}\right),\ \left(\frac{r+\sqrt{\Delta}}{2q^{-1/2}\sqrt{\mathfrak{c}_1\mathfrak{c}_2}},\frac{r+\sqrt{\Delta}}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}\right)\right\}\,,$$

where $\Delta=r^2-4\mathfrak{c}_1\mathfrak{c}_2$. Then $cd\neq 0$. We set $Z_1:=\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(c+d)$ and $Z_2:=q^{1/2}\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(c^{-1}+d^{-1})$. So Z_1 and Z_2 are solutions of (5.62).) Then without loss of generality we may set $r+c_1=Z_1=\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(c+d)$ and so $Z_1+Z_2=(1+q^{-1/2})c_1$ (or $r+c_1=Z_1$ and so $Z_1+Z_2=(1+q^{1/2})r$, if $r+c_1$ is a solution of (5.62)). This implies

$$r=(c+d)\sqrt{\mathfrak{c}_1\mathfrak{c}_2}\frac{1+cdq^{-1/2}}{cd(1+q^{-1/2})},\quad c_1=(c+d)\sqrt{\mathfrak{c}_1\mathfrak{c}_2}\frac{cd-1}{cd(1+q^{-1/2})}\,.$$

Hence (5.60) (or (5.57)) becomes $C_1=(1-q)(1-cd)\mathfrak{c}_1\mathfrak{c}_2$. As a consequence we obtain from (5.58) (or (5.59)) and (5.53) the following

$$B_n=\mathfrak{c}_3+\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(c+d)q^n,\quad C_{n+1}=\mathfrak{c}_1\mathfrak{c}_2(1-q^{n+1})(1-cdq^n),\ \ n=0,1,2,\dots\,, \tag{5.63}$$

together with $k_2=q^{1/2}\frac{c_1}{C_1}=\frac{u(c+d)}{cd\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(1+q^{-1/2})}$. So, (5.55) becomes

$$\phi(z)=-\frac{1}{2u}\Big((z-\mathfrak{c}_3)^2-\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(c+d)(z-\mathfrak{c}_3)+2(cd-1)\mathfrak{c}_1\mathfrak{c}_2\Big)\,,\quad \psi(z)=z-\mathfrak{c}_3-\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(c+d)\,.$$

(For the choice $r+c_1=Z_2$, when $r+c_1$ is a solution of (5.61), we find $C_1=(1-q)(1-c^{-1}d^{-1})\mathfrak{c}_1\mathfrak{c}_2$, and we obtain $B_n=\mathfrak{c}_3-\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(c^{-1}+d^{-1})q^n$ and $C_{n+1}=\mathfrak{c}_1\mathfrak{c}_2(1-q^{n+1})(1-c^{-1}d^{-1}q^n)$, for each $n=0,1,2,\dots$. These coefficients B_n and C_{n+1} give essentially the same OPS as (5.63), but with the parameters c and d replaced by $-1/c$ and $-1/d$, respectively. Similarly, for the choice $r+c_1=Z_2$, when $r+c_1$ is a solution of (5.62), we find $C_1=(1-q)(1-c^{-1}d^{-1}q)\mathfrak{c}_1\mathfrak{c}_2$ and therefore we obtain $B_n=\mathfrak{c}_3+\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(c^{-1}+d^{-1})q^{n+1/2}$ and $C_{n+1}=\mathfrak{c}_1\mathfrak{c}_2(1-q^{n+1})(1-c^{-1}d^{-1}q^{n+1})$, for $n=0,1,2,\dots$. Again, this gives essentially the same OPS as (5.63), but with the parameters c and d replaced by $q^{1/2}/c$ and $q^{1/2}/d$, respectively.) Using (5.63), equation (5.22) now reads as

$$q^{-1}(1+q^{1/2})(C_{n+1}-\mathfrak{c}_1\mathfrak{c}_2)-2(1+\alpha)(C_n-\mathfrak{c}_1\mathfrak{c}_2)+q(1+q^{-1/2})(C_{n-1}-\mathfrak{c}_1\mathfrak{c}_2)$$
$$=2\mathfrak{c}_1\mathfrak{c}_2(\alpha-1)(2\alpha+1)(c+d)^2q^{2n-1}\,, \tag{5.64}$$

for $n=2,3,\ldots$. Noticing that $c^2+d^2=2\alpha cd$, it is not hard to see that the obtained B_n and C_{n+1} in (5.63) satisfy (5.64). Equation (5.23) in this case ($a_n=0$, for $n=0,1,2,\ldots$) reads as

$$2(1-\alpha)b_n(B_n-\mathfrak{c}_3)^2+(1-2\alpha)(c_n+c_{n+1})(B_n-\mathfrak{c}_3)+c_{n+1}(B_{n+1}-\mathfrak{c}_3)+c_n(B_{n-1}-\mathfrak{c}_3)$$
$$+(b_n-b_{n+2})(C_{n+1}-\mathfrak{c}_1\mathfrak{c}_2)+(b_n-b_{n-2})(C_n-\mathfrak{c}_1\mathfrak{c}_2)=0\,, \quad (5.65)$$

where $c_n=t_nC_n=k_2q^{-n/2}C_n$, for $n=1,2,\ldots$. Similarly one can check that (5.65) is also satisfied and, therefore, the system of equations (5.19)–(5.23) is fulfilled.

In a similar way, if $k_2=0$, we obtain

$$\phi(z)=\frac{1}{2u}\Big((z-\mathfrak{c}_3)^2-\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(c+d)(z-\mathfrak{c}_3)+2(cd-1)\mathfrak{c}_1\mathfrak{c}_2\Big)\,,\quad \psi(z)=z-\mathfrak{c}_3-\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(c+d)\,,$$

and since the condition $c^2+d^2-2\alpha cd=0$ holds, we obtain

$$B_n=\mathfrak{c}_3+\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(c+d)q^{-n}\,,\quad C_{n+1}=\mathfrak{c}_1\mathfrak{c}_2\left(1-cdq^{-n}\right)\left(1-q^{-n-1}\right),\quad n=0,1,2,\ldots\,,$$

as solution of the system of difference equations (5.19)–(5.23). Hence

$$P_n(z)=2^n(\mathfrak{c}_1\mathfrak{c}_2)^{n/2}Q_n\left(\frac{z-\mathfrak{c}_3}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}};c,d\middle|q\right)\ \text{or}\ P_n(z)=2^n(\mathfrak{c}_1\mathfrak{c}_2)^{n/2}Q_n\left(\frac{z-\mathfrak{c}_3}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}};c,d\middle|q^{-1}\right)\,, \quad (5.66)$$

for $n=0,1,2,\ldots$, with $c^2+d^2-2\alpha cd=0$, i.e. $c/d=q^{\pm 1/2}$. Thus the proof is concluded.

Remark 5.3.1 *From the result obtained in* (5.66) *we have the following particular case. Let γ be a complex number. Taking $c=q^{(2\gamma+1)/4}$ and $d=q^{(2\gamma+3)/4}$ (respectively $c=q^{(2\gamma+3)/4}$ and $d=q^{(2\gamma+1)/4}$), we have $c/d=q^{-1/2}$ (respectively $c/d=q^{1/2}$) and so we obtain the Continuous q-Laguerre polynomials with the parameter γ (see [34, p.514]).*

5.3.3 Case $deg\pi=2$

In this case we rewrite (5.10) as

$$(z-\mathfrak{c}_3-r)(z-\mathfrak{c}_3-s)D_xP_n(z)=(a_nz+b_n)P_n(z)+c_nP_{n-1}(z)\quad (n=0,1,2,\ldots)\,, \quad (5.67)$$

where $r,s\in\mathbb{C}$ and $c_n\neq 0$ for $n=1,2,3,\ldots$. From (5.26), (5.20) and (5.21) we obtain

$$a_n=\gamma_n,\quad b_n=(\gamma_n-\gamma_{n-1})\sum_{k=0}^{n-1}(B_k-\mathfrak{c}_3)-(r+s+\mathfrak{c}_3)\gamma_n\,, \quad (5.68)$$

$$t_n=\frac{c_n}{C_n}=k_1q^{n/2}+k_2q^{-n/2},\ \text{where}\ k_1=\frac{c_2C_1-q^{-1/2}c_1C_2}{(q-1)C_1C_2},\quad k_2=\frac{c_2C_1-q^{1/2}c_1C_2}{(q^{-1}-1)C_1C_2} \quad (5.69)$$

$$r_n=t_n+\gamma_n-\gamma_{n-1}=\widehat{a}q^{n/2}+\widehat{b}q^{-n/2},\ \text{where}\ \widehat{a}=k_1+u(1-q^{-1/2}),\ \widehat{b}=k_2-u(1-q^{1/2})\,, \quad (5.70)$$

for $n=0,1,2,\ldots$, $u^{-1}=q^{1/2}-q^{-1/2}$. Recall that $t_0=k_1+k_2$ and so, we also define by compatibility

$$r_0:=\widehat{a}+\widehat{b}\,.$$

Lemma 5.3.4 *Let* $(P_n)_{n\geq 0}$ *be a monic OPS satisfying* (5.67). *Then*

$$\widehat{a}\,\widehat{b}(1-2\mathfrak{a}u)(1+2\mathfrak{a}u)\neq 0\,, \tag{5.71}$$

where $\widehat{a}$, $\widehat{b}$ *are defined in* (5.70) *and* $\mathfrak{a}$ *is given in* (5.13).

Proof Assume that (5.71) does not hold. Suppose, for instance, that $\widehat{a}=0$. Then (5.70) reduces to $r_n=\widehat{b}q^{-n/2}$ for each $n=0,1,2,\ldots$. Then (5.21) becomes

$$q^{-3/2}(B_{n+2}-\mathfrak{c}_3)-(q^{-1}+q^{-1/2})(B_{n+1}-\mathfrak{c}_3)+(B_n-\mathfrak{c}_3)=0,\quad n=0,1,2,\ldots\,.$$

Noticing that q and $q^{1/2}$ are the solutions of the associated characteristic equation (for the variable $B_n-\mathfrak{c}_3$), we may write

$$B_n=\mathfrak{c}_3+r_0(1-q^{1/2})q^{n/2}+s_0(1-q)q^n,\quad n=0,1,2,\ldots\,, \tag{5.72}$$

for some complex numbers r_0 and s_0. From (5.13), we also have

$$\mathfrak{a}=\frac{(c_2+2\alpha C_2)C_1}{(c_1+C_1)C_2}-\alpha=\frac{1+r_2}{r_1}-\alpha=-\frac{1}{2u}+\frac{1}{\widehat{b}q^{-1/2}}\,. \tag{5.73}$$

From (5.72), we obtain $S_n=\sum_{j=0}^{n-1}(B_j-\mathfrak{c}_3)=r_0(1-q^{n/2})+s_0(1-q^n)$, for $n=0,1,2,\ldots$. We then apply (4.66) to obtain

$$(r_0q^{n/2}+s_0q^n-r_0-s_0)d_{2n}=\gamma_{n+1}e_n\qquad(n=1,2,3,\ldots)\,. \tag{5.74}$$

Taking into account that in the present context

$$2d_{2n}=(1+2\mathfrak{a}u)q^n+(1-2\mathfrak{a}u)q^{-n},\;2e_n=\Big(\psi(\mathfrak{c}_3)+2u\phi'(\mathfrak{c}_3)\Big)q^{n/2}+\Big(\psi(\mathfrak{c}_3)-2u\phi'(\mathfrak{c}_3)\Big)q^{-n/2}\,,$$

for $n=0,1,2,\ldots$. It is not hard to see that (5.74) implies $r_0=0=s_0$ as well as $\psi(\mathfrak{c}_3)=0=\phi'(\mathfrak{c}_3)$. Hence

$$B_n=\mathfrak{c}_3\quad(n=0,1,2,\ldots)\,.$$

In addition, using (5.12), we obtain $\mathfrak{b}=\mathfrak{a}\mathfrak{c}_3$. Next we apply (4.62) in Theorem 4.3.2 (with $\mathfrak{a}$ given in (5.73), $\mathfrak{b}=\mathfrak{a}\mathfrak{c}_3$ and $B_0=\mathfrak{c}_3$) to obtain

$$C_{n+1}=\frac{(1-q^{n+1})(B-q^n)\Big(\mathfrak{c}_1\mathfrak{c}_2\left(q^{2n+1}+B(1-q^n)\right)+((q+\widehat{b})C_1-q\mathfrak{c}_1\mathfrak{c}_2)q^n\Big)}{(B-q^{2n})(B-q^{2n+2})}\,, \tag{5.75}$$

for $n=0,1,2,\ldots$, with $B=q+\widehat{b}(1-q)$, while (5.22) reduces to the following equation

$$(q^{-1/2}+q^{-1})(C_{n+1}-\mathfrak{c}_1\mathfrak{c}_2)-2(1+\alpha)(C_n-\mathfrak{c}_1\mathfrak{c}_2)+(q^{1/2}+q)(C_{n-1}-\mathfrak{c}_1\mathfrak{c}_2)=0\,,$$

for $n=2,3,\ldots$. Again we observe that q and $q^{1/2}$ are the solutions of the associated characteristic equation (with the variable $C_n-\mathfrak{c}_1\mathfrak{c}_2$) and so we may write $C_{n+1}=\mathfrak{c}_1\mathfrak{c}_2+\bar{r}_0q^{n/2}+\bar{s}_0q^n$, for $n=0,1,2,\ldots$, with $\bar{r}_0,\bar{s}_0\in\mathbb{C}$. This is incompatible with (5.75) and so $\widehat{a}\neq 0$. The case $\widehat{b}=0$ can be treated similarly.

Assume now that $1+2\mathfrak{a}u=0$. Then since

$$\mathfrak{a}=\frac{1+r_2}{r_1}-\alpha\,,$$

we obtain $\widehat{a}=-uq^{-1/2}\neq 0$. On the other hand, we use (4.61) in Theorem 4.3.2 to obtain

$$B_n=\mathfrak{c}_3+\frac{1}{2}q^n(1+q^{-1})\Big((2\mathfrak{b}u+\mathfrak{c}_3)(q^n-1)+2\frac{B_0-\mathfrak{c}_3}{1+q^{-1}}\Big)\,,$$

for $n=0,1,2,\ldots$. This satisfies (5.21) if and only if $\mathfrak{b}=\mathfrak{a}\mathfrak{c}_3$ and $B_0=\mathfrak{c}_3$. So $B_n=\mathfrak{c}_3$. Taking into account this, (4.62) in Theorem 4.3.2 gives

$$C_{n+1}=(1-q^{n+1})\Big((1-q^n)\mathfrak{c}_1\mathfrak{c}_2+\frac{C_1}{1-q}q^n\Big)\quad(n=0,1,2,\ldots)\,.$$

This does not satisfy (5.22) since $\widehat{a}\neq 0$ and $B_n=\mathfrak{c}_3$. Hence $1+2\mathfrak{a}u\neq 0$. The case $1-2\mathfrak{a}u=0$ can be treated similarly. Hence the proof is concluded.

Proposition 5.3.5 *Up to an affine transformation of the variable, the continuous monic Jacobi polynomials are the only OPS satisfying* (5.67).

Proof Taking successively $n=1$ and $n=2$ in (5.67) using (5.11) and (1.65)–(1.68) we obtain the following:

$$B_0=b_1+r+s+2\mathfrak{c}_3,\;\; c_1=(B_0-\mathfrak{c}_3-r)(B_0-\mathfrak{c}_3-s)\,, \tag{5.76}$$

$$b_2=(2\alpha-1)(B_0+B_1-2\mathfrak{c}_3)-2\alpha(r+s+\mathfrak{c}_3)\,, \tag{5.77}$$

$$(r+\mathfrak{c}_3)(s+\mathfrak{c}_3)\Big(2(1-\alpha)\mathfrak{c}_3-B_0-B_1\Big)=-c_2B_0+b_2(B_0B_1-C_1)\,, \tag{5.78}$$

$$c_2=b_2(B_0+B_1)-2\alpha(B_0B_1-C_1)-(r+s+2\mathfrak{c}_3)\Big(2(1-\alpha)\mathfrak{c}_3-B_0-B_1\Big)+2\alpha(r+\mathfrak{c}_3)(s+\mathfrak{c}_3)\,. \tag{5.79}$$

Solving (5.21), with the same technique used to obtain (5.51), we find

$$B_n=\mathfrak{c}_3+\frac{r_0r_1(B_0-\mathfrak{c}_3)q^{n/2}+\widehat{K}_b\Big(\widehat{a}q^n+(\widehat{b}-\widehat{a}q^{-1/2})q^{n/2}-\widehat{b}q^{-1/2}\Big)}{(\widehat{a}q^n+\widehat{b})(\widehat{a}q^{n+1}+\widehat{b})}q^{(n+1)/2}\,, \tag{5.80}$$

for $n=0,1,2,\ldots$, where $\widehat{K}_b:=\Big(r_2(B_1-\mathfrak{c}_3)-r_0(B_0-\mathfrak{c}_3)\Big)/(1-q^{-1/2})$. Since $\widehat{a}\,\widehat{b}\neq 0$ (see (5.71)), assume that $0<q<1$, then $\lim_{n\to\infty}q^{-n/2}(B_n-\mathfrak{c}_3)=-\widehat{K}_b/\widehat{b}$. Then using (4.67), we obtain $\widehat{K}_b=0$ since $1-2\mathfrak{a}u\neq 0$ (also given by (5.71)). If $1<q<\infty$, then we obtain the same result using (4.70) and the fact that $1+2\mathfrak{a}u\neq 0$ (given in (5.71)). Hence (5.80) reduces to

$$B_n=\mathfrak{c}_3+\frac{r_0r_1(B_0-\mathfrak{c}_3)}{r_nr_{n+1}}\quad (n=0,1,2,\dots)\,. \tag{5.81}$$

It is not hard to see that

$$S_n=\sum_{j=0}^{n-1}(B_j-\mathfrak{c}_3)=\frac{r_1(B_0-\mathfrak{c}_3)a_n}{r_n}\quad (n=0,1,2,\dots)\,.$$

Comparing this with (4.66), we arrive at $r_1\psi(\mathfrak{c}_3)d_{2n}=r_{n+1}e_n$, for $n=1,2,3,\dots$. This implies

$$(2\mathfrak{a}ur_1+\widehat{b}q^{-1/2})\psi(\mathfrak{c}_3)=2\widehat{a}uq^{1/2}\phi'(\mathfrak{c}_3)\,, \tag{5.82}$$

$$(2\mathfrak{a}ur_1-\widehat{a}q^{1/2})\psi(\mathfrak{c}_3)=2\widehat{b}uq^{-1/2}\phi'(\mathfrak{c}_3)\,. \tag{5.83}$$

If $\phi'(\mathfrak{c}_3)=0$, then from (5.82)–(5.83) we obtain $r_1\psi(\mathfrak{c}_3)=0$. But from (5.14) we obtain $0\neq c_1+a_1C_1=r_1C_1$, i.e. $r_1\neq 0$. Then we have $\psi(\mathfrak{c}_3)=0$. (Conversely, if $\psi(\mathfrak{c}_3)=0$, then taking into account (5.71), we obtain $\phi'(\mathfrak{c}_3)=0$.) So $B_0=\mathfrak{c}_3$ and $\mathfrak{b}=\mathfrak{a}\mathfrak{c}_3$. With these informations we use (4.61)–(4.62) in Theorem 4.3.2 to obtain

$$B_n=\mathfrak{c}_3\,, \tag{5.84}$$

$$C_{n+1}=\frac{(1-q^{n+1})\Big(2\mathfrak{a}u-1-(1+2\mathfrak{a}u)q^{n-1}\Big)\Big[u\Big(2(\mathfrak{a}+\alpha)C_1-4\mathfrak{a}\mathfrak{c}_1\mathfrak{c}_2\Big)q^n+\mathfrak{c}_1\mathfrak{c}_2\Big(2\mathfrak{a}u-1+(2\mathfrak{a}u+1)q^{2n}\Big)\Big]}{\Big(2\mathfrak{a}u-1-(1+2\mathfrak{a}u)q^{2n-1}\Big)\Big(2\mathfrak{a}u-1-(1+2\mathfrak{a}u)q^{2n+1}\Big)}\,, \tag{5.85}$$

for $n=0,1,2,\dots$. Also taking into account (5.84), (5.22) reads as

$$(r_{n+1}+r_{n+2})(C_{n+1}-\mathfrak{c}_1\mathfrak{c}_2)-2(1+\alpha)r_n(C_n-\mathfrak{c}_1\mathfrak{c}_2)+(r_{n-1}+r_{n-2})(C_{n-1}-\mathfrak{c}_1\mathfrak{c}_2)=0\,. \tag{5.86}$$

This equation is of the same type as (5.40) (with t_n replaced by r_n) and so from (5.41), we may write

$$C_{n+1}=\mathfrak{c}_1\mathfrak{c}_2+\frac{\widehat{\theta}_0\widehat{\theta}_1(C_1-\mathfrak{c}_1\mathfrak{c}_2)q^{n/2}+\widehat{K}_c\Big(\widehat{a}q^n+(\widehat{b}-\widehat{a}q^{-1/2})q^{n/2}-\widehat{b}q^{-1/2}\Big)}{(\widehat{a}q^n+\widehat{b})(\widehat{a}q^{n+1}+\widehat{b})}q^{(n+1)/2}\,, \tag{5.87}$$

for $n=0,1,2,\dots$, for some complex numbers $\widehat{\theta}_0$, $\widehat{\theta}_1$ and $\widehat{K}_c$. Taking into account (5.71) one may see that (5.85) and (5.87) are not compatible. Thus $\phi'(\mathfrak{c}_3)\neq 0$. This implies that the following holds:

$$r_1(B_0-\mathfrak{c}_3)\Big(\widehat{a}q^{1/2}-\widehat{b}q^{-1/2}\Big)\neq 0\,. \tag{5.88}$$

Hence solving (5.82)–(5.83) we obtain

$$\mathfrak{a}=-\frac{1+(q\widehat{a}/\widehat{b})^2}{2u\Big(1-(q\widehat{a}/\widehat{b})^2\Big)},\quad B_0=\mathfrak{c}_3+2u\phi'(\mathfrak{c}_3)\frac{1-(q\widehat{a}/\widehat{b})}{1+(q\widehat{a}/\widehat{b})}\,. \tag{5.89}$$

Indeed the expression of B_0 is obtained by subtracting (5.82) to (5.83), and combining this with one of equations (5.82)–(5.83) yields the expression of $\mathfrak{a}$. Considering $\widehat{a}$, $\widehat{b}$, B_0 and C_1 as free parameters, let us define without loss of generality two complex numbers a and b such that $-q^{a/2}$ and $q^{b/2}$ are solutions of the following quadratic equation

$$Z^2 + \frac{r_1(B_0 - \mathfrak{c}_3)q^{1/4}}{\widehat{b}(1+q^{1/2})\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}\, Z + \frac{\widehat{a}}{\widehat{b}} = 0. \tag{5.90}$$

Then we have the following

$$q^{a/2} - q^{b/2} = \frac{r_1(B_0 - \mathfrak{c}_3)q^{1/4}}{\widehat{b}(1+q^{1/2}\sqrt{\mathfrak{c}_1\mathfrak{c}_2})}\,; \tag{5.91}$$

$$q^{(a+b)/2} = -\frac{\widehat{a}}{\widehat{b}}\,. \tag{5.92}$$

On the other hand, re-writing (5.90) as

$$\left(Z + \frac{r_1(B_0-\mathfrak{c}_3)q^{1/2} - \sqrt{\Delta}}{2\widehat{b}q^{1/4}(1+q^{1/2})\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}\right)\left(Z + \frac{r_1(B_0-\mathfrak{c}_3)q^{1/2} + \sqrt{\Delta}}{2\widehat{b}q^{1/4}(1+q^{1/2})\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}\right) = 0\,,$$

where $\Delta = q(B_0 - \mathfrak{c}_3)^2 r_1^2 - 4\widehat{a}\widehat{b}q^{1/2}(1+q^{1/2})^2\mathfrak{c}_1\mathfrak{c}_2$, we find

$$(q^{a/2},\, q^{b/2}) \in \left\{ \left(\frac{2\widehat{a}(1+q^{1/2})q^{1/4}\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}{r_1(B_0-\mathfrak{c}_3)q^{1/2}+\sqrt{\Delta}},\ -\frac{r_1(B_0-\mathfrak{c}_3)q^{1/2}+\sqrt{\Delta}}{2\widehat{b}(1+q^{1/2})q^{1/4}\sqrt{\mathfrak{c}_1\mathfrak{c}_2}} \right), \left(\frac{2\widehat{a}(1+q^{1/2})q^{1/4}\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}{r_1(B_0-\mathfrak{c}_3)q^{1/2}-\sqrt{\Delta}},\ -\frac{r_1(B_0-\mathfrak{c}_3)q^{1/2}-\sqrt{\Delta}}{2\widehat{b}(1+q^{1/2})q^{1/4}\sqrt{\mathfrak{c}_1\mathfrak{c}_2}} \right) \right\}. \tag{5.93}$$

Hence

$$\begin{aligned}
\mathfrak{a} &= -\frac{1+q^{a+b+2}}{2u(1-q^{a+b+2})}\,,\\
B_0 &= \mathfrak{c}_3 + \frac{\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(1+q^{1/2})q^{1/4}(q^{a/2}-q^{b/2})}{1-q^{(a+b+2)/2}}\,,\\
\mathfrak{b} &= -\frac{(1+q^{a+b+2})\mathfrak{c}_3}{2u(1-q^{a+b+2})} + \frac{\sqrt{\mathfrak{c}_1\mathfrak{c}_2}q^{3/4}(q^{a/2}-q^{b/2})q^{(a+b+2)/2}}{u^2(q^{1/2}-1)(1-q^{a+b+2})(1-q^{(a+b+2)/2})}\,.
\end{aligned}$$

Indeed, the expression of $\mathfrak{a}$ is obtained by putting (5.92) in the first equation appearing in (5.89), the given expression of B_0 is obtained by combining (5.92) and (5.91) and the given expression of $\mathfrak{b}$ is obtained by using (5.12) and the second equation appearing in (5.89).

Note that (5.70) can be written using (5.92) as

$$r_n = \widehat{b}(1-q^{n+(a+b)/2})q^{-n/2} \quad (n=0,1,2,\ldots)\,, \tag{5.94}$$

and also since $t_n = r_n - a_n + a_{n-1}$, we obtain

$$t_n = -\frac{1}{1+q^{-1/2}}\left[1+q^{(2n-1)/2} - \widehat{b}(1+q^{-1/2})(1-q^{n+(a+b)/2})\right]q^{-n/2} \quad (n=0,1,2,\ldots)\,. \tag{5.95}$$

Therefore (5.81) becomes

$$\begin{aligned} B_n &= \mathfrak{c}_3 + q^{1/4}(1+q^{1/2})\sqrt{\mathfrak{c}_1\mathfrak{c}_2}\frac{(1-q^{(a+b)/2})(q^{a/2}-q^{b/2})q^n}{(1-q^{(2n+a+b)/2})(1-q^{(2n+a+b+2)/2})} \\ &= \mathfrak{c}_3 + \sqrt{\mathfrak{c}_1\mathfrak{c}_2}\Big(q^{(2a+1)/4}+q^{-(2a+1)/4}-y_n(a,b)-z_n(a,b)\Big) , \end{aligned} \tag{5.96}$$

for $n=0,1,2,\ldots$, where $y_n(\cdot,\cdot)$ and $z_n(\cdot,\cdot)$ are given in (5.4). Taking into account what is preceding, (5.12) becomes

$$\begin{aligned} \phi(z) &= -\frac{1+q^{a+b+2}}{2u(1-q^{a+b+2})}(z-\mathfrak{c}_3)^2 + q^{1/4}(1+q^{1/2})\sqrt{\mathfrak{c}_1\mathfrak{c}_2}\frac{(q^{a/2}-q^{b/2})}{2u(1+q^{(a+b+2)/2})}(z-\mathfrak{c}_3) \\ &\quad + \mathfrak{c}_1\mathfrak{c}_2\frac{2(1+\alpha)q^{(a+b+4)/2}(q^{a/2}-q^{b/2})^2}{u(1-q^{a+b+2})(1-q^{(a+b+2)/2})^2} - q^{-1/2}\frac{1-q^{a+b+3}}{1-q^{a+b+2}}C_1 , \\ \psi(z) &= z-\mathfrak{c}_3 - q^{1/4}(1+q^{1/2})\sqrt{\mathfrak{c}_1\mathfrak{c}_2}\frac{(q^{a/2}-q^{b/2})}{1-q^{(a+b+2)/2}} . \end{aligned}$$

Let **u** be the regular linear functional with respect to the monic OPS $(P_n)_{n\geq 0}$. Using Theorem 4.3.1, the regularity conditions for **u** are given by

$$\mathfrak{c}_1\mathfrak{c}_2(1-q^{n+a+1})(1-q^{n+b+1})(1-q^{n+a+b+1})C_1 \neq 0 \qquad (n=0,1,2,\ldots) .$$

Also by applying (4.61)–(4.62) in Theorem 4.3.2, we obtain the same expression for B_n and

$$C_{n+1} = -\frac{q^{-1/2}uC_1(1-q^{(a+b+3)/2})(1-q^{(a+b+2)/2})^2(1-q^{n+1})(1-q^{n+a+b+1})(1-q^{n+a+1})(1-q^{n+b+1})}{(1-q^{a+1})(1-q^{b+1})(1+q^{(a+b+1)/2})(1-q^{(2n+a+b+1)/2})(1-q^{(2n+a+b+2)/2})^2(1-q^{(2n+a+b+3)/2})} ,$$

for $n=0,1,2,\ldots$. Using a computer system (Mathematica for example) it is not hard to see that this expression of C_{n+1} together with (5.96) satisfy (5.22) if and only if

$$C_1 = \mathfrak{c}_1\mathfrak{c}_2\frac{(1-q)(1-q^{a+1})(1-q^{b+1})(1+q^{(a+b+1)/2})}{(1-q^{(a+b+3)/2})(1-q^{(a+b+2)/2})^2} .$$

Alternatively, the given expression of C_1 is obtained by taking $n=2$ in (5.22) using expressions of C_2 and C_3 computed from the previous expression of C_{n+1}; and therefore we remark that (5.22) holds for each $n=2,3,\ldots$. Consequently we obtain

$$\begin{aligned} C_{n+1} &= \frac{\mathfrak{c}_1\mathfrak{c}_2(1-q^{n+1})(1-q^{n+a+b+1})(1-q^{n+a+1})(1-q^{n+b+1})}{(1-q^{(2n+a+b+1)/2})(1-q^{(2n+a+b+2)/2})^2(1-q^{(2n+a+b+3)/2})} \\ &= 4\mathfrak{c}_1\mathfrak{c}_2 y_n(a,b)z_{n+1}(a,b) , \end{aligned} \tag{5.97}$$

for $n=0,1,2,\ldots$, where $y_n(\cdot,\cdot)$ and $z_n(\cdot,\cdot)$ are given in (5.4). Taking into account that $c_n=t_nC_n$, we use (5.76)–(5.79) to obtain the following system

$$
\begin{aligned}
&\widehat{b}q^{-1/2}(1-q^{(a+b+2)/2})C_1+(r+s+2\mathfrak{c}_3)B_0-(r+\mathfrak{c}_3)(s+\mathfrak{c}_3)=B_0^2+C_1\,;\\
&\widehat{b}q^{-1}(1-q^{(a+b+4)/2})B_0C_2-(r+\mathfrak{c}_3)(s+\mathfrak{c}_3)(B_0+B_1-2\beta)+2\alpha(B_0B_1-C_1)(r+s+2\mathfrak{c}_3)\\
&\qquad\qquad=(B_0B_1-C_1)\Big[2\alpha\mathfrak{c}_3+(2\alpha-1)(B_0+B_1-2\mathfrak{c}_3)\Big]+(2\alpha-1)B_0C_2\,;\\
&\widehat{b}q^{-1}(1-q^{(a+b+4)/2})C_2-2\alpha(r+\mathfrak{c}_3)(s+\mathfrak{c}_3)+(r+s+2\mathfrak{c}_3-B_0-B_1)\Big((2\alpha-1)(B_0+B_1-2\mathfrak{c}_3)+2\alpha\mathfrak{c}_3\Big)\\
&\qquad\qquad=2\alpha(C_1-B_0B_1)+(2\alpha-1)C_2\,.
\end{aligned}
$$

Solving firstly the above system for $\widehat{b}$, $r+s+2\mathfrak{c}_3$ and $(r+\mathfrak{c}_3)(s+\mathfrak{c}_3)$, and secondly the obtained result for r and s, we obtain

$$\widehat{b}=uq^{1/2}(1+q^{-(a+b+2)/2})\,,$$

and

$$(r,s)\ or\ (s,r)\in\Big\{\Big(\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(q^{(2a+1)/4}+q^{-(2a+1)/4}),\ -\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(q^{(2b+1)/4}+q^{-(2b+1)/4})\Big)\Big\}\,.$$

So (5.94) and (5.95) become

$$
\begin{aligned}
r_n&=u(1+q^{-(a+b+2)/2})(1-q^{n+(a+b)/2})q^{(1-n)/2}\,,\\
t_n&=u(1+q^{-(a+b+1)/2})(1-q^{n+(a+b+1)/2})q^{-n/2}\,,
\end{aligned}
$$

for each $n=0,1,2,\ldots$. Using (5.96), (5.68) becomes

$$b_n=\sqrt{\mathfrak{c}_1\mathfrak{c}_2}(q^{a/2}-q^{b/2})\gamma_n\frac{(1+q^{n+a+b+1/2})}{1-q^{n+(a+b)/2}}q^{-(2a+2b+1)/4}-\mathfrak{c}_3\gamma_n\,, \tag{5.98}$$

for each $n=0,1,2,\ldots$. Also, since $c_n=t_nC_n$, we obtain

$$c_n=-\mathfrak{c}_1\mathfrak{c}_2\gamma_n\frac{(1-q^{n+a})(1-q^{n+b})(1-q^{n+a+b})(1+q^{-(a+b+1)/2})}{(1-q^{n+(a+b-1)/2})(1-q^{n+(a+b)/2})^2}\,, \tag{5.99}$$

for $n=0,1,2,\ldots$. Finally, using Mathematica for e.g., (5.96), (5.97), (5.98) and (5.99), we see that (5.23) is also satisfied. Thus the system of equations (5.19)–(5.23) is satisfied.

Notice that (5.89) can be also written as

$$\mathfrak{a}=\frac{1+(q^{-1}\widehat{b}/\widehat{a})^2}{2u\Big(1-(q^{-1}\widehat{b}/\widehat{a})^2\Big)}\,,\quad B_0=\mathfrak{c}_3-2u\phi'(\mathfrak{c}_3)\frac{1-(q^{-1}\widehat{b}/\widehat{a})}{1+(q^{-1}\widehat{b}/\widehat{a})}\,.$$

Proceeding similarly with the same parameters a and b as defined in (5.93) we obtain the same results with q replaced by q^{-1}. Hence

$$P_n(x)=2^n(\mathfrak{c}_1\mathfrak{c}_2)^{n/2}\widehat{P}_n^{(a,b)}\Big(\frac{x-\mathfrak{c}_3}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}\Big|q\Big)\quad or\quad P_n(x)=2^n(\mathfrak{c}_1\mathfrak{c}_2)^{n/2}\widehat{P}_n^{(a,b)}\Big(\frac{x-\mathfrak{c}_3}{2\sqrt{\mathfrak{c}_1\mathfrak{c}_2}}\Big|q^{-1}\Big)\,,$$

where $\widehat{P}_n^{(a,b)}(\cdot|q)$ is the monic continuous q-Jacobi polynomial defined in (5.4). This concludes the proof.

Chapter 6

Some future directions of research

In this short chapter we introduce some future directions of research problems.

I. Several authors have focused their interest on OPS with respect to Sobolev inner products. The subject is interesting not only from a theoretical point of view, but also since this issue, for instance, brings an important tool in the framework of Approximation Theory. Indeed, this was the main motivation that lead Iserles, Koch, Norsett, and Sanz-Serna [25] to introduce the notion of coherent pairs of measures about 30 years ago. Chapter 3 concerns to new extensions of the notion of coherence. Nevertheless, no connections with Sobolev OPS is considered therein. The study of such connection leads to a challenging research problem.

II. In Chapters 2 and 4 we analyzed regularity conditions for moment linear functionals solutions of a functional equation related with the $\mathbf{D}_{q,\omega}-$classical OPS and the $x-$ classical OPS. Such functional equations are of the form

$$\mathbf{D}(\phi\mathbf{u}) = \mathbf{T}(\psi\mathbf{u})\,, \tag{6.1}$$

where ϕ and ψ are nonzero polynomials such that $deg\phi \leq 2$ and $deg\psi = 1$, $\mathbf{D}$ is an analogue of the derivative operator, and $\mathbf{T}$ is another operator. The problem is to determine necessary and sufficient conditions for the regularity of $\mathbf{u}$, and to give explicit expressions for the coefficients of the TTRR satisfied by the associated monic OPS. Under the regularity of $\mathbf{u}$, it would be interesting to give the classification of the solutions of the above functional equation, as well as of the associated OPS. Notice that the problem deserves analysis even whenever $\mathbf{D} = \mathbf{D}_x$ and $\mathbf{T} = \mathbf{S}_x$.

III. It would be interesting to consider the analogue of **II** in the framework of semiclassical OPS.

IV. In Chapter 5 we solved a conjecture that leads to a characterization of the continuous $q-$Jacobi polynomials and of certain special cases of Al-Salam-Chihara poynomials as the only OPS that satisfy an algebraic structure relation such as

$$\pi(x)D_xP_n(x) = (a_nx+b_n)P_n(x)+c_nP_{n-1}(x) \quad (n=0,1,2\ldots)\,,$$

where π is a polynomial of degree at most 2, and $x = x(s)$ is a $q-$quadratic lattice. It would be interesting to consider the analogous problem for a quadratic lattice.

V. To seek more applications for the developed theory, as we made in Chapter 5. It is worth mentioning that OPS come up in very attractive problems, for instance, in Physics, Operator Theory (Jacobi operators), Random Matrices, PDEs, and Number Theory.

Appendix A

Rodrigues-type formulas

A.1 A Rodrigues-type formula for (q,ω)-classical OPS

Here we provide details on how to obtain (2.45) from (2.40) and (2.41). For this purpose the following identities are useful

$$-1+[2]_{q^{-1}}-q^{-2}\frac{[n-1]_{q^{-1}}}{[n]_{q^{-1}}}=\frac{1}{q[n]_{q^{-1}}}\,, \tag{A.1}$$

$$q[2]_{q^{-1}}-\frac{[n-1]_{q^{-1}}}{[n]_{q^{-1}}}=\frac{[n+1]_q}{[n]_q}\,, \tag{A.2}$$

$$-[n-2]_q q^{2-n}+[2]_{q^{-1}}[n-1]_{q^{-1}}-q^{-n}\frac{[n-1]_{q^{-1}}[n]_q}{[n]_{q^{-1}}}=1\,, \tag{A.3}$$

$$q[n-1]_q d_{n-1}+d_{2n-1}=[n]_q d_n\,, \tag{A.4}$$

for $n=0,1,2,\ldots$. So from (2.40), we may write

$$k_n d_{2n-2}A(x;n)=Y_1(n)x+Y_2(n)\,, \tag{A.5}$$

where, taking into account (A.1),

$$\begin{aligned}
Y_1(n)&=-\frac{[n-1]_{q^{-1}}q^{-n}d_{2n}d_{2n-1}}{[n]_{q^{-1}}}+[2]_{q^{-1}}q^{2-n}d_{2n}d_{2n-1}-q^{2-n}d_{2n}d_{2n-1}\\
&=q^{2-n}d_{2n}d_{2n-1}\left(-1+[2]_{q^{-1}}-q^{-2}\frac{[n-1]_{q^{-1}}}{[n]_{q^{-1}}}\right)\\
&=\frac{q^{1-n}d_{2n}d_{2n-1}}{[n]_{q^{-1}}}\,,
\end{aligned}$$

and by using also (A.2)–(A.3), we obtain

$$\begin{aligned}
Y_2(n) &= \left(q[2]_{q^{-1}} - \frac{[n-1]_{q^{-1}}}{[n]_{q^{-1}}}\right) e_n d_{2n-1} - \frac{d_{2n}d_{2n-1}}{d_{2n-2}} e_{n-1} \\
&\quad - \omega d_{2n}d_{2n-1}\left(-[n-2]_q q^{2-n} + [2]_{q^{-1}}[n-1]_{q^{-1}} - q^{-n}\frac{[n-1]_{q^{-1}}[n]_q}{[n]_{q^{-1}}}\right) \\
&= \frac{[n+1]_q d_{2n-1} e_n}{[n]_q} - \omega d_{2n}d_{2n-1} - \frac{d_{2n}d_{2n-1}}{d_{2n-2}} e_{n-1} \, .
\end{aligned}$$

Therefore, from (A.5) and using the first identity in (2.44), we arrive at

$$\begin{aligned}
A(x;n) &= \frac{k_n^{-1}}{d_{2n-2}}\left(\frac{q^{1-n}d_{2n}d_{2n-1}}{[n]_{q^{-1}}}x + \frac{[n+1]_q d_{2n-1}e_n}{[n]_q} - \omega d_{2n}d_{2n-1} - \frac{d_{2n}d_{2n-1}}{d_{2n-2}}e_{n-1}\right) \\
&= \frac{k_{n+1}^{-1}d_{n-1}}{[n]_{q^{-1}} d2n-2}\left(x + \frac{[n+1]_q e_n}{d_{2n}} - \frac{[n]_q e_{n-1}}{d_{2n-2}} - \omega [n]_q\right) \, .
\end{aligned}$$

Hence the expression of $A(x;n)$ in (2.45) follows taking into account that $[n]_q = q^{n-1}[n]_{q^{-1}}$. Now, from (2.41) and using (2.34), (2.43), (2.35), and (2.42) (with n replaced by $n-1$), it is not hard to see that $B(x;n)$ reduces to the following constant polynomial

$$\begin{aligned}
\frac{k_{n-1}B_n(x;n)}{qd_{2n-2}^{-1}} &= \Big(cd_{2n} + qe_n(e_{n-1} - \omega(1+q)q^{-n}[n-1]_q d_{2n-1}) + \omega^2 q^{1-n} d_{2n}d_{2n-1}[n-1]_q[n]_{q^{-1}}\Big) d_{2n-2} \\
&\quad - \left(e_{n-1} - \omega q^{1-n}[n-1]_q d_{2n-2}\right)\left((1+q)e_n d_{2n-1} - d_{2n}d_{2n-1}\frac{e_{n-1}}{d_{2n-2}} - \omega [n]_q q^{1-n} d_{2n}d_{2n-1}\right) \, .
\end{aligned} \tag{A.6}$$

Taking into account (A.4), (2.33) and the identity $[n]_q - [n-1]_q = q^{n-1}$ (for $n = 0,1,2,\ldots$), we obtain successively

$$\begin{aligned}
B(x;n) &= \frac{qk_{n-1}^{-1}}{d_{2n-2}}\left(cd_{2n}d_{2n-2} - d_{2n}e_{n-1}(e_n - \omega d_{2n-1}) + d_{2n}d_{2n-1}\frac{e_{n-1}^2}{d_{2n-2}}\right) \\
&= \frac{qk_{n-1}^{-1}}{d_{2n-2}}\left(cd_{2n}d_{2n-2} - d_{2n}e_{n-1}(b + qe_{n-1}) + d_{2n}d_{2n-1}\frac{e_{n-1}^2}{d_{2n-2}}\right) \\
&= \frac{qk_{n-1}^{-1}}{d_{2n-2}}\left(cd_{2n}d_{2n-2} - bd_{2n}e_{n-1} + ad_{2n}\frac{e_{n-1}^2}{d_{2n-2}}\right) \\
&= qd_{2n}k_{n-1}^{-1}\phi\left(-\frac{e_{n-1}}{d_{2n-2}}\right) \, .
\end{aligned}$$

Hence using the second identity in (2.44), the expression of $B(x;n)$ in (2.45) follows.

A.2 A Rodrigues-type formula for x-classical OPS

Here we give some key steps on how to obtain (4.48)–(4.49) from (4.46)–(4.47) and (1.81)–(1.82). For this purpose we remark that, from (4.15) and (4.16), the following identities hold:

$$a^{[n-1]} = d_{2n-1} - \alpha d_{2n-2} \,, \tag{A.7}$$

$$b^{[n-1]} = e_n - \alpha e_{n-1} - 2\mathfrak{c}_3 a^{[n-1]} \,, \tag{A.8}$$

$$d_{2n} - 2\alpha d_{2n-1} + d_{2n-2} = 0 \,, \tag{A.9}$$

for each $n = 0, 1, 2, \ldots$. From (4.39), (4.44), and (4.34), and applying (1.81)–(1.82) and (A.8), we obtain

$$(T_{n,0}\psi^{[n]})(z) = \frac{d_{2n}}{\alpha_n}(z - \mathfrak{c}_3) + e_n \,, \tag{A.10}$$

$$(T_{n-1,1}\xi_2)(z;n) = \frac{\gamma_{n-1}}{\alpha_{n-1}}\left(\frac{\alpha^2 d_{2n}d_{2n-1}(\alpha_{n-1}+\alpha\alpha_{n-2})}{\alpha_{n-2}^2}(z-\mathfrak{c}_3) + 2\alpha^3 e_n d_{2n-1}\right), \tag{A.11}$$

$$(T_{n-1,1}\eta_2)(z;n) = \frac{\gamma_{n-1}}{\alpha_{n-1}}\left(\frac{\alpha(\alpha d_{2n-1}-d_{2n})(\alpha_{n-1}+\alpha\alpha_{n-2})}{\alpha_{n-2}^2}(z-\mathfrak{c}_3) + \alpha(e_{n-1}-\alpha e_n)\right). \tag{A.12}$$

Similarly, the relation

$$\begin{aligned}(T_{n-1,1}\eta_2)(z;n) =& \frac{\alpha^2(\alpha d_{2n-1}-d_{2n})}{\alpha_{n-1}\alpha_{n-2}}(z-\mathfrak{c}_3)^2 + \frac{\alpha(e_{n-1}-\alpha e_n)}{\alpha_{n-1}}(z-\mathfrak{c}_3) + \eta_2(\mathfrak{c}_3;n) \\ &+ \frac{4\alpha(1-\alpha^2)\gamma_{n-1}(\alpha d_{2n-1}-d_{2n})}{\alpha_{n-2}}\mathfrak{c}_1\mathfrak{c}_2 \,,\end{aligned} \tag{A.13}$$

holds, as well as

$$\begin{aligned}(T_{n-1,1}\xi_2)(z;n) =& \frac{\alpha^3 d_{2n-1}d_{2n}}{\alpha_{n-1}\alpha_{n-2}}(z-\mathfrak{c}_3)^2 + \frac{2\alpha^3 e_n d_{2n-1}}{\alpha_{n-1}}(z-\mathfrak{c}_3) + \xi_2(\mathfrak{c}_3;n) \\ &+ \frac{4\alpha^2(1-\alpha^2)\gamma_{n-1}d_{2n-1}d_{2n}}{\alpha_{n-2}}\mathfrak{c}_1\mathfrak{c}_2 \,,\end{aligned} \tag{A.14}$$

for each $n = 0, 1, 2, \ldots$.

Now, firstly, we use (A.10), (A.11), and (A.12) together with the identity $\alpha_n + \alpha\gamma_n = \gamma_{n+1}$ $(n = 0, 1, 2, \ldots)$ to see that from (4.46) we have

$$\begin{aligned}\varepsilon_n A(z;n) &= \frac{1}{\gamma_n}(z-\mathfrak{c}_3) - \frac{e_{n-1}}{d_{2n-2}} + \frac{\alpha_n e_n}{\gamma_n d_{2n}} + \alpha e_n \frac{2\alpha d_{2n-1} - d_{2n}}{d_{2n}d_{2n-2}} \\ &= \frac{1}{\gamma_n}(z-\mathfrak{c}_3) - \frac{e_{n-1}}{d_{2n-2}} + \frac{\gamma_{n+1}e_n}{\gamma_n d_{2n}} \,,\end{aligned}$$

for each $n = 0, 1, 2, \ldots$. Hence (4.48) holds.

Secondly, from (A.10)–(A.14), it is straightforward to verify that (4.47) reduces to

$$\varepsilon_n B(z;n) = \eta_2(\mathfrak{c}_3;n) + \frac{(d_{2n} - \alpha d_{2n-1})\xi_2(\mathfrak{c}_3;n)}{\alpha d_{2n} d_{2n-1}} + \frac{\alpha(\alpha e_n - e_{n-1})e_{n-1}}{d_{2n-2}} + \frac{2\alpha^2(\alpha d_{2n-1} - d_{2n})e_n e_{n-1}}{d_{2n} d_{2n-2}} \,.$$

From the definition of $\xi(.;n)$ and $\eta_2(.;n)$ given in (4.38) and (4.42), it is easy to verify that

$$\xi(\mathfrak{c}_3;n) = \alpha^2(e_n e_{n-1} + \phi^{[n-1]}(\mathfrak{c}_3)d_{2n}), \quad \eta_2(\mathfrak{c}_3;n) = -\alpha^2 \phi^{[n-1]}(\mathfrak{c}_3), \quad n = 0,1,2,\dots \,.$$

So, by using (A.7), we obtain

$$\varepsilon_n B(z;n) = -\alpha \frac{d_{2n-2}}{d_{2n-1}} \phi^{[n-1]}(\mathfrak{c}_3) + \alpha e_{n-1} \left(\frac{e_n}{d_{2n-1}} - \frac{e_{n-1}}{d_{2n-2}} \right)$$

for each $n = 0,1,2,\dots$. Then by using successively (A.7) and (A.8), the following holds:

$$\begin{aligned} B(z;n) &= \frac{\alpha^2 \gamma_n d_{2n} d_{2n-2}}{d_{n-1}} \left(d_{2n-1} \frac{e_{n-1}^2}{d_{2n-2}^2} - e_n \frac{e_{n-1}}{d_{2n-2}} + \phi^{[n-1]}(\mathfrak{c}_3) \right) \\ &= \frac{\alpha^2 \gamma_n d_{2n} d_{2n-2}}{d_{n-1}} \left((a^{[n-1]} + \alpha d_{2n-2}) \frac{e_{n-1}^2}{d_{2n-2}^2} - (b^{[n-1]} + \alpha e_{n-1} + 2\mathfrak{c}_3 a^{[n-1]}) \frac{e_{n-1}}{d_{2n-2}} + \phi^{[n-1]}(\mathfrak{c}_3) \right) \\ &= \frac{\alpha^2 \gamma_n d_{2n} d_{2n-2}}{d_{n-1}} \phi^{[n-1]} \left(\mathfrak{c}_3 - \frac{e_{n-1}}{d_{2n-2}} \right) \,. \end{aligned}$$

Hence, (4.49) follows.

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