Nonlinear Analysis 214 (2022) 112590

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Wavelet decomposition and embeddings of generalised Besov–Morrey spaces

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ARTICLE INFO

Article history:
Received 19 November 2020
Accepted 10 September 2021
Communicated by Eugenia Malinnikova

MSC:
46E35

Keywords:
Besov–Morrey spaces
Generalised Morrey spaces
Embeddings
Wavelet decompositions

ABSTRACT

We study embeddings between generalised Besov–Morrey spaces $\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$. Both sufficient and necessary conditions for the embeddings are proved. Embeddings of the Besov–Morrey spaces into the Lebesgue spaces $L_r(\mathbb{R}^d)$ are also considered. Our approach requires a wavelet characterisation of the spaces which we establish for the system of Daubechies wavelets.

1. Introduction

In this paper we study smoothness function spaces built upon generalised Morrey spaces $\mathcal{M}_{\varphi,p}(\mathbb{R}^d)$, $0 < p < \infty$, $\varphi : (0,\infty) \to [0,\infty)$. The generalised version of Morrey spaces $\mathcal{M}_{u,p}(\mathbb{R}^d)$, $0 < p \le u < \infty$, was introduced by T. Mizuhara [16] and E. Nakai [17] in the beginning of the 1990s. The spaces were applied successfully to PDEs, e.g. to nondivergence elliptic differential problems, cf. [3,11] or [29], to parabolic differential equations [33] or Schrödinger equations [13]. We refer to [24] for further information about the spaces and the historical remarks.

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[1] The first and last author were partially supported by the German Research Foundation (DFG), Grant No. Ha 2794/8-1.
[2] The second author was partially supported by the Centre for Mathematics of the University of Coimbra, Portugal - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES.
[3] The author was supported by National Science Center, Poland, Grant No. 2013/10/A/ST1/00091.

https://doi.org/10.1016/j.na.2021.112590
0362-546X/

Also smoothness function spaces built upon Morrey spaces $\mathcal{M}_{u,p}(\mathbb{R}^d)$, in particular Besov–Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$, $0 < p \le u < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, were investigated intensively in recent years. Yu.V. Netrusov was the first who combined the Besov and Morrey norms cf. [19]. He considered function spaces on domains and proved some embedding theorem, but the further attention paid to the spaces was motivated first of all by possible applications to PDEs. The Besov–Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ were introduced by H. Kozono and M. Yamazaki in [12] and used by them to study Navier–Stokes equations. Further applications of the spaces to PDEs can be found e.g. in the papers written by A.L. Mazzucato [15], by L.C.F. Ferreira, M. Postigo [4] or by M. Yang, Z. Fu, J. Sun, [31].

Here we study the Besov spaces $\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$ built upon generalised Morrey spaces. The spaces were introduced and studied by S. Nakamura, T. Noi and Y. Sawano [18], cf. also [1]. In particular they proved the atomic decomposition theorem for the spaces. In the recent paper [10] M. Izuki and T. Noi investigated the spaces on domains. The generalised Besov–Morrey spaces cover Besov–Morrey spaces and local Besov–Morrey spaces considered by H. Triebel [28] as special cases. Our main aim here is to find the sufficient and necessary conditions for the embeddings

$$\mathcal{N}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d).$$

Our main tools are the atomic decomposition and the wavelet characterisation. This approach allows us to consider first embeddings on the level of sequence spaces, cf. Theorem 4.1, and afterwards to transfer the result to function spaces, cf. Theorem 5.1. In particular we regain the characterisation of embeddings of Besov–Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ proved in [7].

The paper is organised as follows. In Section 2 we present some preliminaries. We recall definitions and facts needed later on. In Section 3 we obtain the wavelet characterisation of the generalised Besov–Morrey spaces, cf. Theorem 3.1. Section 4 deals with the sequence spaces $n^s_{\varphi,p,q}$ that correspond to $\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$ via the wavelet characterisation theorem. Theorem 4.1 contains the sufficient and necessary conditions for the embeddings. In the concluding Section 5 we transfer the results to the function spaces. We discuss several concrete examples.

2. Preliminaries

First we fix some notation. By $\mathbb{N}$ we denote the *set of natural numbers*, by $\mathbb{N}_0$ the set $\mathbb{N}\cup\{0\}$, and by $\mathbb{Z}^d$ the *set of all lattice points in $\mathbb{R}^d$ having integer components*. Let $\mathbb{N}_0^d$, where $d \in \mathbb{N}$, be the set of all multi-indices, $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_j \in \mathbb{N}_0$ and $|\alpha| := \sum_{j=1}^d \alpha_j$. If $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, then we put $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. For $a \in \mathbb{R}$, let $\lfloor a \rfloor := \max\{k \in \mathbb{Z} : k \le a\}$, $\lceil a \rceil = \min\{k \in \mathbb{Z} : \ k \ge a\}$, and $a_+ := \max(a, 0)$. Given any $u \in (0, \infty]$, it will be denoted by u' the number, possible ∞, defined by the expression $\frac{1}{u'} = (1 - \frac{1}{u})_+$; in particular when $1 \le u \le \infty$, u' is the same as the conjugate exponent defined through $\frac{1}{u} + \frac{1}{u'} = 1$. All unimportant positive constants will be denoted by C, occasionally the same letter C is used to denote different constants in the same chain of inequalities. By the notation $A \lesssim B$, we mean that there exists a positive constant c such that $A \le c\,B$, whereas the symbol $A \sim B$ stands for $A \lesssim B \lesssim A$. We denote by $|\cdot|$ the Lebesgue measure when applied to measurable subsets of $\mathbb{R}^d$. For each cube $Q \subset \mathbb{R}^d$ we denote its side length by $\ell(Q)$, and, for $a \in (0, \infty)$, we denote by aQ the cube concentric with Q having the side length $a\ell(Q)$. For $x \in \mathbb{R}^d$ and $r \in (0, \infty)$ we denote by $Q(x, r)$ the compact cube centred at x with side length r, whose sides are parallel to the axes of coordinates. We write simply $Q(r) = Q(0, r)$ when $x = 0$. By $\mathcal{Q}$ we denote the collection of all *dyadic cubes* in $\mathbb{R}^d$, namely, $\mathcal{Q} := \{Q_{j,k} := 2^{-j}([0,1)^d + k) : \ j \in \mathbb{Z}, \ k \in \mathbb{Z}^d\}$. Given two (quasi-)Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X into Y is continuous.

Recall first that the classical Morrey space $\mathcal{M}_{u,p}(\mathbb{R}^d)$, $0 < p \le u < \infty$, is defined to be the set of all locally p-integrable functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\|f \mid \mathcal{M}_{u,p}(\mathbb{R}^d)\| := \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{u}-\frac{1}{p}} \left(\int_Q |f(y)|^p \mathrm{d}y\right)^{\frac{1}{p}} < \infty .$$

In this paper we consider generalised Morrey spaces where the parameter u is replaced by a function φ according to the following definition.

Definition 2.1. Let $0 < p < \infty$ and $\varphi : (0, \infty) \to [0, \infty)$ be a function which does not satisfy $\varphi \equiv 0$. Then $\mathcal{M}_{\varphi,p}(\mathbb{R}^d)$ is the set of all locally p-integrable functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^d)$ for which

$$\|f \mid \mathcal{M}_{\varphi,p}(\mathbb{R}^d)\| := \sup_{Q\in\mathcal{Q}} \varphi\big(\ell(Q)\big) \left(\frac{1}{|Q|}\int_Q |f(y)|^p \mathrm{d}y\right)^{\frac{1}{p}} < \infty . \tag{2.1}$$

Remark 2.2. The above definition goes back to [17]. When $\varphi(t) = t^{\frac{d}{u}}$ for $t > 0$ and $0 < p \le u < \infty$, then $\mathcal{M}_{\varphi,p}(\mathbb{R}^d)$ coincides with $\mathcal{M}_{u,p}(\mathbb{R}^d)$, which in turn recovers the Lebesgue space $L_p(\mathbb{R}^d)$ when $u = p$. In the definition of $\|\cdot \mid \mathcal{M}_{\varphi,p}(\mathbb{R}^d)\|$ balls or all cubes with sides parallel to the axes of coordinates can be taken. This change leads to equivalent quasi-norms. Note that for $\varphi_0 \equiv 1$ (which would correspond to $u = \infty$) we obtain

$$\mathcal{M}_{\varphi_0,p}(\mathbb{R}^d) = L_\infty(\mathbb{R}^d), \quad 0 < p < \infty, \quad \varphi_0 \equiv 1, \tag{2.2}$$

due to Lebesgue's differentiation theorem.

When $\varphi(t) = t^{-\sigma}\chi_{(0,1)}(t)$ where $-\frac{d}{p} \le \sigma < 0$, then $\mathcal{M}_{\varphi,p}(\mathbb{R}^d)$ coincides with the local Morrey spaces $\mathcal{L}_p^\sigma(\mathbb{R}^d)$ introduced by H. Triebel in [27], cf. also [28, Section 1.3.4]. If $\sigma = -\frac{d}{p}$, then the space is a uniform Lebesgue space $\mathcal{L}_p(\mathbb{R}^d)$.

For $\mathcal{M}_{\varphi,p}(\mathbb{R}^d)$ it is usually required that $\varphi \in \mathcal{G}_p$, where $\mathcal{G}_p$ is the set of all nondecreasing functions $\varphi : (0, \infty) \to [0, \infty)$ such that $\varphi(t)t^{-d/p}$ is a nonincreasing function, i.e.,

$$1 \le \frac{\varphi(r)}{\varphi(t)} \le \left(\frac{r}{t}\right)^{d/p}, \quad 0 < t \le r < \infty.$$

A justification for the use of the class $\mathcal{G}_p$ comes from the lemma below, cf. e.g. [18, Lemma 2.2]. One can easily check that $\mathcal{G}_{p_2} \subset \mathcal{G}_{p_1}$ if $0 < p_1 \le p_2 < \infty$.

Lemma 2.3 (*[18,24]*)**.** *Let $0 < p < \infty$ and $\varphi : (0, \infty) \to [0, \infty)$ be a function satisfying $\varphi(t_0) \ne 0$ for some $t_0 > 0$.*

(i) *Then $\mathcal{M}_{\varphi,p}(\mathbb{R}^d) \ne \{0\}$ if and only if*

$$\sup_{t>0} \varphi(t) \min(t^{-\frac{d}{p}}, 1) < \infty.$$

(ii) *Assume $\sup_{t>0} \varphi(t) \min(t^{-\frac{d}{p}}, 1) < \infty$. Then there exists $\varphi^* \in \mathcal{G}_p$ such that*

$$\mathcal{M}_{\varphi,p}(\mathbb{R}^d) = \mathcal{M}_{\varphi^*,p}(\mathbb{R}^d)$$

in the sense of equivalent (quasi-)norms.

Remark 2.4. In [5, Thm. 3.3] it is shown that for $1 \le p_2 \le p_1 < \infty$, $\varphi_i \in \mathcal{G}_{p_i}$, $i = 1, 2$, then

$$\mathcal{M}_{\varphi_1,p_1}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\varphi_2,p_2}(\mathbb{R}^d)$$

if and only if there exists some $C > 0$ such that for all $t > 0$, $\varphi_1(t) \le C\varphi_2(t)$. The argument can be immediately extended to $0 < p_2 \le p_1 < \infty$.

In case of $\varphi_i(t) = t^{d/u_i}$, $0 < p_i \le u_i < \infty$, $i = 1, 2$, it is well-known that

$$\mathcal{M}_{u_1,p_1}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{u_2,p_2}(\mathbb{R}^d) \qquad \text{if and only if} \qquad p_2 \le p_1 \le u_1 = u_2,$$

cf. [20] and [21].

We consider the following examples.

Example 2.5.

(i) The function

$$\varphi_{u,v}(t) = \begin{cases} t^{d/u} & \text{if} \quad t \le 1, \\ t^{d/v} & \text{if} \quad t > 1, \end{cases} \tag{2.3}$$

with $0 < u, v < \infty$ belongs to $\mathcal{G}_p$ with $p = \min(u, v)$. In particular, taking $u = v$, the function $\varphi(t) = t^{\frac{d}{u}}$ belongs to $\mathcal{G}_p$ whenever $0 < p \le u < \infty$.

(ii) The function $\varphi(t) = \max(t^{d/v}, 1)$ belongs to $\mathcal{G}_v$. It corresponds to (2.3) with $u = \infty$.

(iii) The function $\varphi(t) = \sup\{s^{d/u}\chi_{(0,1)}(s) : s \le t\} = \min(t^{d/u}, 1)$ defines an equivalent (quasi)-norm in $\mathcal{L}^\sigma_p(\mathbb{R}^d)$, $\sigma = -\frac{d}{u}$, $p \le u$. The function φ belongs to $\mathcal{G}_u \subset \mathcal{G}_p$. It corresponds to (2.3) with $v = \infty$.

(iv) The function $\varphi(t) = t^{d/u}(\log(L + t))^a$, with L being a sufficiently large constant, belongs to $\mathcal{G}_u$ if $0 < u < \infty$ and $a \le 0$.

Other examples can be found e.g. in [24, Ex. 3.15].

Let $\mathcal{S}(\mathbb{R}^d)$ be the set of all Schwartz functions on $\mathbb{R}^d$, endowed with the usual topology, and denote by $\mathcal{S}'(\mathbb{R}^d)$ its topological dual, namely, the space of all bounded linear functionals on $\mathcal{S}(\mathbb{R}^d)$ endowed with the weak $*$-topology. For all $f \in \mathcal{S}(\mathbb{R}^d)$ or $f \in \mathcal{S}'(\mathbb{R}^d)$, we use $\mathcal{F}f$ to denote its Fourier transform, and $\mathcal{F}^{-1}f$ for its inverse. Now let us define the generalised Besov–Morrey spaces introduced in [18].

Let $\eta_0, \eta \in \mathcal{S}(\mathbb{R}^d)$ be nonnegative compactly supported functions satisfying

$$\eta_0(x) > 0 \quad \text{if} \quad x \in Q(2),$$

$$0 \notin \operatorname{supp} \eta \quad \text{and} \quad \eta(x) > 0 \quad \text{if} \quad x \in Q(2) \setminus Q(1).$$

For $j \in \mathbb{N}$, let $\eta_j(x) := \eta(2^{-j}x)$, $x \in \mathbb{R}^d$.

Definition 2.6. Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$. The generalised Besov–Morrey space $\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\big\|f \mid \mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)\big\| := \left(\sum_{j=0}^{\infty} 2^{jsq} \big\|\mathcal{F}^{-1}(\eta_j \mathcal{F}f) \mid \mathcal{M}_{\varphi,p}(\mathbb{R}^d)\big\|^q\right)^{1/q} < \infty,$$

with the usual modification made in case of $q = \infty$.

Remark 2.7. The above spaces have been introduced in [18]. There the authors have proved that those spaces are independent of the choice of the functions η_0 and η considered in the definition, as different choices lead to equivalent quasi-norms, cf. [18, Thm 1.4].

When $\varphi(t) = t^{\frac{d}{u}}$ for $t > 0$ and $0 < p \le u < \infty$, then

$$\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) = \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$$

are the usual Besov–Morrey spaces, which are studied in [32] or in the survey papers by W. Sickel [25,26]. Of course, we can recover the classical Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ for any $0 < p < \infty$, $0 < q \le \infty$, and $s \in \mathbb{R}$, since

$$B^s_{p,q}(\mathbb{R}^d) = \mathcal{N}^s_{p,p,q}(\mathbb{R}^d).$$

When $\varphi(t) = \min(t^{\frac{d}{u}}, 1)$, then we recover the local Besov–Morrey spaces introduced by H. Triebel,

$$\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) = B^s_q(\mathcal{L}^\sigma_p, \mathbb{R}^d), \quad \sigma = -\frac{d}{u}, \quad p \le u,$$

cf. [28, Section 1.3.4].

Besides the elementary embeddings

$$\mathcal{N}^{s+\varepsilon}_{\varphi,p,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{\varphi,p,q_2}(\mathbb{R}^d), \quad \varepsilon > 0,$$

and

$$\mathcal{N}^s_{\varphi,p,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{\varphi,p,q_2}(\mathbb{R}^d), \quad q_1 \le q_2,$$

cf. [18, Prop. 3.3], we can also prove that

$$\mathcal{N}^0_{\varphi,p,\min\{p,2\}}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\varphi,p}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^0_{\varphi,p,\infty}(\mathbb{R}^d) \qquad \text{if} \qquad 1 < p < \infty,$$

when φ satisfies the additional condition

$$c\left(\frac{r}{t}\right)^\varepsilon \le \frac{\varphi(r)}{\varphi(t)}, \quad 0 < t \le r < \infty,$$

for some constants $\varepsilon > 0$ and $c > 0$. This is a consequence of Corollary 6.17 of [18].

The atomic decomposition

An important tool in our later considerations is the characterisation of the generalised Besov–Morrey spaces by means of atomic decompositions. We follow [18] and start by defining the appropriate sequence spaces and atoms.

Definition 2.8. Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$. The generalised Besov–Morrey sequence space $n^s_{\varphi,p,q}(\mathbb{R}^d)$ is the set of all double-indexed sequences $\lambda := \{\lambda_{j,m}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^d} \subset \mathbb{C}$ for which the quasi-norm

$$\|\lambda \mid n^s_{\varphi,p,q}\| := \left(\sum_{j=0}^\infty 2^{jsq} \Big\| \sum_{m\in\mathbb{Z}^d} \lambda_{j,m}\chi_{Q_{j,m}} \,\Big|\, \mathcal{M}_{\varphi,p}(\mathbb{R}^d)\Big\|^q\right)^{1/q} \tag{2.4}$$

is finite (with the usual modification if $q = \infty$).

Remark 2.9. When $\varphi(t) = t^{\frac{d}{u}}$ for $t > 0$ and $0 < p \le u < \infty$, then

$$n^s_{\varphi,p,q}(\mathbb{R}^d) = n^s_{u,p,q}(\mathbb{R}^d)$$

are the usual Besov–Morrey sequence spaces. Moreover if $u = p$, then the space $n^s_{\varphi,p,q}(\mathbb{R}^d)$ coincides with a classical Besov sequence space $b^s_{p,q}(\mathbb{R}^d)$ since $\mathcal{M}_{\varphi,p}(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ in that case.

Definition 2.10. Let $L \in \mathbb{N}_0 \cup \{-1\}$, $K \in \mathbb{N}_0$, and $c > 1$. A C^K-function $a : \mathbb{R}^d \to \mathbb{C}$ is said to be a (K, L, c)-atom centred at $Q_{j,m}$, where $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^d$, if

$$2^{-j|\alpha|} |\mathrm{D}^\alpha a(x)| \le \chi_{cQ_{j,m}}(x) \tag{2.5}$$

for all $x \in \mathbb{R}^d$ and for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le K$, and when

$$\int_{\mathbb{R}^d} x^\beta a(x) \mathrm{d}x = 0, \tag{2.6}$$

for all $\beta \in \mathbb{N}_0^d$ with $|\beta| \le L$ when $L \ge 0$. In the sequel we write $a_{j,m}$ instead of a if the atom is located at $Q_{j,m}$, i.e., $\operatorname{supp} a_{j,m} \subset cQ_{j,m}$.

We use the notation

$$\sigma_p := d\left(\frac{1}{\min(1,p)} - 1\right), \quad 0 < p \le \infty,$$

in the sequel. The following result coincides with [18, Thm. 4.4], cf. also [18, Rmk. 4.3], see also [14, Thm. 10.15].

Theorem 2.11. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$. Let also $c > 1$, $L \in \mathbb{N}_0 \cup \{-1\}$ and $K \in \mathbb{N}_0$ be such that*

$$K \ge \lfloor 1 + s \rfloor_+ \quad \text{and} \quad L \ge \max(-1, \lfloor \sigma_p - s \rfloor).$$

(i) *Let $f \in \mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$. Then there exists a family $\{a_{j,m}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^d}$ of (K, L, c)-atoms and a sequence $\lambda = \{\lambda_{j,m}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^d} \in n^s_{\varphi,p,q}(\mathbb{R}^d)$ such that*

$$f = \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^d} \lambda_{j,m} a_{j,m} \qquad \text{in} \quad \mathcal{S}'(\mathbb{R}^d)$$

and

$$\|\lambda \mid n^s_{\varphi,p,q}(\mathbb{R}^d)\| \lesssim \|f \mid \mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)\|.$$

(ii) *Let $\{a_{j,m}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^d}$ be a family of (K, L, c)-atoms and $\lambda = \{\lambda_{j,m}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^d} \in n^s_{\varphi,p,q}(\mathbb{R}^d)$. Then*

$$f = \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^d} \lambda_{j,m} a_{j,m}$$

converges in $\mathcal{S}'(\mathbb{R}^d)$ and belongs to $\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$. Furthermore

$$\|f \mid \mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)\| \lesssim \|\lambda \mid n^s_{\varphi,p,q}(\mathbb{R}^d)\|.$$

The next lemma will be useful in the sequel and shows that the sequence spaces $n^s_{\varphi,p,q}$ can be defined through a more convenient equivalent norm, extending the result for $n^s_{u,p,q}$ from [7, Prop. 3.1].

Lemma 2.12. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$. Then*

$$n^s_{\varphi,p,q}(\mathbb{R}^d) = \{\lambda = \{\lambda_{j,m}\}_{j,m} : \|\lambda \mid n^s_{\varphi,p,q}\|^* < \infty\}$$

where

$$\|\lambda \mid n^s_{\varphi,p,q}\|^* := \left(\sum_{j=0}^\infty 2^{jsq} \sup_{\substack{\nu:\nu\le j \\ k\in\mathbb{Z}^d}} \varphi(2^{-\nu})^q \, 2^{(\nu-j)\frac{d}{p}q} \Big(\sum_{\substack{m\in\mathbb{Z}^d: \\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^p \Big)^{\frac{q}{p}} \right)^{1/q}$$

with the usual modification if $q = \infty$. Furthermore, $\|\cdot \mid n^s_{\varphi,p,q}\|^*$ is an equivalent quasi-norm in $n^s_{\varphi,p,q}(\mathbb{R}^d)$.

Proof. For each $j \in \mathbb{N}_0$ we calculate the quasi-norm

$$\Big\| \sum_{m\in\mathbb{Z}^d} \lambda_{j,m}\chi_{Q_{j,m}} \,\Big|\, \mathcal{M}_{\varphi,p}(\mathbb{R}^d)\Big\|.$$

Let $Q = Q_{\nu,k}$, $\nu \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, be a dyadic cube. If $j \geq \nu$, then

$$\varphi(\ell(Q))\left(\frac{1}{|Q|}\int_Q \Big|\sum_{m\in\mathbb{Z}^d}\lambda_{j,m}\chi_{Q_{j,m}}(x)\Big|^p\,\mathrm{d}x\right)^{1/p} = \varphi(2^{-\nu})2^{(\nu-j)\frac{d}{p}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}}|\lambda_{j,m}|^p\Bigg)^{1/p}. \tag{2.7}$$

If $j < \nu$, there exists only one $m_0 \in \mathbb{Z}^d$ such that $Q = Q_{\nu,k} \subset Q_{j,m_0}$, and, moreover, since φ is nondecreasing, we obtain

$$\begin{aligned}\varphi(\ell(Q))\left(\frac{1}{|Q|}\int_Q \Big|\sum_{m\in\mathbb{Z}^d}\lambda_{j,m}\chi_{Q_{j,m}}(x)\Big|^p\mathrm{d}x\right)^{1/p} &= \varphi(2^{-\nu})\left(\frac{1}{|Q|}\int_Q|\lambda_{j,m_0}|^p\mathrm{d}x\right)^{1/p}\\ &\leq \varphi(2^{-j})|\lambda_{j,m_0}|. \end{aligned}\tag{2.8}$$

From (2.7) and (2.8) we immediately have

$$\Big\| \sum_{m\in\mathbb{Z}^d} \lambda_{j,m}\chi_{Q_{j,m}} \,\Big|\, \mathcal{M}_{\varphi,p}(\mathbb{R}^d)\Big\| \leq \sup_{\substack{\nu:\nu\leq j\\ k\in\mathbb{Z}^d}}\varphi(2^{-\nu})2^{(\nu-j)\frac{d}{p}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}}|\lambda_{j,m}|^p\Bigg)^{1/p}.$$

The reverse inequality is clear, from the definition of $\|\cdot \mid \mathcal{M}_{\varphi,p}(\mathbb{R}^d)\|$ and (2.7). Therefore

$$\Big\| \sum_{m\in\mathbb{Z}^d} \lambda_{j,m}\chi_{Q_{j,m}} \,\Big|\, \mathcal{M}_{\varphi,p}(\mathbb{R}^d)\Big\| = \sup_{\substack{\nu:\nu\leq j\\ k\in\mathbb{Z}^d}}\varphi(2^{-\nu})2^{(\nu-j)\frac{d}{p}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}}|\lambda_{j,m}|^p\Bigg)^{1/p}. \tag{2.9}$$

The result follows from (2.9) taking into account (2.4). □

3. The wavelet characterisation

We assume that the reader is familiar with the basic notation and assertions of the wavelet theory. There is a variety of excellent books that present general background material on wavelets, we can refer, in particular, to [2,9] and [30]. We will follow the approach presented in [6] and consider here the compactly supported Daubechies wavelets.

Let $L \in \mathbb{N}$ and let $\psi_F, \psi_M \in C^L(\mathbb{R})$ are real-valued compactly supported (L^2-normalised) functions with

$$\int_{\mathbb{R}}\psi_F(t)\mathrm{d}t = C > 0, \qquad \int_{\mathbb{R}}\psi_M(t)t^\ell\mathrm{d}t = 0, \qquad \ell < L. \tag{3.1}$$

The function ψ_F is called scaling function (or father wavelet) and ψ_M is called an associated function (mother wavelet).

Let $G = (G_1, \ldots, G_d) \in G^* = \{F, M\}^{d*}$, where * indicates that at least one of the components of G must be an M. Then we set

$$\psi^G_{j,m} = 2^{jd/2}\prod_{r=1}^d \psi^{G_r}(2^jx_r - m_r), \qquad \psi_m(x) = \prod_{r=1}^d \psi_F(x_r - m_r), \tag{3.2}$$

where $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^d$, $G \in G^*$. The family $\{\psi_m, \psi^G_{j,m} : j \in \mathbb{N}_0,\ m \in \mathbb{Z}^d,\ G \in G^*\}$ is called a (Daubechies) wavelet system.

We will need the following modified version $\tilde{n}^s_{\varphi,p,q}(\mathbb{R}^d)$ of $n^s_{\varphi,p,q}(\mathbb{R}^d)$ spaces. The space $\tilde{n}^s_{\varphi,p,q}(\mathbb{R}^d)$ collects all sequences

$$\lambda = \left\{\lambda_m \in \mathbb{C}, \lambda^G_{j,m} \in \mathbb{C} : \ m \in \mathbb{Z}^d, \ j \in \mathbb{N}_0, \ G \in G^*\right\}$$

quasi-normed by

$$\|\lambda \mid \tilde{n}^s_{\varphi,p,q}(\mathbb{R}^d)\| := \Big\| \sum_{m\in\mathbb{Z}^d} \lambda_m \chi_{Q_m} \Big| \mathcal{M}_{\varphi,p}(\mathbb{R}^d) \Big\| + \sum_{G\in G^*} \Big\| \{\lambda^G_{j,m}\} | n^s_{\varphi,p,q}(\mathbb{R}^d) \Big\|. \tag{3.3}$$

Theorem 3.1. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$. For the wavelets defined in (3.2) we take*

$$L > \max\{\lfloor 1+s \rfloor_+, \frac{d}{p} - s\}. \tag{3.4}$$

Let $f \in \mathcal{S}'(\mathbb{R}^d)$. Then $f \in \mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$ if and only if it can be represented as

$$f = \sum_{m\in\mathbb{Z}^d} \lambda_m \psi_m \ + \sum_{G\in G^*} \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^d} \lambda^G_{jm} 2^{-jd/2} \psi^G_{j,m}, \qquad \lambda \in \tilde{n}^s_{\varphi,p,q}(\mathbb{R}^d),$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^d)$. The representation is unique with

$$\lambda^G_{j,m} = \lambda^G_{j,m}(f) = 2^{jd/2} \left(f, \psi^G_{j,m}\right) \quad \text{and} \quad \lambda_m = \lambda_m(f) = (f, \psi_m),$$

and

$$I : f \mapsto \{\lambda_m(f), 2^{jd/2}(f, \psi^G_{j,m})\}$$

is a linear isomorphism of $\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$ onto $\tilde{n}^s_{\varphi,p,q}(\mathbb{R}^d)$.

Furthermore, $\|I(f) \mid \tilde{n}^s_{\varphi,p,q}(\mathbb{R}^d)\|$ may be used as an equivalent quasi-norm in $\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$.

Proof. *Step 1.* We prove that the theorem follows from Theorem 5.1 in [6]. The space $\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$ is an (isotropic, inhomogeneous) quasi-Banach function space which satisfies

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

Please note that the inequality $L > \frac{d}{p} - s$ implies $L > \sigma_p - s$, therefore $\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$ can be characterised in terms of an L-atomic decomposition with $L = K$ and coefficients in $n^s_{\varphi,p,q}(\mathbb{R}^d)$, cf. Theorem 2.11. So it is sufficient to prove that the sequence space $n^s_{\varphi,p,q} \equiv n^s_{\varphi,p,q}(\mathbb{R}^d)$ is a $\varkappa$-sequence space for some $\varkappa$, $0 < \varkappa < L$, cf. Definition 4.1 in [6], i.e., to prove that

(i) for any $b > 1$, $C_1 > 0$, and all $\mu \in n^s_{\varphi,p,q}$, any sequence $\lambda = \{\lambda_{j,m}\}$ with

$$|\lambda_{j,m}| \le C_1 \sum_{J\in\mathbb{N}_0} 2^{-\varkappa|J-j|} \sum_{M\in I^j_J(m)} 2^{-d(J-j)_+} |\mu_{J,M}|, \quad j \in \mathbb{N}_0, \ m \in \mathbb{Z}^d, \tag{3.5}$$

where

$$I^j_J(m) = \left\{M \in \mathbb{Z}^d : \ bQ_{J,M} \cap C_1 Q_{j,m} \neq \emptyset\right\},$$

belongs to $n^s_{\varphi,p,q}$ and satisfies

$$\|\lambda|n^s_{\varphi,p,q}\| \le C\|\mu|n^s_{\varphi,p,q}\|; \tag{3.6}$$

(ii) for any cube Q there is a constant $c_Q > 0$ such that for all $\mu \in n^s_{\varphi,p,q}$

$$|\mu_{J,M}| \le C_Q 2^{J\varkappa} \|\mu|n^s_{\varphi,p,q}\| \quad \text{for all} \quad J \in \mathbb{N}_0, \ M \in \mathbb{Z}^d \text{ with } Q_{J,M} \subset Q. \tag{3.7}$$

Once these properties are verified, Theorem 5.1 in [6] will imply our Theorem 3.1.

In the next steps we prove the above properties. The following notation and remarks will be useful. For fixed $b>1$, $C_1>0$ we consider two sets of indices $I_J^j(m)$, that were defined above, and $\hat{I}_J^j(M)$ given by the following formula

$$\hat{I}_J^j(M)=\left\{m\in\mathbb{Z}^d:\ bQ_{J,M}\cap C_1Q_{j,m}\neq\emptyset\right\},\quad j,J\in\mathbb{N}_0,\ M\in\mathbb{Z}^d.$$

Please note that the cardinalities of $\#I_J^j(m)$ and $\#\hat{I}_J^j(M)$ satisfy

$$\#I_J^j(m)\sim\begin{cases}1, & J\le j,\\ 2^{d(J-j)}, & J>j,\end{cases}\quad\text{and}\quad \#\hat{I}_J^j(M)\sim\begin{cases}1, & j\le J,\\ 2^{d(j-J)}, & j>J\end{cases}\tag{3.8}$$

and that there is a constant $\eta=\eta(b,C_1)\in\mathbb{N}$ such that if $Q_{j,m}\subset Q_{\nu,k}$ and $bQ_{j,m}\cap C_1Q_{J,M}\neq\emptyset$, then $Q_{J,M}\subset Q_{\nu-\eta,\ell}$ for some dyadic cube $Q_{\nu-\eta,\ell}$, $\ell=\ell(k)$, such that $Q_{\nu,k}\subset Q_{\nu-\eta,\ell}$.

Step 2. We prove the property (i) for $0<p\le 1$. We decompose the sum in (3.5) into two parts for $J\le j$ and for $J>j$. Let $\nu\in\mathbb{Z}$ and $k\in\mathbb{Z}^d$ be fixed, with $\nu\le j$. Then

$$\begin{aligned}\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}}|\lambda_{j,m}|^p &\le C_1\sum_{J=0}^{j}2^{-\varkappa(j-J)p}\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}}\ \sum_{M\in I_J^j(m)}|\mu_{J,M}|^p\\ &\quad+C_1\sum_{J=j+1}^{\infty}2^{-(\varkappa+d)(J-j)p}\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}}\ \sum_{M\in I_J^j(m)}|\mu_{J,M}|^p\\ &\le C_1\sum_{J=0}^{j}2^{-\varkappa(j-J)p}\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu-\eta,\ell}}}\#\hat{I}_J^j(M)\,|\mu_{J,M}|^p\\ &\quad+C_1\sum_{J=j+1}^{\infty}2^{-(\varkappa+d)(J-j)p}\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu-\eta,\ell}}}\#\hat{I}_J^j(M)\,|\mu_{J,M}|^p\\ &\le C_1\sum_{J=0}^{j}2^{-(\varkappa-\frac{d}{p})(j-J)p}\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu-\eta,\ell}}}|\mu_{J,M}|^p\\ &\quad+C_1\sum_{J=j+1}^{\infty}2^{-(\varkappa+d)(J-j)p}\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu-\eta,\ell}}}|\mu_{J,M}|^p,\end{aligned}\tag{3.9}$$

where the last inequality follows from (3.8) and the last but one follows from the definition of the set $\hat{I}_J^j(M)$.

If $\frac{q}{p}\le 1$, then

$$\begin{aligned}\Big(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}}|\lambda_{j,m}|^p\Big)^{\frac{q}{p}} &\le C_1\sum_{J=0}^{j}2^{-(\varkappa-\frac{d}{p})(j-J)q}\Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu-\eta,\ell}}}|\mu_{J,M}|^p\Big)^{\frac{q}{p}}\\ &\quad+C_1\sum_{J=j+1}^{\infty}2^{-(\varkappa+d)(J-j)q}\Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu-\eta,\ell}}}|\mu_{J,M}|^p\Big)^{\frac{q}{p}}.\end{aligned}\tag{3.10}$$

If $\frac{q}{p}>1$, then for any $\varepsilon>0$ we get, using the Hölder inequality,

$$\Big(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}}|\lambda_{j,m}|^p\Big)^{\frac{q}{p}}\le C_1\sum_{J=0}^{j}2^{-(\varkappa-\varepsilon-\frac{d}{p})(j-J)q}\Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu-\eta,\ell}}}|\mu_{J,M}|^p\Big)^{\frac{q}{p}}\tag{3.11}$$

$$+ C_1 \sum_{J=j+1}^{\infty} 2^{-(\varkappa-\varepsilon+d)(J-j)q} \Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu-\eta,\ell}}} |\mu_{J,M}|^p \Big)^{\frac{q}{p}}.$$

So in both cases we have

$$\begin{aligned}
&\sup_{\substack{\nu:\nu\le j\\ k\in\mathbb{Z}^d}} \varphi(2^{-\nu})^q\, 2^{\nu d\frac{q}{p}} \Big(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^p \Big)^{\frac{q}{p}} \qquad (3.12)\\
\le\;& 2^{\eta d\frac{q}{p}} C_1 \sum_{J=0}^{j} 2^{-(\varkappa-\varepsilon-\frac{d}{p})(j-J)q} \sup_{\substack{\nu:\nu\le j\\ \ell\in\mathbb{Z}^d}} \varphi(2^{-\nu+\eta})^q\, 2^{(\nu-\eta)d\frac{q}{p}} \Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu-\eta,\ell}}} |\mu_{J,M}|^p \Big)^{\frac{q}{p}}\\
&+ 2^{\eta d\frac{q}{p}} C_1 \sum_{J=j+1}^{\infty} 2^{-(\varkappa-\varepsilon+d)(J-j)q} \sup_{\substack{\nu:\nu\le j\\ \ell\in\mathbb{Z}^d}} \varphi(2^{-\nu+\eta})^q\, 2^{(\nu-\eta)d\frac{q}{p}} \Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu-\eta,\ell}}} |\mu_{J,M}|^p \Big)^{\frac{q}{p}}\\
\le\;& 2^{\eta d\frac{q}{p}} C_1 \sum_{J=0}^{j} 2^{-(\varkappa-\varepsilon-\frac{d}{p})(j-J)q} \sup_{\substack{\nu:\nu\le J\\ k\in\mathbb{Z}^d}} \varphi(2^{-\nu})^q\, 2^{\nu d\frac{q}{p}} \Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu,k}}} |\mu_{J,M}|^p \Big)^{\frac{q}{p}}\\
&+ 2^{\eta d\frac{q}{p}} C_1 \sum_{J=j+1}^{\infty} 2^{-(\varkappa-\varepsilon+d)(J-j)q} \sup_{\substack{\nu:\nu\le J\\ k\in\mathbb{Z}^d}} \varphi(2^{-\nu})^q\, 2^{\nu d\frac{q}{p}} \Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu,k}}} |\mu_{J,M}|^p \Big)^{\frac{q}{p}}.
\end{aligned}$$

The first inequality follows from (3.10) and (3.11) and the fact that any k appoints one $\ell = \ell(k)$, so the supremum over k can be dominated by the supremum over ℓ. The second inequality follows by rescaling.

In consequence,

$$\begin{aligned}
&\|\lambda|n^s_{\varphi,p,q}\|^q \qquad (3.13)\\
&\le c\sum_{j=0}^{\infty} 2^{j(s-\frac{d}{p})q} \sum_{J=0}^{j} 2^{-(\varkappa-\varepsilon-\frac{d}{p})(j-J)q} \sup_{\substack{\nu:\nu\le J\\ k\in\mathbb{Z}^d}} \varphi(2^{-\nu})^q\, 2^{\nu d\frac{q}{p}} \Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu,k}}} |\mu_{J,M}|^p \Big)^{\frac{q}{p}}\\
&\quad + c\sum_{j=0}^{\infty} 2^{j(s-\frac{d}{p})q} \sum_{J=j+1}^{\infty} 2^{-(\varkappa-\varepsilon+d)(J-j)q} \sup_{\substack{\nu:\nu\le J\\ k\in\mathbb{Z}^d}} \varphi(2^{-\nu})^q\, 2^{\nu d\frac{q}{p}} \Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu,k}}} |\mu_{J,M}|^p \Big)^{\frac{q}{p}}\\
&\le c\sum_{j=0}^{\infty} \sum_{J=0}^{j} 2^{J(s-\frac{d}{p})q} 2^{-(\varkappa-\varepsilon-s)(j-J)q} \sup_{\substack{\nu:\nu\le J\\ k\in\mathbb{Z}^d}} \varphi(2^{-\nu})^q\, 2^{\nu d\frac{q}{p}} \Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu,k}}} |\mu_{J,M}|^p \Big)^{\frac{q}{p}}\\
&\quad + c\sum_{j=0}^{\infty} \sum_{J=j+1}^{\infty} 2^{J(s-\frac{d}{p})q} 2^{-(\varkappa-\varepsilon-\sigma_p+s)(J-j)q} \sup_{\substack{\nu:\nu\le J\\ k\in\mathbb{Z}^d}} \varphi(2^{-\nu})^q\, 2^{\nu d\frac{q}{p}} \Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu,k}}} |\mu_{J,M}|^p \Big)^{\frac{q}{p}}\\
&\le c\sum_{J=0}^{\infty} 2^{J(s-\frac{d}{p})q} \sup_{\substack{\nu:\nu\le J\\ k\in\mathbb{Z}^d}} \varphi(2^{-\nu})^q\, 2^{\nu d\frac{q}{p}} \Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu,k}}} |\mu_{J,M}|^p \Big)^{\frac{q}{p}} \sum_{j=J}^{\infty} 2^{-(\varkappa-\varepsilon-s)(j-J)q}\\
&\quad + c\sum_{J=1}^{\infty} 2^{J(s-\frac{d}{p})q} \sup_{\substack{\nu:\nu\le J\\ k\in\mathbb{Z}^d}} \varphi(2^{-\nu})^q\, 2^{\nu d\frac{q}{p}} \Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu,k}}} |\mu_{J,M}|^p \Big)^{\frac{q}{p}} \sum_{j=0}^{J-1} 2^{-(\varkappa-\varepsilon-\sigma_p+s)(J-j)q}\\
&\le c\|\mu|n^s_{\varphi,p,q}\|^q
\end{aligned}$$

if we choose $\varepsilon > 0$ such that $\varkappa-\varepsilon-\sigma_p+s > 0$ and $\varkappa-\varepsilon-s > 0$. This is always possible if $\varkappa > \max\{\sigma_p-s, s\}$, that is, we need $L > \max\{\sigma_p-s, s\}$ here which is implied by (3.4). This finishes the proof of (i) for $0 < p \le 1$.

Step 3. Now we prove the property (i) for $p > 1$. Applying the Hölder inequality twice yields for some $\varepsilon > 0$,

$$\sum_{J=0}^{j} 2^{-\varkappa(j-J)} \sum_{M\in I_J^j(m)} |\mu_{J,M}| \le c_2 \Big(\sum_{J=0}^{j} 2^{-(\varkappa-\varepsilon)(j-J)p} \sum_{M\in I_J^j(m)} |\mu_{J,M}|^p \Big)^{\frac{1}{p}} \tag{3.14}$$

and

$$\sum_{J=j+1}^{\infty} 2^{-(\varkappa+d)(J-j)} \sum_{M\in I_J^j(m)} |\mu_{J,M}| \le c_2 \Big(\sum_{J=j+1}^{\infty} 2^{-(\varkappa-\varepsilon+\frac{d}{p})(J-j)p} \sum_{M\in I_J^j(m)} |\mu_{J,M}|^p \Big)^{\frac{1}{p}}, \tag{3.15}$$

in view of (3.8). Now using (3.14) and (3.15) and a similar method as in (3.9) we can prove that

$$\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^p \le C_1 \sum_{J=0}^{j} 2^{-(\varkappa-\varepsilon-\frac{d}{p})(j-J)p} \sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu-\eta,\ell}}} |\mu_{J,M}|^p + C_1 \sum_{J=j+1}^{\infty} 2^{-(\varkappa-\varepsilon+\frac{d}{p})(J-j)p} \sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{\nu-\eta,\ell}}} |\mu_{J,M}|^p. \tag{3.16}$$

The rest of the proof goes similarly as in the case $p \le 1$. Now we should choose $\varepsilon > 0$ such that $\varkappa - 2\varepsilon - \sigma_p + s = \varkappa - 2\varepsilon + s > 0$ and $\varkappa - 2\varepsilon - s > 0$. In other words, we need $\varkappa$ to satisfy $L > \varkappa > |s|$, but this is again possible in view of (3.4).

Step 4. The proof of the property (ii) is straightforward. Let Q be some cube, $J \in \mathbb{N}_0$ and $M \in \mathbb{Z}^d$ such that $Q_{J,M} \subset Q$. Then we have

$$\begin{aligned} |\mu_{J,M}| &\le \sup_{k\in\mathbb{Z}^d} \Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}}} \subset Q_{0,k} |\mu_{J,M}|^p \Big)^{1/p} \sim 2^{J\frac{d}{p}} \sup_{k\in\mathbb{Z}^d} \varphi(2^{-0}) 2^{(0-J)\frac{d}{p}} \Big(\sum_{\substack{M\in\mathbb{Z}^d:\\ Q_{J,M}\subset Q_{0,k}}} |\mu_{J,M}|^p \Big)^{\frac{1}{p}} \\ &\le 2^{J(\frac{d}{p}-s)} 2^{Js} \sup_{\substack{\nu:\nu\le J\\ k\in\mathbb{Z}^d}} \varphi(2^{-\nu})\, 2^{(\nu-J)\frac{d}{p}} \Big(\sum_{\substack{N\in\mathbb{Z}^d:\\ Q_{J,N}\subset Q_{\nu,k}}} |\mu_{J,N}|^p \Big)^{\frac{1}{p}} \\ &\le c 2^{J(\frac{d}{p}-s)} \, \|\mu | n^s_{\varphi,p,\infty}\| \le c 2^{J\varkappa} \, \|\mu | n^s_{\varphi,p,q}\|. \end{aligned}$$

So the estimate holds with the same constant for any cube Q and $\varkappa > (\frac{d}{p} - s)_+$. In view of (3.4) it is always possible to find $\varkappa$ such that $L > \varkappa > (\frac{d}{p} - s)_+$. This concludes the proof. □

Remark 3.2. As in the paper [6] we do not claim the condition in (3.4) to be sharp, the assumption on L is just taken for convenience, following the argument in [6]. Moreover, for our purposes, that is, to transfer our sequence space results from Section 4 to the function space counterparts in Section 5, it is absolutely sufficient to find *some* number L satisfying (3.4). But we did not care for minimal assumptions. The result for the 'classical' case $\varphi(t) = t^{d/u}$, $t > 0$, $0 < p \le u < \infty$, can be found in [22, Thm. 4.5, Cor. 4.17] and [23].

4. Embeddings of generalised Besov–Morrey sequence spaces

First we deal with the embeddings of generalised Besov–Morrey sequence spaces $n^s_{\varphi,p,q}$, for the definitions we refer to Section 2. These sequence spaces appear naturally when applying the wavelet decomposition result Theorem 3.1 for generalised Besov–Morrey (function) spaces.

Theorem 4.1. *Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \le \infty$, and $\varphi_i \in \mathcal{G}_{p_i}$, for $i = 1, 2$. We assume without loss of generality that $\varphi_1(1) = \varphi_2(1) = 1$. Let $\varrho = \min(1, \frac{p_1}{p_2})$ and $\alpha_j = \sup_{\nu\le j} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^{\varrho}}$, $j \in \mathbb{N}_0$.*

There is a continuous embedding

$$n^{s_1}_{\varphi_1,p_1,q_1} \hookrightarrow n^{s_2}_{\varphi_2,p_2,q_2} \tag{4.1}$$

if and only if

$$\sup_{\nu\le 0} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^{\varrho}} < \infty, \tag{4.2}$$

and

$$\left\{ 2^{j(s_2-s_1)} \alpha_j \frac{\varphi_1(2^{-j})^{\varrho}}{\varphi_1(2^{-j})} \right\}_j \in \ell_{q^*} \qquad \textit{where} \quad \frac{1}{q^*} = \left(\frac{1}{q_2} - \frac{1}{q_1} \right)_+ . \tag{4.3}$$

The embedding (4.1) is never compact.

Proof. *Step 1.* First we consider the sufficiency of the conditions (4.2)–(4.3). Please note that it follows from (4.2) that the supremum defining α_j is finite, so the sequence $(\alpha_j)_j$ is well defined.

We start by proving some inequalities for any fixed $j \in \mathbb{N}_0$. If $p_2 \le p_1$, i.e., $\varrho = 1$, then we have the following inequality

$$\begin{aligned} &\sup_{\substack{\nu:\nu\le j\\ k\in\mathbb{Z}^d}} \varphi_2(2^{-\nu}) 2^{(\nu-j)\frac{d}{p_2}} \Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{p_2} \Bigg)^{\frac{1}{p_2}} \\ &\qquad \le \alpha_j \sup_{\substack{\nu:\nu\le j\\ k\in\mathbb{Z}^d}} \varphi_1(2^{-\nu}) 2^{(\nu-j)\frac{d}{p_1}} \Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{p_1} \Bigg)^{\frac{1}{p_1}} . \end{aligned} \tag{4.4}$$

Indeed, for any $\nu \le j$ we have

$$\begin{aligned} &\varphi_2(2^{-\nu}) 2^{(\nu-j)\frac{d}{p_2}} \Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{p_2} \Bigg)^{\frac{1}{p_2}} \\ &\qquad \le \varphi_2(2^{-\nu}) 2^{(\nu-j)\frac{d}{p_2}} 2^{(j-\nu)d\left(\frac{1}{p_2}-\frac{1}{p_1}\right)} \Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{p_1} \Bigg)^{\frac{1}{p_1}} \\ &\qquad \le \alpha_j \varphi_1(2^{-\nu}) 2^{(\nu-j)\frac{d}{p_1}} \Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{p_1} \Bigg)^{\frac{1}{p_1}}, \end{aligned}$$

where the first inequality follows by Hölder's inequality. Taking the supremum over $\nu \le j$ and $k \in \mathbb{Z}^d$ we get (4.4).

If $p_1 < p_2$, i.e., $\varrho < 1$, then

$$\begin{aligned} &\sup_{\substack{\nu:\nu\le j\\ k\in\mathbb{Z}^d}} \varphi_2(2^{-\nu}) 2^{(\nu-j)\frac{d}{p_2}} \Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{p_2} \Bigg)^{\frac{1}{p_2}} \\ &\qquad \le \alpha_j \frac{\varphi_1(2^{-j})^{\varrho}}{\varphi_1(2^{-j})} \sup_{\substack{\nu:\nu\le j\\ k\in\mathbb{Z}^d}} \varphi_1(2^{-\nu}) 2^{(\nu-j)\frac{d}{p_1}} \Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{p_1} \Bigg)^{\frac{1}{p_1}} . \end{aligned} \tag{4.5}$$

It is sufficient to prove (4.5) for sequences $(\lambda_{j,m})_m$ satisfying the following assumption

$$\sup_{\substack{\nu:\nu\le j\\ k\in\mathbb{Z}^d}} \varphi_1(2^{-\nu}) 2^{(\nu-j)\frac{d}{p_1}} \Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{p_1} \Bigg)^{\frac{1}{p_1}} = 1. \tag{4.6}$$

In this case $\varphi_1(2^{-j})|\lambda_{j,m}| \le 1$ for any m. So

$$\varphi_1(2^{-j})^{p_2} \sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{p_2} \le \varphi_1(2^{-j})^{p_1} \sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{p_1} \le 2^{(j-\nu)d}\left(\frac{\varphi_1(2^{-j})}{\varphi_1(2^{-\nu})}\right)^{p_1}, \ \nu \le j,$$

and

$$\varphi_2(2^{-\nu})2^{(\nu-j)\frac{d}{p_2}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{p_2}\Bigg)^{\frac{1}{p_2}} \le \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^{\varrho}}\varphi_1(2^{-j})^{\varrho-1} \le \alpha_j\varphi_1(2^{-j})^{\varrho-1}, \ \nu\le j.$$

Taking the supremum we get (4.5).

Step 2. Now we prove sufficiency. The inequality (4.5) coincides with (4.4) if we take $\varrho = 1$, so we can work with (4.5) and $\varrho \le 1$.

From (4.5), for $j \in \mathbb{N}_0$ we have

$$\begin{aligned} &2^{s_2 j}\sup_{\substack{\nu:\nu\le j\\ k\in\mathbb{Z}^d}} \varphi_2(2^{-\nu})2^{(\nu-j)\frac{d}{p_2}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{p_2}\Bigg)^{\frac{1}{p_2}} \\ &\quad\le 2^{j(s_2-s_1)}\alpha_j\frac{\varphi_1(2^{-j})^{\varrho}}{\varphi_1(2^{-j})}2^{s_1 j}\sup_{\substack{\nu:\nu\le j\\ k\in\mathbb{Z}^d}} \varphi_1(2^{-\nu})2^{(\nu-j)\frac{d}{p_1}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu,k}}} |\lambda_{j,m}|^{p_1}\Bigg)^{\frac{1}{p_1}}. \end{aligned} \tag{4.7}$$

If $q_1 = \infty$, thus $q^* = q_2$, by (4.7) and (4.3) we clearly get

$$\|\lambda \mid n^{s_2}_{\varphi_2,p_2,q_2}\| \le \left\|\left\{2^{j(s_2-s_1)}\alpha_j\frac{\varphi_1(2^{-j})^{\varrho}}{\varphi_1(2^{-j})}\right\}_j \mid \ell_{q_2}\right\| \|\lambda \mid n^{s_1}_{\varphi_1,p_1,\infty}\|.$$

If $q_1 < \infty$ and $q_2 \ge q_1$, then $q^* = \infty$, and (4.7) together with (4.3) yield

$$\|\lambda \mid n^{s_2}_{\varphi_2,p_2,q_2}\| \le \|\lambda \mid n^{s_2}_{\varphi_2,p_2,q_1}\| \le \left\|\left\{2^{j(s_2-s_1)}\alpha_j\frac{\varphi_1(2^{-j})^{\varrho}}{\varphi_1(2^{-j})}\right\}_j \mid \ell_{\infty}\right\| \|\lambda \mid n^{s_1}_{\varphi_1,p_1,q_1}\|.$$

Finally, in case of $q_2 < q_1 < \infty$, by (4.7) and Hölder's inequality we obtain

$$\begin{aligned} \|\lambda \mid n^{s_2}_{\varphi_2,p_2,q_2}\| &\le \left\|\left\{2^{j(s_2-s_1)}\frac{\varphi_2(2^{-j})}{\varphi_1(2^{-j})}\right\}_j \mid \ell_{\frac{q_1q_2}{q_1-q_2}}\right\| \|\lambda \mid n^{s_1}_{\varphi_1,p_1,q_1}\| \\ &\le \left\|\left\{2^{j(s_2-s_1)}\alpha_j\frac{\varphi_1(2^{-j})^{\varrho}}{\varphi_1(2^{-j})}\right\}_j \mid \ell_{q^*}\right\| \|\lambda \mid n^{s_1}_{\varphi_1,p_1,q_1}\| \end{aligned} \tag{4.8}$$

thanks to $q^* = \frac{q_1q_2}{q_1-q_2}$, see (4.3).

Step 3. It remains to prove the necessity of the conditions. First we prove that the embedding (4.1) implies (4.2).

Substep 3.1 We fix $j_0 \ge 0$, $\nu_0 \le j_0$ and consider the sequence $\lambda^{(j_0,\nu_0)}$ defined as follows

$$\lambda^{(j_0,\nu_0)}_{j,m} = \begin{cases} \varphi_1(2^{-\nu_0})^{-1} & \text{if} \quad j = j_0 \quad \text{and} \quad Q_{j,m}\subset Q_{\nu_0,0}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

Then

$$\varphi_1(2^{-\nu})2^{(\nu-j)\frac{d}{p_1}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu_0,k}}}|\lambda^{(j_0,\nu_0)}_{j,m}|^{p_1}\Bigg)^{\frac{1}{p_1}} \le \frac{\varphi_1(2^{-\nu})}{\varphi_1(2^{-\nu_0})}\begin{cases}1 & \text{if} \quad j=j_0 \quad \text{and} \quad Q_{\nu,k}\subset Q_{\nu_0,0},\\ 2^{(\nu-\nu_0)\frac{d}{p_1}} & \text{if} \quad j=j_0 \quad \text{and} \quad Q_{\nu_0,0}\subset Q_{\nu,k},\\ 0 & \text{otherwise.}\end{cases} \tag{4.10}$$

The function φ_1 belongs to the class $\mathcal{G}_{p_1}$ therefore $\varphi_1(2^{-\nu})\varphi_1(2^{-\nu_0})^{-1}2^{(\nu-\nu_0)\frac{d}{p_1}} \le 1$ if $Q_{\nu_0,0}\subset Q_{\nu,k}$ and $\varphi_1(2^{-\nu})\varphi_1(2^{-\nu_0})^{-1}\le 1$ if $Q_{\nu,k}\subset Q_{\nu_0,0}$. In consequence

$$\|\lambda^{(j_0,\nu_0)}|n^{s_1}_{\varphi_1,p_1,q_1}\| = 2^{j_0s_1}\sup_{\substack{\nu:\nu\le j_0\\ k\in\mathbb{Z}^d}}\varphi_1(2^{-\nu})\,2^{(\nu-j_0)\frac{d}{p_1}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j_0,m}\subset Q_{\nu,k}}}|\lambda^{(j_0,\nu_0)}_{j_0,m}|^{p_1}\Bigg)^{\frac{1}{p_1}} = 2^{j_0s_1}. \tag{4.11}$$

In a similar way we prove that

$$\begin{aligned}\|\lambda^{(j_0,\nu_0)}|n^{s_2}_{\varphi_2,p_2,q_2}\| &= 2^{j_0s_2}\sup_{\substack{\nu:\nu\le j_0\\ k\in\mathbb{Z}^d}}\varphi_2(2^{-\nu})\,2^{(\nu-j_0)\frac{d}{p_2}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j_0,m}\subset Q_{\nu,k}}}|\lambda^{(j_0,\nu_0)}_{j_0,m}|^{p_2}\Bigg)^{\frac{1}{p_2}}\\ &= 2^{j_0s_2}\frac{\varphi_2(2^{-\nu_0})}{\varphi_1(2^{-\nu_0})}.\end{aligned} \tag{4.12}$$

So if the embedding (4.1) holds, then

$$\frac{\varphi_2(2^{-\nu_0})}{\varphi_1(2^{-\nu_0})} \le C2^{j_0(s_1-s_2)}. \tag{4.13}$$

Moreover the constant C is independent of j_0 and ν_0. So if $p_1\ge p_2$, i.e., $\varrho=1$, we can fix $j_0=0$. This proves (4.2).

Substep 3.2. Let $p_1<p_2$, i.e., $\varrho=\frac{p_1}{p_2}$. Once more we fix $j_0\in\mathbb{N}_0$ and $\nu_0\in\mathbb{Z}$ with $\nu_0\le j_0$. Let $N\in\mathbb{N}$ be such that $1\le N\le 2^{(j_0-\nu_0)d}$. We define a sequence $\lambda^{(N)}=(\lambda^{(N)}_{j,m})$ such that

$$(1)\quad \lambda^{(N)}_{j,m}=1 \quad \text{or} \quad \lambda^{(N)}_{j,m}=0 \quad \text{for any} \quad (j,m), \tag{4.14}$$

$$(2)\quad \lambda^{(N)}_{j,m}=0 \quad \text{if} \quad j\ne j_0 \quad \text{or} \quad Q_{j_0,m}\not\subseteq Q_{\nu_0,0}, \tag{4.15}$$

$$(3)\quad \lambda^{(N)}_{j_0,m}=1 \quad \text{exactly} \quad N \quad \text{times,} \tag{4.16}$$

$$(4)\quad \text{if} \quad Q_{\nu,k}\subset Q_{\nu_0,0}, \quad \nu_0<\nu<j_0, \quad \text{then} \quad Q_{\nu,k} \quad \text{contains at most} \quad 2^{d(\nu_0-\nu)}N+2 \quad \text{cubes} \quad Q_{j_0,m} \quad \text{such that} \quad \lambda^{(N)}_{j_0,m}=1. \tag{4.17}$$

If $N=2^{(j_0-\nu_0)d}$, then we can simply take $\lambda^{(N)}_{j_0,m}=1$ for any cube $Q_{j_0,m}\subset Q_{\nu_0,0}$. So let us assume $N<2^{(j_0-\nu_0)d}$. We put $\lceil x\rceil=\min\{k\in\mathbb{Z}:\ k\ge x\}$, $x\in\mathbb{R}$.

Let $M_1=\lceil 2^{-d}N\rceil$. If $M_1=1$, i.e., $N\le 2^d$, we put $\lambda^{(N)}_{j_0,m}=1$ for at most one cube $Q_{j_0,m}$ in any cube $Q_{\nu_0+1,k}\subset Q_{\nu_0,0}$ in such a way that we do not exceed the total number N and we finish the construction.

Let $M_1>1$ and let $Q_{\nu_0,0}=\bigcup_{i=1}^{2^d}Q_{\nu_0+1,k_i}$. Now we represent N as the following sum

$$N=N^{(1)}_{k_1}+\cdots+N^{(1)}_{k_{2^d}} \tag{4.18}$$

where

$$N^{(1)}_{k_i}=\begin{cases}M_1 & \text{if} \quad iM_1\le N\\ N-(i-1)M_1 & \text{if} \quad (i-1)M_1<N<iM_1,\\ 0 & \text{if} \quad N=N^{(1)}_{k_1}+\cdots+N^{(1)}_{k_{i-1}}.\end{cases} \tag{4.19}$$

We group the elements $\lambda^{(N)}_{j_0,m}$ of the sequence in such a way that exactly $N^{(1)}_{k_i}$ elements related to the cube Q_{ν_0+1,k_i} are equal to 1.

Next we repeat the procedure for any cube Q_{ν_0+1,k_i}. We define $M^{(2)}_{k_i} = \lceil 2^{-d} N^{(1)}_{k_i} \rceil$. If $M^{(2)}_{k_i} = 1$, i.e., $M^{(2)}_{k_i} \le 2^d$, we put $\lambda^{(N)}_{j_0,m} = 1$ for at most one cube $Q_{j_0,m}$ in any cube $Q_{\nu_0+2,k} \subset Q_{\nu_0+1,k_i}$ in such a way that we do not exceed the total number $N^{(1)}_{k_i}$ and we finish the construction on the cube Q_{ν_0+1,k_i}.

If $M^{(2)}_{k_i} > 1$ and $Q_{\nu_0+1,k_i} = \bigcup_{j=1}^{2^d} Q_{\nu_0+2,k_j}$, then we represent $N^{(1)}_{k_i}$ as a sum

$$N^{(1)}_{k_i} = N^{(2)}_{k_i,k_1} + \cdots + N^{(2)}_{k_i,k_j} + \cdots + N^{(2)}_{k_i,k_{2^d}} \tag{4.20}$$

where the numbers $N^{(2)}_{k_i,k_j}$ are defined in a similar way to (4.19) with $N^{(1)}_{k_i}$ instead of N and $M^{(2)}_{k_i}$ instead of M_1.

In the next steps we define $M^{(3)}_{k_i,k_j} = \lceil 2^{-d} N^{(2)}_{k_i,k_j} \rceil$ and so on. The procedure stops after at most $2^{(j_0-\nu_0)}$ steps. One can easily see that for any ν, $\nu_0 < \nu = \nu_0 + \eta < j_0$ we have

$$N^{(\eta)}_{k_i,k_j,\ldots,k_\ell} \le 2^{-d(\nu-\nu_0)} N + \sum_{i=0}^{\eta-1} 2^{-di} \le 2^{-d(\nu-\nu_0)} N + \frac{1}{1-2^{-d}}. \tag{4.21}$$

Now we take $j_0 = 0$ and $N = \lceil 2^{(j_0-\nu_0)d} \varphi_1(2^{-\nu_0})^{-p_1} \rceil \le 2^{(j_0-\nu_0)d}$. If $Q_{\nu,k} \subset Q_{\nu_0,0}$, then

$$\begin{aligned} &\varphi_1(2^{-\nu}) 2^{(\nu-j_0)\frac{d}{p_1}} \Bigg(\sum_{\substack{m\in\mathbb{Z}^d: \\ Q_{j_0,m}\subset Q_{\nu,k}}} |\lambda^{(N)}_{j,m}|^{p_1} \Bigg)^{\frac{1}{p_1}} \\ &\quad \le 2^{\frac{1}{p_1}} \varphi_1(2^{-\nu}) 2^{(\nu-j_0)\frac{d}{p_1}} \max(2^{-d(\nu-\nu_0)} N, 2)^{\frac{1}{p_1}} \\ &\quad \le 2^{\frac{1}{p_1}} \max\left(\frac{\varphi_1(2^{-\nu})}{\varphi_1(2^{-\nu_0})}, 2^{\frac{1}{p_1}} \varphi_1(2^{-\nu}) 2^{(\nu-j_0)\frac{d}{p_1}} \right) \le C \end{aligned} \tag{4.22}$$

since $\nu_0 < \nu \le 0$, $\varphi_1 \in \mathcal{G}_{p_1}$ and $\varphi_1(1) = 1$. The constant C is independent of ν_0. In consequence $\|\lambda^{(N)}|n^{s_1}_{\varphi_1,p_1,q_1}\| \le C$. So if the embedding (4.1) holds, then

$$\begin{aligned} \frac{\varphi_2(2^{-\nu_0})}{\varphi_1(2^{-\nu_0})^{\varrho}} &\le \varphi_2(2^{-\nu_0}) 2^{\nu_0\frac{d}{p_2}} \left(\varphi_1(2^{-\nu_0})^{-p_1} 2^{-\nu_0 d} \right)^{\frac{1}{p_2}} \\ &\le \varphi_2(2^{-\nu_0}) 2^{\nu_0\frac{d}{p_2}} N^{\frac{1}{p_2}} \\ &\le \|\lambda^{(N)}|n^{s_2}_{\varphi_2,p_2,q_2}\| \le C, \end{aligned}$$

proving (4.2) when $p_1 < p_2$.

Step 4. We prove that the assumption (4.3) is necessary. If there exists $\nu_0 \le 0$ such that $\alpha_0 = \frac{\varphi_2(2^{-\nu_0})}{\varphi_1(2^{-\nu_0})^{\varrho}}$, then also for $j \in \mathbb{N}$ we can find $\nu_j \le 0$ such that $\alpha_j = \frac{\varphi_2(2^{-\nu_j})}{\varphi_1(2^{-\nu_j})^{\varrho}}$. If the supremum defining α_0 is not attained, then there exist $\nu_0 \le 0$, and in consequence $\nu_j \le j$, such that

$$\frac{\varphi_2(2^{-\nu_j})}{\varphi_1(2^{-\nu_j})^{\varrho}} \le \alpha_j < 2 \frac{\varphi_2(2^{-\nu_j})}{\varphi_1(2^{-\nu_j})^{\varrho}}, \quad j \in \mathbb{N}_0. \tag{4.23}$$

We used the modified version of the sequences constructed in Substep 3.2.

Substep 4.1 First we assume that $p_1 \ge p_2$, i.e., $\varrho = 1$. Let $q_1 \le q_2$, i.e., $q^* = \infty$, and $i \in \mathbb{N}_0$. We consider the sequence $\lambda^{(i)} = (\lambda^{(i)}_{j,m})$ defined by

$$\lambda^{(i)}_{j,m} = \begin{cases} 2^{-is_1} \alpha_i \varphi_2(2^{-\nu_i})^{-1} & \text{if} \quad j = i \quad \text{and} \quad Q_{i,m} \subset Q_{\nu_i,0}, \\ 0 & \text{otherwise}. \end{cases} \tag{4.24}$$

We prove that there is a positive $C > 0$ such that $\|\lambda^{(i)}|n^{s_1}_{\varphi_1,p_1,q_1}\| \le C$ for any i.

Let $\nu_i \le \nu \le i$. Then

$$\varphi_1(2^{-\nu})2^{(\nu-i)\frac{d}{p_1}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{i,m}\subset Q_{\nu,k}}} \big|\lambda^{(i)}_{i,m}\big|^{p_1}\Bigg)^{\frac{1}{p_1}} \le C2^{-is_1}\frac{\varphi_1(2^{-\nu})}{\varphi_1(2^{-\nu_i})}2^{(\nu-i)\frac{d}{p_1}}2^{(i-\nu)\frac{d}{p_1}} \le C2^{-is_1}. \tag{4.25}$$

If $\nu < \nu_i$, then

$$\begin{aligned}
\varphi_1(2^{-\nu})2^{(\nu-i)\frac{d}{p_1}}&\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{i,m}\subset Q_{\nu,k}}} \big|\lambda^{(i)}_{i,m}\big|^{p_1}\Bigg)^{\frac{1}{p_1}}\\
&\le \varphi_1(2^{-\nu_i})\frac{\varphi_1(2^{-\nu})}{\varphi_1(2^{-\nu_i})}2^{(\nu-i)\frac{d}{p_1}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{i,m}\subset Q_{\nu_i,0}}} \big|\lambda^{(i)}_{i,m}\big|^{p_1}\Bigg)^{\frac{1}{p_1}}\\
&\le \varphi_1(2^{-\nu_i})2^{(\nu_i-i)\frac{d}{p_1}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{i,m}\subset Q_{\nu_i,0}}} \big|\lambda^{(i)}_{i,m}\big|^{p_1}\Bigg)^{\frac{1}{p_1}} \le C2^{-is_1}
\end{aligned} \tag{4.26}$$

where the last but one inequality follows from the inclusion $\varphi_1 \in \mathcal{G}_{p_1}$. The inequalities (4.25) and (4.26) give us $\|\lambda^{(i)}|n^{s_1}_{\varphi_1,p_1,q_1}\| \le C$. So if the embedding (4.1) holds, then

$$2^{i(s_2-s_1)}\alpha_i = 2^{is_2}\varphi_2(2^{-\nu_i})2^{(\nu_i-i)\frac{d}{p_2}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{i,m}\subset Q_{\nu_i,0}}} \big|\lambda^{(i)}_{i,m}\big|^{p_2}\Bigg)^{\frac{1}{p_2}} \le C\|\lambda^{(i)}|n^{s_1}_{\varphi_1,p_1,q_1}\| \le C. \tag{4.27}$$

Thus $\|\{2^{i(s_2-s_1)}\alpha_i\}_i|\ell_\infty\| \le C$ which is (4.3) in this case.

Now let $q_2 < q_1$, i.e., $q^* < \infty$. Let $\mu = (\mu_j)_j \in \ell_{q_1}$ and $\|\mu|\ell_{q_1}\| = 1$. We consider the sequence $\lambda = (\lambda_{j,m})$ defined by the formula

$$\lambda_{j,m} = \begin{cases} 2^{-js_1}\alpha_j\varphi_2(2^{-\nu_j})^{-1}\mu_j & \text{if} \quad Q_{j,m}\subset Q_{\nu_j,0},\\ 0 & \text{otherwise.}\end{cases}$$

In the same way as above we show that $\|\lambda|n^{s_1}_{\varphi_1,p_1,q_1}\| \le C\|\mu|\ell_{q_1}\|$. So if the embedding (4.1) holds, then

$$\begin{aligned}
\sum_{j=0}^{\infty} 2^{j(s_2-s_1)q_2}\alpha_j^{q_2}|\mu_j|^{q_2} &= \sum_{j=0}^{\infty} 2^{js_2q_2}\varphi_2(2^{-\nu_j})^{q_2}2^{q_2(\nu_j-j)\frac{d}{p_2}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu_j,0}}} |\lambda_{j,m}|^{p_2}\Bigg)^{\frac{q_2}{p_2}}\\
&\le \|\lambda|n^{s_2}_{\varphi_2,p_2,q_2}\|^{q_2} \le C\|\lambda|n^{s_1}_{\varphi_1,p_1,q_1}\|^{q_2} \le C.
\end{aligned}$$

Any element of the sequence $\big(2^{j(s_2-s_1)}\alpha_j\big)_j$ is positive therefore

$$\begin{aligned}
\|2^{j(s_2-s_1)}\alpha_j|\ell_{q^*}\|^{q_2} &= \|(2^{j(s_2-s_1)}\alpha_j)^{q_2}|\ell_{q^*/q_2}\|\\
&= \sup_{\|\varkappa|\ell_{(q^*/q_2)'}\|=1;\varkappa_j\ge 0} \sum_{j=0}^{\infty} 2^{j(s_2-s_1)q_2}\alpha_j^{q_2}\varkappa_j\\
&\le \sup_{\|\mu|\ell_{q_1}\|=1} \sum_{j=0}^{\infty} 2^{j(s_2-s_1)q_2}\alpha_j^{q_2}|\mu_j|^{q_2} \le C,
\end{aligned}$$

since $(\frac{q^*}{q_2})' = \frac{q_1}{q_2}$.

Substep 4.2 We deal now with the case $p_1 < p_2$, i.e., $\varrho = \frac{p_1}{p_2}$. Let $q_1 \le q_2$, i.e., $q^* = \infty$, and $i \in \mathbb{N}$. Consider the construction explained in Substep 3.2, with j_0 and ν_0 replaced by i and ν_i, respectively, where ν_i satisfies (4.23). Moreover, let $N_i = \lceil 2^{(i-\nu_i)d}\varphi_1(2^{-\nu_i})^{-p_1}\varphi_1(2^{-i})^{p_1}\rceil$ and let $\lambda^{(N_i)}$ be the sequence described in the above mentioned Substep 3.2. Define the sequence $\beta^{(i)}$ by

$$\beta^{(i)}_{j,m} = \begin{cases} 2^{-is_1}\alpha_i\varphi_2(2^{-\nu_i})^{-1}\varphi_1(2^{-\nu_i})^{\rho}\varphi_1(2^{-i})^{-1}\lambda^{(N_i)}_{j,m} & \text{if } j=i \text{ and } Q_{i,m}\subset Q_{\nu_i,0}, \\ 0 & \text{otherwise.}\end{cases} \tag{4.28}$$

We prove that there is a positive $C > 0$ such that $\|\beta^{(i)}|n^{s_1}_{\varphi_1,p_1,q_1}\| \le C$ for any i.

By using (4.23) and the fact that $\varphi_1 \in \mathcal{G}_{p_1}$, we have, in case of $\nu_i \le \nu \le i$, that

$$\begin{aligned}\varphi_1(2^{-\nu})2^{(\nu-i)\frac{d}{p_1}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{i,m}\subset Q_{\nu,k}}}|\beta^{(i)}_{i,m}|^{p_1}\Bigg)^{\frac{1}{p_1}} &\le 2\frac{\varphi_1(2^{-\nu})}{\varphi_1(2^{-i})}2^{-is_1}2^{(\nu-i)\frac{d}{p_1}}\left(2^{-d(\nu-\nu_i)}N_i + \frac{1}{1-2^{-d}}\right)^{\frac{1}{p_1}} \\ &\le C2^{-is_1},\end{aligned}$$

and, in case of $\nu < \nu_i$,

$$\varphi_1(2^{-\nu})2^{(\nu-i)\frac{d}{p_1}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{i,m}\subset Q_{\nu,k}}}|\beta^{(i)}_{i,m}|^{p_1}\Bigg)^{\frac{1}{p_1}} \le 2\frac{\varphi_1(2^{-\nu})}{\varphi_1(2^{-i})}2^{-is_1}2^{(\nu-i)\frac{d}{p_1}}N_i^{\frac{1}{p_1}} \le C2^{-is_1}.$$

The above inequalities show that $\|\beta^{(i)}|n^{s_1}_{\varphi_1,p_1,q_1}\| \le C$. So, if the embedding (4.1) holds, then

$$\begin{aligned}C \ge \|\beta^{(i)}|n^{s_2}_{\varphi_2,p_2,q_2}\| &\ge 2^{s_2 i}\varphi_2(2^{-\nu_i})2^{(\nu_i-i)\frac{d}{p_2}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{i,m}\subset Q_{\nu_i,0}}}|\beta^{(i)}_{i,m}|^{p_2}\Bigg)^{\frac{1}{p_2}} \\ &\ge 2^{(s_2-s_1)i}2^{(\nu_i-i)\frac{d}{p_2}}\alpha_i N_i^{\frac{1}{p_2}} \ge 2^{(s_2-s_1)i}\alpha_i\frac{\varphi_1(2^{-i})^{\varrho}}{\varphi_1(2^{-i})}.\end{aligned}$$

Thus $\left\|\left\{2^{(s_2-s_1)i}\alpha_i\frac{\varphi_1(2^{-i})^{\varrho}}{\varphi_1(2^{-i})}\right\}_i|\ell_\infty\right\| \le C$.

Now let $q_2 < q_1$, i.e., $q^* < \infty$. Let $\mu = (\mu_j)_j \in \ell_{q_1}$ and consider the sequence $\beta = (\beta_{j,m})$ defined by

$$\beta_{j,m} = \begin{cases} 2^{-js_1}\alpha_j\varphi_2(2^{-\nu_j})^{-1}\varphi_1(2^{-\nu_j})^{\rho}\varphi_1(2^{-j})^{-1}\lambda^{(N_j)}_{j,m}\,\mu_j & \text{if} \quad Q_{j,m}\subset Q_{\nu_j,0} \\ 0 & \text{otherwise.}\end{cases}$$

In the same way as above we show that $\|\beta|n^{s_1}_{\varphi_1,p_1,q_1}\| \le C\|\mu|\ell_{q_1}\|$. So if the embedding (4.1) holds, then

$$\begin{aligned}\sum_{j=0}^{\infty}2^{j(s_2-s_1)q_2}\alpha_j^{q_2}\frac{\varphi_1(2^{-j})^{\varrho q_2}}{\varphi_1(2^{-j})^{q_2}}|\mu_j|^{q_2} &= \sum_{j=0}^{\infty}2^{js_2q_2}\varphi_2(2^{-\nu_j})^{q_2}2^{q_2(\nu_j-j)\frac{d}{p_2}}\Bigg(\sum_{\substack{m\in\mathbb{Z}^d:\\ Q_{j,m}\subset Q_{\nu_j,0}}}|\beta_{j,m}|^{p_2}\Bigg)^{\frac{q_2}{p_2}} \\ &\le \|\beta|n^{s_2}_{\varphi_2,p_2,q_2}\|^{q_2} \le C.\end{aligned}$$

Thus

$$\left\{2^{j(s_2-s_1)q_2}\alpha_j^{q_2}\frac{\varphi_1(2^{-j})^{\varrho q_2}}{\varphi_1(2^{-j})^{q_2}}\mu_j{}^{q_2}\right\}_j \in \ell_1 \qquad \text{for all } (\mu_j)_j \in \ell_{q_1}$$

which is equivalent to

$$\left\{2^{j(s_2-s_1)q_2}\alpha_j^{q_2}\frac{\varphi_1(2^{-j})^{\varrho q_2}}{\varphi_1(2^{-j})^{q_2}}\eta_j\right\}_j \in \ell_1 \quad \text{for all } (\eta_j)_j \in \ell_{r_1}$$

with $r_1 = \frac{q_1}{q_2}$. But this implies that

$$\left\{ 2^{j(s_2-s_1)q_2} \alpha_j^{q_2} \frac{\varphi_1(2^{-j})^{\varrho q_2}}{\varphi_1(2^{-j})^{q_2}} \right\}_j \in \ell_{r_1'}$$

which means that

$$\left\{ 2^{j(s_2-s_1)} \alpha_j \frac{\varphi_1(2^{-j})^{\varrho}}{\varphi_1(2^{-j})} \right\}_j \in \ell_{q^*}.$$

Step 5. It remains to prove the non-compactness of (4.1). This follows by the same method as in [7, Thm. 3.2], i.e., we can take a sequence $\lambda^{(\mu)} = \{\lambda_{j,m}^{(\mu)}\}_{j,m}$, $\mu \in \mathbb{N}$,

$$\lambda_{j,m}^{(\mu)} = \begin{cases} 1, & \text{if} \quad j = 0 \text{ and } \quad m = (\mu, 0, 0, \ldots, 0), \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|\lambda^{(\mu)}|n^{s_1}_{\varphi_1,p_1,q_1}\| = 1$ and $\|\lambda^{(\mu_1)} - \lambda^{(\mu_2)}|n^{s_2}_{\varphi_2,p_2,q_2}\| \geq 1$ if $\mu_1 \neq \mu_2$. □

Remark 4.2. Following the above proof one observes, that $n^{s_1}_{\varphi_1,p_1,q_1} \hookrightarrow n^{s_2}_{\varphi_2,p_2,q_2}$ if and only if $\tilde{n}^{s_1}_{\varphi_1,p_1,q_1} \hookrightarrow \tilde{n}^{s_2}_{\varphi_2,p_2,q_2}$, where the latter spaces have been introduced in Section 3. The first term in (3.3) can just be treated as the term with $j = 0$ in the argument above.

Example 4.3. We explicate Theorem 4.1 for a few settings as mentioned in Example 2.5.

(a) In the particular case of $\varphi_i(t) = t^{\frac{d}{u_i}}$, $0 < p_i \leq u_i < \infty$, $i = 1, 2$, condition (4.2) means $\frac{1}{u_2} \leq \frac{1}{u_1} \min(1, \frac{p_1}{p_2})$ that is equivalent to

$$u_1 \leq u_2 \qquad \text{and} \qquad \frac{p_2}{u_2} \leq \frac{p_1}{u_1}. \tag{4.29}$$

Moreover, since

$$\left\{ 2^{j(s_2-s_1)} \frac{\varphi_2(2^{-j})}{\varphi_1(2^{-j})} \right\}_j = \left\{ 2^{j(s_2-s_1+\frac{d}{u_1}-\frac{d}{u_2})} \right\}_j ,$$

we recover exactly the conditions for classical Besov–Morrey sequence spaces in Theorem 3.2 of [7].

(b) Besides the 'classical' example given above, we consider the functions φ_{u_i,v_i} defined by (2.3), $0 < u_i, v_i < \infty$, $i = 1, 2$. Now one can easily calculate that condition (4.2) is equivalent to $\frac{v_1}{v_2} \leq \varrho$ and the condition (4.3) is equivalent to

$$\begin{cases} \frac{s_1-s_2}{d} > \max\left\{0, \frac{1}{u_1} - \frac{1}{u_2}, \frac{p_1}{u_1}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right\}, & \text{if} \quad q_1 > q_2, \\ \frac{s_1-s_2}{d} \geq \max\left\{0, \frac{1}{u_1} - \frac{1}{u_2}, \frac{p_1}{u_1}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right\}, & \text{if} \quad q_1 \leq q_2. \end{cases} \tag{4.30}$$

Please note that (4.30) coincides with the conditions formulated in [8] for embeddings of Besov–Morrey spaces defined on bounded domains.

(c) Finally we return to the setting in Example 2.5(iii),

$$\varphi_i(t) = \begin{cases} t^{\frac{d}{u_i}}, & 0 < t < 1, \\ 1, & t \geq 1, \end{cases}$$

where $u_i \geq p_i$, $i = 0, 1$. Formally this can be seen as an extension of the previous example to $v_i = \infty$, $i = 1, 2$. Please note that the sequence spaces correspond via Theorem 3.1 to the local Besov–Morrey spaces, cf. Remark 2.7. Since $\sup_{t>1} \varphi_2(t)/\varphi_1(t)^{\varrho} = 1$, (4.2) is satisfied, thus it remains to deal with the condition (4.3), which leads to (4.30) again.

Next we collect a number of interesting and useful implications of Theorem 4.1.

Corollary 4.4. *Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \leq \infty$, and $\varphi_i \in \mathcal{G}_{p_i}$, for $i = 1, 2$. Let $\varphi_1(1) = \varphi_2(1) = 1$ and $\varrho = \min(1, \frac{p_1}{p_2})$. We assume that the sequence $\alpha_j = \sup_{\nu \leq j} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^\varrho}$ converges to some $\alpha \geq 1$ and that (4.2) is satisfied.*

(i) *If $p_1 \geq p_2$, then the embedding (4.1) is continuous if and only if $s_1 > s_2$ or $s_1 = s_2$ and $q_1 \leq q_2$.*
(ii) *If $p_1 < p_2$, then the embedding (4.1) is continuous if and only if*

$$\left\{2^{j(s_2-s_1)} \varphi_1(2^{-j})^{\frac{p_1}{p_2}-1}\right\}_j \in \ell_{q^*} \quad \text{where} \quad \frac{1}{q^*} = \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+ . \tag{4.31}$$

We recall that $b^s_{p,q}$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$, denote the classical Besov sequence spaces, cf. Remark 2.9. Now we extend the above definition to the case $p = \infty$.

We have the following observation from Lemma 2.12.

Corollary 4.5. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$.*

(i) *If $\inf_{t>0} \varphi(t) > 0$, then $n^s_{\varphi,p,q} \hookrightarrow b^s_{\infty,q}$.*
(ii) *If $\sup_{t>0} \varphi(t) < \infty$, then $b^s_{\infty,q} \hookrightarrow n^s_{\varphi,p,q}$.*

In particular, if $0 < \inf_{t>0} \varphi(t) \leq \sup_{t>0} \varphi(t) < \infty$, then $n^s_{\varphi,p,q} = b^s_{\infty,q}$ (in the sense of equivalent norms).

Remark 4.6. This can be seen as some sequence space counterpart of Remark 2.2.

Corollary 4.7. *Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \leq \infty$, and $\varphi_i \in \mathcal{G}_{p_i}$, for $i = 1, 2$. Then*

$$n^{s_1}_{\varphi_1,p_1,q_1} = n^{s_2}_{\varphi_2,p_2,q_2} \qquad \text{(in the sense of equivalent norms)} \tag{4.32}$$

if and only if

$$s_1 = s_2 \qquad \text{and} \qquad q_1 = q_2, \tag{4.33}$$

and one of the two conditions

$$0 < \inf_{t>0} \varphi_i(t) \leq \sup_{t>0} \varphi_i(t) < \infty, \quad i = 1, 2, \tag{4.34}$$

or

$$p_1 = p_2 \qquad \text{and} \qquad \varphi_1(t) \sim \varphi_2(t), \quad t > 0, \tag{4.35}$$

holds.

Proof. *Step 1.* Clearly (4.33) and (4.35) imply (4.32), but also (4.33) and (4.34) lead to (4.32) which can be seen as follows: either one checks directly the conditions (4.2) and (4.3), or one uses Corollary 4.5 and observes that (4.34) leads to $n^{s_1}_{\varphi_1,p_1,q_1} = b^{s_1}_{\infty,q_1}$ and $n^{s_2}_{\varphi_2,p_2,q_2} = b^{s_2}_{\infty,q_2}$. Hence (4.33) completes the proof of the sufficiency for (4.32).

Step 2. Now we deal with the necessity. Here we apply Theorem 4.1 twice, that is, for $n^{s_1}_{\varphi_1,p_1,q_1} \hookrightarrow n^{s_2}_{\varphi_2,p_2,q_2}$ and $n^{s_2}_{\varphi_2,p_2,q_2} \hookrightarrow n^{s_1}_{\varphi_1,p_1,q_1}$. Thus (4.2) in both cases leads to

$$\varphi_2(2^{-\nu}) \leq c\varphi_1(2^{-\nu})^{\min(1,\frac{p_1}{p_2})} \leq c'\varphi_2(2^{-\nu})^{\min(\frac{p_1}{p_2},\frac{p_2}{p_1})}, \quad \nu \leq 0,$$

such that $\varphi_2(2^{-\nu})^{1-\min(\frac{p_1}{p_2},\frac{p_2}{p_1})} \leq c''$, $\nu \leq 0$. This requires either $p_1 = p_2$ and thus $\varphi_1(t) \sim \varphi_2(t)$, $t \geq 1$, or $\sup_{t>0} \varphi_i(t) < \infty$, $i = 1, 2$.

If $p_1 = p_2$ and $\varphi_1(t) \sim \varphi_2(t)$, $t \geq 1$, then $\alpha_j \sim \max_{\nu=0,\dots,j} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})}$, likewise $\widetilde{\alpha}_j \sim \max_{\nu=0,\dots,j} \frac{\varphi_1(2^{-\nu})}{\varphi_2(2^{-\nu})}$. Thus (4.3) leads, in particular, to

$$\max_{\nu=0,\dots,j} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})} \leq c\, 2^{j(s_1-s_2)} \leq \min_{\nu=0,\dots,j} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})}$$

for all $j \in \mathbb{N}_0$, i.e., $\varphi_1(t) \sim \varphi_2(t)$, $0 < t \leq 1$, $s_1 = s_2$, which finally implies $q_1 = q_2$, as desired.

Assume now $\sup_{t>0} \varphi_i(t) < \infty$, where we may restrict ourselves to the case $p_1 \neq p_2$. It is sufficient to show that there appears a contradiction if (4.34) is not satisfied, as then – again in view of Corollary 4.5–(4.34) and (4.32) yield $b^{s_1}_{\infty,q_1} = b^{s_2}_{\infty,q_2}$ which is known to imply (4.33) finally. So let us assume $p_1 < p_2$, hence $\min(1, \frac{p_1}{p_2}) = \frac{p_1}{p_2}$, $\min(1, \frac{p_2}{p_1}) = 1$. Let $\varepsilon > 0$, then there exists some $j_0 = j_0(\varepsilon) \in \mathbb{N}$ such that $\varphi_1(2^{-j})^{\frac{p_1}{p_2}-1} \geq (\inf_{t>0} \varphi_1(t) + \varepsilon)^{\frac{p_1}{p_2}-1} \geq c > 0$ for $j \geq j_0$. If (4.34) is not satisfied, then at least one of the sequences $(\alpha_j)_j$ or $(\widetilde{\alpha}_j)_j$ diverges,

$$\alpha_j = \sup_{\nu \leq j} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^{\frac{p_1}{p_2}}} \xrightarrow[j\to\infty]{} \infty \qquad \text{or} \qquad \widetilde{\alpha}_j = \sup_{\nu \leq j} \frac{\varphi_1(2^{-\nu})}{\varphi_2(2^{-\nu})} \xrightarrow[j\to\infty]{} \infty.$$

Let us assume that $\varphi_2(t) \to 0$ for $t \to 0$. Then $\alpha_j \geq 1$, $j \in \mathbb{N}_0$, and $\widetilde{\alpha}_j \to \infty$ for $j \to \infty$. Consequently (4.3) applied to $n^{s_1}_{\varphi_1,p_1,q_1} \hookrightarrow n^{s_2}_{\varphi_2,p_2,q_2}$ leads to $s_1 \geq s_2$, but the second embedding requires $s_2 > s_1$. This is a contradiction. □

Next we study the special situation when $\varphi_1 = \varphi_2 = \varphi$.

Corollary 4.8. *Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \leq \infty$ for $i = 1, 2$, $\varphi \in \mathcal{G}_{\max(p_1,p_2)}$, and $\frac{1}{q^*} = (\frac{1}{q_2} - \frac{1}{q_1})_+$.*

(i) *Let $p_1 \geq p_2$. Then*

$$n^{s_1}_{\varphi,p_1,q_1} \hookrightarrow n^{s_2}_{\varphi,p_2,q_2} \tag{4.36}$$

if and only if

$$\begin{cases} s_1 > s_2, & \text{if } q_1 > q_2, \\ s_1 \geq s_2, & \text{if } q_1 \leq q_2. \end{cases} \tag{4.37}$$

(ii) *Let $p_1 < p_2$. Then*

$$n^{s_1}_{\varphi,p_1,q_1} \hookrightarrow n^{s_2}_{\varphi,p_2,q_2} \tag{4.38}$$

if and only if

$$\sup_{t>0} \varphi(t) < \infty \qquad \text{and} \qquad \left\{ 2^{j(s_2-s_1)} \varphi(2^{-j})^{\frac{p_1}{p_2}-1} \right\}_j \in \ell_{q^*}\ . \tag{4.39}$$

Proof. If $p_1 \geq p_2$, then (using the notation of Theorem 4.1) $\varrho = 1$ and $\alpha_j \equiv 1$, and (4.2) is automatically satisfied. Moreover, (4.3) reduces to the question whether $\{2^{j(s_2-s_1)}\}_j \in \ell_{q^*}$, i.e., (4.37), which completes the proof of (i).

In case of $p_1 < p_2$, $\varrho = \frac{p_1}{p_2} < 1$ and (4.2) is obviously equivalent to the first condition in (4.39), that is, $\sup_{t>0}\ \varphi(t) < \infty$. Moreover, in that case $\alpha_j \sim 1$, such that (4.3) can be rewritten as the second part of (4.39). □

Remark 4.9. Note that a sufficient condition for φ in (ii) is – in addition to $\sup_{t>0}\ \varphi(t) < \infty$ – that

$$\frac{s_1 - s_2}{d} > \frac{p_1}{p_2} \left(\frac{1}{p_1} - \frac{1}{p_2} \right)$$

using the properties of $\varphi \in \mathcal{G}_{\max(p_1,p_2)} = \mathcal{G}_{p_2}$.

Example 4.10. We explicate Corollary 4.8 for some function φ. We restrict ourselves to the situation $p_1 < p_2$, i.e., Corollary 4.8(ii). Let

$$\varphi(t) = \begin{cases} t^{\frac{d}{p_2}} \left(1 + |\log t|\right)^a, & 0 < t < 1, \\ 1, & t \geq 1, \end{cases}$$

where $a \in \mathbb{R}$. Since $\sup_{t>0} \varphi(t) < \infty$, we deal with the second condition in (4.39), which leads to

$$\begin{cases} \frac{s_1 - s_2}{d} > \frac{p_1}{p_2} \left(\frac{1}{p_1} - \frac{1}{p_2}\right), & & \text{or,} \\ \frac{s_1 - s_2}{d} = \frac{p_1}{p_2} \left(\frac{1}{p_1} - \frac{1}{p_2}\right) & \text{and} \quad a \geq 0, & \text{if} \quad q_1 \leq q_2, \quad \text{or,} \\ \frac{s_1 - s_2}{d} = \frac{p_1}{p_2} \left(\frac{1}{p_1} - \frac{1}{p_2}\right) & \text{and} \quad a > \frac{1}{q^*} \frac{p_2}{p_2 - p_1}, & \text{if} \quad q_1 > q_2. \end{cases}$$

We used that in this case

$$\left\{2^{j(s_2-s_1)} \varphi(2^{-j})^{\frac{p_1}{p_2}-1}\right\}_j = \left\{2^{j(s_2 - s_1 - d\frac{p_1}{p_2}(\frac{1}{p_1} - \frac{1}{p_2}))} (1+j)^{a(\frac{p_1}{p_2}-1)}\right\}_j .$$

Now we focus on embeddings where either the target or the source space is a Besov sequence space, for what we recall that

$$b^s_{p,q} = n^s_{p,p,q}, \quad 0 < p < \infty, \quad 0 < q \leq \infty, \quad s \in \mathbb{R}.$$

Corollary 4.11. *Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \leq \infty$ for $i = 1, 2$, and $\varphi_1 \in \mathcal{G}_{p_1}$. Denote again $\frac{1}{q^*} = (\frac{1}{q_2} - \frac{1}{q_1})_+$. Then*

$$n^{s_1}_{\varphi_1, p_1, q_1} \hookrightarrow b^{s_2}_{p_2, q_2} \tag{4.40}$$

if and only if

$$p_1 \leq p_2, \qquad \varphi_1(t) \sim t^{\frac{d}{p_1}}, \ t \geq 1, \qquad \text{and} \quad \left\{2^{j(s_2-s_1)} \varphi_1(2^{-j})^{\frac{p_1}{p_2}-1}\right\}_j \in \ell_{q^*}. \tag{4.41}$$

Proof. We apply Theorem 4.1 (and its notation) with $\varphi_2(t) = t^{\frac{d}{p_2}}$, $t > 0$. Thus (4.2) is equivalent to

$$\varphi_1(t) \geq \ c \, t^{\frac{d}{p_2}\frac{1}{\varrho}}, \quad t \geq 1.$$

On the other hand, $\varphi_1 \in \mathcal{G}_{p_1}$ implies $\varphi_1(t) \leq t^{\frac{d}{p_1}}$, $t \geq 1$, hence this results in $p_1 \leq p_2$ and $\varphi_1(t) \sim t^{\frac{d}{p_1}}$, $t \geq 1$, which is the first part of (4.41). We concentrate on (4.3) and observe that α_j is bounded, since

$$1 \leq \alpha_j = \sup_{\nu \leq j} \varphi_1(2^{-\nu})^{-\frac{p_1}{p_2}} 2^{-\nu\frac{d}{p_2}} \sim \max_{\nu=0,\dots,j} \varphi_1(2^{-\nu})^{-\frac{p_1}{p_2}} 2^{-\nu\frac{d}{p_2}} \leq \max_{\nu=0,\dots,j} 2^{\nu\frac{d}{p_2}} 2^{-\nu\frac{d}{p_2}} = 1,$$

where we used again $\varphi_1 \in \mathcal{G}_{p_1}$, this time leading to $\varphi_1(2^{-\nu}) \geq 2^{-\nu\frac{d}{p_1}}$, $\nu \in \mathbb{N}_0$. Thus (4.3) corresponds to the second part in (4.41). ☐

Example 4.12. We illustrate Corollary 4.11 for φ_1 given by

$$\varphi_1(t) = \begin{cases} t^{d/u} & \text{if} \quad t \leq 1, \\ t^{d/p_1} & \text{if} \quad t > 1, \end{cases}$$

with $p_1 \leq u < \infty$, a special case of (2.3). It turns out that in such a case (4.40) holds if and only if $p_1 \leq p_2$ and

$$\begin{cases} \frac{s_1 - s_2}{d} > \frac{p_1}{u} \left(\frac{1}{p_1} - \frac{1}{p_2}\right), & \text{if} \quad q_1 > q_2, \quad \text{or,} \\ \frac{s_1 - s_2}{d} \geq \frac{p_1}{u} \left(\frac{1}{p_1} - \frac{1}{p_2}\right), & \text{if} \quad q_1 \leq q_2. \end{cases}$$

When $u = p_1$ this is the sequence space counterpart of [7, Cor. 3.7].

Corollary 4.13. *Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \le \infty$ for $i = 1,2$, and $\varphi_2 \in \mathcal{G}_{p_2}$. Denote again $\frac{1}{q^*} = (\frac{1}{q_2} - \frac{1}{q_1})_+$.*

(i) *Let $p_1 \le p_2$. Then*

$$b^{s_1}_{p_1,q_1} \hookrightarrow n^{s_2}_{\varphi_2,p_2,q_2}$$

if and only if

$$\{2^{j(s_2-s_1+\frac{d}{p_1})}\varphi_2(2^{-j})\} \in \ell_{q^*}. \tag{4.42}$$

(ii) *Let $p_1 > p_2$. Then*

$$b^{s_1}_{p_1,q_1} \hookrightarrow n^{s_2}_{\varphi_2,p_2,q_2}$$

if and only if

$$\sup_{t\ge 1} t^{-\frac{d}{p_1}}\varphi_2(t) < \infty, \tag{4.43}$$

and

$$\left\{2^{j(s_2-s_1)} \sup_{0\le\nu\le j} 2^{\nu\frac{d}{p_1}}\varphi_2(2^{-\nu})\right\}_j \in \ell_{q^*}. \tag{4.44}$$

Proof. Part (ii) exactly corresponds to Theorem 4.1 with $\varphi_1(t) = t^{\frac{d}{p_1}}$, $t > 0$, and $\varrho = 1$. As for part (i), now with $\varrho = \frac{p_1}{p_2}$, (4.2) reads as

$$\sup_{\nu\le 0} 2^{\nu\frac{d}{p_2}}\varphi_2(2^{-\nu}) \le c,$$

but this is always true in view of $\varphi_2 \in \mathcal{G}_{p_2}$. By the same argument,

$$\alpha_j \sim \max_{\nu=0,\dots,j} 2^{\nu\frac{d}{p_2}}\varphi_2(2^{-\nu}) = \varphi_2(2^{-j})2^{j\frac{d}{p_2}},$$

which leads to

$$2^{j(s_2-s_1)}\alpha_j\varphi_1(2^{-j})^{\varrho-1} = 2^{j(s_2-s_1+\frac{d}{p_2})}\varphi_2(2^{-j})2^{-j\frac{d}{p_1}(\frac{p_1}{p_2}-1)} = 2^{j(s_2-s_1+\frac{d}{p_1})}\varphi_2(2^{-j}),$$

such that (4.3) coincides with (4.42). □

Example 4.14. We consider a model function for part (ii), i.e., when $p_1 > p_2$. Recall that in this case there is no continuous embedding for classical Besov-(Morrey) spaces on $\mathbb{R}^d$. Let

$$\varphi_2(t) = \begin{cases} t^{\frac{d}{u_1}}, & t \ge 1, \\ t^{\frac{d}{u_2}}, & 0 < t \le 1, \end{cases}$$

where $u_1 \ge p_1$ and $u_2 \ge p_2$. We may even admit $u_1 = \infty$ with the understanding that $\varphi_2(t) = 1$ for $t \ge 1$. Then $\varphi_2 \in \mathcal{G}_{p_2}$, (4.43) is satisfied, and (4.44) leads to

$$\begin{cases} \frac{s_1-s_2}{d} \ge \left(\frac{1}{p_1} - \frac{1}{u_2}\right)_+ & \text{if} \quad q_1 \le q_2, \\ \frac{s_1-s_2}{d} > \left(\frac{1}{p_1} - \frac{1}{u_2}\right)_+ & \text{if} \quad q_1 > q_2. \end{cases}$$

In particular, $u_1 = u_2 = p_1$ is admitted, such that $\varphi_1(t) = \varphi_2(t) = t^{\frac{d}{p_1}}$ then and we recover our result from Corollary 4.8(i) for this case.

5. Embeddings of generalised Besov–Morrey function spaces

Now we deal with embeddings of the function spaces. We benefit from our sequence space result Theorem 4.1 and the wavelet characterisation of the function spaces, cf. Theorem 3.1. The following statement is the immediate consequence of the just mentioned theorems, recall also Remark 4.2.

Theorem 5.1. *Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \le \infty$, and $\varphi_i \in \mathcal{G}_{p_i}$, for $i = 1, 2$. We assume without loss of generality that $\varphi_1(1) = \varphi_2(1) = 1$.*

There is a continuous embedding

$$\mathcal{N}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d) \tag{5.1}$$

if and only if (4.2) and (4.3) *are satisfied, using the notation of Theorem 4.1.*

The embedding (5.1) is never compact.

Remark 5.2. If $\varphi_i(t) = t^{\frac{d}{u_i}}$, $i = 1, 2$, then Theorem 5.1 coincides with [7, Theorem 3.3].

Corollary 5.3. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, and $\varphi \in \mathcal{G}_r$, $r = \frac{p^2}{p_1}$. If $0 < p_1 \le p$, then*

$$\mathcal{N}^{s+\frac{d}{p}}_{p,p_1,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{s}_{\varphi,p,q}(\mathbb{R}^d).$$

Proof. The statement follows directly from Theorem 5.1 with $\varphi_1(t) = t^{\frac{d}{p}}$ and $\varphi_2(t) = \varphi(t)$. □

Now we collect further consequences of Theorem 5.1 parallel to our approach in Section 4.

Corollary 5.4. *Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \le \infty$, and $\varphi_i \in \mathcal{G}_{p_i}$, for $i = 1, 2$. Then*

$$\mathcal{N}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) = \mathcal{N}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d) \qquad \text{(in the sense of equivalent norms)} \tag{5.2}$$

if and only if we have the equalities (4.33) and one of the two conditions (4.34) or (4.35) holds.

Proof. This is the function space version of Corollary 4.7. □

Corollary 5.5. *Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \le \infty$ for $i = 1, 2$, $\varphi \in \mathcal{G}_{\max(p_1,p_2)}$, and $\frac{1}{q^*} = (\frac{1}{q_2} - \frac{1}{q_1})_+$. Then*

$$\mathcal{N}^{s_1}_{\varphi,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{s_2}_{\varphi,p_2,q_2}(\mathbb{R}^d) \tag{5.3}$$

if and only if $p_1 \ge p_2$ and (4.37) holds or $p_1 < p_2$ and (4.39) holds.

Proof. This is the counterpart for function spaces of Corollary 4.8. □

Corollary 5.6. *Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \le \infty$ for $i = 1, 2$, and $\varphi_1 \in \mathcal{G}_{p_1}$. Denote again $\frac{1}{q^*} = (\frac{1}{q_2} - \frac{1}{q_1})_+$. Then*

$$\mathcal{N}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^d)$$

if and only if the conditions (4.41) hold.

Proof. This corresponds to Corollary 4.11. □

In combination with the well-known embedding $B^0_{r,1}(\mathbb{R}^d) \hookrightarrow L_r(\mathbb{R}^d)$, $1 \le r < \infty$, we thus obtain from Corollary 5.6 the following result.

Corollary 5.7. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, $\varphi \in \mathcal{G}_p$, with $\varphi(t) \sim t^{\frac{d}{p}}$, $t \ge 1$. Assume $1 \le r < \infty$ with $r \ge p$, and let $\frac{1}{q'} = (1 - \frac{1}{q})_+$. Then*

$$\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) \hookrightarrow L_r(\mathbb{R}^d)$$

if $\left\{2^{-js}\varphi(2^{-j})^{\frac{p}{r}-1}\right\}_j \in \ell_{q'}$.

Finally we return to the situation studied in Corollary 4.13.

Corollary 5.8. *Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \le \infty$ for $i = 1, 2$, and $\varphi_2 \in \mathcal{G}_{p_2}$. Denote again $\frac{1}{q^*} = (\frac{1}{q_2} - \frac{1}{q_1})_+$. Then*

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$$

if and only if

(i) *$p_1 \le p_2$ and the condition (4.42) holds*
or
(ii) *$p_1 > p_2$ and the conditions (4.43)–(4.44) hold.*

Remark 5.9. Obviously one can also explicate Theorem 5.1 for the example functions, similar to Examples 4.3, 4.10 etc. For instance, we can prove that the formula (4.30) gives sufficient and necessary conditions for the embedding of two local Besov–Morrey spaces.

In the end we study some endpoint situations in Corollaries 5.6–5.8, recall also the sequence space counterpart in Corollary 4.5. We begin with an extension of Corollary 5.7 to $r = \infty$. Recall our notation $\frac{1}{q'} = (1 - \frac{1}{q})_+$ for $0 < q \le \infty$.

Corollary 5.10. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, and $\varphi \in \mathcal{G}_p$. Assume that*

$$\left\{2^{-js}\varphi(2^{-j})^{-1}\right\}_{j\in\mathbb{N}_0} \in \ell_{q'} \ . \tag{5.4}$$

Then

$$\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d).$$

Proof. We apply Theorem 5.1 with $\varphi_1 = \varphi$, $\varphi_2 \equiv 1$, $s_1 = s$, $s_2 = 0$, $p_1 = p_2 = p$, $q_1 = q$, $q_2 = 1$ and hence $q^* = q'$. Thus, in view of (2.2),

$$\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) \hookrightarrow B^0_{\infty,1}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d),$$

where the latter embedding is well-known. □

Remark 5.11. In case of $\varphi(t) = t^{\frac{d}{u}}$, $0 < p \le u < \infty$, (5.4) reads as

$$\begin{cases} s > \frac{d}{u}, & \text{if} \quad 1 < q \le \infty, \\ s \ge \frac{d}{u}, & \text{if} \quad 0 < q \le 1, \end{cases}$$

and this is even known to be also necessary for the embedding $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$, cf. [8, Prop. 5.5].

Next we return to Sawano's observation that for $\inf_{t>0}\varphi(t) > 0$, then $\mathcal{M}_{\varphi,p}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$, while $\sup_{t>0}\varphi(t) < \infty$ implies $L_\infty(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\varphi,p}(\mathbb{R}^d)$, leading as a special case to (2.2), cf. [24]. For convenience, let us denote these conditions by

$$\begin{aligned} &(\mathbf{I}) \qquad \inf_{t>0}\varphi(t) > 0, \qquad \text{and} \\ &(\mathbf{S}) \qquad \sup_{t>0}\varphi(t) < \infty. \end{aligned}$$

Corollary 5.12. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, and $\varphi \in \mathcal{G}_p$.*

(i) *If φ satisfies $(\mathbf{I})$, then*

$$\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) \hookrightarrow B^s_{\infty,q}(\mathbb{R}^d).$$

(ii) *If φ satisfies $(\mathbf{S})$, then*

$$B^s_{\infty,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d).$$

Hence if φ satisfies $(\mathbf{I})$ and $(\mathbf{S})$, i.e., $\varphi \sim 1$, then

$$\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) = B^s_{\infty,q}(\mathbb{R}^d)$$

(in the sense of equivalent norms).

Proof. The first statement is also a consequence of Theorem 5.1 with $\varphi_1(t) = \varphi(t)$ and $\varphi_2(t) = \varphi_0(t) \equiv 1$, since $\mathcal{M}_{\varphi_0,p}(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$, recall (2.2). Moreover if $\sup_{t>0}\varphi(t) < \infty$, then $L_\infty(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\varphi,p}(\mathbb{R}^d)$. This implies the second embedding. □

We conclude our paper with a closer look on the consequences of $(\mathbf{I})$ and $(\mathbf{S})$ for the standard embedding

$$\mathcal{N}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d). \tag{5.5}$$

Corollary 5.13. *Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \le \infty$, and $\varphi_i \in \mathcal{G}_{p_i}$, for $i = 1, 2$. We assume without loss of generality that $\varphi_1(1) = \varphi_2(1) = 1$.*

- **(i)** *Assume that φ_1 satisfies $(\mathbf{I})$. Then (5.5) holds if and only if (4.2) is satisfied and $\{2^{j(s_2-s_1)}\}_j \in \ell_{q^*}$.*
- **(ii)** *Assume that φ_2 satisfies $(\mathbf{I})$ while φ_1 does not. Then (5.5) holds if and only if (4.2) is satisfied and $\{2^{j(s_2-s_1)}\varphi_1(2^{-j})^{-1}\}_j \in \ell_{q^*}$.*
- **(iii)** *Assume that φ_2 satisfies $(\mathbf{S})$ while φ_1 does not satisfy $(\mathbf{I})$. Then (5.5) holds if and only if (4.3) holds. In particular (5.5) holds if $\{2^{j(s_2-s_1)}\varphi_1(2^{-j})^{-1}\}_j \in \ell_{q^*}$.*
- **(iv)** *Assume that φ_1 satisfies $(\mathbf{S})$. Then (5.5) holds if and only if also φ_2 satisfies $(\mathbf{S})$ and (4.3) holds.*

Proof. We begin with (i). In view of Theorem 5.1 it remains to verify that $(\mathbf{I})$ for φ_1 together with $\{2^{j(s_2-s_1)}\}_j \in \ell_{q^*}$ is equivalent to (4.3). However, since φ_2 is nondecreasing and φ_1 satisfies $(\mathbf{I})$, we get that $1 \le \alpha_j \le c\,\alpha_0$ and $\varphi_1(2^{-j})^{\varrho-1} \le c'$ such that (4.3) follows.

Next we deal with (ii). This time we need to show that the assumptions on φ_2 and $\{2^{j(s_2-s_1)}\varphi_1(2^{-j})^{-1}\}_j \in \ell_{q^*}$ is equivalent to (4.3). But using the boundedness of φ_2 and the monotonicity of φ_1 we obtain $c\varphi_1(2^{-j})^{-\varrho} \le \alpha_j \le \varphi_1(2^{-j})^{-\varrho}$ for sufficiently large j since φ_1 does not satisfy $(\mathbf{I})$. But this together with our assumption leads to (4.3).

First observe that the assumed boundedness of φ_2 from above in (iii) together with the boundedness of φ_1 from below already imply (4.2). The boundedness of $\alpha_j \le \varphi_1(2^{-j})^{-\varrho}$ follows in the same way as in (ii).

It remains to deal with (iv). If φ_2 satisfies $(\mathbf{S})$, then (4.2) is a consequence of $\varphi_1(2^{-\nu}) \ge \varphi_1(1) = 1$ for $\nu \le 0$. The rest follows by Theorem 5.1. □

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